Periodic solutions for a class of non-autonomous differential delay equations

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Abstract. By applying symplectic transformation, Floquet theory and some results in critical point theory, we establish the existence of periodic solutions for a class of non-autonomous differential delay equations, which can be changed to Hamiltonian systems.

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1. Introduction and main result

In 1974, Kaplan and Yorke [1] studied a class of differential delay equations

$$x'(t) = -f(x(t-1)) \quad (1.1)$$

and

$$x'(t) = -[f(x(t-1)) + f(x(t-2))], \quad (1.2)$$

where $f(x) \in C(\mathbb{R}, \mathbb{R})$ is odd, $xf(x) > 0$ for $x \neq 0$, and f satisfies some asymptotically linear conditions both at origin and at infinity. They proved that the Eqs. 1.1 and 1.2 have periodic solutions with period 4 and 6, respectively. They also conjectured that similar results should be true for (1.1) or (1.2) with more than two delays under similar conditions on f . This is called the Kaplan–Yorke’s conjecture.

Since then the two equations and their general cases have been studied by many authors through various methods. In 1978, Nussbaum [3] obtained the existence of a $2n$ -periodic solution for a class of differential delay equations which include the Eqs. 1.1 and 1.2 as two special cases, where n is a positive integer.

Thus, the Kaplan–Yorke’s conjecture was solved. But the method used in [3] is quite different from that in [1].

Recently many authors [4–9] made use of Kaplan and Yorke’s original idea to study some general differential delay equations by changing them into Hamiltonian systems. Li and He [4, 5] studied the existence of multiple periodic solutions of the following equation

$$x'(t) = -[f(x(t - r_1)) + f(x(t - r_2)) + \cdots + f(x(t - r_{n-1}))] \quad (1.3)$$

where r_i ($i = 1, 2, \dots, n - 1$) are positive constants. By variational methods they proved that the system (1.3) has multiple periodic solutions.

According to the variational methods used in [4, 5], Cima and his collaborators [6] applied qualitative theory of ordinary differential equations to study the following equation

$$x'(t) = -[f(x(t - r_1))g(x(t - r_2)) + f(x(t - r_2))g(x(t - r_1))].$$

They found a suitable condition such that the above equation has periodic solutions.

Then Llibre and Tarta [7] generalized the work in [6]. They considered the following system

$$\begin{aligned} x'(t) = & af(x(t - r_1))g(x(t - r_2)) + bf(x(t - r_1))g(x(t - r_3)) \\ & - af(x(t - r_2))g(x(t - r_1)) + cf(x(t - r_2))g(x(t - r_3)) \\ & - bf(x(t - r_3))g(x(t - r_1)) - cf(x(t - r_1))g(x(t - r_2)). \end{aligned}$$

They got a sufficient condition for the existence of periodic solutions of the system, where a, b, c take the values 1 or -1 .

When the delays r_i ($i = 1, 2, \dots, n - 1$) take positive integers, Jekel and Johnston [8] studied the following system

$$x'(t) = -[f(x(t - 1)) + f(x(t - 2)) + \cdots + f(x(t - (n - 1)))], \quad (1.4)$$

and by using some known results on the existence of closed orbits for Hamiltonian vector fields they showed that the Eq. 1.4 has a periodic solution. Almost at the same time, Fei [9] took a different approach from those employed in previous papers to study the existence of multiple periodic solutions for the Eq. 1.4. He applied the pseudo index theory established in [10] to obtain the multiple periodic solutions of (1.4) when $f(x)$ is asymptotically linear both at zero and at infinity with respect to x .

The previous work is mainly concerned with autonomous differential delay equations. In this paper, we study the existence of periodic solutions for a class of non-autonomous differential delay equations with the following form

$$x'(t) = -[f(t, x(t - \tau)) + f(t, x(t - 2\tau)) + \cdots + f(t, x(t - (n - 1)\tau))], \quad (1.5)$$

where $\tau = \pi/n$ and $f(t, x)$ is odd and asymptotically linear both at zero and at infinity with respect to x . We shall apply symplectic transformation and some well known results in critical point theory to obtain the existence of periodic solutions for the Eq. 1.5.

Before introducing our main results, we make the following assumptions.

- (H_1) $f(t, x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x and periodic in t with period τ .
 (H'_1) $f(t, x) \in C^1(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x and periodic in t with period τ .
 (H_2) For all $t \in [0, \tau]$,

$$\lim_{x \rightarrow 0} \frac{f(t, x)}{x} = \beta_0(t), \quad \lim_{x \rightarrow \infty} \frac{f(t, x)}{x} = \beta_\infty(t),$$

i.e., the two limits are uniformly in t . It is obvious that $\beta_0(t)$ and $\beta_\infty(t)$ are periodic functions with period τ .

Since there is much difference between $n = 2N$ ($N \in \mathbb{Z}^+$) and $n = 2N + 1$, we consider the two cases, respectively. In this paper, we always assume that $n = 2N$ is even. Let I_{2N} be the identity matrix in $\mathbb{R}^{2N}$ and $\alpha \in \mathbb{R}$. We denote the matrix αI_{2N} by α for convenience. Let A_{2N} be a $2N \times 2N$ skew symmetric matrix defined by

$$A_{2N} = \begin{pmatrix} 0 & -1 & \cdots & -1 & -1 \\ 1 & 0 & \cdots & -1 & -1 \\ 1 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Let $\sigma(iA_{2N}^{-1})$ denotes the spectrum of the matrix iA_{2N}^{-1} . Then from Lemma 4 of [5] and a direct computation, we have

$$\sigma(iA_{2N}^{-1}) = \left\{ \text{ctan} \frac{2l-1}{4N} \pi : l = 1, 2, \dots, 2N \right\}.$$

Write $\alpha_0 = \frac{1}{\tau} \int_0^\tau \beta_0(t) dt$ and $\alpha_\infty = \frac{1}{\tau} \int_0^\tau \beta_\infty(t) dt$. Then our main results read as follows.

Theorem 1.1. *Let (H_1) and (H_2) hold. Suppose that*

$$(\mathcal{H}_3) \quad \alpha_0 \neq k \text{ctan} \frac{2l-1}{4N} \pi, \quad \alpha_\infty \neq k \text{ctan} \frac{2l-1}{4N} \pi, \quad \text{for all } k \in \mathbb{Z}^+ \text{ and } 1 \leq l \leq 2N,$$

and

$$(\mathcal{H}_4) \quad \text{There exist at least an integer } l_0 \text{ with } 1 \leq l_0 \leq 2N \text{ and some } k_0 \in \mathbb{Z}^+ \text{ such that}$$

$$\min\{\alpha_0, \alpha_\infty\} < k_0 \text{ctan} \frac{2l_0-1}{4N} \pi < \max\{\alpha_0, \alpha_\infty\}.$$

Then the equation 1.5 has at least one nontrivial periodic solution.

Theorem 1.2. *Let (H'_1) and (H_2) hold. Suppose that*

(H_5) $\alpha_\infty \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$. There exist $k_i \in \mathbb{Z}^+$ and $l_i \in \mathbb{Z}^+$ with either $1 \leq l_1 < l_2 < \dots < l_{n_0} \leq N$ or $N+1 \leq l_1 < l_2 < \dots < l_{n_0} \leq 2N$ such that

$$\alpha_0 = k_1 \tan \frac{2l_1-1}{4N} \pi = k_2 \tan \frac{2l_2-1}{4N} \pi = \dots = k_{n_0} \tan \frac{2l_{n_0}-1}{4N} \pi,$$

where $1 \leq i \leq n_0$ and $1 \leq n_0 \leq N$.

and

(H_6) There are at least $n_0 + 1$ integers $m_1, m_2, \dots, m_{n_0+1}$ with $1 \leq m_1 < m_2 < \dots < m_{n_0+1} \leq 2N$ and $n_0 + 1$ integers $k_1, k_2, \dots, k_{n_0+1}$ such that

$$\max\{\alpha_0, \alpha_\infty\} \geq k_1 \tan \frac{2m_1-1}{4N} \pi > \dots > k_{n_0+1} \tan \frac{2m_{n_0+1}-1}{4N} \pi > \min\{\alpha_0, \alpha_\infty\}.$$

Then the equation 1.5 has at least one nontrivial periodic solution.

Remark 1.3. In the autonomous case as considered in [4, 5, 9], multiple periodic solutions are obtained. In non-autonomous case, we can only obtain single periodic solution since the functional is not S^1 -invariant.

Remark 1.4. As we pointed out above, Theorems 1.1 and 1.2 are concerned with the existence of periodic solutions for non-autonomous differential delay Eq. 1.5. Therefore, our results generalize the results obtained in the references.

2. Proof of the main results

We will prove our main results by several steps. We first transform the system (1.5) into an equivalent Hamiltonian system. Then we construct a symplectic transformation which reduces the linear parts of the Hamiltonian system to constant coefficients. Subsequently, some lemmas will be given. Finally, we use two critical point results and Galerkin approximation method to prove Theorems 1.1 and Theorem 1.2.

2.1. An equivalent Hamiltonian system

We first change the Eq. 1.5 to a form of Hamiltonian system. Suppose that $x(t)$ is a solution for the Eq. 1.5 satisfying $x(t) = -x(t - n\tau)$. Let

$$x_1(t) = x(t), \quad x_2(t) = x(t - \tau), \dots, x_n(t) = x(t - (n-1)\tau). \quad (2.1)$$

Then $X(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ is a solution of the following system

$$\frac{d}{dt} X(t) = A_{2N} F(t, X(t)), \quad (2.2)$$

where $F(t, X) = (f(t, x_1), f(t, x_2), \dots, f(t, x_n))^T$ and A_{2N} is defined in Sect. 1. Conversely, if a solution $X(t)$ of the system (2.2) has the following symmetric structure

$$x_1(t) = -x_n(t - \tau), x_2(t) = x_1(t - \tau), \dots, x_n(t) = x_{n-1}(t - \tau), \quad (2.3)$$

then $x(t) = x_1(t)$ is a solution of the Eq. 1.5 satisfying $x(t - n\tau) = -x(t)$.

Thus, solutions of the system (2.2) with the symmetric structure (2.3) give solutions of the differential delay equation (1.5). Hence, solving the Eq. 1.5 is equivalent to finding solutions of the Eq. 2.2 with the symmetric structure (2.3).

Since A_{2N} is a nonsingular skew symmetric Hamiltonian matrix, i.e., it satisfies $JA_{2N} + A_{2N}^\top J = 0$, where $J = \begin{pmatrix} 0 & -I_{2N} \\ I_{2N} & 0 \end{pmatrix}$ is the standard symplectic matrix, the system (2.2) can be written as the following Hamiltonian system

$$y'(t) = A_{2N} \nabla_y H(t, y), \quad (2.4)$$

where $H(t, y) = \int_0^{y_1} f(t, x) dx + \cdots + \int_0^{y_{2N}} f(t, x) dx$, $\forall y = (y_1, y_2, \dots, y_{2N})^\top \in \mathbb{R}^{2N}$, and $\nabla_y H(t, y)$ denotes the gradient of $H(t, y)$ with respect to y . The matrix A_{2N} is called the symplectic structure associated to the Hamiltonian system (2.4). Below, we turn to study the Hamiltonian system (2.4).

2.2. Reducibility of the linear parts

The problem now is equivalent to finding periodic solutions of the Hamiltonian system (2.4) with the symmetric structure (2.3). By the assumption (H_1) and (H_2) , the system (2.4) is asymptotically linear with respect to y both at origin and at infinity with coefficients of the linear parts depending on t . If we can find a global symplectic transformation in the sense of the symplectic structure A_{2N} which transforms (2.4) into an equivalent Hamiltonian system with constant coefficients of the linear parts both at zero and at infinity, then we can apply some well known results in [12, 13] to prove our theorems. Since the coefficients of the linear parts at origin and infinity may be different, the Floquet theory cannot be applied directly. We want to use Hamiltonian flow mapping to construct such global symplectic transformation. For this purpose, we only need to construct the corresponding Hamiltonian function.

By the assumption (H_2) , we have

$$f(t, x) = \begin{cases} \beta_0(t)x + o(|x|), & \text{as } |x| \rightarrow 0, \\ \beta_\infty(t)x + o(|x|), & \text{as } |x| \rightarrow \infty. \end{cases}$$

Then $H(t, y)$ is even with respect to y and satisfies

$$\nabla_y H(t, y) = \begin{cases} \beta_0(t)y + o(|y|), & \text{as } |y| \rightarrow 0, \\ \beta_\infty(t)y + o(|y|), & \text{as } |y| \rightarrow \infty. \end{cases}$$

And the corresponding Hamiltonian system (2.4) satisfies

$$y'(t) = A_{2N} \beta_0(t)y + o(|y|) \quad \text{as } |y| \rightarrow 0, \quad (2.5)$$

$$y'(t) = A_{2N} \beta_\infty(t)y + o(|y|) \quad \text{as } |y| \rightarrow \infty. \quad (2.6)$$

For the system (2.5), let $y = P_1(t, z)$, where $P_1(t, z) = e^{Q_1(t)} z$, $Q_1(t) = A_{2N} \int_0^t \gamma_0(\xi) d\xi$ and $\gamma_0(t) = \beta_0(t) - \alpha_0$. Then the transformation $y = P_1(t, z)$ is symplectic.

Then with the transformation $y = P_1(t, z)$ the system (2.5) is transformed into the following Hamiltonian system

$$z'(t) = \alpha_0 A_{2N} z + o(|z|) \quad \text{as } |z| \rightarrow 0. \quad (2.7)$$

Similarly, let $y = P_2(t, z)$, where $P_2(t, z) = e^{Q_2(t)} z$, $Q_2(t) = A_{2N} \int_0^t \gamma_\infty(\xi) d\xi$ and $\gamma_\infty(t) = \beta_\infty(t) - \alpha_\infty$. In the same way we have that the transformation $y = P_2(t, z)$ is also symplectic. Then the differential equation (2.6) can be transformed to the following Hamiltonian system

$$z'(t) = \alpha_\infty A_{2N} z + o(|z|) \quad \text{as } |z| \rightarrow \infty. \quad (2.8)$$

Noting that $Q_i(t)$ ($i = 1, 2$) are τ -periodic, the functions $e^{-Q_i(t)}$ ($i = 1, 2$) are bounded. Thus, there exist two positive constants $\bar{r}$ and $\bar{R}$ with $\bar{r} < \bar{R}$ such that

$$\bar{r}|y| \leq |e^{-Q_i(t)} y| \leq \bar{R}|y| \quad (i = 1, 2). \quad (2.9)$$

We now manage to construct a global symplectic transformation $\Psi(t, z)$ such that

$$y = \Psi(t, z) = \begin{cases} P_1(t, z), & |z| < \bar{r}/\bar{R} \\ P_2(t, z), & |z| > \bar{R}/\bar{r}. \end{cases} \quad (2.10)$$

Let $\Gamma_0(t) = \int_0^t \gamma_0(\xi) d\xi$ and $\Gamma_\infty(t) = \int_0^t \gamma_\infty(\xi) d\xi$. Then $\Gamma_0(t)$ and $\Gamma_\infty(t)$ are two τ -periodic functions. Let $\rho_1(w)$ and $\rho_2(w)$ be two smooth functions satisfying

$$\rho_1(|w|^2) = \begin{cases} 1, & \text{as } |w| < \bar{r} \\ 0, & \text{as } |w| > \bar{R}, \end{cases} \quad \rho_2(|w|^2) = \begin{cases} 0, & \text{as } |w| < \bar{r} \\ 1, & \text{as } |w| > \bar{R}. \end{cases}$$

Let $\tilde{H}_t(w) = \frac{1}{2}\Gamma_0(t)|w|^2\rho_1(|w|^2) + \frac{1}{2}\Gamma_\infty(t)|w|^2\rho_2(|w|^2)$, where t is regarded as a parameter. It is obvious that $\tilde{H}_t(w)$ is a τ -periodic function with respect to the parameter t .

Consider the following Hamiltonian system

$$\frac{dw}{ds} = A_{2N} \nabla_w \tilde{H}_t(w). \quad (2.11)$$

Note that the above Hamiltonian system (2.11) is autonomous with respect to “ s ” and t is regarded as a parameter. Let $\Psi_t(s, w)$ be the general solution of (2.11). By the uniqueness of solutions of ordinary differential equations, $\Psi_t(s, w)$ is τ -periodic with respect to the parameter. Now we prove that the flow mapping $\Psi_t(s, w)$ is symplectic with respect to the symplectic form “ A_{2N} ”, that is,

$$\frac{\partial \Psi_t(s, w)}{\partial w} A_{2N} \left(\frac{\partial \Psi_t(s, w)}{\partial w} \right)^\top = A_{2N}.$$

Since A_{2N} is skew symmetric and non-degenerate, there is a non-degenerate matrix S such that

$$S A_{2N} S^\top = J,$$

where J is the standard symplectic matrix. Let $w = S^{-1}\tilde{z}$. Then the system (2.11) is transformed to the following standard Hamiltonian system

$$\frac{d\tilde{z}}{ds} = J\nabla_{\tilde{z}}\tilde{H}_t(\tilde{z}), \quad (2.12)$$

where $\tilde{H}_t(\tilde{z}) = \tilde{H}_t(S^{-1}\tilde{z})$. Let $\tilde{\Psi}_t(s, \tilde{z}) = S\Psi_t(s, S^{-1}\tilde{z})$. Then $\tilde{\Psi}_t(s, \tilde{z})$ is the general solution of the Hamiltonian system (2.12). Thus, we have that $\frac{\partial\tilde{\Psi}_t}{\partial\tilde{z}}J\left(\frac{\partial\tilde{\Psi}_t}{\partial\tilde{z}}\right)^\top = J$, i.e.,

$$\begin{aligned} S\frac{\partial\Psi_t}{\partial w}S^{-1}J\left(S\frac{\partial\Psi_t}{\partial w}S^{-1}\right)^\top &= J \\ \Rightarrow S\frac{\partial\Psi_t}{\partial w}(S^{-1}J(S^{-1})^\top)\left(\frac{\partial\Psi_t}{\partial w}\right)^\top S^\top &= J \\ \Rightarrow \frac{\partial\Psi_t}{\partial w}A_{2N}\left(\frac{\partial\Psi_t}{\partial w}\right)^\top &= A_{2N}. \end{aligned}$$

Let $y = \Psi(t, z) = \Psi_t(s, w)|_{s=1, w=z} = \Psi_t(1, z)$. Then we have

$$\frac{\partial\Psi}{\partial z}A_{2N}\left(\frac{\partial\Psi}{\partial z}\right)^\top = A_{2N}.$$

For $|w| < \bar{r}$, the system (2.11) becomes $dw/ds = \Gamma_0(t)A_{2N}w$. So, $w(s) = e^{\Gamma_0(t)A_{2N}s}z$. If $|s| < 1, |z| < \bar{r}/\bar{R}$, then $|w(s)| = |e^{\Gamma_0(t)A_{2N}s}z| \leq \bar{R}|z| < \bar{r}$. Thus for $|s| < 1, |z| < \bar{r}/\bar{R}$, we have $w(s) = e^{\Gamma_0(t)A_{2N}s}z$. Therefore, $\Psi(t, z) = P_1(t, z)$ as $|z| < \bar{r}/\bar{R}$. In the same way we can prove that $\Psi(t, z) = P_2(t, z)$ as $|z| > \bar{R}/\bar{r}$.

Let $z = \Phi(t, y)$ be the inverse mapping of the transformation $y = \Psi(t, z)$. Then the system (2.4) is changed to the following system

$$\frac{dz}{dt} = A_{2N}\nabla_z\hat{H}(t, z) + \frac{\partial\Phi}{\partial t}, \quad (2.13)$$

where $\hat{H}(t, z) = H(t, \Psi(t, z))$. By a direct computation, $A_{2N}^{-1}\frac{\partial^2\Phi}{\partial t\partial y}$ is symmetric. Then according to [15], there is a smooth function R such that

$$A_{2N}\nabla_zR = \frac{\partial\Phi}{\partial t}.$$

Then $H^*(t, z) = \hat{H}(t, z) + R(t, z)$ is the Hamiltonian function of the system (2.13) and the Hamiltonian system (2.13) can be rewritten as

$$\frac{dz}{dt} = A_{2N}\nabla_zH^*(t, z). \quad (2.14)$$

By (2.7) and (2.8), $H^*(t, z)$ satisfies the following properties.

$$\nabla_zH^*(t, z) = \alpha_0z + o(|z|) \quad \text{as } |z| \rightarrow 0 \quad \text{uniformly in } t, \quad (2.15)$$

$$\nabla_zH^*(t, z) = \alpha_\infty z + o(|z|) \quad \text{as } |z| \rightarrow \infty \quad \text{uniformly in } t. \quad (2.16)$$

Now in order to prove the main results of this paper, we only need to consider the Hamiltonian system (2.14). In the following of this paper, we prove the following two theorems.

Theorem 2.1. *Under the assumptions of Theorem 1.1, the system 2.14 has at least one nontrivial periodic solution.*

Theorem 2.2. *Under the assumptions of Theorem 1.2, the system 2.14 has at least one nontrivial periodic solution.*

2.3. Some lemmas

We now have constructed the global symplectic transformation $y = \Psi(t, z)$ which transforms the system (2.4) into the Hamiltonian system (2.14). Note that the system (2.14) satisfies the two properties (2.15) and (2.16). We shall use two well known critical point results given in [12, 13] to prove the two theorems.

We work in the Hilbert space $E = W^{\frac{1}{2}, 2}(S^1, \mathbb{R}^{2N})$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. The space E consists of all $z(t)$ in $L^2(S^1, \mathbb{R}^{2N})$ whose Fourier series

$$z(t) = a_0 + \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt)$$

satisfies

$$|a_0|^2 + \frac{1}{2} \sum_{k=1}^{+\infty} k(|a_k|^2 + |b_k|^2) < +\infty.$$

The inner product on E is defined by

$$\langle z_1, z_2 \rangle = (a_0^1, a_0^2) + \frac{1}{2} \sum_{k=1}^{\infty} k[(a_k^1, a_k^2) + (b_k^1, b_k^2)],$$

where $z_i = a_0^i + \sum_{k=1}^{+\infty} (a_k^i \cos kt + b_k^i \sin kt)$ ($i = 1, 2$) and $(\cdot, \cdot)$ denotes the inner product in $\mathbb{R}^{2N}$.

Recall that A_{2N}^{-1} is also a nonsingular skew symmetric matrix. For $\forall z, y \in E$, we define an operator A on E by extending the bilinear form

$$\langle Az, y \rangle = \int_0^{2\pi} (A_{2N}^{-1} z'(t), y(t)) dt. \quad (2.17)$$

It is not difficult to compute that A is a bounded self-adjoint linear operator on E and

$$Az = \sum_{k=1}^{+\infty} (A_{2N}^{-1} b_k \cos kt - A_{2N}^{-1} a_k \sin kt). \quad (2.18)$$

For each $z \in E$, we define a functional ψ on E by

$$\psi(z) = \frac{1}{2} \langle Az(t), z(t) \rangle - \int_0^{2\pi} H^*(t, z(t)) dt.$$

It is well known that critical points of ψ are solutions of the system (2.14). Hence, to seek periodic solutions of the system (2.14) is equivalent to seeking critical points of ψ .

For any $\alpha \in \mathbb{R}$ and $z, y \in E$, we define another operator B_α on E by

$$\langle B_\alpha z, y \rangle = - \int_0^{2\pi} (\alpha z(t), y(t)) dt. \quad (2.19)$$

Then B_α is also a bounded self-adjoint linear operator on E . Moreover, B_α is compact and for any $z \in E$,

$$B_\alpha z = -\alpha a_0 + \sum_{k=1}^{+\infty} \frac{1}{k} (-\alpha a_k \cos kt - \alpha b_k \sin kt). \quad (2.20)$$

For any $z(t) \in E$ and by (2.19) and (2.20), we have

$$(A + B_\alpha)z(t) = -\alpha a_0 + \sum_{k=1}^{+\infty} \left(\left(A_{2N}^{-1} b_k - \frac{1}{k} \alpha a_k \right) \cos kt + \left(-A_{2N}^{-1} a_k - \frac{1}{k} \alpha b_k \right) \sin kt \right). \quad (2.21)$$

Lemma 2.3. Suppose that (H_1) and (H_2) hold. Then the functional ψ satisfies

$$\|\psi'(z) - (A + B_{\alpha_0})z\| = o(\|z\|) \text{ as } \|z\| \rightarrow 0, \quad (2.22)$$

$$\|\psi'(z) - (A + B_{\alpha_\infty})z\| = o(\|z\|) \text{ as } \|z\| \rightarrow \infty. \quad (2.23)$$

Proof. From (2.15), for any $\tilde{r} > 0$, there exists a constant $C(\tilde{r})$ (here and in the following C denotes various constants) such that

$$|\nabla_x H^*(t, x) - \alpha_0 x| \leq \tilde{r}|x| + C(\tilde{r})|x|^2, \text{ for each } x \in \mathbb{R}^{2N}.$$

Notice that

$$\psi(z) = \frac{1}{2} \langle (A + B_{\alpha_0})z(t), z(t) \rangle - \int_0^{2\pi} \left(H^*(t, z(t)) - \frac{1}{2} (\alpha_0 z(t), z(t)) \right) dt.$$

Thus, we have

$$\|\psi'(z) - (A + B_{\alpha_0})z\| \leq C \|\nabla_z H^*(t, z) - \alpha_0 z\| \leq C(\tilde{r})\|z\| + C(\tilde{r})\|z\|^2,$$

which means (2.22).

By (2.16), we have $|\nabla_x H^*(t, x) - \alpha_\infty x| \leq \tilde{r}|x| + C(\tilde{r})$, for each $x \in \mathbb{R}^{2N}$. Then (2.23) holds similarly. $\square$

In order to obtain solutions of (2.14) with the symmetric structure (2.3), we define a $2N \times 2N$ matrix T_{2N} with the following form

$$T_{2N} = \begin{pmatrix} 0 & 0 & \cdots & 0 & -1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then by T_{2N} , for any $z(t) \in E$, define an action on z by

$$\delta z(t) = T_{2N} z(t - \tau). \quad (2.24)$$

Then by a direct computation we have that $\delta^{2N} z(t) = -z(t - 2N\tau)$, $\delta^{4N} z(t) = z(t)$ and $G = \{\delta, \delta^2, \dots, \delta^{4N}\}$ is a compact group action over E . If $\delta z(t) = z(t)$ holds, then through a direct check, we have that $z(t)$ has the symmetric structure (2.3). Write $SE = \{z \in E : \delta z(t) = z(t)\}$. Then from $z(t) = -z(t - 2N\tau)$ and a direct computation, we have

$$SE = \left\{ z \in E : z(t) = \sum_{k=1}^{\infty} (a_k \cos(2k-1)t + b_k \sin(2k-1)t) \right\}.$$

Let $\psi|_{SE}$ be the restriction of ψ on SE . We have the following lemma.

Lemma 2.4. *Critical points of $\psi|_{SE}$ over SE are critical points of ψ on E .*

Proof. We first check that $T_{2N}\Psi_t(s, w) = \Psi_t(s, T_{2N}w)$, where $\Psi_t(s, w)$ is the flow mapping generated by the system (2.11). In fact, by the definition of $\tilde{H}$, we get

$$\begin{aligned} w'(s) &= A_{2N} \{ \Gamma_0(t) w \rho_1(|w|^2) + \Gamma_0(t) |w|^2 \rho_1'(|w|^2) w \\ &\quad + \Gamma_\infty(t) w \rho_2(|w|^2) + \Gamma_\infty(t) |w|^2 \rho_2'(|w|^2) w \}. \end{aligned}$$

Notice that $T_{2N}T_{2N}^\top = I$ and T_{2N} commutes with A_{2N} and J . Set $\tilde{z}(s) = \Psi_t(s, T_{2N}w)$, $\tilde{y}(s) = \Psi_t(s, w)$, then

$$\begin{aligned} \frac{d\tilde{z}(s)}{ds} &= A_{2N} \{ \Gamma_0(t) T_{2N} w \rho_1(|w|^2) + \Gamma_0(t) |w|^2 \rho_1'(|w|^2) T_{2N} w \\ &\quad + \Gamma_\infty(t) T_{2N} w \rho_2(|w|^2) + \Gamma_\infty(t) |w|^2 \rho_2'(|w|^2) T_{2N} w \} \\ &= T_{2N} A_{2N} \{ \Gamma_0(t) w \rho_1(|w|^2) + \Gamma_0(t) |w|^2 \rho_1'(|w|^2) w \\ &\quad + \Gamma_\infty(t) w \rho_2(|w|^2) + \Gamma_\infty(t) |w|^2 \rho_2'(|w|^2) w \} \\ &= T_{2N} \frac{d\tilde{y}(s)}{ds}, \end{aligned}$$

this, jointly with $\Psi_t(0, T_{2N}w) = T_{2N}w = T_{2N}\Psi_t(0, w)$ shows that $T_{2N}\Psi_t(s, w) = \Psi_t(s, T_{2N}w)$. The definition of H^* , together with a direct computation implies that

$$H^*(t, T_{2N}z) = H^*(t, z), \quad \nabla_z H^*(t, T_{2N}z) = T_{2N} \nabla_z H^*(t, z).$$

Combining these with (2.24) and the fact that any $z(t) \in E$ is 2π -periodic, we can verify that

$$\psi(\delta z) = \psi(z), \quad \psi'(\delta z) = \delta\psi'(z) = \psi'(z).$$

That means ψ is G -invariant and ψ' is G -equivariant. Moreover, $\psi'(z) \in SE$. Therefore, if $\psi'(z) = 0$ on SE , then $\psi'(z) = 0$ on E . Hence, the conclusion of Lemma 2.4 holds. $\square$

Remark 2.5. It follows from Lemma 2.4 that we only need to find critical points of $\psi|_{SE}$ on SE . In the following, we use Galerkin approximation method and two critical point results to obtain critical points of $\psi|_{SE}$. We first analyze the spectrum of the operator $A + B_\alpha$.

Observing (2.21), we only need to analyze the spectrum of the following matrix

$$T_k(\alpha) = \begin{pmatrix} -\alpha I_{2N} & -kA_{2N}^{-1} \\ kA_{2N}^{-1} & -\alpha I_{2N} \end{pmatrix}.$$

Let λ_0 be an eigenvalue of A_{2N} . By Lemma 4 of [5], Proposition 3.6 of [16] and a direct computation, we have

$$\det(\lambda_0 I_{2N} - A_{2N}) = \frac{1}{2}((1 - \lambda_0)^{2N} + (1 + \lambda_0)^{2N}) = 0. \quad (2.25)$$

Then (2.25) gives

$$\sigma(A_{2N}) = \left\{ \pm i \tan \frac{2l-1}{4N} \pi : l = 1, 2, \dots, N \right\}, \quad (2.26)$$

i.e., A_{2N} has N pairs of pure imaginary eigenvalues. Let λ be an eigenvalue of the matrix $T_k(\alpha)$. Then

$$\begin{aligned} \det(\lambda I_{4N} - T_k(\alpha)) &= \det((\lambda + \alpha)^2 I_{2N} + (kA_{2N}^{-1})^2) \\ &= \det((\lambda + \alpha)^2 I_{2N} - (ikA_{2N}^{-1})^2) \\ &= \det((\lambda + \alpha)I_{2N} - ikA_{2N}^{-1}) \det((\lambda + \alpha)I_{2N} + ikA_{2N}^{-1}) \\ &= \det(\lambda I_{2N} - (ikA_{2N}^{-1} - \alpha I_{2N})) \det(\lambda I_{2N} - (-ikA_{2N}^{-1} - \alpha I_{2N})). \end{aligned}$$

Notice that $ikA_{2N}^{-1} - \alpha I_{2N}$ and $-ikA_{2N}^{-1} - \alpha I_{2N}$ have the same spectrum. It is easy to see that if λ takes the eigenvalue of $ikA_{2N}^{-1} - \alpha I_{2N}$, then $\det(\lambda I_{4N} - T_k(\alpha)) = 0$. By (2.26), one has

$$\sigma(ikA_{2N}^{-1}) = \left\{ \pm i \tan \frac{2l-1}{4N} \pi : l = 1, 2, \dots, N \right\}, \quad (2.27)$$

i.e. ikA_{2N}^{-1} has N pairs of real eigenvalues. In order to prove our main results we need to use the following index. By $M^-(\cdot)$ and $M^0(\cdot)$ we denote the negative

and the zero Morse indices of the symmetric matrix, respectively. For a constant diagonal matrix αI_{2N} , we define our index as

$$i^-(\alpha) = \sum_{k=1}^{\infty} (M^-(T_k(\alpha)) - 2N)$$

and

$$i^0(\alpha) = \sum_{k=1}^{\infty} M^0(T_k(\alpha)).$$

Since for k large enough, one has $M^-(T_k(\alpha)) = 2N$ and $M^0(T_k(\alpha)) = 0$. Hence $i^-(\alpha)$ and $i^0(\alpha)$ are well defined. The following lemma gives the relation between $\alpha \in \mathbb{R}$ and the indices $i^0(\alpha), i^-(\alpha)$.

- Lemma 2.6.** (1) Suppose that (H_3) hold. Then $i^0(\alpha_0) = i^0(\alpha_\infty) = 0$.
 (2) Suppose that (H_3) and (H_4) hold. Then $i^-(\alpha_0) \neq i^-(\alpha_\infty)$.
 (3) Suppose that (H_5) and (H_6) hold. Then $i^-(\alpha_\infty) \notin [i^-(\alpha_0), i^-(\alpha_0) + i^0(\alpha_0)]$.

Proof. Case (1). From (2.27), if $\alpha_0 \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$, then $\lambda \neq 0$, where λ is the eigenvalue of $T_k(\alpha_0)$. That means $M^0(T_k(\alpha_0)) = 0$ for $k \geq 1$. Thus $i^0(\alpha_0) = \sum_{k=1}^{\infty} M^0(T_k(\alpha_0)) = 0$. Similarly, we have $i^0(\alpha_\infty) = 0$.

Case (2). Without loss of generality, we suppose that $\alpha_0 < \alpha_\infty$. Then by the conditions (H_3) and (H_4) ,

$$\alpha_0 < k_0 \tan \frac{2l_0 - 1}{4N} \pi < \alpha_\infty.$$

Then by a direct check, we have

$$M^-(T_{k_0}(\alpha_0)) + 2 \leq M^-(T_{k_0}(\alpha_\infty)).$$

For each $k \neq k_0$, one can check easily that $M^-(T_k(\alpha_0)) \leq M^-(T_k(\alpha_\infty))$. Hence, one has $\sum_{k=1}^{\infty} (M^-(T_k(\alpha_0)) - 2N) < \sum_{k=1}^{\infty} (M^-(T_k(\alpha_\infty)) - 2N)$, since $M^-(T_k(\alpha)) = 2N$ for k large enough. This yields that $i^-(\alpha_0) < i^-(\alpha_\infty)$. Then case (2) holds.

Case (3). By the assumption (H_5) , $i^0(\alpha_0) = 2n_0$. Without loss of generality, let $\alpha_0 < \alpha_\infty$. Then from (H_6) and with a similar argument to case (2), we get

$$M^-(T_{k_j}(\alpha_\infty)) \geq 2 + M^-(T_{k_j}(\alpha_0)), \quad \text{for } j = 1, 2, \dots, n_0 + 1.$$

Moreover, $M^-(T_k(\alpha_\infty)) \geq M^-(T_k(\alpha_0))$ for $k \neq k_j (j = 1, 2, \dots, n_0 + 1)$. Thus, $i^-(\alpha_\infty) - (i^-(\alpha_0) + i^0(\alpha_0)) \geq 2$. That means $i^-(\alpha_\infty) \notin [i^-(\alpha_0), i^-(\alpha_0) + i^0(\alpha_0)]$. $\square$

For $k \geq 1$, set

$$SE(k) = \{z \in E : z(t) = a_k \cos(2k-1)t + b_k \sin(2k-1)t\}$$

and $SE_m = SE(1) \oplus SE(2) \cdots \oplus SE(m)$.

Let $\Omega = \{P_m : m = 1, 2, \dots\}$ be a sequence of orthogonal projections. Ω is called a Galerkin approximation scheme with respect to the operator $A + B_\alpha$, if it satisfies the following properties:

- (1) The image of P_m , as a subspace of SE , has finite dimension;
- (2) $P_m z \rightarrow z$ as $m \rightarrow \infty$ for any $z \in SE$;
- (3) P_m commutes with $A + B_\alpha$.

Let $P_m : SE \rightarrow SE_m$ be the orthogonal projection. One may check easily that P_m satisfies the above properties (1–3). Thus, Ω is a Galerkin approximation scheme with respect to the operator $A + B_\alpha$.

Let A_m and $\psi|_{SE_m}$ be the restrictions of A and $\psi|_{SE}$ on SE_m , respectively. We have the following important lemma.

Lemma 2.7. *Let (H_1) and (H_2) hold. If $\alpha_\infty \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$, then*

- (1) $\psi|_{SE}$ satisfies $(PS)^*$ condition over SE , i.e., every sequence $\{z_j\} \subset SE$ with $z_j \in SE_j$, $\psi'|_{SE_j}(z_j) \rightarrow 0$ and $\psi|_{SE}(z_j)$ being bounded, possesses a convergent subsequence.
- (2) $\psi|_{SE_j}$ satisfies (PS) condition, i.e., every sequence $\{z_i\} \subset SE_j$, $\psi'|_{SE_j}(z_i) \rightarrow 0$ and $\psi|_{SE_j}(z_i)$ being bounded, possesses a convergent subsequence.

Proof. From Lemma 2.6, $\alpha_\infty \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$ means that $i^0(\alpha_\infty) = 0$ which yields that $A + B_{\alpha_\infty}$ has a bounded inverse. Then the proof is standard. $\square$

Observe that our functional $\psi|_{SE_m}$ acts on the finite dimensional space SE_m . The following two lemmas are well known results in critical point theory. Please consult [2, 12, 13].

Lemma 2.8. *Let $\psi|_{SE_m}$ be a C^1 function satisfying (2.22) and (2.23). If $M^0(A_m + P_m B_{\alpha_0} P_m) = M^0(A_m + P_m B_{\alpha_\infty} P_m) = 0$ and $M^-(A_m + P_m B_{\alpha_0} P_m) \neq M^-(A_m + P_m B_{\alpha_\infty} P_m)$, then $\psi|_{SE_m}$ has at least one nontrivial critical point.*

Lemma 2.9. *Let $\psi|_{SE_m}$ be a C^2 function satisfying (2.22) and (2.23). If $M^0(A_m + P_m B_{\alpha_\infty} P_m) = 0$ and $M^-(A_m + P_m B_{\alpha_\infty} P_m) \notin [M^-(A_m + P_m B_{\alpha_0} P_m), M^-(A_m + P_m B_{\alpha_0} P_m) + M^0(A_m + P_m B_{\alpha_0} P_m)]$, then $\psi|_{SE_m}$ has at least one nontrivial critical point.*

To apply Lemmas 2.8 and 2.9 to get critical points of $\psi|_{SE}$, we introduce the following lemma.

Lemma 2.10. *If $\alpha \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$, then*

$$M^-(A_m + P_m B_\alpha P_m) - M^-(A_m) = M^-(A_k + P_k B_\alpha P_k) - M^-(A_k)$$

for m, k large enough.

Note that $\alpha \neq k \tan \frac{2l-1}{4N} \pi$, for all $k \in \mathbb{Z}^+$ and $1 \leq l \leq 2N$, means that $i^0(\alpha) = 0$. Then the proof is the same as in [13]. For the linear compact operator B_α , we define

$$I^-(B_\alpha) = \{k = M^-(A_m + P_m B_\alpha P_m) - M^-(A_m), \text{ for infinitely many } m\},$$

$$I^0(B_\alpha) = \{k = M^0(A_m + P_m B_\alpha P_m), \text{ for infinitely many } m\}.$$

Let

$$\begin{aligned} I^-(\psi, \infty) &= I^-(B_{\alpha_\infty}), & I^-(\psi, 0) &= I^-(B_{\alpha_0}); \\ I^0(\psi, \infty) &= I^0(B_{\alpha_\infty}), & I^0(\psi, 0) &= I^0(B_{\alpha_0}). \end{aligned}$$

By Lemma 2.10, the index above is singleton and well defined. Now we prove the following lemma.

Lemma 2.11. *With respect to the approximation scheme Ω , one has*

$$\begin{aligned} I^-(\psi|_{SE}, 0) &= \sum_{k=1}^{\infty} (M^-(T_k(\alpha_0)) - 2N) = i^-(\alpha_0), \\ I^-(\psi|_{SE}, \infty) &= \sum_{k=1}^{\infty} (M^-(T_k(\alpha_\infty)) - 2N) = i^-(\alpha_\infty), \\ I^0(\psi|_{SE}, 0) &= \sum_{k=1}^{\infty} M^0(T_k(\alpha_0)) = i^0(\alpha_0). \end{aligned}$$

Proof. It follows from the definition of the negative Morse index of $T_k(\alpha_0)$ and (2.21) that the dimension of the negative eigenspace of the operator $A_m + P_m B_{\alpha_0} P_m$ on $SE_m = \bigoplus_{j=1}^m SE(j)$ is equal to $\sum_{k=1}^m (M^-(T_k(\alpha_0)))$. For $\alpha_0 = 0$, $M^-(T_k(\alpha_0)) = 2N$. Therefore, the formula

$$I^-(\psi|_{SE}, 0) = \sum_{k=1}^{\infty} (M^-(T_k(\alpha_0)) - 2N) = i^-(\alpha_0)$$

holds. Similarly, we have the other formulas. $\square$

2.4. Proof of Theorem 2.1

We are now ready to prove the main results. We first give the proof of Theorem 2.1.

Proof of Theorem 2.1 Under the assumptions of Theorem 2.1 and by (2.23) and Lemma 2.6, for $\epsilon = \frac{1}{2} \|(A + B_{\alpha_\infty})^{-1}\|^{-1}$, there is a large $R > 0$ such that

$$\|\psi'|_{SE}(z) - (A + B_{\alpha_\infty})(z)\| < \epsilon \|z\| \quad \text{for } \|z\| > R.$$

That yields

$$\begin{aligned} \|\psi'|_{SE}(z)\| &\geq \|(A + B_{\alpha_\infty})^{-1}\|^{-1} \|z\| - \|\psi'|_{SE}(z) - (A + B_{\alpha_\infty})(z)\| \\ &> \epsilon \|z\| \quad \text{for } \|z\| > R. \end{aligned}$$

This means $\psi|_{SE}$ has no critical points outside the ball B_R .

Since $A + B_{\alpha_0}$ has bounded inverse and $\|(I - P_m)B_{\alpha_0}\| \rightarrow 0$ as $m \rightarrow 0$, there exists a constant C such that $\|(A_m + P_m B_{\alpha_0} P_m)^{-1}\| < C$. By (2.22), we can take r small enough with $r < R$ such that

$$\|\psi'|_{SE}(z) - (A + B_{\alpha_0})(z)\| < \frac{1}{2C} \|z\| \quad \text{for } \|z\| < r.$$

Thus, for m large enough and for $z \in SE_m$ with $\|z\| < r$, one has

$$\begin{aligned} & \|\psi'|_{SE}(z) - (A_m + P_m B_{\alpha_0} P_m)(z)\| \\ & \leq \|\psi'|_{SE}(z) - (A + B_{\alpha_0})(z)\| \\ & < \frac{1}{2C} \|z\| \\ & < \frac{1}{2} \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1} \|z\|. \end{aligned}$$

From the above inequality, we get

$$\begin{aligned} \|\psi'|_{SE}(z)\| & > \|(A_m + P_m B_{\alpha_0} P_m)(z)\| - \frac{1}{2} \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1} \|z\| \\ & > \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1} \|z\| - \frac{1}{2} \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1} \|z\| \\ & = \frac{1}{2} \|(A_m + P_m B_{\alpha_0} P_m)^{-1}\|^{-1} \|z\|. \end{aligned}$$

Therefore, 0 is the unique critical point of $\psi|_{SE}$ inside B_r .

By Lemmas 2.6 and 2.11, $M^-(A_m + P_m B_{\alpha_0} P_m) \neq M^-(A_m + P_m B_{\alpha_\infty} P_m)$. Then Lemmas 2.3 and 2.8 yield $\psi|_{SE_m}$ has a nontrivial critical point z_m with $r < \|z_m\| < R$. By Lemma 2.7, $\psi|_{SE}$ satisfies the $(PS)^*$ condition. So, $\{z_m\}$ has a subsequence convergent to a point z , which is just a critical point of $\psi|_{SE}$ with $r < \|z\| < R$. $\square$

2.5. Proof of Theorem 2.2

In this section, we prove Theorem 2.2.

Proof of Theorem 2.2 Since $M^0(T_m(\alpha_0)) = 0$ for m large enough, the null space of $(A + B_{\alpha_0})|_{SE}$ is included in SE_m , where $(A + B_{\alpha_0})|_{SE}$ is the restriction of $A + B_{\alpha_0}$ on SE . Thus, there exists a constant C such that

$$\|(A_m + P_m B_{\alpha_0} P_m)^\sharp\| < C \text{ for } m \text{ large enough,}$$

where $(A_m + P_m B_{\alpha_0} P_m)^\sharp$ denotes the inverse of $A_m + P_m B_{\alpha_0} P_m$ restricted in the range of $A_m + P_m B_{\alpha_0} P_m$. By (2.22) we can take a small r such that

$$\|\psi''|_{SE_m}(z) - (A + B_{\alpha_0})\| \leq \frac{1}{2C} \quad \text{as } \|z\| < r.$$

As $\|z\| < 2r$, one has

$$\begin{aligned} & \|\psi''|_{SE_m}(z) - (A_m + P_m B_{\alpha_0} P_m)\| \\ & \leq \|\psi''|_{SE_m}(z) - (A + B_{\alpha_0})\| \\ & \leq \frac{1}{2C} \\ & < \frac{1}{2} \|(A_m + P_m B_{\alpha_0} P_m)^\sharp\|^{-1}. \end{aligned}$$

Then by a similar argument with [13, Theorem 1.3], we have the fact that if $\psi|_{SE_m}$ has critical points, then at least one of them is outside B_r . By Lemmas 2.6 and

2.11, we have for m large enough

$$M^-(A_m + P_m B_{\alpha_\infty} P_m) \notin \left[M^-(A_m + P_m B_{\alpha_0} P_m), M^-(A_m + P_m B_{\alpha_0} P_m) + M^0(A_m + P_m B_{\alpha_0} P_m) \right].$$

Hence, Lemma 2.9, jointly with Lemma 2.3 and the proof of Theorem 2.1 yields that $\psi|_{SE_m}$ has a nontrivial critical point z_m with $r < \|z_m\| < R$. By Lemma 2.7, $\psi|_{SE}$ satisfies the $(PS)^*$ condition. So, $\{z_m\}$ has a subsequence convergent to a point z , which is just a critical point of $\psi|_{SE}$ with $r < \|z\| < R$. $\square$

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