Stationary solutions for the non-linear Hartree equation with a slowly varying potential

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Abstract. We consider the non-linear Hartree equation with a slowly varying external potential V_ε and a short range, attractive two-body interaction W . We prove the existence of stationary solutions which are approximatively given by a superposition of several Hartree solitons with their center of mass positions behaving, at the leading order, as classical particles at rest in the background potential V_ε .

Mathematics Subject Classification (2000). 35Q55 (35J10).

Keywords. Elliptic equation, Shadowing lemma.

1. Introduction

The nonlinear Hartree equation arises in the mean-field limit of large systems of weakly interacting non-relativistic bosonic atoms or molecules (see [5, 12, 17]). In a recent paper [10] the authors consider bosons systems with an attractive two-body interactions W . In this case the system has bound states, namely the bosons may condense into a spatially localized cluster that, in the mean-field regime, is described by a “one-particle” wave function corresponding to a *solitary wave solution or soliton* of the nonlinear Hartree equation in $\mathbb{R}^N$

$$i\partial_t\psi = -\Delta\psi - (W * |\psi|^2)\psi. \quad (1.1)$$

Here $N \geq 3$ and $\psi = \psi(t, x)$, $\psi: I \times \mathbb{R}^N \subset \mathbb{R}^{N+1} \rightarrow \mathbb{C}$ is the complex wave function and $*$ denotes convolution.

A particular class of stationary solutions of this equation are those of the form $\psi(t, x) = e^{iE_0 t}U(x)$ where the couple (E_0, U) is a solution of the nonlinear eigenvalue problem

$$-\Delta U - (W * U^2)U = -E_0 U \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

Solutions of this nonlinear eigenvalue problem can be found as a non-trivial local minimizer of the Hartree (static) energy functional

$$\frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi|^2 dx - \frac{1}{4} \int_{\mathbb{R}^N} (W * |\psi|^2)^2 dx$$

constrained on the $L^2(\mathbb{R}^N)$ sphere $\|u\|^2 = \mu_0$. Such minimizers U_0 are called *Hartree ground states* and E_0 is the corresponding Lagrange multiplier. We will call *Hartree solitons* the corresponding stationary solutions $\psi_0(t, x) = e^{iE_0 t} U_0(x)$ of (1.1).

We recall that the existence of *Hartree ground states* goes back to Lieb [13] for the Coulomb two-body potential and can be obtained for any $W \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, with $p \geq \frac{N}{2}$ as an application of the concentration-compactness method developed by Lions [14, 15] for all $\mu > \mu_*$ for some $\mu_* = \mu_*(W) \geq 0$. See in particular [9] for a discussion of existence and also of stability of the *Hartree solitons*.

In [10] the authors investigate the motion of Hartree solitons U_0 when a (slowly varying) potential V_ε is added. They show that the center of mass at time t of a solution starting close to U_0 remains close to the solution $r(t)$ of Newton's equation of motion $\ddot{r} = V'_\varepsilon(r)$ which starts close to the center of mass of U_0 .

More precisely, in [10] the authors study

$$i\partial_t \psi = -\Delta \psi + V_\varepsilon \psi - (W * |\psi|^2) \psi \quad (1.3)$$

where $V_\varepsilon(x) = V(\varepsilon x)$, with V smooth, bounded and non-negative, $\varepsilon > 0$ is a small perturbation parameter and W is a positive, radially symmetric two-body potential, $W \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, with $p \geq \frac{N}{2}$ and $W(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.

They consider any initial condition sufficiently near to a *Hartree (ground state) soliton* $\psi(0, x) \approx U_0(x)$, where U_0 is a minimizer of the Hartree energy on the L^2 -sphere, for some suitable μ_0 (see Lemma 2.1 in [10]) and E_0 is the relative Lagrange multiplier.

They prove that there exists a solution of (1.3) of the form

$$\psi(x, t) = U_0(x - r(t)) e^{i\theta(x, t)} + h_\varepsilon(x, t),$$

for $|t| < O(\varepsilon^{-1})$, where $r(t)$ solves, at the leading order, Newton's equation of motion $\ddot{r} = -V'_\varepsilon(r)$, $\theta(x, t) = \dot{r}(t) \cdot (x - r(t)) + E_0 t + \theta_0(t)$ and h_ε is a small dispersive correction term.

Their result requires the following *key assumption* (see also (1.26), (3.12) in [10])

- *Non-degeneracy condition or null space condition*

$$(H) \quad \text{Ker } L_0 = \text{span}\{\partial_i U_0, i = 1, \dots, N\}$$

$$\text{where } L_0 h = -\Delta h + E_0 h - 2(W * (U_0 h)) U_0 - (W * U_0^2) h.$$

This assumption says that no extra degeneracy of the kernel appear besides the expected one due to the invariance under spatial translations.

Note that for local nonlinearity (i.e., nonlinearities of the form $f(|u|)u$), such assumption (H) is known to be satisfied in dimension one (see [18]), and in higher

dimensions for pure powers $|u|^{p-1}u$ with $p > 1$ (see e.g. [16]) and for generic f 's. We refer to the Appendix C in [7] for a detailed discussion.

Very little is known in the non-local case. The techniques used to prove such a non-degeneracy assumption in the local case do not apply.

Very recently, in [8], the authors state that this assumption is verified numerically for some value of $E_0 > 0$ for the Coulomb two-body interaction.

In [10] the authors also investigate at the formal level a k -particles (Newtonian) limit of the Hartree equation (1.3) with a W rapidly decreasing at infinity. They consider an initial configuration of k Hartree ground state solitons $\{U_0(\cdot - p_j)\}_{j=1}^k$, centered at some positions $p_j \in \mathbb{R}^N$, sufficiently far-separated ($|p_i - p_j| > O(\varepsilon^{-1}), i \neq j$). Then, they claim the existence of a solution of the form

$$\psi(x, t) = \sum_{j=1}^k U_0(x - r_j(t))e^{i\theta_j(x, t)} + h_\varepsilon(x, t) \quad (1.4)$$

for $|t| < O(\varepsilon^{-1})$, showing, only heuristically, that the limit $\varepsilon \rightarrow 0$ corresponds to the point-particles limit in which the trajectories of the particles $r_j(t)$ are solutions of the classical Newton's equations of motion for a system of k -particles subject to the background potential V_ε , the phases satisfy $\theta_j(x, t) = \dot{r}_j(t) \cdot (x - r_j(t)) + \theta_j(t)$ and $h_\varepsilon(\cdot, t) = o(\varepsilon)$ is a small dispersive correction term.

Aim of the present paper is to establish a first step towards a proof of the existence of solutions of the form (1.4) for $k > 1$. We restrict our attention to the stationary case and we consider an external potential V with a multiplicity of (non-degenerate) critical points $\{p_j\}_{j=1}^k$. We prove the existence of stationary solutions of equation (1.3) of the form $\psi(x, t) = e^{iE_0 t}U(x)$ with

$$U(x) = \sum_{j=1}^k U_0(x - p_\varepsilon^{(j)}) + h_\varepsilon(x) \quad (1.5)$$

where $p_\varepsilon^{(j)} = \frac{p_j}{\varepsilon} + o(1)$ and $h_\varepsilon = o(\varepsilon)$, in an appropriately chosen norm.

More precisely, let us assume:

- (h1) V is smooth, non-negative, 1-periodic in each variable and with a non-degenerate minimum in $x = 0$, with $V(0) = 0$ and $V''(0)$ positive definite.
- (h2) $W \in L^p(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$, with $p \geq \frac{N}{2}$, is positive, radially symmetric, non-increasing, and $W(x) \rightarrow 0$ as $|x| \rightarrow +\infty$.
- (h3) By the results in [10, Lemma 2.1] (or in [9, Theorem 2]) we know that the Hartree ground state U_0 of given L^2 norm $\mu_0 > \mu_* \geq 0$ exists, is smooth, positive, radially symmetric and exponentially decaying at infinity and solve equation (1.2) for a certain $E_0 > 0$. We assume that such U_0 and E_0 satisfy assumption (H).

Given V , W and E_0 as above, we let $V_\varepsilon(x) = V(\varepsilon x)$ and consider the following elliptic equation with a nonlocal nonlinearity

$$-\Delta u + (E_0 + V_\varepsilon)u - (W * u^2)u = 0 \quad \text{on } \mathbb{R}^N. \quad (1.6)$$

As a starting point in our analysis we prove the existence of a solution U_ε of the equation (1.6) (for $\varepsilon > 0$ sufficiently small) which is a small perturbation of the Hartree ground state U_0 solution of (1.2).

To prove the existence of a solution U_ε of the equation (1.6) such that $U_\varepsilon \rightarrow U_0$, as $\varepsilon \rightarrow 0$, we use a slight modification of a perturbation argument introduced by Floer and Weinstein [6] for the Schrödinger equation with a cubic local nonlinearity. We have:

Theorem 1.1. *Let V , W , U_0 and E_0 be such that (h1), (h2) and (h3) hold.*

Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ there exists a smooth, positive solution U_ε of the equation (1.6) satisfying the following properties:

- (i) $U_\varepsilon(\cdot + p_\varepsilon) \rightarrow U_0$, as $\varepsilon \rightarrow 0$, in $C^k(\mathbb{R}^N)$, for any $k \in \mathbb{N}$, with $|p_\varepsilon| = O(\varepsilon^\alpha)$, for $\alpha \in (0, 1)$; furthermore

$$U_\varepsilon(x) = U_0(x - p_\varepsilon) + h_\varepsilon(x)$$

with $\|h_\varepsilon\|_{C^k} = O(\varepsilon^2)$.

- (ii) Given $R > 0$ there exist positive constants C and c such that

$$|U_\varepsilon(x)| + |\nabla U_\varepsilon(x)| \leq C e^{-c|x|}$$

for any $|x| \geq R$ and $\varepsilon \in (0, \bar{\varepsilon})$.

We mention that, in a non-perturbative setting, the existence and multiplicity of solutions for the non-linear Hartree equation with a periodic external potential is studied in [1] by variational methods.

In this paper our aim is to prove the existence of a solution of (1.6) that is approximatively equal to the superposition of k -“one-particle” solutions sufficiently far-separated. To this purpose we use a PDE version of a “shadowing-type lemma”, as introduced in [3]. Let us point out that for the existence of such a superposition solution it is essential that the k -“one-particle” solutions do not interact too much. We remind that in the case of local non-linearities this fact is a direct consequence of the exponential decay of the “one-particle” solution. Here, we are dealing with a non-local nonlinearity, hence we need to assume some decay at infinity of the two-body potential W . More precisely, we assume

(h4) $W(x) = O(\frac{1}{|x|^\beta})$ as $|x| \rightarrow +\infty$, with $\beta > \max\{4, N\}$.

and we have the following result:

Theorem 1.2. *Let (h1), (h2), (h3) and (h4) hold. Then there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any $\mathcal{P} \subseteq \mathbb{N}$ and for any set of points $\{p_j\}_{j \in \mathcal{P}} \subset \mathbb{Z}^N$ (minimum points of V), there exists u_ε , a smooth solution of (1.6) such that*

$$u_\varepsilon(x) = \sum_{j \in \mathcal{P}} U_0(x - p_\varepsilon^{(j)}) + h_\varepsilon(x) \quad (1.7)$$

where $p_\varepsilon^{(j)} = \frac{p_j}{\varepsilon} + O(\varepsilon^\alpha)$, with $\alpha \in (0, 1)$, and $\|h_\varepsilon\|_{C^k} = O(\varepsilon^2)$, for any $k \in \mathbb{N}$.

Among the solutions given by Theorem 1.2, for $\mathcal{P}$ countable, we can select solutions with periodic boundary conditions. Indeed, we have the following result:

Theorem 1.3. *Let (h1), (h2), (h3) and (h4) hold. Then, there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any set of points $\{p_j\}_{j \in \mathcal{P}} \subset \mathbb{Z}^N$ periodically arranged in $\mathbb{R}^N$ (see (6.2)), there exists u_ε a smooth periodic solution of (1.6), with periodic cell of order ε^{-1} and satisfying (1.7).*

We remark that in the case that $\mathcal{P}$ is a finite set, namely for “ k -particles” solutions (the only one with finite energy) it is possible to weaken the assumption (h4), asking only the decay exponent $\beta > 4$ (see Remark 5.1). However, on one hand the physical interesting case is for dimension space $N = 3$, hence (h4) is not more restrictive, on the other hand the integrability condition at infinity of the two-body potential W (namely $\beta > N$) guarantees that the critical parameter $\bar{\varepsilon}$ given in Theorem 1.2 does not depend on the number k of points $\{p_j\}_{j=1}^k$.

Let us point out that the Theorem 1.2 can be easily generalized to any smooth bounded background potential V with a multiplicity of non-degenerate minima (or maxima) at some points $\{p_j\}_j \in \mathbb{R}^N$. However we prefer to consider the potential V periodic on one hand to avoid heavy notation and to simplify estimates, thanks to the uniformity around to each minimum point, on the other hand because in the periodic case we prove the existence of periodic solutions of (1.6) (see Theorem 1.3), a result which we believe interesting in itself.

Besides that, following a truncation argument already developed in the case of non-linear Schrödinger equation (see e.g. [4]), it is possible to extend Theorem 1.2 also to unbounded external potential V .

Finally, it is also possible to generalize these results to a potential V with possibly degenerate minimum (or maximum) points, that is for instance $V'(0) = V''(0) = \dots = V^{(2m-1)}(0) = 0$ and $V^{(2m)}(0)$ positive or negative definite. In particular, to extend Theorem 1.1 to this case, one should generalize to nonlocal nonlinearities the argument used in [2] for the case of nonlinear Schrödinger equation with a power (local) non-linearity and then for Theorem 1.2 one should assume a faster power decay at infinity of the two-body potential W ($\beta > \max\{4m, N\}$).

Beyond these possible extensions of Theorem 1.2, more interesting is the question of dynamics for the equation (1.3). As already mentioned it is reasonable (see [10]) to expect the existence of solutions which approximate the trajectories of (classical) point particles. In this direction it would be important to investigate the stability properties of the stationary solutions obtained in Theorem 1.2. In particular, it would be interesting to investigate the “basins of attraction” of the stationary solutions obtained in Theorem 1.2, namely to determine the set of initial conditions $p_j^{(0)} \in \mathbb{R}^N$ around the minima of the external potential V for which the solution of (1.3) remains close to a stationary “multiparticle” solution.

2. Preliminaries

In this section we state some properties about the nonlinear term which is non local due to the presence of a convolution.

From now on we denote by C any positive constant, possibly different from line to line.

First of all we note that by Hölder inequality it follows that

$$\|W * (uv)\|_{L^\infty} \leq \|W\|_{L^s} \|uv\|_{L^{s/(s-1)}} \leq \|W\|_{L^s} \|u\|_{L^{\frac{2s}{s-1}}} \|v\|_{L^{\frac{2s}{s-1}}}. \quad (2.1)$$

Hence in particular, for $W \in L^s(\mathbb{R}^N)$ with $s \geq N/2$, by Sobolev embedding's theorems, there exists a constant $C > 0$ such that for any $u, v \in H^1(\mathbb{R}^N)$,

$$\|W * (uv)\|_{L^\infty} \leq C \|u\|_{H^1} \|v\|_{H^1}. \quad (2.2)$$

Indeed, $H^1(\mathbb{R}^N)$ is continuously embedded in $L^{\frac{2s}{s-1}}(\mathbb{R}^N)$ for all $s \geq N/2$.

In our setting we deal mostly with the function space $H^2(\mathbb{R}^N)$. Clearly, for $W \in L^s(\mathbb{R}^N)$ with $s \geq N/2$ we have

$$\|W * (uv)\|_{L^\infty} \leq C \|u\|_{H^2} \|v\|_{H^2}. \quad (2.3)$$

for any $u, v \in H^2(\mathbb{R}^N, \mathbb{R})$.

Moreover, if $\text{supp}(uv) = A$, with $A \subset \mathbb{R}^N$, then we have

$$\|W * (uv)\|_{L^\infty} \leq C \|u\|_{L^{\frac{2s}{s-1}}(A)} \|v\|_{L^{\frac{2s}{s-1}}(A)} \leq C \|u\|_{H^2(A)} \|v\|_{H^2(A)}. \quad (2.4)$$

Recall also that, for bounded set $\Omega \subset \mathbb{R}^N$, $H^2(\Omega)$ is compactly embedded in $L^{\frac{2s}{s-1}}(\Omega)$ for all $s \geq N/2$ (indeed, the condition to be satisfied is $s > \frac{N}{4}$).

In particular, in view of (2.4) the proof of the following lemma is standard (see e.g. [1, Lemma 3.1] for a more general result). We give the details of the proof for completeness.

Lemma 2.1. *Let (h2) hold. Let $\{u_n\}, \{v_n\}, \{w_n\}$ be bounded sequences in $H^2(\mathbb{R}^N)$. Then, if $v_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$ and $\{w_n\}$ is strongly convergent in $H^2(\mathbb{R}^N)$ or vice-versa, in both case, we have, up to subsequence,*

$$\|[W * (u_n v_n)] w_n\|_{L^2} \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

Proof. First let us assume that $v_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$ and $w_n \rightarrow w_0$ strongly in $H^2(\mathbb{R}^N)$. For $R > 0$, we have

$$\|[W * (u_n v_n)] w_n\|_{L^2} \leq \|[W * (u_n v_n)] w_n\|_{L^2(B_R)} + \|W * (u_n v_n)\|_{L^\infty} \|w_n\|_{L^2(\mathbb{R}^N \setminus B_R)}$$

Now, set $W = W_1 + W_2$, with $W_1 \in L^s(\mathbb{R}^N)$ ($s \geq \frac{N}{2}$) and $W_2 \in L^\infty(\mathbb{R}^N)$. Let us consider the compact set $K = \bar{B}_{R+2d}$, for some $d > 0$, and its characteristic set function χ_K . In view of (2.4), we have

$$\begin{aligned} \|[W * (u_n v_n)] w_n\|_{L^2(B_R)} &\leq \|[W * (\chi_K u_n v_n)] w_n\|_{L^2(B_R)} \\ &\quad + \|[W * ((1 - \chi_K) u_n v_n)] w_n\|_{L^2(B_R)} \\ &\leq \|w_n\|_{L^2} \|W_1\|_{L^s} \|u_n\|_{L^{\frac{2s}{s-1}}(K)} \|v_n\|_{L^{\frac{2s}{s-1}}(K)} \\ &\quad + \|w_n\|_{L^2} \|W_2\|_{L^\infty} \|u_n\|_{L^2(K)} \|v_n\|_{L^2(K)} \\ &\quad + \sup_{|\xi| > d} W(\xi) \|w_n\|_{L^2} \|u_n\|_{L^2} \|v_n\|_{L^2} \end{aligned}$$

Hence, we have

$$\| [W * (u_n v_n)] w_n \|_{L^2(B_R)} \leq C (\|v_n\|_{L^{\frac{2s}{s-1}}(K)} + \|v_n\|_{L^2(K)} + \sup_{|\xi|>d} W(\xi)).$$

Since by the compact Sobolev embedding $\|v_n\|_{L^{\frac{2s}{s-1}}(K)} \rightarrow 0$ and $\|v_n\|_{L^2(K)} \rightarrow 0$, up to subsequence, and $W(x) \rightarrow 0$ as $|x| \rightarrow +\infty$, each term in the right hand side can be made arbitrarily small, for $n \in \mathbb{N}$ and $d > 0$ sufficiently large. Moreover, since by (2.3) $\|W * (u_n v_n)\|_{L^\infty}$ is uniformly bounded, we have

$$\|W * (u_n v_n)\|_{L^\infty} \|w_n\|_{L^2(\mathbb{R}^N \setminus B_R)} \leq C (\|w_n - w_0\|_{L^2} + \|w_0\|_{L^2(\mathbb{R}^N \setminus B_R)})$$

and again each term in the right hand side can be made arbitrarily small, for $n \in \mathbb{N}$ and $R > 0$ sufficiently large. Therefore the first case of the lemma is proved.

In the second case, we assume vice-versa that $v_n \rightarrow v_0$ strongly in $H^2(\mathbb{R}^N)$ and $w_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$. For $K = \bar{B}_{R+2d}$ we have,

$$\| [W * (u_n v_n)] w_n \|_{L^2} \leq \| [W * (u_n v_n)] w_n \|_{L^2(K)} + \| [W * (u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)}.$$

Then, since again by (2.3) $\|W * (u_n v_n)\|_{L^\infty}$ is uniformly bounded,

$$\| [W * (u_n v_n)] w_n \|_{L^2(K)} \leq \|W * (u_n v_n)\|_{L^\infty} \|w_n\|_{L^2(K)} \leq C \|w_n\|_{L^2(K)}$$

and by the compact Sobolev embedding theorem, $\|w_n\|_{L^2(K)} \rightarrow 0$, up to subsequence.

Now let us estimate $\| [W * (u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)}$. Denoting by χ_{B_R} the characteristic function of the set B_R , we have

$$\begin{aligned} \| [W * (u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)} &\leq \| [W * (\chi_{B_R} u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)} \\ &\quad + \| [W * ((1 - \chi_{B_R}) u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)}. \end{aligned}$$

Then,

$$\| [W * (\chi_{B_R} u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)} \leq C \sup_{|\xi|>d} W(\xi) \|u_n v_n\|_{L^1} \|w_n\|_{L^2} \leq C \sup_{|\xi|>d} W(\xi)$$

and $\sup_{|\xi|>d} W(\xi)$ is arbitrarily small for $d > 0$ sufficiently large. Finally, by (2.4) we have

$$\begin{aligned} \| [W * ((1 - \chi_{B_R}) u_n v_n)] w_n \|_{L^2(\mathbb{R}^N \setminus K)} &\leq C \|w_n\|_{L^2} (\|W_1\|_{L^s} \|u_n\|_{H^2} \|v_n\|_{H^2(\mathbb{R}^N \setminus B_R)} \\ &\quad + \|W_2\|_{L^\infty} \|u_n\|_{L^2} \|v_n\|_{L^2(\mathbb{R}^N \setminus B_R)}) \\ &\leq C (\|v_n - v_0\|_{H^2} + \|v_0\|_{H^2(\mathbb{R}^N \setminus B_R)}) \end{aligned}$$

where $\|v_n - v_0\|_{H^2} \rightarrow 0$ as $n \rightarrow +\infty$ and $\|v_0\|_{H^2(\mathbb{R}^N \setminus B_R)}$ is arbitrarily small for $R > 0$ sufficiently large. Therefore the second case of the lemma is proved. $\square$

We conclude this section with the following definition that will be used in the sequel. Given any function $h : \mathbb{R}^N \rightarrow \mathbb{R}$, we define $\tau_y h = h(\cdot + y)$; in particular, note that

$$\tau_y (W * h) = W * (\tau_y h). \quad (2.5)$$

3. Existence of “one-particle” solution for (1.6)

To prove the existence of a solution of (1.6) for $\varepsilon > 0$ small, we use a slight modification of a perturbation argument introduced by Floer and Weinstein [6] for the Schrödinger equation with a cubic local nonlinearity.

We introduce the nonlinear maps F_0 and $F_\varepsilon \in C^1(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ defined by

$$\begin{aligned} F_0(u) &= -\Delta u + E_0 u - (W * u^2)u \\ F_\varepsilon(u) &= -\Delta u + V_\varepsilon u - (W * u^2)u \end{aligned}$$

where, to simplify the notation, from now on the “new” potential $V_\varepsilon(x) = V(\varepsilon x)$ will be the potential of assumption (h1) plus the positive constant E_0 . Hence in particular from now on we have $V_\varepsilon(x) \geq E_0$ and $V(0) = E_0$.

For any $u \in H^2(\mathbb{R}^N)$ we consider the differential operators $DF_0(u)$ and $DF_\varepsilon(u) : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ given by

$$\begin{aligned} DF_0(u)h &= -\Delta h + E_0 h - 2(W * (uh))u - (W * u^2)h \\ DF_\varepsilon(u)h &= -\Delta h + V_\varepsilon h - 2(W * (uh))u - (W * u^2)h. \end{aligned}$$

Let us denote $U_{p,\varepsilon}(x) = U_0(x - \frac{p}{\varepsilon})$, with $p \in \mathbb{R}^N$, $\varepsilon > 0$ and U_0 a smooth, positive, radially symmetric solution of (1.2) (with $E_0 = V(0)$). We have

Lemma 3.1. *There exist positive constants C and μ such that for any $R > 0$ we have*

$$\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}^2 + \|DF_\varepsilon(U_{p,\varepsilon})\nabla U_{p,\varepsilon}\|_{L^2}^2 \leq C(e^{-\mu R} + \max_{x \in B_{\varepsilon R}(p)} |V(x) - V(0)|^2).$$

In particular $\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2} \rightarrow 0$ and $\|DF_\varepsilon(U_{p,\varepsilon})\nabla U_{p,\varepsilon}\|_{L^2} \rightarrow 0$ as $(p, \varepsilon) \rightarrow 0$.

Proof. Indeed, since $F_0(U_{p,\varepsilon}) = 0$, we have the following estimate

$$\begin{aligned} \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}^2 &= \int_{\mathbb{R}^N} (V_\varepsilon - V(0))^2 U_{p,\varepsilon}^2 dx = \int_{\mathbb{R}^N} (V(\varepsilon x + p) - V(0))^2 U_0^2 dx \\ &\leq \max_{x \in B_{\varepsilon R}(p)} |V(x) - V(0)|^2 \int_{B_R(0)} U_0^2 dx + 4\|V\|_\infty \int_{\mathbb{R}^N \setminus B_R(0)} U_0^2 dx \\ &\leq C \left(\max_{x \in B_{\varepsilon R}(p)} |V(x) - V(0)|^2 + e^{-\mu R} \right) \end{aligned}$$

where we use the exponential decay of U_0 . Analogously, in view of the exponential decay of ∇U_0 , we have

$$\begin{aligned} \|DF_\varepsilon(U_{p,\varepsilon})\nabla U_{p,\varepsilon}\|_{L^2}^2 &= \int_{\mathbb{R}^N} (V(\varepsilon x + p) - V(0))^2 (\nabla U_0)^2 dx \\ &\leq C \left(\max_{x \in B_{\varepsilon R}(p)} |V(x) - V(0)|^2 + e^{-\mu R} \right). \end{aligned}$$

Then, to conclude the proof of the lemma, it is sufficient to choose $R = \frac{1}{\sqrt{\varepsilon}}$. $\square$

Now we consider the following expansion around the solution $U_{p,\varepsilon}$ of the unperturbed problem (1.2) (recall that $F_0(U_{p,\varepsilon}) = 0$ for any $p \in \mathbb{R}^N$ and $\varepsilon > 0$)

$$F_\varepsilon(U_{p,\varepsilon} + h) = F_\varepsilon(U_{p,\varepsilon}) + DF_\varepsilon(U_{p,\varepsilon})h + N_{p,\varepsilon}(h) \quad (3.1)$$

where

$$N_{p,\varepsilon}(h) = -2(W * (U_{p,\varepsilon}h))h - (W * h^2)(U_{p,\varepsilon} + h).$$

We look for functions $h \in H^2(\mathbb{R}^N)$ solving $F_\varepsilon(U_{p,\varepsilon} + h) = 0$. To this aim we follow a Liapunov-Schmidt type reduction as in [6].

We denote $K_{p,\varepsilon} = \text{Ker}[DF_0(U_{p,\varepsilon})] = \text{span}\{\partial_j U_{p,\varepsilon}, j = 1, \dots, N\}$. We consider the orthogonal decomposition $L^2(\mathbb{R}^N) = K_{p,\varepsilon} \oplus K_{p,\varepsilon}^\perp$ and we look for $h \in K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)$ solving the equation $F_\varepsilon(U_{p,\varepsilon} + h) = 0$.

We introduce the orthogonal projections $\pi_{p,\varepsilon} : L^2(\mathbb{R}^N) \rightarrow K_{p,\varepsilon}$ and $\pi_{p,\varepsilon}^\perp : L^2(\mathbb{R}^N) \rightarrow K_{p,\varepsilon}^\perp$ and we define the operator

$$T_{p,\varepsilon} = \pi_{p,\varepsilon}^\perp DF_\varepsilon(U_{p,\varepsilon}) \Big|_{K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)}$$

which satisfies the following properties:

Lemma 3.2. *There exist positive constants $C, \bar{r}$ and $\bar{\varepsilon}$ such that for any $p \in B_{\bar{r}}(0)$, any $\varepsilon \in (0, \bar{\varepsilon})$ and any $h \in K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)$,*

$$\|T_{p,\varepsilon}h\|_{L^2} \geq C\|h\|_{H^2}.$$

Furthermore, the operator $T_{p,\varepsilon} : K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N) \rightarrow K_{p,\varepsilon}^\perp$ is invertible with a bounded inverse (uniformly w.r.t. $p \in B_{\bar{r}}(0)$ and any $\varepsilon \in (0, \bar{\varepsilon})$).

Proof. Arguing by contradiction, we suppose that there exist sequences $|p_n| \downarrow 0$, $\varepsilon_n \downarrow 0$ and $h_n \in K_{p_n,\varepsilon_n}^\perp \cap H^2(\mathbb{R}^N)$ with $\|h_n\|_{H^2} = 1$ such that

$$\|T_{p_n,\varepsilon_n}h_n\|_{L^2} \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

Then, we consider the sequence $z_n = h_n(\cdot + \frac{p_n}{\varepsilon_n})$, $z_n \in K_0^\perp \cap H^2(\mathbb{R}^N)$, and $\|z_n\|_{H^2} = 1$. We have $z_n \rightharpoonup z_\infty$ weakly in H^2 , with $z_\infty \in K_0^\perp \cap H^2(\mathbb{R}^N)$.

Now we claim that $DF_0(U_0)z_\infty = 0$, which implies $z_\infty \in K_0 \cap K_0^\perp$, that is $z_\infty \equiv 0$.

Indeed, let us denote $V_n(x) = V_{\varepsilon_n}(x + \frac{p_n}{\varepsilon_n}) = V(\varepsilon_n x + p_n)$. For $h \in H^2(\mathbb{R}^N) \cap K_0^\perp$ we define

$$\begin{aligned} T_n h &= \pi_0^\perp(-\Delta h + V_n(x)h - 2(W * (U_0 h))U_0 - (W * U_0^2)h) \\ &= \pi_0^\perp(DF_0(U_0)h + (V_n(x) - V(0))h). \end{aligned}$$

Using (2.5) and since $\pi_0^\perp(\tau_{p_n/\varepsilon_n} f) = \tau_{p_n/\varepsilon_n}(\pi_{p_n,\varepsilon_n}^\perp f)$ an easy calculation shows that

$$T_n z_n = (T_{p_n,\varepsilon_n} h_n)(\cdot + \frac{p_n}{\varepsilon_n});$$

hence by the contradiction assumption $\|T_n z_n\|_{L^2} \rightarrow 0$.

Then, for any compact set K , we have

$$\begin{aligned} \|DF_0(U_0)z_n\|_{L^2(K)} &= \|\pi_0^\perp DF_0(U_0)z_n\|_{L^2(K)} \leq \|T_n z_n\|_{L^2} \\ &\quad + \|\pi_0^\perp (V_n - V(0))z_n\|_{L^2(K)} \end{aligned}$$

and, since $(V_n - V(0)) \rightarrow 0$ uniformly on compact sets and $\|z_n\|_{L^2}$ is bounded, we may conclude that $DF_0(U_0)z_n \rightarrow 0$ in L^2_{loc} . Moreover, $z_n \rightharpoonup z_\infty$ weakly in H^2 implies $DF_0(U_0)z_n \rightharpoonup DF_0(U_0)z_\infty$ weakly in L^2 , namely $DF_0(U_0)z_\infty = 0$ on any compact set, that is $DF_0(U_0)z_\infty = 0$, which implies $z_\infty \in K_0 \cap K_0^\perp = \{0\}$, as claimed.

Then, by Lemma 2.1 we have, up to subsequences, $(W * (U_0 z_n))U_0 \rightarrow 0$ and $(W * U_0^2)z_n \rightarrow 0$ in L^2 . Hence, if we denote $B_n = -\Delta + V_n(x)$ we have $\pi_0^\perp B_n z_n \rightarrow 0$ in L^2 , up to subsequences.

On the other hand, since B_n has a uniformly bounded inverse, there exists a positive constant γ such that $\|B_n z_n\|_{L^2} \geq \gamma$. Therefore, we get $\|\pi_0 B_n z_n\|_{L^2} \geq \frac{\gamma}{2}$, for all n sufficiently large, namely there exists $\alpha \in \mathbb{R}^N$ such that, up to subsequences,

$$\langle \alpha \cdot \nabla U_0, B_n z_n \rangle_{L^2} \geq 1. \quad (3.2)$$

We are now ready to derive the contradiction. Indeed, since $DF_0(U_0)(\alpha \cdot \nabla U_0) = 0$, we have

$$\begin{aligned} \langle \alpha \cdot \nabla U_0, B_n z_n \rangle_{L^2} &= \langle B_n (\alpha \cdot \nabla U_0), z_n \rangle_{L^2} = \langle (V_n - V(0))\alpha \cdot \nabla U_0, z_n \rangle_{L^2} \\ &\quad + \langle (2W * (U_0 \alpha \cdot \nabla U_0))U_0, z_n \rangle_{L^2} \\ &\quad + \langle (W * U_0^2)\alpha \cdot \nabla U_0, z_n \rangle_{L^2}. \end{aligned}$$

Since $\|(V_n - V(0))\alpha \cdot \nabla U_0\|_{L^2} \rightarrow 0$, as $n \rightarrow +\infty$ and $z_n \rightharpoonup 0$ weakly in L^2 , we get $\langle \alpha \cdot \nabla U_0, B_n z_n \rangle_{L^2} \rightarrow 0$, as $n \rightarrow +\infty$ that is in contradiction with (3.2).

Therefore we may conclude that the operator $T_{p,\varepsilon}$ is injective. Furthermore, $T_{p,\varepsilon}$ is a Fredholm-type operator. Indeed, a straightforward consequence of Lemma 2.1 is that the linear operator $\hat{W}_{p,\varepsilon} : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ given by

$$\hat{W}_{p,\varepsilon} h = 2(W * (U_{p,\varepsilon} h))U_{p,\varepsilon} + (W * U_{p,\varepsilon}^2)h$$

is compact. Therefore, we may conclude that $T_{p,\varepsilon}$ is invertible and the inverse $T_{p,\varepsilon}^{-1}$ is uniformly bounded. $\square$

Now we use the Banach fixed point theorem to find a function $h \in K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)$ such that

$$\pi_{p,\varepsilon}^\perp F_\varepsilon(U_{p,\varepsilon} + h) = 0$$

for any $p \in B_r(0)$, with $r > 0$ sufficiently small and $\varepsilon > 0$ sufficiently small.

By the expansion (3.1), we have to solve the fixed point equation $f_{p,\varepsilon}(h) = h$ where $f_{p,\varepsilon} : H^2(\mathbb{R}^N) \rightarrow H^2(\mathbb{R}^N)$ is defined by

$$f_{p,\varepsilon}(h) = -T_{p,\varepsilon}^{-1} \pi_{p,\varepsilon}^\perp (F_\varepsilon(U_{p,\varepsilon}) + N_{p,\varepsilon}(h)). \quad (3.3)$$

To this aim we need first some estimates for the nonlinear map $N_{p,\varepsilon}$; we have

Lemma 3.3. *There exists a positive constant C (independent of p and ε) such that*

$$\begin{aligned} \|N_{p,\varepsilon}(h)\|_{L^2} &\leq C\|h\|_{H^2}^2 \\ \|N_{p,\varepsilon}(h) - N_{p,\varepsilon}(k)\|_{L^2} &\leq C \max\{\|h\|_{H^2}, \|k\|_{H^2}\} \|h - k\|_{H^2} \end{aligned}$$

for any $h, k \in H^2(\mathbb{R}^N)$ such that $\max\{\|h\|_{H^2}, \|k\|_{H^2}\} \leq 1$.

Proof. We have

$$N_{p,\varepsilon}(h) = -2(W * (U_{p,\varepsilon}h))h - (W * h^2)(U_{p,\varepsilon} + h).$$

By (2.3), we have the following estimates:

$$\begin{aligned} \|(W * (U_{p,\varepsilon}h))h\|_{L^2} &\leq \|W * (U_{p,\varepsilon}h)\|_{L^\infty} \|h\|_{L^2} \leq C\|U_0\|_{H^2} \|h\|_{H^2} \|h\|_{L^2}, \\ \|(W * h^2)U_{p,\varepsilon}\|_{L^2} &\leq \|W * h^2\|_{L^\infty} \|U_{p,\varepsilon}\|_{L^2} \leq C\|h\|_{H^2}^2 \|U_0\|_{L^2} \end{aligned}$$

and

$$\|(W * h^2)h\|_{L^2} \leq \|W * h^2\|_{L^\infty} \|h\|_{L^2} \leq C\|h\|_{H^2}^2 \|h\|_{L^2}.$$

Moreover, again by (2.3) we have

$$\begin{aligned} \|(W * (U_{p,\varepsilon}k))k - (W * (U_{p,\varepsilon}h))h\|_{L^2} \\ \leq \|W * (U_{p,\varepsilon}(k - h))\|_{L^\infty} \|k\|_{L^2} + \|W * (U_{p,\varepsilon}h)\|_{L^\infty} \|h - k\|_{L^2} \\ \leq C(\|U_0\|_{H^2} \|h - k\|_{H^2} \|k\|_{L^2} + \|U_0\|_{H^2} \|h\|_{H^2} \|h - k\|_{L^2}), \end{aligned}$$

$$\begin{aligned} \|(W * k^2)U_{p,\varepsilon} - (W * h^2)U_{p,\varepsilon}\|_{L^2} &\leq \|W * (k^2 - h^2)\|_{L^\infty} \|U_{p,\varepsilon}\|_{L^2} \\ &\leq C\|h + k\|_{H^2} \|h - k\|_{H^2} \|U_0\|_{L^2} \end{aligned}$$

and

$$\begin{aligned} \|(W * k^2)k - (W * h^2)h\|_{L^2} &\leq \|W * (k^2 - h^2)\|_{L^\infty} \|k\|_{L^2} \|W * h^2\|_{L^\infty} \|k - h\|_{L^2} \\ &\leq C(\|h + k\|_{H^2} \|h - k\|_{H^2} \|k\|_{L^2} + \|h\|_{H^2}^2 \|k - h\|_{L^2}). \end{aligned}$$

Then, in view of all the estimates above the lemma easily follows. $\square$

Now we apply the Banach Theorem to the fixed point equation $f_{p,\varepsilon}(h) = h$.

Proposition 3.4. *There exist positive constants C , $\bar{r}$, $\bar{\varepsilon} > 0$ such that for any $p \in B_{\bar{r}}(0)$ and $\varepsilon \in (0, \bar{\varepsilon})$ there exists unique $h_{p,\varepsilon} \in K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)$ such that $F_\varepsilon(U_{p,\varepsilon} + h_{p,\varepsilon}) \in K_{p,\varepsilon}$. Moreover, $h_{p,\varepsilon}$ is continuous in $p \in B_{\bar{r}}(0)$ and*

$$\|h_{p,\varepsilon}\|_{H^2} \leq C\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}. \quad (3.4)$$

Proof. First we claim that the map $f_{p,\varepsilon}$ maps the closed ball $\mathcal{B}_\rho = \{h \in H^2(\mathbb{R}^N) : \|h\|_{H^2} \leq \rho\}$ into itself, for some $\rho \in (0, 1)$.

Indeed, for $h \in \mathcal{B}_\rho$, by Lemmas 3.2 and 3.3, we have

$$\begin{aligned} \|f_{p,\varepsilon}(h)\|_{H^2} &\leq \|T_{p,\varepsilon}^{-1}\| \|\pi_{p,\varepsilon}^\perp(F_\varepsilon(U_{p,\varepsilon}) + N_{p,\varepsilon}(h))\|_{L^2} \\ &\leq C(\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2} + \|h\|_{H^2}^2). \end{aligned}$$

Hence, choosing $\rho > 0$ sufficiently small and by Lemma 3.1, there exist $\bar{r} > 0$ and $\bar{\varepsilon} > 0$ such that for $p \in B_{\bar{r}}(0)$ and $\varepsilon \in (0, \bar{\varepsilon})$ we get

$$\|f_{p,\varepsilon}(h)\|_{H^2} \leq C(\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2} + \|h\|_{H^2}^2) \leq \rho$$

for any $\|h\|_{H^2} \leq \rho$.

Moreover, by Lemmas 3.2 and 3.3, for any $h, k \in \mathcal{B}_\rho$, we have

$$\|f_{p,\varepsilon}(h) - f_{p,\varepsilon}(k)\|_{H^2} \leq \|T_{p,\varepsilon}^{-1}\| \|N_{p,\varepsilon}(h) - N_{p,\varepsilon}(k)\|_{L^2} \leq C\rho \|h - k\|_{H^2};$$

therefore choosing $\rho > 0$ eventually smaller, we get

$$\|f_{p,\varepsilon}(h) - f_{p,\varepsilon}(k)\|_{H^2} \leq \frac{1}{2} \|h - k\|_{H^2}$$

and we may conclude that $f_{p,\varepsilon}$ is a contraction in $\mathcal{B}_\rho$; hence, by the Banach fixed point theorem, there exists unique $h_{p,\varepsilon} \in \mathcal{B}_\rho$ such that $f_{p,\varepsilon}(h_{p,\varepsilon}) = h_{p,\varepsilon}$. In particular, since for any $h \in H^2(\mathbb{R}^N)$ we have $f_{p,\varepsilon}(h) \in K_{p,\varepsilon}^\perp \cap H^2(\mathbb{R}^N)$, we have $h_{p,\varepsilon} \in K_{p,\varepsilon}^\perp \cap \mathcal{B}_\rho$.

Now, given $p_0, p \in B_{\bar{r}}(0)$ we have

$$\begin{aligned} \|h_{p,\varepsilon} - h_{p_0,\varepsilon}\|_{H^2} &= \|f_{p,\varepsilon}(h_{p,\varepsilon}) - f_{p_0,\varepsilon}(h_{p_0,\varepsilon})\|_{H^2} \\ &\leq \|f_{p,\varepsilon}(h_{p,\varepsilon}) - f_{p,\varepsilon}(h_{p_0,\varepsilon})\|_{H^2} + \|f_{p,\varepsilon}(h_{p_0,\varepsilon}) - f_{p_0,\varepsilon}(h_{p_0,\varepsilon})\|_{H^2} \\ &\leq \frac{1}{2} \|h_{p,\varepsilon} - h_{p_0,\varepsilon}\|_{H^2} + \|f_{p,\varepsilon}(h_{p_0,\varepsilon}) - f_{p_0,\varepsilon}(h_{p_0,\varepsilon})\|_{H^2}. \end{aligned}$$

Hence, we get

$$\|h_{p,\varepsilon} - h_{p_0,\varepsilon}\|_{H^2} \leq 2\|f_{p,\varepsilon}(h_{p_0,\varepsilon}) - f_{p_0,\varepsilon}(h_{p_0,\varepsilon})\|_{H^2}.$$

Therefore the continuity of $h_{p,\varepsilon}$ with respect to $p \in B_{\bar{r}}(0)$ follows directly from the continuity of $f_{p,\varepsilon}(h)$. To verify that $f_{p,\varepsilon}(h)$ is in fact continuous, note that the continuity w.r.t. p of $U_{p,\varepsilon}$ and its derivatives implies the continuity of $F_\varepsilon(U_{p,\varepsilon})$, $N_{p,\varepsilon}(h)$, $DF_\varepsilon(U_{p,\varepsilon})$ and of the projections $\pi_{p,\varepsilon}^\perp, \pi_{p,\varepsilon}$.

Finally we have the following estimate on the fixed point,

$$\begin{aligned} \|h_{p,\varepsilon}\|_{H^2} &= \|f_{p,\varepsilon}(h_{p,\varepsilon})\|_{H^2} \leq \|f_{p,\varepsilon}(h_{p,\varepsilon}) - f_{p,\varepsilon}(0)\|_{H^2} + \|f_{p,\varepsilon}(0)\|_{H^2} \\ &\leq \frac{1}{2} \|h_{p,\varepsilon}\|_{H^2} + \|T_{p,\varepsilon}^{-1}\| \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2} \end{aligned}$$

and we may conclude

$$\|h_{p,\varepsilon}\|_{H^2} \leq C\|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}.$$

Consequently, by Lemma 3.1, we get $\|h_{p,\varepsilon}\|_{H^2} \rightarrow 0$ as $(p, \varepsilon) \rightarrow 0$. $\square$

To complete the proof of the existence of a solution for (1.6) we have to study the following finite dimensional problem: for any $\varepsilon \in (0, \bar{\varepsilon})$ find $p \in B_{\bar{r}}(0)$ such that $\pi_{p,\varepsilon} F_\varepsilon(U_{p,\varepsilon} + h_{p,\varepsilon}) = 0$, where $\pi_{p,\varepsilon}$ is the projection into the kernel $K_{p,\varepsilon}$.

To this aim, for any $\varepsilon \in (0, \bar{\varepsilon})$ we consider the vector valued function $s_\varepsilon : B_{\bar{r}}(0) \rightarrow \mathbb{R}^N$ given by

$$\begin{aligned} s_\varepsilon(p) &= \frac{1}{\varepsilon} \langle F_\varepsilon(U_{p,\varepsilon} + h_{p,\varepsilon}), \nabla U_{p,\varepsilon} \rangle_{L^2} \\ &= \frac{1}{\varepsilon} (\langle F_\varepsilon(U_{p,\varepsilon} + h_{p,\varepsilon}), \partial_1 U_{p,\varepsilon} \rangle_{L^2}, \dots, \langle F_\varepsilon(U_{p,\varepsilon} + h_{p,\varepsilon}), \partial_N U_{p,\varepsilon} \rangle_{L^2}) \end{aligned}$$

and the rescaled function v_ε defined on $B_1(0) \subset \mathbb{R}^N$ given by

$$v_\varepsilon(p) = \varepsilon^{-\alpha} s_\varepsilon(\varepsilon^\alpha p),$$

with $1 < \alpha < 2$.

We look for $p \in B_1(0)$ such that $v_\varepsilon(p) = 0$. To this aim, let us introduce the linear vector valued function $v_0 : \mathbb{R}^N \rightarrow \mathbb{R}^N$ given by

$$v_0(p) = -\frac{1}{2} \|U_0\|_{L^2}^2 V''(0)p,$$

where $V''(0)$ is the Hessian matrix of V at the origin. We have the following result:

Proposition 3.5. *The functions v_ε converge uniformly to v_0 on $B_1(0)$.*

Proof. By the expansion (3.1), we have

$$\begin{aligned} \varepsilon s_\varepsilon(p) &= \langle F_\varepsilon(U_{p,\varepsilon}), \nabla U_{p,\varepsilon} \rangle_{L^2} + \langle DF_\varepsilon(U_{p,\varepsilon})h_{p,\varepsilon}, \nabla U_{p,\varepsilon} \rangle_{L^2} \\ &\quad + \langle N_{p,\varepsilon}(h_{p,\varepsilon}), \nabla U_{p,\varepsilon} \rangle_{L^2}. \end{aligned} \quad (3.5)$$

First, we consider the second term in (3.5), $e_2(p) = \langle DF_\varepsilon(U_{p,\varepsilon})h_{p,\varepsilon}, \nabla U_{p,\varepsilon} \rangle_{L^2}$; since $DF_\varepsilon(U_{p,\varepsilon})$ is a self adjoint operator, we have

$$|e_2(p)| = |\langle h_{p,\varepsilon}, DF_\varepsilon(U_{p,\varepsilon}) \nabla U_{p,\varepsilon} \rangle_{L^2}| \leq \|DF_\varepsilon(U_{p,\varepsilon}) \nabla U_{p,\varepsilon}\|_{L^2} \|h_{p,\varepsilon}\|_{L^2},$$

and by Proposition 3.4,

$$|e_2(p)| \leq C \|DF_\varepsilon(U_{p,\varepsilon}) \nabla U_{p,\varepsilon}\|_{L^2} \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}.$$

Moreover, by Lemma 3.3 and Proposition 3.4, we have

$$|e_3(p)| = |\langle N_{p,\varepsilon}(h_{p,\varepsilon}), \nabla U_{p,\varepsilon} \rangle_{L^2}| \leq C \|h_{p,\varepsilon}\|_{H^2}^2 \leq C \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2}^2$$

where $e_3(p)$ denotes the third term in (3.5).

Therefore, we have

$$|e_2(p) + e_3(p)| \leq C \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2} (\|DF_\varepsilon(U_{p,\varepsilon}) \nabla U_{p,\varepsilon}\|_{L^2} + \|F_\varepsilon(U_{p,\varepsilon})\|_{L^2});$$

hence by Lemma 3.1, for any $\rho > 0$

$$|e_2(p) + e_3(p)| \leq C(e^{-\mu\rho} + \max_{x \in B_{\varepsilon\rho}(p)} |V(x) - V(0)|^2).$$

Now, since $V'(0) = 0$, we have

$$\max_{x \in B_{\varepsilon\rho}(p)} |V(x) - V(0)|^2 \leq C(\varepsilon\rho + |p|)^4 \quad (3.6)$$

and by rescaling, we may conclude that

$$|e_2(\varepsilon^\alpha p) + e_3(\varepsilon^\alpha p)| \leq C(e^{-\mu\rho} + (\varepsilon\rho + \varepsilon^\alpha |p|)^4).$$

Finally, let us consider the first term in (3.5) $e_1(p) = \langle F_\varepsilon(U_{p,\varepsilon}), \nabla U_{p,\varepsilon} \rangle_{L^2}$. Recalling that $F_0(U_{p,\varepsilon}) = 0$ and by integration by parts, we have

$$\begin{aligned} e_1(p) &= \langle (V_\varepsilon - V(0))U_{p,\varepsilon}, \nabla U_{p,\varepsilon} \rangle_{L^2} = -\frac{1}{2} \langle (\nabla V_\varepsilon)U_{p,\varepsilon}, U_{p,\varepsilon} \rangle_{L^2} \\ &= -\frac{\varepsilon}{2} \int_{\mathbb{R}^N} \nabla V(\varepsilon x + p) |U_0(x)|^2 dx. \end{aligned}$$

Then, recalling that U_0 is a radial function we have

$$\begin{aligned} \frac{1}{\varepsilon} e_1(p) - v_0(p) &= \frac{1}{2} \int_{\mathbb{R}^N} [V''(0)p - \nabla V(\varepsilon x + p)] |U_0(x)|^2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^N} [V''(0)(\varepsilon x + p) - \nabla V(\varepsilon x + p)] |U_0(x)|^2 dx; \end{aligned}$$

hence, by Taylor's expansion, since V and its derivatives are bounded and U_0 decays exponentially, we get

$$\begin{aligned} \left| \frac{1}{\varepsilon} e_1(p) - v_0(p) \right| &\leq C \int_{\mathbb{R}^N} |\varepsilon x + p|^2 |U_0(x)|^2 dx \leq C \int_{B_\rho(0)} |\varepsilon x + p|^2 |U_0(x)|^2 dx \\ &\quad + C \int_{\mathbb{R}^N \setminus B_\rho(0)} |\varepsilon x + p|^2 |U_0(x)|^2 dx \leq C((\varepsilon\rho + |p|)^2 + e^{-\mu\rho}) \end{aligned}$$

where $\mu > 0$ is a suitable constant.

Then by rescaling we obtain

$$\left| \frac{1}{\varepsilon} e_1(\varepsilon^\alpha p) - v_0(\varepsilon^\alpha p) \right| \leq C((\varepsilon\rho + \varepsilon^\alpha |p|)^2 + e^{-\mu\rho}).$$

Collecting all the estimates above, we get

$$\begin{aligned} |v_\varepsilon(p) - v_0(p)| &\leq |\varepsilon^{-\alpha}(s_\varepsilon(\varepsilon^\alpha p) - v_0(\varepsilon^\alpha p))| \\ &\leq \varepsilon^{-\alpha} \left| \frac{1}{\varepsilon} e_1(\varepsilon^\alpha p) - v_0(\varepsilon^\alpha p) \right| + \varepsilon^{-\alpha-1} |e_2(\varepsilon^\alpha p) + e_3(\varepsilon^\alpha p)| \\ &\leq C_1 \varepsilon^{-\alpha} [(\varepsilon^\alpha |p| + \rho\varepsilon)^2 + e^{-\mu\rho}] + C_2 \varepsilon^{-\alpha-1} [e^{-\mu\rho} + (\varepsilon^\alpha |p| + \rho\varepsilon)^4]. \end{aligned}$$

Choosing $\rho = \varepsilon^{-\beta}$, with $\beta > 0$ and recalling that $|p| \leq 1$, we get

$$\begin{aligned} |v_\varepsilon(p) - v_0(p)| &\leq C_1 \varepsilon^{-\alpha} [(\varepsilon^\alpha + \varepsilon^{1-\beta})^2 + e^{-\mu\varepsilon^{-\beta}}] \\ &\quad + C_2 \varepsilon^{-\alpha-1} [(\varepsilon^\alpha + \varepsilon^{1-\beta})^4 + e^{-\mu\varepsilon^{-\beta}}]. \end{aligned}$$

Since we choose $1 < \alpha < 2$, for $\beta > 0$ sufficiently small we may conclude that $\max_{p \in B_1(0)} |v_\varepsilon(p) - v_0(p)| \rightarrow 0$ as $\varepsilon \rightarrow 0$. $\square$

Proof of Theorem 1.1. To conclude the proof of the existence of a solution for equation (1.6) we use some simple facts of degree's theory. Indeed, since the Hessian matrix $V''(0)$ is positive definite, we have that $\deg(B_1(0), v_0, 0) = 1$; therefore by Proposition 3.5 we may conclude that $\deg(B_1(0), v_\varepsilon, 0) = 1$, for any $\varepsilon > 0$ sufficiently small. Hence v_ε must have a zero in $\bar{p}_\varepsilon \in B_1(0)$, for $\varepsilon > 0$ sufficiently

small. By definition of the function v_ε , such a zero identifies a point $p_\varepsilon = \varepsilon^\alpha \bar{p}_\varepsilon$ such that

$$\langle F_\varepsilon(U_{p_\varepsilon, \varepsilon} + h_{p_\varepsilon, \varepsilon}), \partial_j U_{p_\varepsilon, \varepsilon} \rangle_{L^2} = 0 \quad j = 1, \dots, N$$

and $|p_\varepsilon| = O(\varepsilon^\alpha)$, with $1 < \alpha < 2$.

Hence, in view of Proposition 3.4, we may conclude that $F_\varepsilon(U_{p_\varepsilon, \varepsilon} + h_{p_\varepsilon, \varepsilon}) = 0$, namely $U_\varepsilon(x) = U_{p_\varepsilon, \varepsilon}(x) + h_{p_\varepsilon, \varepsilon}(x) = U_0(x - \hat{p}_\varepsilon) + h_{p_\varepsilon, \varepsilon}(x)$ is a solution of (1.6), where $\hat{p}_\varepsilon = \frac{p_\varepsilon}{\varepsilon}$, $|\hat{p}_\varepsilon| = O(\varepsilon^{\alpha-1})$, with $\alpha \in (1, 2)$.

Moreover, by Proposition 3.4 we have

$$\begin{aligned} \|h_{p_\varepsilon, \varepsilon}\|_{H^2}^2 &\leq C \|F_\varepsilon(U_{p_\varepsilon, \varepsilon})\|_{L^2}^2 = C \left(\int_{\mathbb{R}^N} V_\varepsilon(x) - V(0) \right)^2 U_0^2(x - \hat{p}_\varepsilon) dx \\ &\leq C \varepsilon^4 \int_{\mathbb{R}^N} |x|^4 U_0^2(x - \hat{p}_\varepsilon) dx \leq C \varepsilon^4. \end{aligned} \quad (3.7)$$

Therefore, we may conclude in particular that $U_\varepsilon(\cdot + \hat{p}_\varepsilon) \rightarrow U_0$ strongly in $H^2(\mathbb{R}^N)$ and $\|U_\varepsilon - U_{p_\varepsilon, \varepsilon}\|_{H^2} = \|h_{p_\varepsilon, \varepsilon}\|_{H^2}^2 = O(\varepsilon^2)$.

Moreover, using the equations (1.2) and (1.6) for U_ε and $U_{p_\varepsilon, \varepsilon}$ respectively, we have that $h_{p_\varepsilon, \varepsilon}$ satisfies an elliptic equation of the type $(-\Delta + V_\varepsilon)h_{p_\varepsilon, \varepsilon} = f_\varepsilon$, and in view of (2.2), we have $\|f_\varepsilon\|_{L^2} \leq \|(V(0) - V_\varepsilon)U_{p_\varepsilon, \varepsilon}\|_{L^2} + C\|h_{p_\varepsilon, \varepsilon}\|_{H^1}$. Then, by induction, we have for any $k \in \mathbb{N}$

$$\|f_\varepsilon\|_{H^k} \leq \|(V(0) - V_\varepsilon)U_{p_\varepsilon, \varepsilon}\|_{H^k} + C\|h_{p_\varepsilon, \varepsilon}\|_{H^{k+1}},$$

hence, in particular $f_\varepsilon \in H^{k-1}(\mathbb{R}^N)$ whenever $h_{p_\varepsilon, \varepsilon} \in H^k(\mathbb{R}^N)$.

Moreover, by standard elliptic estimates (see e.g. [11]), we get

$$\|h_{p_\varepsilon, \varepsilon}\|_{H^{k+2}} \leq C(\|(V(0) - V_\varepsilon)U_{p_\varepsilon, \varepsilon}\|_{H^k} + \|h_{p_\varepsilon, \varepsilon}\|_{H^{k+1}}).$$

Therefore, noting that, as $\varepsilon \rightarrow 0$, the leading order term of $\|(V(0) - V_\varepsilon)U_{p_\varepsilon, \varepsilon}\|_{H^k}$ is in fact $\|(V(0) - V_\varepsilon)U_{p_\varepsilon, \varepsilon}\|_{L^2} = \|F_\varepsilon(U_{p_\varepsilon, \varepsilon})\|_{L^2}$, in view of (3.7) and by iteration, we may conclude $\|h_{p_\varepsilon, \varepsilon}\|_{H^k} = O(\varepsilon^2)$, for any $k \in \mathbb{N}$, that is, by Sobolev embedding $\|h_{p_\varepsilon, \varepsilon}\|_{C^k} = O(\varepsilon^2)$, for any $k \in \mathbb{N}$. Consequently, $U_\varepsilon(\cdot + \hat{p}_\varepsilon) \rightarrow U_0$ strongly in $C^k(\mathbb{R}^N)$, for any $k \in \mathbb{N}$.

To conclude the theorem we show that U_ε and ∇U_ε decay exponentially to zero as $|x| \rightarrow +\infty$, uniformly w.r.t. ε , for $\varepsilon > 0$ sufficiently small.

To this aim, we first note that also $-U_\varepsilon$ is a solution of (1.6), and hence in particular $|U_\varepsilon|$ solves (1.6). Therefore we can always assume U_ε to be a non negative function, and by the maximum principle U_ε is in fact strictly positive.

Moreover, we claim that for any $\delta > 0$ sufficiently small, there exists $R_1 > 0$ and $\varepsilon_1 > 0$ such that $-\Delta U_\varepsilon + (V(0) - \delta)U_\varepsilon \leq 0$, for $|x| \geq R$, with $R > R_1$ and $\varepsilon \in (0, \varepsilon_1)$. Indeed, it is enough to note that, since $W(x) \rightarrow 0$ as $|x| \rightarrow +\infty$, $\sup_{|x| \geq R} |W * U_\varepsilon^2|$ can be made arbitrarily small, for $\varepsilon > 0$ sufficiently small and $R > 0$ sufficiently large.

Then, we consider a comparison function $f(x) = Ce^{-c|x|}$, where $c = \sqrt{V(0) - \delta}$ and $C = U_0|_{|x|=R}e^{cR} + 1$. We have $-\Delta f + (V(0) - \delta)f \geq 0$.

Now, if we apply the maximum principle to the function $g = f - U_\varepsilon$, since $g(x) \geq 0$ for $|x| = R$ and $\varepsilon > 0$ sufficiently small, we may conclude that

$$0 < U_\varepsilon(x) \leq f(x) = Ce^{-c|x|} \quad \forall |x| \geq R. \quad (3.8)$$

Finally, since $\|W * U_\varepsilon^2\|_\infty$ is uniformly bounded, the exponential decay of the gradient ∇U_ε follows by standard elliptic estimates (see e.g. Th. 3.9 in [11]). $\square$

4. Linear properties of the “one-particle” solution U_ε

In this section we study the differential operator $DF_\varepsilon(U_\varepsilon) : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$. We introduce the following notation

$$L_\varepsilon h \equiv DF_\varepsilon(U_\varepsilon)h = -\Delta h + V_\varepsilon h - 2(W * (U_\varepsilon h))U_\varepsilon - (W * (U_\varepsilon)^2)h$$

and $L_0 : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ defined by

$$L_0 h \equiv DF_0(U_0)h = -\Delta h + V(0)h - 2(W * (U_0 h))U_0 - (W * U_0^2)h.$$

Lemma 4.1. *There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ the operator L_ε is invertible. Moreover,*

- (i) *there exists a constant $\gamma > 0$ (independent of ε) such that*

$$\frac{1}{\varepsilon^2} \|L_\varepsilon h\|_{L^2} \geq \gamma \|h\|_{H^2}$$

for any $\varepsilon \in (0, \bar{\varepsilon})$ and any $h \in H^2(\mathbb{R}^N)$;

- (ii) *there exists a constant $K > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any $f \in L^2(\mathbb{R}^N)$ such that $\sup_j |\langle f, \partial_j U_\varepsilon \rangle_{L^2}| = o(\varepsilon^2) \|f\|_{L^2}$ we have*

$$\|L_\varepsilon^{-1} f\|_{H^2} \leq K \|f\|_{L^2}.$$

Proof. First we prove that L_ε is a Fredholm-type operator. Since the operator $B_\varepsilon = -\Delta + V_\varepsilon$ is invertible with a uniformly bounded inverse, it is convenient to write $L_\varepsilon = B_\varepsilon(\mathbb{I} - K_\varepsilon)$ where $K_\varepsilon : H^2(\mathbb{R}^N) \rightarrow H^2(\mathbb{R}^N)$ is defined by $K_\varepsilon h = B_\varepsilon^{-1}[2(W * (U_\varepsilon h))U_\varepsilon + (W * (U_\varepsilon)^2)h]$. Hence it is enough to prove that the operator K_ε is compact. Indeed, let us consider a sequence $w_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$, we have

$$\|K_\varepsilon w_n\|_{H^2} \leq C(\|(W * (U_\varepsilon w_n))U_\varepsilon\|_{L^2} + \|(W * (U_\varepsilon)^2)w_n\|_{L^2})$$

and by Lemma 2.1 we may conclude that $K_\varepsilon w_n \rightarrow 0$ in $H^2(\mathbb{R}^N)$.

Now we prove (i) which in particular implies that L_ε is injective and hence invertible. Arguing by contradiction, let us assume that there exist sequences $\varepsilon_n \downarrow 0$ and $\{h_n\} \subset H^2(\mathbb{R}^N)$, $\|h_n\|_{H^2} = 1$, such that

$$\frac{1}{\varepsilon_n^2} \|L_{\varepsilon_n} h_n\|_{L^2} \rightarrow 0 \quad (4.1)$$

and, up to subsequence, $h_n \rightharpoonup h_0$ weakly in H^2 .

We claim that, up to subsequence, $L_{\varepsilon_n} h_n \rightharpoonup L_0 h_0$ weakly in L^2 .

Indeed, for any $f \in L^2(\mathbb{R}^N)$, we have

$$\begin{aligned} |\langle L_{\varepsilon_n} h_n - L_0 h_0, f \rangle_{L^2}| &\leq |\langle \Delta(h_n - h_0), f \rangle_{L^2}| \\ &\quad + |\langle V(0)(h_n - h_0), f \rangle_{L^2}| + |\langle (V_{\varepsilon_n} - V(0))h_n, f \rangle_{L^2}| \\ &\quad + 2\|(W * (U_{\varepsilon_n} h_n))U_{\varepsilon_n} - (W * (U_0 h_0))U_0\|_{L^2} \|f\|_{L^2} \\ &\quad + \|(W * ((U_{\varepsilon_n})^2)h_n - (W * U_0^2)h_0)\|_{L^2} \|f\|_{L^2}. \end{aligned} \quad (4.2)$$

Since $h_n \rightharpoonup h_0$ weakly in H^2 , the first and the second term in the right hand side go to zero as $n \rightarrow +\infty$.

Let us consider the third term; we have

$$|\langle (V_{\varepsilon_n} - V(0))h_n, f \rangle_{L^2}| \leq \|(V_{\varepsilon_n} - V(0))f\|_{L^2},$$

and for any fixed $f \in L^2(\mathbb{R}^N)$, as $n \rightarrow +\infty$,

$$\|(V_{\varepsilon_n} - V(0))f\|_{L^2} \rightarrow 0. \quad (4.3)$$

Indeed, we have

$$\begin{aligned} \|(V_{\varepsilon_n} - V(0))f\|_{L^2}^2 &= \|(V_{\varepsilon_n} - V(0))f\|_{L^2(\bar{B}_R)}^2 + \|(V_{\varepsilon_n} - V(0))f\|_{L^2(\mathbb{R}^N \setminus \bar{B}_R)}^2 \\ &\leq \|V_{\varepsilon_n} - V(0)\|_{L^\infty(\bar{B}_R)}^2 \|f\|_{L^2(\bar{B}_R)}^2 + 4\|V\|_\infty^2 \|f\|_{L^2(\mathbb{R}^N \setminus \bar{B}_R)}^2 \end{aligned}$$

and each term in the right hand side can be made arbitrarily small fixing $R > 0$ sufficiently large and letting $n \rightarrow +\infty$.

Let us estimate the last two terms. We have

$$\begin{aligned} \|(W * (U_{\varepsilon_n} h_n))U_{\varepsilon_n} - (W * (U_0 h_0))U_0\|_{L^2} &\leq \|W * (U_{\varepsilon_n} h_n)\|_{L^\infty} \|U_{\varepsilon_n} - U_0\|_{L^2} \\ &\quad + \|(W * ((U_{\varepsilon_n} - U_0)h_n))U_0\|_{L^2} + \|(W * (U_0(h_n - h_0)))U_0\|_{L^2}. \end{aligned}$$

In view of (2.3) and Lemma 2.1 each term in the previous sum goes to zero as $n \rightarrow +\infty$ proving that

$$\|(W * (U_{\varepsilon_n} h_n))U_{\varepsilon_n} - (W * (U_0 h_0))U_0\|_{L^2} \rightarrow 0, \quad \text{as } n \rightarrow +\infty. \quad (4.4)$$

Finally, we have

$$\begin{aligned} \|(W * (U_{\varepsilon_n})^2)h_n - (W * U_0^2)h_0\|_{L^2} &\leq \|(W * ((U_{\varepsilon_n})^2 - U_0^2))h_n\|_{L^2} \\ &\quad + \|(W * U_0^2)(h_n - h_0)\|_{L^2} \end{aligned} \quad (4.5)$$

and again by Lemma 2.1 each term goes to zero as $n \rightarrow +\infty$.

Then we may conclude that $L_{\varepsilon_n} h_n \rightharpoonup L_0 h_0$ weakly in L^2 . Therefore, by (4.1), we have $L_0 h_0 = 0$, namely $h_0 \in \text{Ker } L_0$. Hence by assumption (H) $h_0 = a \cdot \nabla U_0$ for some $a \in \mathbb{R}^N$.

Now we claim that, up to subsequence, $h_n \rightarrow h_0$ strongly in H^2 , which in particular implies that $h_0 \neq 0$, namely that $a \neq 0$.

We consider

$$\begin{aligned} L_{\varepsilon_n} h_n - L_0 h_0 &= -\Delta(h_n - h_0) + V_{\varepsilon_n} h_n - V(0)h_0 - 2(W * (U_{\varepsilon_n} h_n))U_{\varepsilon_n} \\ &\quad + 2(W * (U_0 h_0))U_0 - (W * (U_{\varepsilon_n})^2)h_n + (W * U_0^2)h_0; \end{aligned}$$

then, since $L_0 h_0 = 0$, we have

$$\begin{aligned} \|(-\Delta + V_{\varepsilon_n})(h_n - h_0)\|_{L^2} &\leq \|L_{\varepsilon_n} h_n\|_{L^2} + \|(V(0) - V_{\varepsilon_n})h_0\|_{L^2} \\ &\quad + 2\|(W * (U_{\varepsilon_n} h_n))U_{\varepsilon_n} - (W * (U_0 h_0))U_0\|_{L^2} \\ &\quad + \|(W * (U_{\varepsilon_n})^2)h_n - (W * U_0^2)h_0\|_{L^2}; \end{aligned} \quad (4.6)$$

in view of (4.1), (4.3), (4.4) and (4.5) each term in the right side of (4.6) goes to zero, as $n \rightarrow +\infty$.

Therefore, since the operator $-\Delta + V_{\varepsilon_n}$ is invertible, with a uniformly bounded inverse, we may conclude that, as $n \rightarrow +\infty$,

$$\|h_n - h_0\|_{H^2} \leq C\|(-\Delta + V_{\varepsilon_n})(h_n - h_0)\|_{L^2} \rightarrow 0,$$

proving the claim.

Now we introduce the sequence $z_n = a \cdot \nabla U_{\varepsilon_n}$, we have $L_{\varepsilon_n} z_n = -(a \cdot \nabla V_{\varepsilon_n})U_{\varepsilon_n}$. Moreover, since W is a radial function, L_{ε_n} is a self-adjoint operator in L^2 , hence we have

$$\begin{aligned} |\langle L_{\varepsilon_n} h_n, z_n \rangle_{L^2}| &= |\langle z_n, L_{\varepsilon_n} z_n \rangle_{L^2} + \langle h_n - z_n, L_{\varepsilon_n} z_n \rangle_{L^2}| \\ &\geq |\langle a \cdot \nabla U_{\varepsilon_n}, (a \cdot \nabla V_{\varepsilon_n})U_{\varepsilon_n} \rangle_{L^2}| - \|h_n - z_n\|_{L^2} \|L_{\varepsilon_n} z_n\|_{L^2} \\ &= \frac{1}{2} |\langle U_{\varepsilon_n}, V_{\varepsilon_n}''[a, a]U_{\varepsilon_n} \rangle_{L^2}| - \|h_n - z_n\|_{L^2} \|L_{\varepsilon_n} z_n\|_{L^2} \end{aligned}$$

where we used the notation $V_{\varepsilon_n}''[a, a] = \sum_{i,j} a_i a_j \frac{\partial^2 V_{\varepsilon_n}}{\partial x_i \partial x_j}$.

Since, as $n \rightarrow +\infty$,

$$\|h_n - z_n\|_{L^2} \leq \|h_n - h_0\|_{L^2} + \|a \cdot \nabla (U_0 - U_{\varepsilon_n})\|_{L^2} \rightarrow 0$$

and, by a Taylor's expansion,

$$\|L_{\varepsilon_n} z_n\|_{L^2} = \|(a \cdot \nabla V_{\varepsilon_n})U_{\varepsilon_n}\|_{L^2} = \varepsilon_n^2 \|U_{\varepsilon_n} V''(0)[a, \cdot]\|_{L^2} + o(\varepsilon_n^2),$$

then, using the contradiction assumption, we get

$$\frac{1}{2} |\langle U_{\varepsilon_n}, V_{\varepsilon_n}''[a, a]U_{\varepsilon_n} \rangle_{L^2}| \leq \|L_{\varepsilon_n} h_n\|_{L^2} \|z_n\|_{L^2} + \|h_n - z_n\|_{L^2} \|L_{\varepsilon_n} z_n\|_{L^2} = o(\varepsilon_n^2).$$

Then, we may conclude that, as $n \rightarrow +\infty$,

$$\frac{1}{\varepsilon_n^2} |\langle U_{\varepsilon_n}, V_{\varepsilon_n}''[a, a]U_{\varepsilon_n} \rangle_{L^2}| \rightarrow 0.$$

On the other hand, since U_{ε_n} decays exponentially, and using a Taylor's expansion for $V_{\varepsilon_n}''(x)$, we have, for any $r > 0$

$$\begin{aligned} \frac{1}{\varepsilon_n^2} |\langle U_{\varepsilon_n}, V_{\varepsilon_n}''[a, a]U_{\varepsilon_n} \rangle_{L^2}| &\geq \frac{1}{\varepsilon_n^2} \left| \int_{B_{r/\varepsilon_n}} V_{\varepsilon_n}''(x)[a, a]U_{\varepsilon_n}^2 dx \right| + o(1) \\ &\geq \left| \int_{B_{r/\varepsilon_n}} V''(0)[a, a]U_{\varepsilon_n}^2 dx \right| + o(1). \end{aligned} \quad (4.7)$$

Then, since $V''(0)$ is positive definite, passing to the limit as $n \rightarrow +\infty$, we get

$$0 \geq \int_{\mathbb{R}^N} V''(0)[a, a]U_0^2 dx > 0, \quad (4.8)$$

a contradiction.

To prove (ii), assume by contradiction that there exist sequences $\varepsilon_n \downarrow 0$ and $\{h_n\} \subset H^2(\mathbb{R}^N)$, $\|h_n\|_{H^2} = 1$ such that $\frac{1}{\varepsilon_n^2} |\langle L_{\varepsilon_n} h_n, \nabla U_{\varepsilon_n} \rangle_{L^2}| \rightarrow 0$ and $\|L_{\varepsilon_n} h_n\|_{L^2} \rightarrow 0$ as $n \rightarrow +\infty$.

Arguing exactly as above we may conclude that $L_{\varepsilon_n} h_n \rightharpoonup L_0 h_0$ weakly in L^2 . Therefore, by the contradiction assumption, we have again $L_0 h_0 = 0$, which implies that $h_0 \in \text{Ker } L_0$, namely that $h_0 = a \cdot \nabla U_0$ for some $a \in \mathbb{R}^N$ and again as before, we may conclude that, up to subsequence, $h_n \rightarrow h_0$ strongly in H^2 , which in particular implies again that $a \neq 0$. Again, arguing as before, we introduce $z_n = a \cdot \nabla U_{\varepsilon_n}$ and we have

$$\begin{aligned} |\langle L_{\varepsilon_n} h_n, z_n \rangle_{L^2}| &\geq \frac{1}{2} |\langle U_{\varepsilon_n}, V''_{\varepsilon_n}[a, a]U_{\varepsilon_n} \rangle_{L^2}| - \|h_n - z_n\|_{L^2} \|L_{\varepsilon_n} z_n\|_{L^2}, \\ \|h_n - z_n\|_{L^2} &\rightarrow 0 \quad \text{as } n \rightarrow +\infty \end{aligned}$$

and, by a Taylor's expansion,

$$\|L_{\varepsilon_n} z_n\|_{L^2} = \|(a \cdot \nabla V_{\varepsilon_n})U_{\varepsilon_n}\|_{L^2} = \varepsilon_n^2 \|U_{\varepsilon_n} V''(0)[a, \cdot]\|_{L^2} + o(\varepsilon_n^2).$$

In view of (4.7) and (4.8) we get for any $n \in \mathbb{N}$ sufficiently large

$$\frac{1}{\varepsilon_n^2} |\langle L_{\varepsilon_n} h_n, z_n \rangle| \geq \frac{1}{2} |\langle U_{\varepsilon_n}, V''_{\varepsilon_n}[a, a]U_{\varepsilon_n} \rangle_{L^2}| + o(1) > \text{const.} > 0.$$

On the other hand, by the contradiction assumption, we have, as $n \rightarrow +\infty$,

$$\frac{1}{\varepsilon_n^2} |\langle L_{\varepsilon_n} h_n, z_n \rangle| \leq \frac{|a|}{\varepsilon_n^2} \sup_j |\langle L_{\varepsilon_n} h_n, \partial_j U_{\varepsilon_n} \rangle| \rightarrow 0$$

obtaining a contradiction which proves (ii). $\square$

As a direct consequence of Lemma 4.1-(ii) in view of the (uniform w.r.t. ε) exponential decay of ∇U_{ε} , we have an uniform estimate on L_{ε}^{-1} acting on functions supported sufficiently far away from the origin. More precisely, we have

Lemma 4.2. *Let $\rho > 0$, there exist $\bar{\varepsilon} > 0$ and $K > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and for any $f \in L^2(\mathbb{R}^N)$ with $\text{supp } f \subset \mathbb{R}^N \setminus B_{\rho/\varepsilon}(0)$, then $\|L_{\varepsilon}^{-1} f\|_{H^2} \leq K \|f\|_{L^2}$.*

Proof. By Hölder inequality and the (uniform w.r.t. ε) exponential decay of ∇U_{ε} , we have

$$\sup_j |\langle f, \partial_j U_{\varepsilon} \rangle_{L^2}| \leq \|f\|_{L^2} \|\nabla U_{\varepsilon}\|_{L^2(\mathbb{R}^N \setminus B_{\rho/\varepsilon}(0))} \leq C e^{-\frac{\rho}{\varepsilon}} \|f\|_{L^2}, \quad (4.9)$$

hence the assumption of (ii)-Lemma 4.1 holds. $\square$

5. “Multi-particles” solutions

In this section we prove the existence of a solution u of equation (1.6) of “multi-particle” type. We introduce the following notation $\hat{p}_j = p_j/\varepsilon$, where $p_j \in \mathbb{Z}^N$ are minimum points of the external potential V , and $\hat{U}_j(x) = U_\varepsilon(x - \hat{p}_j)$. Note that by the $\mathbb{Z}^N$ -periodicity of the potential V , each function $\hat{U}_j$ is a solution of equation (1.6).

We look for solutions to equation (1.6) of the following form:

$$u = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \quad (5.1)$$

where $\mathcal{P} \subseteq \mathbb{N}$, the functions $\hat{\varphi}_j$ belong to a suitable partition of unity and z is a small function in a suitable norm.

To this aim we assume in addition that $W(x) = O(\frac{1}{|x|^\beta})$, as $|x| \rightarrow +\infty$, with $\beta > \max\{4, N\}$ ((h4)).

5.1. Partition of unity and Banach spaces

Let us consider a set $\{p_j\}_{j \in \mathcal{P}} \subset \mathbb{Z}^N$ of nondegenerate minimum points of V . Since V is smooth and periodic, there exists $r_0 > 0$ such that $V''(x)$ is positive definite for any $x \in B_{r_0}(p_j)$ and any $j \in \mathcal{P}$.

Then, we consider a small N -cube centered at the origin $K = (-l, l)^N$, $0 < l \leq \min\{r_0, \frac{1}{32}\}$, and we set $P_j = K + p_j$. We have $P_j \cap P_{j'} = \emptyset$, for every $j' \neq j$. Moreover, we denote by $\{q_k\}_{k \in \mathcal{Q}}$, for $\mathcal{Q} \subseteq \mathbb{N}$, the collection of points of the rescaled lattice $\frac{1}{m}\mathbb{Z}^N \setminus \{p_j\}_{j \in \mathcal{P}}$, for some $m \in \mathbb{N}$ such that $\frac{8}{15l} < m < \frac{4}{7l}$, and we set $Q_k = K + q_k$. The collection of sets $\{P_j, Q_k\}_{\mathcal{P} \times \mathcal{Q}}$ is a locally finite open covering of $\mathbb{R}^N$. We consider an associated partition of unity $\{\varphi_j, \psi_k\}_{\mathcal{P} \times \mathcal{Q}}$ defined as follows: let $0 \leq \xi \in C_c^\infty(K)$ be such that $\sum_{a \in \frac{1}{m}\mathbb{Z}^N} \xi(x + a) = 1$, we set $\varphi_j(x) = \xi(x - p_j)$, for $j \in \mathcal{P}$, and $\psi_k(x) = \xi(x - q_k)$, for $k \in \mathcal{Q}$. Hence we have the following properties:

- (i) $\text{supp } \varphi_j \subset P_j$, $\varphi_j(p_j) = 1$, $0 \leq \varphi_j(x) \leq 1$.
- (ii) $\text{supp } \psi_k \subset Q_k$, $\psi_k(q_k) = 1$, $0 \leq \psi_k(x) \leq 1$.
- (iii) $\sum_{j \in \mathcal{P}} \varphi_j(x) + \sum_{k \in \mathcal{Q}} \psi_k(x) = 1 \quad \forall x \in \mathbb{R}^N$.

We also define the rescaled covering,

$$\hat{P}_j = \{x : \varepsilon x \in P_j\}, \quad \hat{Q}_k = \{x : \varepsilon x \in Q_k\}.$$

Then $\{\hat{\varphi}_j, \hat{\psi}_k\}_{\mathcal{P} \times \mathcal{Q}}$ defined by

$$\hat{\varphi}_j(x) = \varphi_j(\varepsilon x), \quad \hat{\psi}_k(x) = \psi_k(\varepsilon x)$$

is a partition of unity associated with $\{\hat{P}_j, \hat{Q}_k\}_{\mathcal{P} \times \mathcal{Q}}$.

We introduce also the sets

$$\hat{C}_j = \{x \in \hat{P}_j : \hat{\varphi}_j(x) = 1\}.$$

Note that $\text{supp}\{\nabla\hat{\varphi}_j, D^2\hat{\varphi}_j\} \subset \hat{P}_j \setminus \hat{C}_j$ and

$$\begin{aligned} \sup_{\mathcal{P} \times \mathcal{Q}} \{ \|\partial_\alpha \hat{\varphi}_j\|_\infty, \|\partial_\alpha \hat{\psi}_k\|_\infty, \alpha = 1, \dots, N \} &\leq C\varepsilon, \\ \sup_{\mathcal{P} \times \mathcal{Q}} \{ \|\partial_{\alpha,\beta}^2 \hat{\varphi}_j\|_\infty, \|\partial_{\alpha,\beta}^2 \hat{\psi}_k\|_\infty, \alpha, \beta = 1, \dots, N \} &\leq C\varepsilon^2. \end{aligned} \quad (5.2)$$

It will be convenient also to consider the following functions:

$$\hat{g}_j(x) = \frac{\hat{\varphi}_j(x)}{(\sum_{k \in \mathcal{P}} \hat{\varphi}_k^2 + \sum_{k \in \mathcal{Q}} \hat{\psi}_k^2)^{1/2}}, \quad \hat{h}_k(x) = \frac{\hat{\psi}_k(x)}{(\sum_{j \in \mathcal{P}} \hat{\varphi}_j^2 + \sum_{j \in \mathcal{Q}} \hat{\psi}_j^2)^{1/2}}; \quad (5.3)$$

we have $\sum_{j \in \mathcal{P}} \hat{g}_j(x)^2 + \sum_{k \in \mathcal{Q}} \hat{h}_k(x)^2 = 1$ for all $x \in \mathbb{R}^N$.

We also define the following sets of indices:

$$\begin{aligned} \mathcal{Q}_P(j) &= \{h \in \mathcal{Q} : \hat{Q}_h \cap \hat{P}_j \neq \emptyset\}, \\ \mathcal{Q}_Q(k) &= \{h \in \mathcal{Q} : \hat{Q}_h \cap \hat{Q}_k \neq \emptyset\}, \\ \mathcal{P}_Q(k) &= \{i \in \mathcal{P} : \hat{P}_i \cap \hat{Q}_k \neq \emptyset\}. \end{aligned} \quad (5.4)$$

Since the covering is locally finite, $\sup_{j \in \mathcal{P}} |\mathcal{Q}_P(j)| + \sup_{k \in \mathcal{Q}} (|\mathcal{P}_Q(k)| + |\mathcal{Q}_Q(k)|) < C$, for some constant $C > 0$.

Now let us introduce the following Banach spaces

$$\begin{aligned} \hat{X}_\varepsilon &= \{u \in H_{\text{loc}}^2(\mathbb{R}^N) : \|u\|_{\hat{X}_\varepsilon} \equiv \sup_{\mathcal{P} \times \mathcal{Q}} \{\|u\|_{H^2(\hat{P}_j)}, \|u\|_{H^2(\hat{Q}_k)}\} < +\infty\}, \\ \hat{Y}_\varepsilon &= \{u \in L_{\text{loc}}^2(\mathbb{R}^N) : \|u\|_{\hat{Y}_\varepsilon} \equiv \sup_{\mathcal{P} \times \mathcal{Q}} \{\|u\|_{L^2(\hat{P}_j)}, \|u\|_{L^2(\hat{Q}_k)}\} < +\infty\}. \end{aligned}$$

Remark 5.1. Note that the use of the Banach space $\hat{X}_\varepsilon$ and $\hat{Y}_\varepsilon$ forces to assume the integrability condition at infinity of W , namely $W(x) = O(\frac{1}{|x|^\beta})$, as $|x| \rightarrow +\infty$, with $\beta > N$, even if $\mathcal{P}$ is a finite set (see (5.12)). However, for $\mathcal{P}$ finite, one could simply use the space $H^2(\mathbb{R}^N)$ and $L^2(\mathbb{R}^N)$ assuming only $\beta > 4$ (see the proof of theorem 5.6); on the other hand, to obtain the independence of the parameter ε on the number of points p_j (see Theorem 1.2) we need to use the spaces $\hat{X}_\varepsilon$ and $\hat{Y}_\varepsilon$.

We have the following result:

Lemma 5.2. *There exists a constant $C > 0$ such that*

$$\|[W * (uv)]w\|_{\hat{Y}_\varepsilon} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon} \|w\|_{\hat{X}_\varepsilon}, \quad (5.5)$$

for any $u, v, w \in \hat{X}_\varepsilon$.

Furthermore, the following estimates hold

$$\sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{P}_j)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}, \quad (5.6)$$

$$\sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_j uv)\|_{L^\infty(\hat{P}_i)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}, \quad (5.7)$$

$$\sum_{j \in \mathcal{P}} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon}, \quad (5.8)$$

for some constants $C, c > 0$, for any $u, v \in \hat{X}_\varepsilon$ and for any $j \in \mathcal{P}$ and $k \in \mathcal{Q}$.

Proof. First, let us prove (5.6) and (5.7). Since $\text{dist}(\hat{P}_i, \hat{P}_j) \geq \frac{C}{\varepsilon} |p_i - p_j|$, for $i \neq j$, and $W = O(\frac{1}{|x|^\beta})$ as $|x| \rightarrow +\infty$, we have

$$\begin{aligned} \sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{P}_j)} &\leq \sum_{i \in \mathcal{P} \setminus \{j\}} \frac{C\varepsilon^\beta}{|p_i - p_j|^\beta} \|u\|_{L^2(\hat{P}_i)} \|v\|_{L^2(\hat{P}_i)} \\ &\leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon} \sum_{n=1}^{\infty} \sum_{i: n \leq |p_i - p_j| < n+1} \frac{1}{|p_i - p_j|^\beta} \\ &\leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon} \sum_{n=1}^{\infty} n^{N-1-\beta}. \end{aligned}$$

Therefore, since $\beta > N$, we may conclude

$$\sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{P}_j)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}.$$

To prove (5.7), we can estimate exactly as before

$$\begin{aligned} \sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_j uv)\|_{L^\infty(\hat{P}_i)} &\leq \|u\|_{L^2(\hat{P}_j)} \|v\|_{L^2(\hat{P}_j)} \sum_{i \in \mathcal{P} \setminus \{j\}} \frac{C\varepsilon^\beta}{|p_i - p_j|^\beta} \\ &\leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}. \end{aligned} \quad (5.9)$$

Hence, (5.6) and (5.7) hold.

Now to prove (5.5), we need to estimate

$$\begin{aligned} \|W * (uv)\|_{L^\infty(\hat{P}_j)} &= \left\| W * \left[\left(\sum_{i \in \mathcal{P}} \hat{\varphi}_i + \sum_{k \in \mathcal{Q}} \hat{\psi}_k \right) uv \right] \right\|_{L^\infty(\hat{P}_j)} \\ &\leq \|W * (\hat{\varphi}_j uv)\|_{L^\infty(\hat{P}_j)} + \sum_{k \in \mathcal{Q}_P(j)} \|W * (\hat{\psi}_k uv)\|_{L^\infty(\hat{P}_j)} \\ &\quad + \sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{P}_j)} \\ &\quad + \sum_{k \in \mathcal{Q} \setminus \mathcal{Q}_P(j)} \|W * (\hat{\psi}_k uv)\|_{L^\infty(\hat{P}_j)} \end{aligned} \quad (5.10)$$

and

$$\begin{aligned}
 \|W * (uv)\|_{L^\infty(\hat{Q}_k)} &= \left\| W * \left[\left(\sum_{i \in \mathcal{P}} \hat{\varphi}_i + \sum_{h \in \mathcal{Q}} \hat{\psi}_h \right) uv \right] \right\|_{L^\infty(\hat{Q}_k)} \\
 &\leq \sum_{i \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{Q}_k)} + \sum_{h \in \mathcal{Q}_Q(k)} \|W * (\hat{\psi}_h uv)\|_{L^\infty(\hat{Q}_k)} \\
 &\quad + \sum_{i \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{Q}_k)} + \sum_{h \in \mathcal{Q} \setminus \mathcal{Q}_Q(k)} \|W * (\hat{\psi}_h uv)\|_{L^\infty(\hat{Q}_k)}. \quad (5.11)
 \end{aligned}$$

In view of (2.4), we can estimate all the finite sum in (5.10) and (5.11) as follows

$$\begin{aligned}
 \|W * (\hat{\varphi}_j uv)\|_{L^\infty(\hat{P}_j)} &\leq C \|u\|_{H^2(\hat{P}_j)} \|v\|_{H^2(\hat{P}_j)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}, \\
 \sum_{k \in \mathcal{Q}_P(j)} \|W * (\hat{\psi}_k uv)\|_{L^\infty(\hat{P}_j)} &\leq C |\mathcal{Q}_P(j)| \|u\|_{H^2(\hat{Q}_k)} \|v\|_{H^2(\hat{Q}_k)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}, \\
 \sum_{i \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_i uv)\|_{L^\infty(\hat{Q}_k)} &\leq C |\mathcal{P}_Q(k)| \|u\|_{H^2(\hat{P}_i)} \|v\|_{H^2(\hat{P}_i)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}
 \end{aligned}$$

and

$$\sum_{h \in \mathcal{Q}_Q(k)} \|W * (\hat{\psi}_h uv)\|_{L^\infty(\hat{Q}_k)} \leq C |\mathcal{Q}_Q(k)| \|u\|_{H^2(\hat{Q}_h)} \|v\|_{H^2(\hat{Q}_h)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}.$$

Then, since for any $k \in \mathcal{Q} \setminus \mathcal{Q}_P(j)$ or $j \in \mathcal{P} \setminus \mathcal{P}_Q(k)$, $\text{dist}(\hat{Q}_k, \hat{P}_j) \geq \frac{c}{\varepsilon} |q_k - p_j|$, arguing as we did to prove (5.6), we have

$$\begin{aligned}
 \sum_{k \in \mathcal{Q} \setminus \mathcal{Q}_P(j)} \|W * (\hat{\psi}_k uv)\|_{L^\infty(\hat{P}_j)} &\leq \sum_{k \in \mathcal{Q} \setminus \mathcal{Q}_P(j)} \frac{C \varepsilon^\beta}{|p_j - q_k|^\beta} \|u\|_{L^2(\hat{Q}_k)} \|v\|_{L^2(\hat{Q}_k)} \\
 &\leq C \varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}, \\
 \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j uv)\|_{L^\infty(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \frac{C \varepsilon^\beta}{|p_j - q_k|^\beta} \|u\|_{L^2(\hat{P}_j)} \|v\|_{L^2(\hat{P}_j)} \\
 &\leq C \varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon},
 \end{aligned}$$

and finally, since for $h \in \mathcal{Q} \setminus \mathcal{Q}_Q(k)$, $\text{dist}(\hat{Q}_h, \hat{Q}_k) \geq \frac{c}{\varepsilon} |q_h - q_k|$, we have

$$\begin{aligned}
 \sum_{h \in \mathcal{Q} \setminus \mathcal{Q}_Q(k)} \|W * (\hat{\psi}_h uv)\|_{L^\infty(\hat{Q}_k)} &\leq \sum_{h \in \mathcal{Q} \setminus \mathcal{Q}_Q(k)} \frac{C \varepsilon^\beta}{|q_h - q_k|^\beta} \|u\|_{L^2(\hat{Q}_h)} \|v\|_{L^2(\hat{Q}_h)} \\
 &\leq C \varepsilon^\beta \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon}. \quad (5.12)
 \end{aligned}$$

In view of all the previous inequalities, we may conclude that

$$\|W * (uv)\|_{L^\infty(\hat{P}_j)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon} \quad \forall j \in \mathcal{P}$$

and

$$\|W * (uv)\|_{L^\infty(\hat{Q}_k)} \leq C \|u\|_{\hat{X}_\varepsilon} \|v\|_{\hat{X}_\varepsilon} \quad \forall k \in \mathcal{Q}.$$

Then, since we have

$$\|[W * (uv)]w\|_{L^2(\hat{P}_j)} \leq \|W * (uv)\|_{L^\infty(\hat{P}_j)} \|w\|_{L^2(\hat{P}_j)}$$

and

$$\|[W * (uv)]w\|_{L^2(\hat{Q}_k)} \leq \|W * (uv)\|_{L^\infty(\hat{Q}_k)} \|w\|_{L^2(\hat{Q}_k)},$$

(5.5) immediately follows.

Finally we prove (5.8). We have

$$\begin{aligned} \sum_{j \in \mathcal{P}} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} \\ &\quad + \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)}. \end{aligned}$$

Since for any $j \in \mathcal{P} \setminus \mathcal{P}_Q(k)$, $\text{dist}(\hat{P}_j, \hat{Q}_k) \geq \frac{\varepsilon}{\varepsilon} |p_j - q_k|$, arguing as above to prove (5.6) and recalling that $\|\hat{U}_j\|_{L^2(\hat{P}_j)} \leq C$, we have

$$\begin{aligned} \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \frac{C\varepsilon^\beta}{|p_j - q_k|^\beta} \|\hat{U}_j\|_{L^2(\hat{P}_j)} \|u\|_{L^2(\hat{P}_j)} \\ &\leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon}. \end{aligned}$$

Now, let $r > 0$ be such that $B_{\frac{r}{2\varepsilon}}(\hat{p}_j) \subset \hat{C}_j \subset \hat{P}_j \setminus \hat{Q}_k$ and let us consider a cut-off function $\hat{\beta}_j \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \hat{\beta}_j \leq 1$, $\hat{\beta}_j(x) = 1$ for $x \in B_{\frac{r}{4\varepsilon}}(\hat{p}_j)$ and $\hat{\beta}_j(x) = 0$ for $x \in \mathbb{R}^N \setminus B_{\frac{r}{2\varepsilon}}(\hat{p}_j)$. We have

$$\begin{aligned} \sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\beta}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} \\ &\quad + \sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j (1 - \hat{\beta}_j) \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)}. \end{aligned}$$

Since $\text{dist}(B_{\frac{r}{2\varepsilon}}(\hat{p}_j), \hat{Q}_k) \geq \frac{\varepsilon}{\varepsilon} |p_j - q_k|$ for all $j \in \mathcal{P}_Q(k)$, arguing again as to prove (5.6) we have

$$\sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\beta}_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon}.$$

Finally, since $|\mathcal{P}_Q(k)| \leq C$, for any $k \in \mathcal{Q}$, using (2.4) and the exponential decay of $\hat{U}_j$ (uniform w.r.t. $j \in \mathcal{P}$ and $\varepsilon > 0$, sufficiently small), we have

$$\begin{aligned} \sum_{j \in \mathcal{P}_Q(k)} \|W * (\hat{\varphi}_j (1 - \hat{\beta}_j) \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} &\leq C \|u\|_{\hat{X}_\varepsilon} \sum_{j \in \mathcal{P}_Q(k)} \|\hat{U}_j\|_{H^2(\hat{P}_j \setminus B_{\frac{r}{4\varepsilon}}(\hat{p}_j))} \\ &\leq C e^{-c/\varepsilon} \|u\|_{\hat{X}_\varepsilon}. \end{aligned}$$

Therefore, we may conclude

$$\sum_{j \in \mathcal{P}} \|W * (\varphi_j \hat{U}_j u)\|_{L^\infty(\hat{Q}_k)} \leq C\varepsilon^\beta \|u\|_{\hat{X}_\varepsilon}$$

and (5.8) is proved. $\square$

5.2. The shadowing lemma

We introduce the functional $\mathcal{F}_\varepsilon : \hat{X}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ defined as follows

$$\begin{aligned} \mathcal{F}_\varepsilon(z) = & -\Delta z + V_\varepsilon z + \sum_{j \in \mathcal{P}} \hat{\varphi}_j [W * (\hat{U}_j)^2] \hat{U}_j \\ & - \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right)^2 \right] \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right) - \sum_{j \in \mathcal{P}} [\Delta, \hat{\varphi}_j] \hat{U}_j; \end{aligned}$$

where $[\Delta, \hat{\varphi}_j] = \Delta \hat{\varphi}_j + 2\nabla \hat{\varphi}_j \nabla$.

Using (5.5), it is easy to verify that $\mathcal{F}_\varepsilon \in C^1(\hat{X}_\varepsilon, \hat{Y}_\varepsilon)$. Moreover if $z \in \hat{X}_\varepsilon$ satisfies $\mathcal{F}_\varepsilon(z) = 0$, then, using $\sum_{j \in \mathcal{P}} \hat{\varphi}_j F_\varepsilon(\hat{U}_j) = 0$, it is easy to check that $u = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z$ is a solution for (1.6). In this subsection we prove the existence of such a $z \in \hat{X}_\varepsilon$. To begin we observe that the function $\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \in \hat{X}_\varepsilon$ is an approximate solution for (1.6). Indeed, we have

Lemma 5.3. *There exists a constant $C > 0$ such that, for any $\varepsilon > 0$ sufficiently small, we have*

$$\|\mathcal{F}_\varepsilon(0)\|_{\hat{Y}_\varepsilon} \leq C\varepsilon^\beta.$$

Proof. In view of (2.4), we have

$$\begin{aligned} \|\mathcal{F}_\varepsilon(0)\|_{L^2(\hat{P}_j)} & \leq \left\| \hat{\varphi}_j \hat{U}_j \left[W * \left(\hat{U}_j^2 - \sum_{i \in \mathcal{P}} \hat{\varphi}_i^2 \hat{U}_i^2 \right) \right] \right\|_{L^2} + \|[\Delta, \hat{\varphi}_j] \hat{U}_j\|_{L^2} \\ & \leq C \left(\|\hat{U}_j\|_{L^{\frac{2s}{s-1}}(\mathbb{R}^N \setminus \hat{C}_j)}^2 + \sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i \hat{U}_i)^2\|_{L^\infty(\hat{P}_j)} \right. \\ & \quad \left. + \|\hat{U}_j\|_{L^2(\hat{P}_j \setminus \hat{C}_j)} \right). \end{aligned}$$

Hence, by (5.6) and the exponential decay of $\hat{U}_j$, we get

$$\|\mathcal{F}_\varepsilon(0)\|_{L^2(\hat{P}_j)} \leq C\varepsilon^\beta.$$

for some constant $C > 0$ (independent of $j \in \mathcal{P}$).

Moreover, since $(\hat{Q}_k \cap \hat{P}_j) \subset \hat{P}_j \setminus \hat{C}_j$ and $|\mathcal{P}_Q(k)| \leq C$, for any $k \in \mathcal{Q}$, using again (2.4) and the exponential decay of $\hat{U}_j$, we get the following estimate

$$\begin{aligned} \|\mathcal{F}_\varepsilon(0)\|_{L^2(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P}_Q(k)} \left\| \hat{\varphi}_j \hat{U}_j \left[W * \left(\hat{U}_j^2 - \sum_{i \in \mathcal{P}} \hat{\varphi}_i^2 \hat{U}_i^2 \right) \right] \right\|_{L^2(\hat{Q}_k)} \\ &\quad + \sum_{j \in \mathcal{P}_Q(k)} \|\Delta, \hat{\varphi}_j\| \hat{U}_j\|_{L^2(\hat{Q}_k)} \leq C \sum_{j \in \mathcal{P}_Q(k)} \|\hat{U}_j\|_{L^2(\hat{P}_j \setminus \hat{C}_j)} \\ &\leq C e^{-c/\varepsilon}, \end{aligned}$$

for some constants $C, c > 0$ (independent of $k \in \mathcal{Q}$).

Therefore the lemma is proved recalling that by definition

$$\|\mathcal{F}_\varepsilon(0)\|_{\hat{Y}_\varepsilon} = \sup_{(j,k) \in \mathcal{P} \times \mathcal{Q}} \{\|\mathcal{F}_\varepsilon(0)\|_{L^2(\hat{P}_j)}, \|\mathcal{F}_\varepsilon(0)\|_{L^2(\hat{Q}_k)}\}.$$

□

Now for $z \in \hat{X}_\varepsilon$ we consider the differential operator $D\mathcal{F}_\varepsilon(z) : \hat{X}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ given by

$$\begin{aligned} D\mathcal{F}_\varepsilon(z)h &= -\Delta h + V_\varepsilon h - 2 \left[W * \left(\left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right) h \right) \right] \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right) \\ &\quad - \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right)^2 \right] h \end{aligned}$$

and we denote $\mathcal{A}_\varepsilon = D\mathcal{F}_\varepsilon(0) : \hat{X}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ given by

$$\mathcal{A}_\varepsilon h = -\Delta h + V_\varepsilon h - 2 \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j h \right) \right] \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j - \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \right)^2 \right] h.$$

We will prove that $\mathcal{A}_\varepsilon$ is invertible, for $\varepsilon > 0$ sufficiently small. To this purpose it is convenient to consider the operator $B_\varepsilon : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ defined by

$$B_\varepsilon h = -\Delta h + V_\varepsilon h$$

and, for all $j \in \mathcal{P}$, the operator $A_{\varepsilon,j} : H^2(\mathbb{R}^N) \rightarrow L^2(\mathbb{R}^N)$ defined by

$$A_{\varepsilon,j} h = -\Delta h + V_\varepsilon h - 2[W * (\hat{U}_j h)] \hat{U}_j - [W * (\hat{U}_j)^2] h.$$

Following a gluing technique introduced by Angenent [3], we consider an “approximate inverse” S_ε for $\mathcal{A}_\varepsilon$ using as building blocks the operators $A_{\varepsilon,j}$ and B_ε . To this aim we preliminarily explore the properties of the operators $A_{\varepsilon,j}$. Note that in view of (2.5) and by the periodicity of V , for all $h \in H^2(\mathbb{R}^N)$ and any $j \in \mathcal{P}$, we have

$$A_{\varepsilon,j} h = \tau_{-\hat{p}_j} L_\varepsilon(\tau_{\hat{p}_j} h), \quad (5.13)$$

where L_ε is defined in Sect. 4. Then, as a direct consequence of Lemmas 4.1 and 4.2, we have

Lemma 5.4. *There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and $j \in \mathcal{P}$, the operator $A_{\varepsilon,j}$ is invertible and $\|A_{\varepsilon,j}^{-1}\| \leq \frac{C}{\varepsilon^2}$, for some constant $C > 0$ (independent of $j \in \mathcal{P}$).*

Furthermore, there exists $K > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ and $j \in \mathcal{P}$, if $f \in L^2(\mathbb{R}^N)$ is such that $|\sup_k \langle f, \partial_k \hat{U}_j \rangle_{L^2}| = o(\varepsilon^2) \|f\|_{L^2}$, then

$$\|A_{\varepsilon,j}^{-1} f\|_{H^2} \leq K \|f\|_{L^2}. \quad (5.14)$$

In particular, (5.14) holds for any $f \in L^2(\mathbb{R}^N)$ with $\text{supp } f \subset \mathbb{R}^N \setminus \hat{C}_j$.

Now, we have the following result:

Proposition 5.5. *There exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$, the differential operator $\mathcal{A}_\varepsilon : \hat{X}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ has an inverse and $\|\mathcal{A}_\varepsilon^{-1}\| \leq \frac{C}{\varepsilon^2}$, for some constant $C > 0$. Furthermore, there exists a constant $K > 0$ (independent of ε) such that for all $f \in \hat{Y}_\varepsilon$ with $\text{supp } f \subset \mathbb{R}^N \setminus \bigcup_{j \in \mathcal{P}} \hat{C}_j$ it results $\|\mathcal{A}_\varepsilon^{-1} f\|_{\hat{X}_\varepsilon} \leq K \|f\|_{\hat{Y}_\varepsilon}$.*

Proof. Let us introduce a linear operator $S_\varepsilon : \hat{Y}_\varepsilon \rightarrow \hat{X}_\varepsilon$ defined as follows

$$S_\varepsilon = \sum_{j \in \mathcal{P}} \hat{g}_j A_{\varepsilon,j}^{-1} \hat{g}_j + \sum_{k \in \mathcal{Q}} \hat{h}_k B_\varepsilon^{-1} \hat{h}_k.$$

We claim that

$$\|S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq \frac{C}{\varepsilon^2} \|f\|_{\hat{Y}_\varepsilon}.$$

Indeed for any $f \in \hat{Y}_\varepsilon$, in view of Lemma 5.4 and since the operator B_ε has a (uniformly w.r.t. ε) bounded inverse, we have

$$\begin{aligned} \|S_\varepsilon f\|_{H^2(\hat{P}_j)} &\leq \|\hat{g}_j A_{\varepsilon,j}^{-1} \hat{g}_j f\|_{H^2(\hat{P}_j)} + \left\| \sum_{k \in \mathcal{Q}_P(j)} \hat{h}_k B_\varepsilon^{-1} \hat{h}_k f \right\|_{H^2(\hat{P}_j)} \\ &\leq C \|A_{\varepsilon,j}^{-1}\| \|\hat{g}_j f\|_{L^2} + C \sum_{k \in \mathcal{Q}_P(j)} \|B_\varepsilon^{-1}\| \|\hat{h}_k f\|_{L^2} \leq \frac{C}{\varepsilon^2} \|f\|_{\hat{Y}_\varepsilon}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|S_\varepsilon f\|_{H^2(\hat{Q}_k)} &\leq \sum_{j \in \mathcal{P}_Q(k)} \|\hat{g}_j A_{\varepsilon,j}^{-1} \hat{g}_j f\|_{H^2(\hat{Q}_k)} + \sum_{l \in \mathcal{Q}_Q(k)} \|\hat{h}_l B_\varepsilon^{-1} \hat{h}_l f\|_{H^2(\hat{Q}_k)} \\ &\leq \frac{C}{\varepsilon^2} \|f\|_{\hat{Y}_\varepsilon} \end{aligned}$$

and the claim is proved.

Now we claim that for $\varepsilon > 0$ sufficiently small both the operators $S_\varepsilon \mathcal{A}_\varepsilon : \hat{X}_\varepsilon \rightarrow \hat{X}_\varepsilon$ and $\mathcal{A}_\varepsilon S_\varepsilon : \hat{Y}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ are invertible. Indeed, we have

$$\begin{aligned} S_\varepsilon \mathcal{A}_\varepsilon &= \mathbb{I}_{\hat{X}_\varepsilon} + \sum_{j \in \mathcal{P}} \hat{g}_j A_{\varepsilon,j}^{-1} (\hat{g}_j \mathcal{A}_\varepsilon - A_{\varepsilon,j} \hat{g}_j) + \sum_{k \in \mathcal{Q}} \hat{h}_k B_\varepsilon^{-1} (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) \\ \mathcal{A}_\varepsilon S_\varepsilon &= \mathbb{I}_{\hat{Y}_\varepsilon} + \sum_{j \in \mathcal{P}} (\mathcal{A}_\varepsilon \hat{g}_j - \hat{g}_j A_{\varepsilon,j}) A_{\varepsilon,j}^{-1} \hat{g}_j + \sum_{k \in \mathcal{Q}} (\mathcal{A}_\varepsilon \hat{h}_k - \hat{h}_k B_\varepsilon) B_\varepsilon^{-1} \hat{h}_k. \end{aligned}$$

Hence it suffices to prove that the differences $S_\varepsilon \mathcal{A}_\varepsilon - \mathbb{I}_{\hat{X}_\varepsilon}$ and $\mathcal{A}_\varepsilon S_\varepsilon - \mathbb{I}_{\hat{Y}_\varepsilon}$ are small in norm, for $\varepsilon > 0$ sufficiently small.

For all $h \in \hat{X}_\varepsilon$ we estimate

$$\left\| \sum_{j \in \mathcal{P}} \hat{g}_j A_{\varepsilon,j}^{-1} (\hat{g}_j \mathcal{A}_\varepsilon - A_{\varepsilon,j} \hat{g}_j) h \right\|_{\hat{X}_\varepsilon} \leq C \sup_{j \in \mathcal{P}} \|A_{\varepsilon,j}^{-1} (\hat{g}_j \mathcal{A}_\varepsilon - A_{\varepsilon,j} \hat{g}_j) h\|_{H^2(\hat{P}_j)}.$$

In view of Lemma 5.4, recalling that $\text{supp}([\hat{g}_j, \Delta]h) \subset \hat{P}_j \setminus \hat{C}_j$, we have

$$\begin{aligned} \|A_{\varepsilon,j}^{-1} (\hat{g}_j \mathcal{A}_\varepsilon - A_{\varepsilon,j} \hat{g}_j) h\|_{H^2(\hat{P}_j)} &\leq C \|[\hat{g}_j, \Delta]h\|_{L^2} \\ &\quad + \frac{C}{\varepsilon^2} \left\| \left[W * \left(\sum_{i \in \mathcal{P}} (\hat{\varphi}_i \hat{U}_i)^2 - \hat{U}_j^2 \right) \right] \hat{g}_j h \right\|_{L^2} \\ &\quad + \frac{C}{\varepsilon^2} \left\| \left[W * \left(\sum_{i \in \mathcal{P}} \hat{\varphi}_i \hat{U}_i h \right) \right] \hat{g}_j \hat{\varphi}_j \hat{U}_j \right. \\ &\quad \left. - [W * (\hat{U}_j \hat{g}_j h)] \hat{U}_j \right\|_{L^2}. \end{aligned} \quad (5.15)$$

Now, by the properties of $\hat{g}_j$, we have

$$\|[\hat{g}_j, \Delta]h\|_{L^2(\hat{P}_j)} = \|h \Delta \hat{g}_j + 2 \nabla \hat{g}_j \nabla h\|_{L^2(\hat{P}_j)} \leq C \varepsilon \|h\|_{\hat{X}_\varepsilon}.$$

Then, let us consider the second term in the r.h.s. of (5.15). In view of (2.4), the exponential decay of $\hat{U}_j$ and (5.6), we have

$$\begin{aligned} \left\| \left[W * \left(\sum_{i \in \mathcal{P}} (\hat{\varphi}_i \hat{U}_i)^2 - \hat{U}_j^2 \right) \right] \hat{g}_j h \right\|_{L^2} &\leq \|h\|_{L^2(\hat{P}_j)} \left\| W * \left[\sum_{i \in \mathcal{P}} (\hat{\varphi}_i \hat{U}_i)^2 - \hat{U}_j^2 \right] \right\|_{L^\infty(\hat{P}_j)} \\ &\leq \|h\|_{L^2(\hat{P}_j)} \left(\|\hat{U}_j\|_{L^{\frac{2s}{s-1}}(\mathbb{R}^N \setminus \hat{C}_j)}^2 \right. \\ &\quad \left. + \sum_{i \in \mathcal{P} \setminus \{j\}} \|W * (\hat{\varphi}_i \hat{U}_i)^2\|_{L^\infty(\hat{P}_j)} \right) \\ &\leq C \varepsilon^\beta \|h\|_{\hat{X}_\varepsilon}. \end{aligned}$$

Finally let us consider the last term in the r.h.s. of (5.15); we have

$$\begin{aligned} & \left\| \left[W * \left(\sum_{i \in \mathcal{P}} \hat{\varphi}_i \hat{U}_i h \right) \right] \hat{g}_j \hat{\varphi}_j \hat{U}_j - [W * (\hat{U}_j \hat{g}_j h)] \hat{U}_j \right\|_{L^2} \\ & \leq \| [W * (\hat{\varphi}_j \hat{U}_j h)] \hat{g}_j \hat{\varphi}_j \hat{U}_j - [W * (\hat{U}_j \hat{g}_j h)] \hat{U}_j \|_{L^2} \\ & \quad + \left\| \left[W * \left(\sum_{i \in \mathcal{P} \setminus \{j\}} \hat{\varphi}_i \hat{U}_i h \right) \right] \hat{g}_j \hat{\varphi}_j \hat{U}_j \right\|_{L^2}. \end{aligned}$$

Now, since $\hat{U}_j$ decays exponentially, and in view of (2.4), we have

$$\| [W * (\hat{\varphi}_j \hat{U}_j h)] \hat{g}_j \hat{\varphi}_j \hat{U}_j - [W * (\hat{U}_j \hat{g}_j h)] \hat{U}_j \|_{L^2} \leq C e^{-c/\varepsilon} \|h\|_{\hat{X}_\varepsilon};$$

moreover, in view of (5.6), we have

$$\begin{aligned} \left\| \left[W * \left(\sum_{i \in \mathcal{P} \setminus \{j\}} \hat{\varphi}_i \hat{U}_i h \right) \right] \hat{g}_j \hat{\varphi}_j \hat{U}_j \right\|_{L^2} & \leq C \sum_{i \in \mathcal{P} \setminus \{j\}} \| [W * (\hat{\varphi}_i \hat{U}_i h)] \|_{L^\infty(\hat{P}_j)} \\ & \leq C \varepsilon^\beta \|h\|_{\hat{X}_\varepsilon}; \end{aligned}$$

and collecting all the estimates above (since $\beta > 4$) we get

$$\left\| \sum_{j \in \mathcal{P}} \hat{g}_j A_{\varepsilon,j}^{-1} (\hat{g}_j \mathcal{A}_\varepsilon - A_{\varepsilon,j} \hat{g}_j) h \right\|_{\hat{X}_\varepsilon} \leq C(\varepsilon + o(\varepsilon)) \|h\|_{\hat{X}_\varepsilon}. \quad (5.16)$$

Then, since B_ε has a uniformly bounded inverse, we have

$$\begin{aligned} \left\| \sum_{k \in \mathcal{Q}} \hat{h}_k B_\varepsilon^{-1} (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) h \right\|_{\hat{X}_\varepsilon} & \leq C \sup_{k \in \mathcal{Q}} \| B_\varepsilon^{-1} (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) h \|_{H^2(\hat{Q}_k)} \\ & \leq C \sup_{k \in \mathcal{Q}} \| (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) h \|_{L^2}. \end{aligned}$$

Now, we have

$$\begin{aligned} \| (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) h \|_{L^2} & \leq \| [\hat{h}_k, \Delta] h \|_{L^2} + \left\| \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \right)^2 \right] \hat{h}_k h \right\|_{L^2} \\ & \quad + 2 \left\| \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j h \right) \right] \hat{h}_k \sum_{i \in \mathcal{P}} \hat{\varphi}_i \hat{U}_i \right\|_{L^2}. \end{aligned}$$

As before, by the properties of $\hat{h}_k$ we have

$$\| [\hat{h}_k, \Delta] h \|_{L^2} = \| h \Delta \hat{h}_k + 2 \nabla \hat{h}_k \nabla h \|_{L^2} \leq C \varepsilon \|h\|_{\hat{X}_\varepsilon}.$$

Moreover, in view of (5.8), we have

$$\begin{aligned} \left\| \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \right)^2 \right] \hat{h}_k h \right\|_{L^2} &\leq \sum_{j \in \mathcal{P}} \|W * (\hat{\varphi}_j \hat{U}_j)^2\|_{L^\infty(\hat{Q}_k)} \|h\|_{L^2(\hat{Q}_k)} \\ &\leq C \varepsilon^\beta \|h\|_{\hat{X}_\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \left\| \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j h \right) \right] \hat{h}_k \sum_{i \in \mathcal{P}_Q(k)} \hat{\varphi}_i \hat{U}_i \right\|_{L^2} &\leq C \sum_{j \in \mathcal{P}} \|W * (\hat{\varphi}_j \hat{U}_j h)\|_{L^\infty(\hat{Q}_k)} \\ &\leq C \varepsilon^\beta \|h\|_{\hat{X}_\varepsilon} \end{aligned}$$

Again, collecting all the estimates above, we get

$$\left\| \sum_{k \in \mathcal{Q}} \hat{h}_k B_\varepsilon^{-1} (\hat{h}_k \mathcal{A}_\varepsilon - B_\varepsilon \hat{h}_k) h \right\|_{\hat{X}_\varepsilon} \leq C(\varepsilon + o(\varepsilon)) \|h\|_{\hat{X}_\varepsilon}. \quad (5.17)$$

Therefore we may conclude that there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon})$ we get

$$\|S_\varepsilon \mathcal{A}_\varepsilon - \mathbb{I}_{\hat{X}_\varepsilon}\| \leq \frac{1}{2}$$

and, with similar estimates, we may also obtain

$$\|\mathcal{A}_\varepsilon S_\varepsilon - \mathbb{I}_{\hat{Y}_\varepsilon}\| \leq \frac{1}{2}.$$

Hence both the operators $S_\varepsilon \mathcal{A}_\varepsilon : \hat{X}_\varepsilon \rightarrow \hat{X}_\varepsilon$ and $\mathcal{A}_\varepsilon S_\varepsilon : \hat{Y}_\varepsilon \rightarrow \hat{Y}_\varepsilon$ are invertible with (uniformly w.r.t. $\varepsilon \in (0, \bar{\varepsilon})$) bounded inverse. Now we obtain the inverse of $\mathcal{A}_\varepsilon$ setting $\mathcal{A}_\varepsilon^{-1} \equiv (S_\varepsilon \mathcal{A}_\varepsilon)^{-1} S_\varepsilon = S_\varepsilon (\mathcal{A}_\varepsilon S_\varepsilon)^{-1}$. Then, for any $f \in \hat{Y}_\varepsilon$, we have

$$\|\mathcal{A}_\varepsilon^{-1} f\|_{\hat{X}_\varepsilon} = \|(S_\varepsilon \mathcal{A}_\varepsilon)^{-1} S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq C \|S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq \frac{C}{\varepsilon^2} \|f\|_{\hat{Y}_\varepsilon}$$

and the first part of the lemma is proved.

Now, let $f \in \hat{Y}_\varepsilon$ such that $\text{supp } f \subset \mathbb{R}^N \setminus \cup_{j \in \mathcal{P}} \hat{C}_j$. Then, in view of Lemma 5.4(5.14) we have

$$\begin{aligned} \|S_\varepsilon f\|_{H^2(\hat{P}_j)} &\leq C \|A_{\varepsilon,j}^{-1} \hat{g}_j f\|_{H^2} + C \sum_{k \in \mathcal{Q}_P(j)} \|B_\varepsilon^{-1} \hat{h}_k f\|_{H^2} \\ &\leq C \|\hat{g}_j f\|_{L^2} + C \sum_{k \in \mathcal{Q}_P(j)} \|\hat{h}_k f\|_{L^2} \leq C \|f\|_{\hat{Y}_\varepsilon} \end{aligned}$$

and

$$\begin{aligned} \|S_\varepsilon f\|_{H^2(\hat{Q}_k)} &\leq C \sum_{j \in \mathcal{P}_Q(k)} \|A_{\varepsilon,j}^{-1} \hat{g}_j f\|_{H^2} + C \sum_{l \in \mathcal{Q}_Q(k)} \|B_\varepsilon^{-1} \hat{h}_l f\|_{H^2} \\ &\leq C \sum_{j \in \mathcal{P}_Q(k)} \|\hat{g}_j f\|_{L^2} + C \sum_{l \in \mathcal{Q}_Q(k)} \|\hat{h}_l f\|_{L^2} \leq C \|f\|_{\hat{Y}_\varepsilon}; \end{aligned}$$

hence we get $\|S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq C \|f\|_{\hat{Y}_\varepsilon}$ and we may conclude that

$$\|\mathcal{A}_\varepsilon^{-1} f\|_{\hat{X}_\varepsilon} = \|(S_\varepsilon \mathcal{A}_\varepsilon)^{-1} S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq C \|S_\varepsilon f\|_{\hat{X}_\varepsilon} \leq C \|f\|_{\hat{Y}_\varepsilon}.$$

□

Now let $\bar{\varepsilon} > 0$ such that Proposition 5.5 holds for all $\varepsilon \in (0, \bar{\varepsilon})$ and let us introduce the nonlinear map $\mathcal{G}_\varepsilon \in C^1(\hat{X}_\varepsilon, \hat{X}_\varepsilon)$ defined by

$$\mathcal{G}_\varepsilon(z) = z - \mathcal{A}_\varepsilon^{-1} \mathcal{F}_\varepsilon(z).$$

If z_ε is a fixed point of $\mathcal{G}_\varepsilon$, then $\mathcal{F}_\varepsilon(z_\varepsilon) = 0$ and $u_\varepsilon = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z_\varepsilon$ is a solution of (1.6). We use the Banach fixed point argument applied to the map $\mathcal{G}_\varepsilon$ defined on $\mathcal{B}_\rho^\varepsilon = \{z \in \hat{X}_\varepsilon : \|z\|_{\hat{X}_\varepsilon} \leq \rho\}$ for a suitable $\rho > 0$.

Theorem 5.6. *There exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$ there exist $\rho_\varepsilon > 0$ and $z_\varepsilon \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon$ unique fixed point of $\mathcal{G}_\varepsilon$ in $\mathcal{B}_{\rho_\varepsilon}^\varepsilon$. Furthermore $\rho_\varepsilon = O(\varepsilon^2)$.*

Proof. We have

$$D\mathcal{G}_\varepsilon(z) = \mathbb{I}_{\hat{X}_\varepsilon} - \mathcal{A}_\varepsilon^{-1} D\mathcal{F}_\varepsilon(z),$$

hence, in particular, $D\mathcal{G}_\varepsilon(0) = 0$.

For $z \in \hat{X}_\varepsilon$ and $h \in \hat{X}_\varepsilon$ we have

$$\|D\mathcal{G}_\varepsilon(z)h\|_{\hat{X}_\varepsilon} = \|\mathcal{A}_\varepsilon^{-1}(D\mathcal{F}_\varepsilon(z) - D\mathcal{F}_\varepsilon(0))h\|_{\hat{X}_\varepsilon} \leq \frac{C}{\varepsilon^2} \|(D\mathcal{F}_\varepsilon(z) - D\mathcal{F}_\varepsilon(0))h\|_{\hat{Y}_\varepsilon}$$

and we estimate

$$\begin{aligned} \|(D\mathcal{F}_\varepsilon(z) - D\mathcal{F}_\varepsilon(0))h\|_{\hat{Y}_\varepsilon} &\leq \| [W * z^2] h \|_{\hat{Y}_\varepsilon} + 2 \left\| [W * (zh)] \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z \right) \right\|_{\hat{Y}_\varepsilon} \\ &\quad + 2 \left\| \left[W * \left(\sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j h \right) \right] z \right\|_{\hat{Y}_\varepsilon} \\ &\quad + 2 \left\| \left[W * \left(z \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \right) \right] h \right\|_{\hat{Y}_\varepsilon}; \end{aligned}$$

in view of Lemma 5.2-(5.5), we can estimate all the terms in the r.h.s., getting for $\|z\|_{\hat{X}_\varepsilon} \leq 1$

$$\begin{aligned} \|(D\mathcal{F}_\varepsilon(z) - D\mathcal{F}_\varepsilon(0))h\|_{\hat{Y}_\varepsilon} &\leq C\|h\|_{\hat{X}_\varepsilon} \left(\left\| \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j \right\|_{\hat{X}_\varepsilon} \|z\|_{\hat{X}_\varepsilon} + \|z\|_{\hat{X}_\varepsilon}^2 \right) \\ &\leq C\|z\|_{\hat{X}_\varepsilon} \|h\|_{\hat{X}_\varepsilon}; \end{aligned}$$

then we may conclude that

$$\|D\mathcal{G}_\varepsilon(z)h\|_{\hat{X}_\varepsilon} \leq \frac{C}{\varepsilon^2} \|z\|_{\hat{X}_\varepsilon} \|h\|_{\hat{X}_\varepsilon}.$$

Therefore there exists $\bar{\rho} > 0$ such that setting $\rho_\varepsilon = \varepsilon^2 \bar{\rho}$ we have

$$\|D\mathcal{G}_\varepsilon(z)\| < \frac{1}{2} \quad \forall z \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon.$$

Then, for every $z \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon$, we have

$$\|\mathcal{G}_\varepsilon(z)\|_{\hat{X}_\varepsilon} \leq \|\mathcal{G}_\varepsilon(z) - \mathcal{G}_\varepsilon(0)\|_{\hat{X}_\varepsilon} + \|\mathcal{G}_\varepsilon(0)\|_{\hat{X}_\varepsilon} \leq \frac{1}{2} \|z\|_{\hat{X}_\varepsilon} + \|\mathcal{G}_\varepsilon(0)\|_{\hat{X}_\varepsilon}.$$

Now in view of Proposition 5.5 and Lemma 5.3, we have

$$\|\mathcal{G}_\varepsilon(0)\|_{\hat{X}_\varepsilon} = \|\mathcal{A}_\varepsilon^{-1} \mathcal{F}_\varepsilon(0)\|_{\hat{X}_\varepsilon} \leq \frac{C}{\varepsilon^2} \|\mathcal{F}_\varepsilon(0)\|_{\hat{Y}_\varepsilon} \leq C\varepsilon^{\beta-2}.$$

Then, since $\beta > 4$, taking $\bar{\varepsilon} > 0$ eventually smaller, we have that $\|\mathcal{G}_\varepsilon(z)\|_{\hat{X}_\varepsilon} \leq \rho_\varepsilon$ for any $z \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon$ and $\varepsilon \in (0, \bar{\varepsilon})$.

Therefore $\mathcal{G}_\varepsilon$ is a strict contraction in $\mathcal{B}_{\rho_\varepsilon}^\varepsilon$ for any $\varepsilon \in (0, \bar{\varepsilon})$ and, by the Banach fixed point theorem, there exists, unique, $z_\varepsilon \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon$ such that $\mathcal{G}_\varepsilon(z_\varepsilon) = z_\varepsilon$ for any $\varepsilon \in (0, \bar{\varepsilon})$. $\square$

6. Proof of Theorems 1.2 and 1.3

Proof of Theorem 1.2. By Theorem 5.6 there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ there exists $z_\varepsilon \in \hat{X}_\varepsilon$ such that the function u_ε defined by $u_\varepsilon = \sum_{j \in \mathcal{P}} \hat{\varphi}_j \hat{U}_j + z_\varepsilon$ is a solution of (1.6). It results

$$u_\varepsilon(x) = \sum_{j \in \mathcal{P}} U_0(x - p_\varepsilon^{(j)}) + \omega_\varepsilon(x)$$

where

$$\begin{aligned} \omega_\varepsilon(x) &= \sum_{j \in \mathcal{P}} [\hat{\varphi}_j(x) \hat{U}_j(x) - U_0(x - p_\varepsilon^{(j)})] + z_\varepsilon(x) \\ p_\varepsilon^{(j)} &= \frac{p_j}{\varepsilon} + O(\varepsilon^\alpha), \quad \alpha \in (0, 1). \end{aligned} \tag{6.1}$$

By Theorem 1.1 we have $\|\hat{U}_j - U_0(\cdot - p_\varepsilon^{(j)})\|_{C^k} = O(\varepsilon^2)$ for any $k \in \mathbb{N}$ and $j \in \mathcal{P}$. Hence, for all $i \in \mathcal{P}$, using the exponential decay of $\hat{U}_i$, we estimate

$$\begin{aligned} & \left\| \sum_{j \in \mathcal{P}} [\hat{\varphi}_j \hat{U}_j - U_0(\cdot - p_\varepsilon^{(j)})] \right\|_{H^2(\hat{P}_i)} \leq \|(\hat{\varphi}_i - 1)\hat{U}_i\|_{H^2(\hat{P}_i)} \\ & \quad + \|\hat{U}_i - U_0(\cdot - p_\varepsilon^{(i)})\|_{H^2} + \sum_{j \in \mathcal{P} \setminus \{i\}} \|U_0(\cdot - p_\varepsilon^{(j)})\|_{H^2(\hat{P}_i)} \\ & \leq C e^{-c/\varepsilon} + C \varepsilon^2 + \sum_{n=1}^{\infty} \sum_{j: n \leq |p_j - p_i| \leq n+1} \|U_0(\cdot - p_\varepsilon^{(j)})\|_{H^2(\hat{P}_i)} \\ & \leq O(\varepsilon^2) + C \sum_{n=1}^{\infty} n^{N-1} e^{-\frac{c}{\varepsilon}n} = O(\varepsilon^2). \end{aligned}$$

Similarly, for all $k \in \mathcal{Q}$, we have

$$\begin{aligned} & \left\| \sum_{j \in \mathcal{P}} [\hat{\varphi}_j \hat{U}_j - U_0(\cdot - p_\varepsilon^{(j)})] \right\|_{H^2(\hat{Q}_k)} \leq \sum_{j \in \mathcal{P}_Q(k)} \|\hat{\varphi}_j \hat{U}_j - U_0(\cdot - p_\varepsilon^{(j)})\|_{H^2(\hat{Q}_k)} \\ & \quad + \sum_{j \in \mathcal{P} \setminus \mathcal{P}_Q(k)} \|U_0(\cdot - p_\varepsilon^{(j)})\|_{H^2(\hat{Q}_k)} \leq C e^{-c/\varepsilon} + C \sum_{n=1}^{\infty} \sum_{j: n \leq |p_j - q_k| \leq n+1} e^{-c \frac{|q_k - p_j|}{\varepsilon}} \\ & \leq C e^{-c/\varepsilon} + C \sum_{n=1}^{\infty} n^{N-1} e^{-c \frac{n}{\varepsilon}} = O(e^{-c/\varepsilon}). \end{aligned}$$

Moreover, by Theorem 5.6 we know that $\|z_\varepsilon\|_{\hat{X}_\varepsilon} \leq \rho_\varepsilon = O(\varepsilon^2)$, hence we may conclude that $\|\omega_\varepsilon\|_{\hat{X}_\varepsilon} = O(\varepsilon^2)$.

Finally, noting that z_ε satisfies an elliptic equation of the type $(-\Delta + V_\varepsilon)z_\varepsilon = f_\varepsilon$, by standard elliptic estimates on compact sets (see proof of Theorem 1.1), we may conclude that $\|\omega_\varepsilon\|_{C^k} = O(\varepsilon^2)$, for any $k \in \mathbb{N}$. $\square$

Proof of Theorem 1.3. Consider a set of points $\{p_j\}_{j \in \mathcal{P}}$ periodically arranged, that is there exist $p_1, \dots, p_s \in \mathbb{Z}^N$ and $a_1, \dots, a_N \in \mathbb{N}$ such that

$$\{p_j\}_{j \in \mathcal{P}} = \{p_k + \sum_{i=1}^N m_i a_i \underline{e}_i : k = 1, \dots, s; (m_1, \dots, m_N) \in \mathbb{Z}^N\} \quad (6.2)$$

where $(\underline{e}_1, \dots, \underline{e}_N)$ are the unit vectors of $\mathbb{R}^N$. We claim that the solution u_ε given by Theorem 1.2 is periodic, that is

$$u_\varepsilon \left(x + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i \right) = u_\varepsilon(x) \quad \text{for any } (m_1, \dots, m_N) \in \mathbb{Z}^N.$$

Indeed, we have $\hat{\varphi}_j(x + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i) = \hat{\varphi}_{j'}(x)$ and $\hat{U}_j(x + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i) = \hat{U}_{j'}(x)$, with $\hat{p}_{j'} = \hat{p}_j - \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i$, namely $j' \in \mathcal{P}$. Therefore, we have

$$u_\varepsilon \left(x + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i \right) = \sum_{j \in \mathcal{P}} \hat{\varphi}_j(x) \hat{U}_j(x) + z_\varepsilon \left(x + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i \right).$$

In view of (2.5) and the periodicity of V , we have $\mathcal{F}_\varepsilon(z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)) = 0$, namely $z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)$ is a fixed point of $\mathcal{G}_\varepsilon$. Hence, to conclude, it is sufficient to prove that $z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i) \in \mathcal{B}_{\rho_\varepsilon}^\varepsilon$.

In fact, for any $(j, k) \in \mathcal{P} \times \mathcal{Q}$ we have $\|z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)\|_{H^2(\hat{P}_j)} = \|z_\varepsilon\|_{H^2(\hat{P}_j)}$ and $\|z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)\|_{H^2(\hat{Q}_k)} = \|z_\varepsilon\|_{H^2(\hat{Q}_k)}$, for suitable $(\bar{j}, \bar{k}) \in \mathcal{P} \times \mathcal{Q}$. Hence we may conclude $\|z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)\|_{\dot{X}_\varepsilon} = \|z_\varepsilon\|_{\dot{X}_\varepsilon} \leq \rho_\varepsilon$.

Therefore $z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i)$ is a fixed point of $\mathcal{G}_\varepsilon$ in $\mathcal{B}_{\rho_\varepsilon}^\varepsilon$ and, by uniqueness, $z_\varepsilon(\cdot + \frac{1}{\varepsilon} \sum_{i=1}^N m_i a_i \underline{e}_i) = z_\varepsilon$. $\square$

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Received: 25 August 2008.

Accepted: 22 May 2009.