Effect of topology on the multiplicity of solutions for some semilinear elliptic systems with critical Sobolev exponent

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Abstract. In this paper, we are concerned with the effect of the domain topology on the multiplicity of solutions for some semilinear elliptic systems involving critical Sobolev exponent. We show that if the interaction term is sufficiently small, then the number of solutions of the system is estimated from below by $2 \operatorname{cat} \Omega$. The proof of this fact requires analysis of the structure of the Nehari manifold associated with the system.

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1. Problem and a main result

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 4$) be a bounded domain and let $a_{11}, a_{12}, a_{22} \in \mathbb{R}$.

In this paper, we are concerned with the multiplicity of solutions of the following elliptic systems involving critical Sobolev exponent:

$$(ES) \quad \begin{cases} -\Delta u = a_{11}u + a_{12}v + u|u|^{p-2} & \text{in } \Omega, \\ -\Delta v = a_{12}u + a_{22}v + v|v|^{p-2} & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $p = 2^* = 2N/(N - 2)$ denotes the critical Sobolev exponent.

Recall that the Sobolev embedding $H_0^1 \hookrightarrow L^p$ is continuous for $p \leq 2^*$, but is not compact for $p = 2^*$. Reflecting this lack of compactness, the elliptic equation

involving critical Sobolev exponent

$$(P) \quad \begin{cases} -\Delta u = \gamma u + u|u|^{2^*-2} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

exhibits various interesting phenomena, where $\gamma \in \mathbb{R}$. Among other things, the most striking fact is that the existence and the multiplicity of solutions of (P) highly depend on the topological and the geometrical structure of Ω as well as the sign of γ . For example, for star-shaped domains Ω , Pohozaev [14] proved that there are no solutions for (P) with $\gamma \leq 0$. On the other hand, Brezis and Nirenberg [6] showed that, for $\gamma \in (0, \lambda_1)$, there exists a positive solution for (P), where λ_1 denotes the first eigenvalue of $-\Delta$ with the homogeneous Dirichlet boundary condition. The proof of this existence result highly depends on the positivity of γ .

As for the relationship between the topology of Ω and the existence or the multiplicity of solutions, Bahri and Coron [4] proved that if $H_k(\Omega)$, the k -th homology group of Ω with $\mathbb{Z}_2$ -coefficient, is nontrivial for some k , then there exists at least one nontrivial solution for (P) with $\gamma = 0$. With the aid of a version of the Lyapunov-Schmidt reduction, Rey [15] proved that the number of positive solutions of (P) is estimated from below by $\text{cat } \Omega$ if γ is a sufficiently small positive number, where $\text{cat } \Omega$ denotes the Ljusternik-Schnirelman category (see e.g. [16, p. 100, Definition 5.12]). Later, by using the photography method introduced by Benci and Cerami [5], Lazzo [11] gave a short proof of this result.

In this paper, we shall investigate whether such an effect of the domain topology on the multiplicity of solutions is also observed in the system (ES) or not. The existence of at least one nontrivial solution for (ES) is treated in [3]. The existence and the multiplicity of (positive) solutions for another type of elliptic systems involving critical Sobolev exponent are also treated in [1, 2, 8–10]. Particularly, in [10], Han obtained the existence of $\text{cat } \Omega$ positive solutions for critical elliptic systems with the product type nonlinearity by using the photography method.

Let μ_1 (resp. μ_2) be the smallest (resp. largest) eigenvalue of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$.

Our main result reads as follows.

Theorem 1.1. *There exists $\bar{\mu} > 0$ such that for $A = (a_{ij})$ with $0 < \mu_1 \leq \mu_2 \leq \bar{\mu}$, (ES) has at least $2 \text{cat } \Omega$ nontrivial solutions. These solutions are positive if $a_{12} \geq 0$.*

The proof needs the analysis of the structure of the Nehari manifold associated with (ES). We show that the level set of the functional associated with (ES) in the Nehari manifold splits into two disjoint parts if the energy is sufficiently small, see Proposition 2.13. This fact and the method of photography initiated by Benci and Cerami [5] yields the theorem.

Notations. The smallest (resp. the largest) eigenvalue of $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$ is denoted by μ_1 (resp. μ_2). S denotes the best Sobolev constant defined by $S = \inf_{\{u \in H_0^1; \|u\|_p=1\}} \|\nabla u\|_2^2$.

$E = H_0^1 \times H_0^1$, $z = (u, v) \in E$, $E_1 = \{z \in E; v = 0, \|\nabla u\|_2^2 \geq S^{N/2}/2\}$ and $E_2 = \{z \in E; u = 0, \|\nabla v\|_2^2 \geq S^{N/2}/2\}$. The standard metric on E is denoted by $d(z_1, z_2) := \sqrt{\|\nabla u_1 - \nabla u_2\|_2^2 + \|\nabla v_1 - \nabla v_2\|_2^2}$, where $z_1 = (u_1, v_1)$, $z_2 = (u_2, v_2) \in E$.

Let $\varphi_\gamma(u) = \|\nabla u\|_2^2/2 - \gamma\|u\|_2^2/2 - \|u\|_p^p/p$ be the energy functional associated with (P). Also let $Q(u, v) = a_{11}\|u\|_2^2 + 2a_{12} \int uv + a_{22}\|v\|_2^2$, $I(u, v) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{2}Q(u, v) - \frac{1}{p}\|u\|_p^p - \frac{1}{p}\|v\|_p^p$ and $J(u, v) = \|\nabla u\|_2^2 + \|\nabla v\|_2^2 - Q(u, v) - \|u\|_p^p - \|v\|_p^p$. It is easy to see that $J, I \in C^2(E; \mathbb{R})$.

For a set X , a functional K on X and $a \in \mathbb{R}$, we define $[K \leq a]_X := \{x \in X; K(x) \leq a\}$.

The Nehari manifold we investigate is denoted by $M = \{z \in E; z \neq 0, J(z) = 0\}$. It is easy to see that M is a complete C^2 Banach Finsler manifold modeled on E .

Let $\beta(z) = \int x(|u(x)|^p + |v(x)|^p dx) / (\|u\|_p^p + \|v\|_p^p)$.

I and M for a sequence of matrix $A_n = \begin{pmatrix} a_{11}^n & a_{12}^n \\ a_{12}^n & a_{22}^n \end{pmatrix}$ is denoted by I_n and M_n , respectively.

2. Preliminaries

On category. In order to prove Theorem 1.1, we have to compare the category of Ω and that of a subset of M . For this purpose, we introduce the following comparison lemma for category. Though this lemma is somewhat well-known (see e.g. [17, Proof of Lemma 5.25]), we shall give the proof for the convenience of the reader.

Lemma 2.1. *Let A (resp. B) be a topological space and let $a \subset A$ (resp. $b \subset B$) be a closed subset. Also let $\Phi \in C(a; b)$ and $\beta \in C(B; A)$ be maps such that $\beta \circ \Phi$ is homotopically equivalent to the canonical injection from a to A . Then $\text{cat}_B b \geq \text{cat}_A a$ holds.*

Proof. Let $m = \text{cat}_B b$. Then there exist a family of closed subsets (b_j) of B , a family of mappings $\eta_j \in C([0, 1] \times b_j; B)$ and $u_j \in B$ such that $\eta_j(0, u) = u$ and $\eta_j(1, u) = u_j$ for any $u \in b_j$ and $j = 1, \dots, m$. Let $\overline{\Phi(a)}$ denotes the closure of $\Phi(a)$ in B and set

$$a_j = \begin{cases} \Phi^{-1}(b_j \cap \overline{\Phi(a)}) & \text{if } b_j \cap \overline{\Phi(a)} \neq \emptyset, \\ \emptyset & \text{if } b_j \cap \overline{\Phi(a)} = \emptyset. \end{cases}$$

Then it is easy to see that $a = \bigcup_{j=1}^m a_j$ and a_j is closed in A .

Since $\beta \circ \Phi$ is homotopically equivalent to the canonical injection from a to A , there exists $f \in C([0, 1] \times a; A)$ such that for all $x \in a$,

$$f(0, x) = x, \quad f(1, x) = \beta \circ \Phi(x).$$

Let us define $g_j \in C([0, 1] \times a_j; A)$ by

$$g_j(t, x) = \begin{cases} f(2t, x) & \text{for } (t, x) \in [0, 1/2] \times a_j, \\ \beta \circ \eta_j(2t - 1, \Phi(x)) & \text{for } (t, x) \in [1/2, 1] \times a_j. \end{cases}$$

Then for all $x \in a_j$, $g_j(0, x) = f(0, x) = x$ and $g_j(1, x) = \beta \circ \eta_j(1, \Phi(x)) = \beta(u_j) \in A$. Therefore a_j is a closed contractible subset of A . Hence, by the definition of the category $\text{cat}_A a$, we have $\text{cat}_A a \leq m$ and the proof is finished. $\square$

Proposition 2.2. *Let L be a C^1 Banach Finsler manifold and let L_0 , L_1 and L_2 be closed subsets of L such that $L_0 = L_1 \cup L_2$ and $L_1 \cap L_2 = \emptyset$. Then $\text{cat } L_0 = \text{cat } L_1 + \text{cat } L_2$.*

Proof. Let $n_j = \text{cat } L_j$ ($j = 0, 1, 2$). Then there exists a family of closed subsets $(A_j)_{j=1}^{n_0}$ of L_0 such that $L_0 = \bigcup_{j=1}^{n_0} A_j$ and A_j is contractible in L_0 . Let $\mathcal{A}_k := \{A_j; A_j \cap L_k \neq \emptyset\}$ and $\#\mathcal{A}_k =: m_k$ for $k = 1, 2$.

Since $L_1 \cap L_2 = \emptyset$ and since A_j is contractible in L_0 , $A_j \subset L_1$ (resp. $A_j \subset L_2$) implies $A_j \cap L_2 = \emptyset$ (resp. $A_j \cap L_1 = \emptyset$). Therefore for any j , we have $A_j \in \mathcal{A}_1$ or $A_j \in \mathcal{A}_2$ exclusively and $n_0 = m_1 + m_2$.

Moreover it is obvious that $\mathcal{A}_1$ is a covering of L_1 which consists of closed contractible sets in L_1 . Hence by the definition of the category of L_1 , we have $n_1 \leq m_1$. Similarly, $n_2 \leq m_2$. Thus we have $n_1 + n_2 \leq m_1 + m_2 = n_0$.

Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be a covering of L_1 (resp. L_2) which consists of n_1 (resp. n_2) closed and contractible sets in L_1 (resp. L_2) whose existence follows from the definition of $n_1 = \text{cat } L_1$ (resp. $n_2 = \text{cat } L_2$). Then members of $\mathcal{B}_1$ and $\mathcal{B}_2$ are closed contractible in L_0 . Therefore the union of $\mathcal{B}_1$ and $\mathcal{B}_2$ gives the closed contractible covering of L_0 , so we have $n_0 \leq n_1 + n_2$. Thus $n_0 = n_1 + n_2$. $\square$

We next recall the Ljusternik-Schnirelman theorem (category version). At first we introduce the Palais-Smale condition.

Definition 2.3. ((PS)-condition) Let L be a C^1 Banach Finsler manifold, $K \in C^1(L; \mathbb{R})$ and $c \in \mathbb{R}$. Then

- (a) $(z_n) \subset L$ is called a Palais-Smale sequence at level c ((PS) $_c$ -sequence) if $K(z_n) \rightarrow c$ and $(dK)_{z_n} \rightarrow 0$, where $(dK)_{z_n}$ denotes the first derivative of K on the manifold L at z_n .
- (b) K is said to satisfy the Palais-Smale condition at level c ((PS) $_c$ -condition) if every (PS) $_c$ -sequence has a convergent subsequence.

In this paper, we rely on the following abstract critical point theorem, see e.g. [17, p. 90, Theorem 5.20].

Proposition 2.4. (Ljusternik-Schnirelman theorem) *Let L be a complete C^2 Banach Finsler manifold and let K be a C^1 functional on L which is bounded from below. Let $a < a'$ and suppose that K satisfies (PS) $_c$ for all $c \leq a'$ and $\{z \in L; (dK)_z = 0, K(z) = a\} = \emptyset$.*

Then $[K \leq a]_L$ contains at least $\text{cat}[K \leq a]_L$ critical points.

Properties of I on M . Here we collect some facts concerning the behavior of I on M .

Lemma 2.5. *For $z \in M$, there holds*

$$\begin{aligned} I(z) &= (1/2 - 1/p)(\|u\|_p^p + \|v\|_p^p) \\ &= (1/2 - 1/p)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 - Q(u, v)). \end{aligned}$$

Proof. Directly follows from $J(z) = 0$ for $z \in M$. □

Lemma 2.6. *For any $A = (a_{ij})$ with $\mu_2 < \lambda_1$,*

$$S^{N/2} \left(1 - \frac{\mu_2}{\lambda_1}\right)^{(N-2)/2} \leq \|\nabla u\|_2^2 + \|\nabla v\|_2^2, \quad (2.1)$$

$$S^{N/2} \left(1 - \frac{\mu_2}{\lambda_1}\right)^{N/2} \leq \|u\|_p^p + \|v\|_p^p \quad (2.2)$$

for any $z = (u, v) \in M$.

Proof. From $J(z) = 0$, we have

$$\begin{aligned} \|\nabla u\|_2^2 + \|\nabla v\|_2^2 &= Q(u, v) + \|u\|_p^p + \|v\|_p^p \\ &\leq \frac{\mu_2}{\lambda_1}(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \|u\|_p^p + \|v\|_p^p, \end{aligned}$$

thus

$$\left(1 - \frac{\mu_2}{\lambda_1}\right)(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \leq \|u\|_p^p + \|v\|_p^p. \quad (2.3)$$

Then the Sobolev inequality gives

$$\begin{aligned} (\text{RHS of (2.3)}) &\leq \|\nabla u\|_2^p / S^{p/2} + \|\nabla v\|_2^p / S^{p/2} \\ &\leq (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)^{p/2} / S^{p/2}. \end{aligned} \quad (2.4)$$

Recall that $z \neq 0$ since $z \in M$. Then by (2.3) and (2.4), we have (2.1). From this relation together with (2.3), (2.2) follows. □

Lemma 2.7. *Let $A_n = (a_{ij}^n)$ satisfies $\mu_{2,n} \downarrow 0$ and let $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$. Then for any $(z_n) \subset [I_n \leq S^{N/2}/N + \varepsilon_n]_{M_n}$, we have*

$$\|\nabla u_n\|_2^2 = \|u_n\|_p^p + o(1) = S^{N/2}, \quad \|\nabla v_n\|_2^2 = \|v_n\|_p^p + o(1) = o(1) \quad (2.5)$$

or

$$\|\nabla u_n\|_2^2 = \|u_n\|_p^p + o(1) = o(1), \quad \|\nabla v_n\|_2^2 = \|v_n\|_p^p + o(1) = S^{N/2} \quad (2.6)$$

along some subsequence.

Proof. It is obvious that

$$S^{N/2}/N + \varepsilon_n \geq I_n(z_n) = (1/2 - 1/p)(\|u_n\|_p^p + \|v_n\|_p^p)$$

holds in view of Lemma 2.5 and $(z_n) \subset [I_n \leq S^{N/2}/N + \varepsilon_n]_{M_n}$. Hence

$$\|u_n\|_p^p + \|v_n\|_p^p \leq S^{N/2} + o(1).$$

Also note that from (2.2) and $\mu_{2,n} \downarrow 0$,

$$\|u_n\|_p^p + \|v_n\|_p^p \geq S^{N/2}(1 - \mu_{2,n}/\lambda_1)^{N/2} = S^{N/2} + o(1).$$

Hence we have

$$\|u_n\|_p^p + \|v_n\|_p^p = S^{N/2} + o(1). \quad (2.7)$$

This relation together with (2.3) and $\mu_{2,n} \downarrow 0$ implies that

$$\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 \leq \frac{\|u_n\|_p^p + \|v_n\|_p^p}{\left(1 - \frac{\mu_{2,n}}{\lambda_1}\right)} = S^{N/2} + o(1).$$

Hence, combining this relation with $S^{N/2} + o(1) \leq \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2$ which is obtained from (2.1), we have

$$\|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 = S^{N/2} + o(1). \quad (2.8)$$

From (2.7), we can assume that

$$\|u_n\|_p^p \rightarrow \alpha S^{N/2}, \quad \|v_n\|_p^p \rightarrow (1 - \alpha)S^{N/2} \quad (2.9)$$

holds for some $\alpha \in [0, 1]$ along an appropriate subsequence. The Sobolev inequality and (2.8) imply that

$$S\|u_n\|_p^2 + S\|v_n\|_p^2 \leq \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 = S^{N/2} + o(1). \quad (2.10)$$

Also note that from (2.9),

$$\begin{aligned} S\|u_n\|_p^2 + S\|v_n\|_p^2 + o(1) &= S(\alpha S^{N/2})^{2/p} + S((1 - \alpha)S^{N/2})^{2/p} \\ &= S^{N/2}(\alpha^{2/p} + (1 - \alpha)^{2/p}) \end{aligned} \quad (2.11)$$

holds. Relations (2.10) and (2.11) imply that

$$\alpha^{2/p} + (1 - \alpha)^{2/p} \leq 1.$$

Then, by the convexity of $f(\alpha) = \alpha^{2/p} + (1 - \alpha)^{2/p}$, we have $\alpha = 0$ or $\alpha = 1$. When $\alpha = 1$, $\|u_n\|_p^p = S^{N/2} + o(1)$ and $\|v_n\|_p^p = o(1)$ holds. This relation together with the Sobolev inequality implies that $\|\nabla u_n\|_2^2 \geq S^{N/2}$. Hence by (2.8), we have $\|\nabla v_n\|_2^2 = o(1)$, thus (2.5). Similarly, (2.6) follows from the case where $\alpha = 0$. $\square$

Palais-Smale properties of I in M . Here we collect some facts concerning the Palais-Smale property of I on M .

Lemma 2.8. *Every $(PS)_c$ -sequence of I in M is a $(PS)_c$ -sequence of I in E . Especially, every critical point of I in M is a critical point of I in E .*

Proof. Let (z_n) be a $(PS)_c$ -sequence of I in M . It is easy to see that $(z_n) \subset M$ implies $(dJ)_{z_n} \neq 0$ in E^* . Hence, by the Lagrange multiplier rule, we can find $(\rho_n) \subset \mathbb{R}$ such that

$$(dI)_{z_n} = \rho_n(dJ)_{z_n} + o(1) \text{ in } E^*. \quad (2.12)$$

Then testing (2.12) by z_n , we have

$$\begin{aligned} & \|\nabla u_n\|_2^2 + \|\nabla v_n\|_2^2 - a_{11}\|u_n\|_2^2 - 2a_{12} \int u_n v_n - a_{22}\|v_n\|_2^2 \\ & - \|u_n\|_p^p - \|v_n\|_p^p \\ & = \rho_n(2\|\nabla u_n\|_2^2 + 2\|\nabla v_n\|_2^2 - 2a_{11}\|u_n\|_2^2 - 4a_{12} \int u_n v_n - 2a_{22}\|v_n\|_2^2 \\ & - p\|u_n\|_p^p - p\|v_n\|_p^p) + o(1). \end{aligned} \quad (2.13)$$

Since the left hand side of (2.13) is nothing but $J(z_n)$, it should be equal to 0 in view of $z_n \in M$. Similarly, again by $J(z_n) = 0$, we observe that the right hand side of (2.13) is equal to $\rho_n(2-p)(\|u_n\|_p^p + \|v_n\|_p^p)$. Note that (z_n) is a bounded sequence in E in view of Lemma 2.5 and $I(z_n) = c + o(1)$. Hence, by (2.13), (2.2) and $\mu_2 < \lambda_1$, we have

$$\rho_n = o(1). \quad (2.14)$$

The boundedness of (z_n) in E implies that the boundedness of $\|(dJ)_{z_n}\|_{E^*}$. Hence (2.14) together with (2.12) implies the conclusion. $\square$

Proposition 2.9. *I satisfies $(PS)_c$ -condition in M for c with $c < S^{N/2}/N$.*

Proof. Let $(z_n) = (u_n, v_n) \subset M$ be a nonconvergent $(PS)_c$ -sequence of I in M . Without loss of generality, we assume that (u_n) is nonconvergent in H_0^1 . As we frequently discussed above, (u_n) and (v_n) are bounded in H_0^1 . Therefore we can further assume that

$$u_n \rightharpoonup u, \quad v_n \rightharpoonup v \text{ weakly in } H_0^1 \quad (2.15)$$

for some $u, v \in H_0^1$. Let $w_n := u_n - u$. Then by (2.15) and by the Brezis-Lieb Lemma (see e.g. [17, p. 21, Lemma 1.32]), we have

$$\|\nabla u_n\|_2^2 = \|\nabla u\|_2^2 + \|\nabla w_n\|_2^2 + o(1), \quad \|u_n\|_p^p = \|u\|_p^p + \|w_n\|_p^p + o(1). \quad (2.16)$$

Note that by Lemma 2.8, (z_n) is a (PS)-sequence of I in E . Therefore the boundedness of (z_n) in E , (2.15) and (2.16) assure that

$$\begin{aligned} o(1) &= (dI)_{z_n}(u_n, 0) \\ &= \|\nabla u_n\|_2^2 - a_{11}\|u_n\|_2^2 - a_{12} \int u_n v_n - \|u_n\|_p^p \\ &= \|\nabla w_n\|_2^2 + \|\nabla u\|_2^2 - a_{11}\|u\|_2^2 - a_{12} \int uv \\ &\quad - \|w_n\|_p^p - \|u\|_p^p \end{aligned} \quad (2.17)$$

holds. Also by (2.15), we see that (u, v) is a solution for the first equation of (ES). Hence multiplying u to the first equation of (ES) and integrating it over Ω , we obtain

$$\|\nabla u\|_2^2 - a_{11}\|u\|_2^2 - a_{12} \int uv - \|u\|_p^p = 0.$$

This relation together with (2.17) implies that $\alpha = \|\nabla w_n\|_2^2 + o(1) = \|w_n\|_p^p + o(1)$ for some $\alpha \in \mathbb{R}$. Note that our assumption $w_n = u_n - u \not\rightarrow 0$ in H_0^1 assures that $\alpha > 0$. Therefore by the Sobolev inequality, we get $\alpha \geq S\alpha^{2/p}$, so $\alpha \geq S^{N/2}$. Then Lemma 2.5 and (2.16) yield

$$\begin{aligned} c &= I(z_n) + o(1) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|_p^p + o(1) \\ &= \left(\frac{1}{2} - \frac{1}{p}\right) (\|u\|_p^p + \|w_n\|_p^p) + o(1) \geq \left(\frac{1}{2} - \frac{1}{p}\right) S^{N/2}, \end{aligned}$$

thus we have the conclusion. $\square$

Method of Photography. Let $(\Omega)_r^- = \{x \in \Omega; d_{\mathbb{R}^N}(x, \partial\Omega) \geq r\}$, $(\Omega)_r^+ = \{x \in \mathbb{R}^N; d_{\mathbb{R}^N}(x, \Omega) \leq r\}$, where $d_{\mathbb{R}^N}(\cdot, \cdot)$ stands for the standard Euclidean distance.

Since Ω is a smooth bounded domain, there exists $r > 0$ such that $(\Omega)_r^+$, $(\Omega)_r^-$ and Ω are homotopically equivalent. Hereafter we fix such $r > 0$.

Proposition 2.10. *For $r > 0$ above, there exist $\varepsilon > 0$ and $\bar{\mu} > 0$ such that for any $A = (a_{ij})$ with $0 < \mu_1 \leq \mu_2 < \bar{\mu}$,*

$$\beta \left([I \leq S^{N/2}/N + \varepsilon]_M \right) \subset (\Omega)_r^+.$$

Proof. Assume that the conclusion is not true. Then there exist $\varepsilon_n \downarrow 0$, $A_n = (a_{ij}^n)$ with $\mu_{2,n} \downarrow 0$ and $(z_n) \subset [I_n \leq S^{N/2}/N + \varepsilon_n]_{M_n}$ such that

$$\beta(z_n) \not\subset (\Omega)_r^+. \quad (2.18)$$

By Lemma 2.7, we have

$$\|\nabla u_n\|_2^2 = \|u_n\|_p^p + o(1) = S^{N/2}, \quad \|\nabla v_n\|_2^2 = \|v_n\|_p^p + o(1) = o(1) \quad (2.19)$$

or

$$\|\nabla u_n\|_2^2 = \|u_n\|_p^p + o(1) = o(1), \quad \|\nabla v_n\|_2^2 = \|v_n\|_p^p + o(1) = S^{N/2}. \quad (2.20)$$

Suppose that (2.19) holds. Then it is easy to see that $w_n := S^{-(N-2)/4}u_n$ satisfies $\|\nabla w_n\|_2^2 \rightarrow S$ and $\|w_n\|_p^2 = 1$, namely, (w_n) is a minimizing sequence for the minimization problem $\inf_{\|w\|_p^2=1} \|\nabla w\|_2^2$. Here we recall that this minimization problem has no solution if $\Omega \neq \mathbb{R}^N$ (see [6], see also [16, p. 171, Theorem 1.2]). Therefore by the concentration compactness argument [12, 13], we see that (w_n) should concentrate on some $x_0 \in \Omega$, i.e., $|w_n|^p \rightharpoonup \delta_{x_0}$ weakly in $\mathcal{M}(\Omega)$, where $\mathcal{M}(\Omega)$ denotes the set consists of Radon measures on Ω . Combining this fact with $\|v_n\|_p^p \rightarrow 0$, we have $\beta(z_n) = \int x|u_n|^p / \|u_n\|_p^p + o(1) = \int x|w_n|^p / \|w_n\|_p^p + o(1) = x_0 + o(1) \in (\Omega)_r^+$, which is absurd in view of (2.18). If (2.20) holds, then the same argument as above (u_n) replaced by (v_n) leads to the similar contradiction. $\square$

Let $\gamma \in (0, \lambda_1)$ and let u_γ be a function which attain the minimum of φ_γ on $\{u \in H_0^1(B_{r/2}(0)) \setminus \{0\}; \|\nabla u\|_2^2 = \gamma\|u\|_2^2 + \|u\|_p^p\}$, where $B_{r/2}(0) = \{x \in \mathbb{R}^N; |x| < r/2\}$ and $\varphi_\gamma(w) = \|\nabla w\|_2^2/2 - \gamma\|w\|_2^2/2 - \|w\|_p^p/p$. The function u_γ is nothing but the least energy solution (mountain pass solution) of the equation $-\Delta u = \gamma u + u|u|^{p-2}$ in $B_{r/2}(0)$ with the homogeneous Dirichlet boundary condition. The existence and the radial symmetry of such a minimizer is well known [6, 7]. Let $a_{11}, a_{22} \in (0, \lambda_1)$ and also let $\Phi_j: (\Omega)_r^- \rightarrow E$ be

$$\Phi_1(y) = (u_{a_{11}}(\cdot - y), 0), \quad \Phi_2(y) = (0, u_{a_{22}}(\cdot - y)).$$

Clearly $\Phi_j \in C((\Omega)_r^-; E)$. Set $S_{a_{11}} := S_{a_{11}}(B_{r/2}(0))$ and $S_{a_{22}} := S_{a_{22}}(B_{r/2}(0))$. Note that since $a_{11}, a_{22} \in (0, \lambda_1)$, we have $S_{a_{11}}, S_{a_{22}} < S$. Moreover, we have $\varphi_{a_{11}}(u_{a_{11}}) = S_{a_{11}}^{N/2}/N$ and $\varphi_{a_{22}}(u_{a_{22}}) = S_{a_{22}}^{N/2}/N$ (see [6]).

Proposition 2.11. *There exists $\bar{a} > 0$ such that for any $a_{11}, a_{22} \in (0, \bar{a})$ and $a_{12} \in \mathbb{R}$, $\Phi_j((\Omega)_r^-) \subset [I \leq d_0]_M \cap E_j$ for $j = 1$ and 2 , where $d_0 := \max(S_{a_{11}}, S_{a_{22}})^{N/2}/N (< S^{N/2}/N)$.*

Proof. Since $J \circ \Phi_1(y) = \|\nabla u_{a_{11}}\|_2^2 - a_{11}\|u_{a_{11}}\|_2^2 - \|u_{a_{11}}\|_p^p = 0$, we have $\Phi_1((\Omega)_r^-) \subset M$. Moreover, for sufficiently small $\bar{a}$, $S_{a_{11}}^{N/2}, S_{a_{22}}^{N/2} \in (S^{N/2}/2, S^{N/2})$ for $a_{11}, a_{22} \in (0, \bar{a})$ since $S_\gamma \rightarrow S$ as $\gamma \rightarrow 0$ (see [6]). Hence

$$I \circ \Phi_1(y) = \varphi_{a_{11}}(u_{a_{11}}) = S_{a_{11}}^{N/2}/N \leq d_0,$$

$$\|\nabla \Phi_1(y)\|_2^2 = \|\nabla u_{a_{11}}\|_2^2 \geq \|\nabla u_{a_{11}}\|_2^2 - a_{11}\|u_{a_{11}}\|_2^2 = S_{a_{11}}^{N/2} \geq S^{N/2}/2$$

in view of $\varphi_{a_{11}}(u_{a_{11}}) = S_{a_{11}}^{N/2}/N$ and $\|\nabla u_{a_{11}}\|_2^2 - a_{11}\|u_{a_{11}}\|_2^2 = \|u_{a_{11}}\|_p^p$. The conclusion for Φ_2 follows from the similar argument. $\square$

On the structure of M . In order to obtain multiple solutions, we have to investigate the fine structure of the Nehari manifold M . We start with the following lemma. Recall that $E_1 = \{z \in E; v = 0, \|\nabla u\|_2^2 \geq S^{N/2}/2\}$ and $E_2 = \{z \in E; u = 0, \|\nabla v\|_2^2 \geq S^{N/2}/2\}$.

Proposition 2.12. *There holds $\overline{(E_1)_\eta} \cap \overline{(E_2)_\eta} = \emptyset$ for $\eta = S^{N/4}/6$, where $(E_i)_\eta := \{z \in E; d(z, E_i) < \eta\}$ and $\overline{(E_i)_\eta}$ stands for the closure of $(E_i)_\eta$ in E .*

Proof. Assume that $\overline{(E_1)_\eta} \cap \overline{(E_2)_\eta} \neq \emptyset$. Let $z = (u, v) \in \overline{(E_1)_\eta} \cap \overline{(E_2)_\eta}$. Then by $z \in \overline{(E_1)_\eta}$, there exists $z_1 = (x, 0) \in E_1$ such that $d(z_1, z) < 3\eta/2$. Similarly, $z \in \overline{(E_2)_\eta}$ implies the existence of $z_2 = (0, y) \in E_2$ with $d(z_2, z) < 3\eta/2$. Therefore

$$d(z_1, z_2) \leq d(z_1, z) + d(z, z_2) \leq 3\eta = S^{N/4}/2. \quad (2.21)$$

On the other hand, the definition of E_1 and that of E_2 yield

$$d(z_1, z_2) = \sqrt{\|\nabla x\|_2^2 + \|\nabla y\|_2^2} \geq S^{N/4},$$

which is absurd in view of (2.21). $\square$

Proposition 2.13. *For any $\eta > 0$, there exist $\varepsilon > 0$ and $\bar{\mu} > 0$ such that for any $A = (a_{ij})$ with $0 < \mu_1 \leq \mu_2 \leq \bar{\mu}$,*

$$[I \leq S^{N/2}/N + \varepsilon]_M \subset \overline{(E_1)_\eta} \cup \overline{(E_2)_\eta}.$$

Proof. Assume that the conclusion is not true. Then there exist $\eta > 0$,

$$\varepsilon_n \downarrow 0, \quad A_n = (a_{ij}^n) \text{ with } \mu_{2,n} \downarrow 0, \quad (z_n) \subset [I_n \leq S^{N/2}/N + \varepsilon_n]_{M_n} \quad (2.22)$$

such that

$$z_n \notin \overline{(E_1)_\eta} \cup \overline{(E_2)_\eta}. \quad (2.23)$$

By (2.22) and Lemma 2.7, we have $\|\nabla u_n\|_2^2 \rightarrow \alpha S^{N/2}$ and $\|\nabla v_n\|_2^2 \rightarrow (1 - \alpha) S^{N/2}$ for $\alpha = 0$ or $\alpha = 1$ taking subsequence if necessary.

Suppose that $\alpha = 0$. Then

$$\|\nabla u_n\|_2^2 = o(1), \quad \|\nabla v_n\|_2^2 = S^{N/2} + o(1). \quad (2.24)$$

Hence, for large n , we have $\|\nabla v_n\|_2^2 \geq S^{N/2}/2$, thus $(0, v_n) \in E_2$. Therefore by (2.24), we have $d(z_n, E_2) \leq d(z_n, (0, v_n)) = \|\nabla u_n\|_2 \rightarrow 0$, thus $z_n \in \overline{(E_2)_\eta}$ for large n . This fact contradicts (2.23).

In the case where $\alpha = 1$, we also obtain the conclusion by the similar argument. $\square$

3. Proof of Theorem 1.1

Now we are in the position to give the proof of Theorem 1.1.

Since a critical point of I in E is a solution of (ES), we shall establish the existence of multiple critical points of I in E . For this purpose, it is enough to prove the existence of multiple critical points of I in M in view of Lemma 2.8.

Let $d_0 = \max(S_{a_{11}}, S_{a_{22}})^{N/2}/N$. Without loss of generality, we can assume that the existence of a regular value $d \in (d_0, S^{N/2}/N)$ since otherwise we already have infinitely many critical points.

At first, we estimate the category of some level set of I (in M) by the category of Ω from below.

Claim. There exist $\bar{\mu} > 0$ such that for any $A = (a_{ij})$ with $0 < \mu_1 \leq \mu_2 \leq \bar{\mu}$,

$$\text{cat}[I \leq d]_M \geq 2 \text{cat } \Omega.$$

Proof of Claim. Take any $r > 0$ such that $(\Omega)_r^+$, Ω and $(\Omega)_r^-$ are all homotopically equivalent. Then by Proposition 2.10, there exist $\varepsilon_1 > 0$ and $\bar{\mu}_1 > 0$ which satisfy the conclusion of Proposition 2.10. Also for $\eta = S^{N/4}/6$, Proposition 2.13 yields the existence of $\varepsilon_2 > 0$ and $\bar{\mu}_2 > 0$ with the conclusion of Proposition 2.13.

Let $\varepsilon := \min(\varepsilon_1, \varepsilon_2)$ and take $A = (a_{ij})$ which satisfies $0 < \mu_1 \leq \mu_2 < \bar{\mu} := \min(\bar{\mu}_1, \bar{\mu}_2)$. Taking $\bar{\mu}$ small if necessary, we can also assume that $a_{11}, a_{22} \in (0, \bar{a})$, where $\bar{a}$ is a number which appears in Proposition 2.11. Let $M_j := [I \leq d]_M \cap \overline{(E_j)_\eta}$.

Since $d_0 < d < S^{N/2}/N + \varepsilon \leq S^{N/2}/N + \varepsilon_1$, Proposition 2.10 and Proposition 2.11 yield

$$(\Omega)_r^- \xrightarrow{\Phi_j} M_j \xrightarrow{\beta} (\Omega)_r^+.$$

Moreover since $u_{a_{jj}}(\cdot)$ is a radial function, we see that $\beta \circ \Phi_j$ is a canonical injection from $(\Omega)_r^-$ to $(\Omega)_r^+$ for $j = 1, 2$. Then by Lemma 2.1 with $a = (\Omega)_r^-$, $A = (\Omega)_r^+$ and $b = B = M_j$,

$$\text{cat } M_j \geq \text{cat}_{(\Omega)_r^+} (\Omega)_r^- = \text{cat } \Omega \quad (3.1)$$

holds for $j = 1, 2$.

Since $d < S^{N/2}/N + \varepsilon \leq S^{N/2}/N + \varepsilon_2$, Proposition 2.12 and Proposition 2.13 imply that $M_1 \cap M_2 = \emptyset$ and $[I \leq d]_M = M_1 \cup M_2$. Hence Proposition 2.2 (with $L := M$, $L_0 := M_1 \cup M_2$, $L_1 := M_1$ and $L_2 := M_2$) yields

$$\text{cat}[I \leq d]_M = \text{cat } M_1 + \text{cat } M_2. \quad (3.2)$$

The conclusion follows from (3.1) and (3.2). $\square$

By Proposition 2.4 and Proposition 2.9, the number of critical points in $[I \leq d]_M$ is equal to or greater than $\text{cat}[I \leq d]_M$. Hence combining this fact with Claim above, we have the first assertion in Theorem 1.1.

By construction, every solution (u, v) obtained above satisfies

$$I(u, v) \leq d < \frac{S^{N/2}}{N}. \quad (3.3)$$

We show that all solutions (u, v) of (ES) satisfying (3.3) are positive solutions if μ_2 is sufficiently small and $a_{12} \geq 0$. Indeed, assume that there exists a solution (u, v) which changes its sign and satisfies (3.3). Let $u^+(x) := \max(0, u(x))$, $v^+(x) := \max(0, v(x))$, $u^-(x) := \min(0, u(x))$ and $v^-(x) := \min(0, v(x))$. Multiplying u^+ (resp. v^+) to the first equation (resp. the second equation) of (ES), we obtain

$$\begin{aligned} \|\nabla u^+\|_2^2 &= a_{11}\|u^+\|_2^2 + a_{12} \int_{\Omega} v u^+ dx + \|u^+\|_p^p, \\ \|\nabla v^+\|_2^2 &= a_{12} \int_{\Omega} u v^+ dx + a_{22}\|v^+\|_2^2 + \|v^+\|_p^p, \end{aligned}$$

thus

$$\begin{aligned}
 \|\nabla u^+\|_2^2 + \|\nabla v^+\|_2^2 &= a_{11}\|u^+\|_2^2 + a_{12} \int_{\Omega} (vu^+ + uv^+)dx + a_{22}\|v^+\|_2^2 \\
 &\quad + \|u^+\|_p^p + \|v^+\|_p^p \\
 &\leq a_{11}\|u^+\|_2^2 + 2a_{12} \int_{\Omega} u^+v^+dx + a_{22}\|v^+\|_2^2 \\
 &\quad + \|u^+\|_p^p + \|v^+\|_p^p
 \end{aligned}$$

if $a_{12} \geq 0$. This relation together with the Sobolev inequality and $a_{12} \geq 0$ yields

$$\begin{aligned}
 \left(1 - \frac{\mu_2}{\lambda_1}\right) S(\|u^+\|_p^p + \|v^+\|_p^p)^{2/p} &\leq \left(1 - \frac{\mu_2}{\lambda_1}\right) S(\|u^+\|_p^2 + \|v^+\|_p^2) \\
 &\leq \left(1 - \frac{\mu_2}{\lambda_1}\right) (\|\nabla u^+\|_2^2 + \|\nabla v^+\|_2^2) \\
 &\leq \|\nabla u^+\|_2^2 + \|\nabla v^+\|_2^2 - a_{11}\|u^+\|_2^2 - 2a_{12} \int_{\Omega} u^+v^+dx - a_{22}\|v^+\|_2^2 \\
 &\leq \|u^+\|_p^p + \|v^+\|_p^p.
 \end{aligned}$$

Hence

$$\left(1 - \frac{\mu_2}{\lambda_1}\right)^{N/2} S^{N/2} \leq \|u^+\|_p^p + \|v^+\|_p^p \quad (3.4)$$

if $\|u^+\|_p^p + \|v^+\|_p^p \neq 0$. Similarly, we have

$$\left(1 - \frac{\mu_2}{\lambda_1}\right)^{N/2} S^{N/2} \leq \|u^-\|_p^p + \|v^-\|_p^p$$

if $a_{12} \geq 0$ and $\|u^-\|_p^p + \|v^-\|_p^p \neq 0$. This relation, Lemma 2.5 and (3.4) imply that

$$I(z) \geq 2 \left(1 - \frac{\mu_2}{\lambda_1}\right)^{N/2} \frac{S^{N/2}}{N} \geq \frac{3}{2} \frac{S^{N/2}}{N}$$

for μ_2 sufficiently small, which is absurd in view of (3.3). $\square$

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