A weighted eigenvalue problem for the p-Laplacian plus a potential

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Abstract. Let Δ_p denote the p-Laplacian operator and Ω be a bounded domain in $\mathbb{R}^N$. We consider the eigenvalue problem

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega)$$

for a potential V and a weight function m that may change sign and be unbounded. Therefore the functional to be minimized is indefinite and may be unbounded from below. The main feature here is the introduction of a value $\alpha(V, m)$ that guarantees the boundedness of the energy over the weighted sphere $M = \{u \in W_0^{1,p}(\Omega); \int_{\Omega} m|u|^p dx = 1\}$. We show that the above equation has a principal eigenvalue if and only if either $m \geq 0$ and $\alpha(V, m) > 0$ or m changes sign and $\alpha(V, m) \geq 0$. The existence of further eigenvalues is also treated here, mainly a second eigenvalue (to the right) and their dependence with respect to V and m .

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1. Introduction

Our aim is to study the following eigenvalue problem

$$(P) \quad \begin{cases} -\Delta_p u + V(x)|u|^{p-2}u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (1)$$

where Ω is a bounded domain of $\mathbb{R}^N$, $p > 1$ and V, m are functions possibly indefinite and unbounded. The weight m is assumed to have a nontrivial positive part. The case $V \equiv 0$ has been studied, among others, by [2, 11] under different hypothesis on m . They prove that there exists a first positive eigenvalue, denoted by $\lambda_1(m)$ defined by

$$\lambda(m) \stackrel{\text{def}}{=} \inf \left\{ E(u) \stackrel{\text{def}}{=} \int_{\Omega} |\nabla u|^p dx; \quad u \in W_0^{1,p}(\Omega), \quad \int_{\Omega} m|u|^p dx = 1 \right\}, \quad (2)$$

which is simple, isolated and principal. By a *principal* eigenvalue we mean that it is associated to a positive eigenfunction. Moreover $\lambda_1(m)$ is the unique positive principal eigenvalue. A sequence of positive eigenvalues $\lambda_k(m)$ can be obtained by the Ljusternik–Schnirelman theory (cf. [14]). Moreover there holds

$$\lambda_1(m) < \lambda_2(m) \leq \lambda_3(m) \leq \cdots \leq \lambda_k(m) \rightarrow \infty \text{ as } k \rightarrow +\infty$$

and (cf. [3])

$$\lambda_2(m) = \min\{\lambda > \lambda_1(m); \lambda \text{ is an eigenvalue}\}.$$

A second characterization of $\lambda_2(m)$ has been given in [4] as a particular case of a Fučík problem:

$$\lambda_2(m) = \inf_{\gamma \in \Upsilon} \max_{u \in \gamma_{[-1,1]}} \int_{\Omega} |\nabla u|^p dx$$

where $\Upsilon \stackrel{\text{def}}{=} \{\gamma \in C([-1, 1], M); \gamma(\pm 1) = \pm \varphi_m\}$. Here

$$M \stackrel{\text{def}}{=} \left\{ u \in W_0^{1,p}(\Omega); \int_{\Omega} m|u|^p dx = 1 \right\} \quad (3)$$

and φ_m is the positive eigenfunction associated to $\lambda_1(m)$ with $\varphi_m \in M$.

The case of a potential $V \geq 0$ is technically very similar to the case $V \equiv 0$, as the energy associated to problem (P)

$$E_V(u) \stackrel{\text{def}}{=} \int_{\Omega} (|\nabla u|^p + V|u|^p) dx \quad (4)$$

and E have similar properties. In this case the existence of a first positive principal simple eigenvalue was proved by [20]. Problem (P) with V changing its sign, satisfying condition (H1) below and $m \equiv 1$ has been recently considered by [12]. Therein the authors prove, among other results, the existence of a unique principal eigenvalue characterized by $\inf\{E_V(u); u \in W_0^{1,p}(\Omega), \int_{\Omega} |u|^p dx = 1\}$. When V and m are indefinite some new difficulties arise. For instance, in [15] it has been observed, when $p = 2$, that for some choices of V and m the functional E_V is unbounded from below on M . Therefore it is natural to look for additional conditions on V and m that guarantee the existence of principal eigenvalues. In [7, 8] Binding and Huang consider problem (P) for bounded and possibly indefinite V, m . They introduce the notion of eigencurve $\mu(\lambda)$ (that we also use here) and give a necessary and sufficient condition for the existence of principal eigenvalues. Some sufficient conditions are also stated in [19] and [13] in the case $p = 2$. We first extend those results to more general V, m by introducing the value

$$\alpha(V, m) \stackrel{\text{def}}{=} \inf \left\{ E_V(u); \|u\|_p = 1, \int_{\Omega} m|u|^p dx = 0 \right\}.$$

When $m \geq 0$, this infimum is somehow close to the first eigenvalue of

$$-\Delta_p u + V(x)|u|^{p-2}u = \lambda|u|^{p-2}u \quad \text{in } \Omega_0; \quad u = 0 \quad \text{on } \partial\Omega_0, \quad (5)$$

where $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus \text{supp}(m)$. In this sense, we extend the results in [19] to sign-changing weights. Thereafter, we initiate the study of higher eigenvalues, focusing on the first non-principal one.

In Sect. 2 we show that (P) has a principal eigenvalue if and only if either $m \geq 0$ and $\alpha(V, m) > 0$ or m changes sign and $\alpha(V, m) \geq 0$. We also prove that E_V is unbounded from below on M whenever (P) has no principal eigenvalue. We characterize in Sect. 3 the second eigenvalue (to the right) of (P). To the best of our knowledge, the existence of further eigenvalues for (P) seems to be new. In Sect. 4 we deal with some continuity problems related to the first two eigenvalues with respect to either V or m .

The Lebesgue norm in $L^r(\Omega)$ will be denoted by $\|\cdot\|_r$ and the usual norm of $W_0^{1,p}(\Omega)$ by $\|\cdot\|$. The weak convergence in $W_0^{1,p}(\Omega)$ is denoted by $\rightharpoonup$. The positive and negative part of a function u are denoted by $u^+ \stackrel{\text{def}}{=} \max\{u, 0\}$ and $u^- \stackrel{\text{def}}{=} \max\{-u, 0\}$. Equalities (and inequalities) between two functions must be understood *a.e.*. The support of a function is always understood in the measurable sense. If A is a measurable set of $\mathbb{R}^N$, $|A|$ stands for its Lebesgue measure.

2. Existence of principal eigenvalues

Let V and m satisfy the following assumption:

$$(H1) \quad m, V \in L^r(\Omega) \quad \text{where} \quad \begin{cases} r > N/p & \text{if } 1 < p \leq N, \\ r = 1 & \text{if } p > N. \end{cases}$$

Given $m \in L^r(\Omega)$ we set

$$\Omega_+ \stackrel{\text{def}}{=} \{x \in \Omega; m(x) > 0\} \text{ and } \Omega_- \stackrel{\text{def}}{=} \{x \in \Omega; m(x) < 0\}.$$

We assume throughout that $|\Omega_+| > 0$.

We start by proving the following result:

Theorem 1. *Assume (H1) and $m > 0$ in Ω . Consider the manifold M defined in (3), the functional E_V defined in (4) and define*

$$\lambda_1(V, m) \stackrel{\text{def}}{=} \inf\{E_V(u); u \in M\}. \quad (6)$$

Then $\lambda_1(V, m) > -\infty$ is the lowest eigenvalue of problem (P). Moreover $\lambda_1(V, m)$ is principal, simple and there is no other principal eigenvalue of (P).

The proof of Theorem 1 relies on the following lemma:

Lemma 2. *Let ω satisfy (H1) and let $B \subset L^r(\Omega)$ be a bounded set. If $\omega > 0$ on Ω then there exists two positive constants C_1, C_2 such that*

$$\|u\|^p \leq C_1 E_V(u) + C_2 \int_{\Omega} \omega |u|^p dx$$

for every $V \in B$ and every $u \in W_0^{1,p}(\Omega)$.

Proof of Lemma 2. By Hölder inequality and (H1) we have

$$\left| \int_{\Omega} V |u|^p dx \right| \leq \|V\|_r \|u\|_{pr'}^p. \quad (7)$$

On the other hand we claim that for all $\varepsilon > 0$ there exists $M_\varepsilon > 0$ such that

$$\|u\|_{pr'}^p \leq \varepsilon \|u\|^p + M_\varepsilon \int_{\Omega} \omega |u|^p dx. \quad (8)$$

This inequality can be deduced from a general result stated in [18, Prop. 4.1]. We include the proof here for the sake of completeness. Assume by contradiction that there exists $\epsilon_0 > 0$ and a sequence $u_n \in W_0^{1,p}(\Omega)$ such that $\|u_n\|_{pr'} = 1$, and $\epsilon_0 \|u_n\|^p + n \int_{\Omega} \omega |u_n|^p dx < 1$. Then (u_n) is a bounded sequence in $W_0^{1,p}(\Omega)$ and therefore, up to a subsequence, (u_n) converges weakly to some $u_0 \in W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. Thus $\|u_0\|_{pr'} = 1$ and $\int_{\Omega} \omega |u_0|^p dx \leq 0$, a contradiction. Now let $T > 0$ be such that $\|V\|_r \leq T$ for all $V \in B$ and let ε satisfy $0 < \varepsilon < T^{-1}$. By combining (7) and (8) we get

$$(1 - \varepsilon T) \|u\|^p \leq E_V(u) + M_\varepsilon T \int_{\Omega} \omega |u|^p dx$$

and the result follows. $\square$

Proof of Theorem 1. By Lemma 2 we have that $\lambda_1(V, m) > -\infty$. Furthermore any minimizing sequence (u_n) is bounded. Thus there exists $u_0 \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, (u_n) converges weakly to u_0 in $W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. Hence $E_V(u_0) \leq \lim_{n \rightarrow +\infty} E_V(u_n) = \lambda_1(V, m)$, $u_0 \in M$ and consequently $E_V(u_0) = \lambda_1(V, m)$. By the Lagrange Multipliers rule, $\lambda_1(V, m)$ is an eigenvalue for Problem (P). Moreover $E_V(|u|) = E_V(u)$ for any u , so that $\lambda_1(V, m)$ possesses a nonnegative eigenfunction. Harnack Inequality from [21] allows us to conclude that the eigenvalue is principal. Simplicity of $\lambda_1(V, m)$ and non existence of further principal eigenvalues are consequences of Lemma 4 below. $\square$

Remark 3. By the regularity results of [17] and [21] all solutions u of Problem (P) belong to $L^\infty(\Omega) \cap C^\alpha(\Omega)$ for some $0 < \alpha < 1$.

Lemma 4. *Let V, m satisfy (H1) and assume that $u, v \in W_0^{1,p}(\Omega)$ are two eigenfunctions of Problem (P) associated respectively to λ and β . Assume also that $u \geq 0$, $v > 0$ in Ω . If $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) when $\int_{\Omega} m |u|^p dx \geq 0$ (resp. when $\int_{\Omega} m |u|^p dx \leq 0$) then $u = cv$ for some $c > 0$ and $\lambda = \beta$.*

Notice that in Lemma 4 both V and m can change sign.

Proof. Let L, R be defined, for any $\xi \geq 0$ and $\phi > 0$ two a.e. differentiable functions, as

$$\begin{aligned} L(\xi, \phi) &\stackrel{\text{def}}{=} |\nabla \xi|^p + (p-1) \frac{\xi^p}{\phi^p} |\nabla \phi|^p - p \frac{\xi^{p-1}}{\phi^{p-1}} |\nabla \phi|^{p-2} \nabla \phi \nabla \xi, \\ R(\xi, \phi) &\stackrel{\text{def}}{=} |\nabla \xi|^p - |\nabla \phi|^{p-2} \nabla \left(\frac{\xi^p}{\phi^{p-1}} \right) \nabla \phi. \end{aligned}$$

Picone's Identity (cf. [1]) claims that $0 \leq L = R$ and that, if $\frac{\xi}{\phi} \in W_{loc}^{1,1}(\Omega)$, then $L(\xi, \phi) = 0$ if and only if ξ and ϕ are colinear (the regularity condition $\frac{\xi}{\phi} \in W_{loc}^{1,1}(\Omega)$ is needed to ensure that $\nabla(\frac{\xi}{\phi}) = 0 \Rightarrow \frac{\xi}{\phi} = cte$). We apply Picone's Identity to u and $v + \varepsilon$ for any $\varepsilon > 0$. After integration we find

$$\begin{aligned} 0 &\leq \int_{\Omega} L(u, v + \varepsilon) dx = \int_{\Omega} R(u, v + \varepsilon) dx \\ &= \int_{\Omega} \left[|\nabla u|^p - |\nabla v|^{p-2} \nabla \left(\frac{u^p}{(v + \varepsilon)^{p-1}} \right) \nabla v \right] dx \\ &= \lambda \int_{\Omega} m |u|^p dx - \int_{\Omega} V |u|^p dx + \int_{\Omega} (V - \beta m) |v|^{p-1} \frac{u^p}{(v + \varepsilon)^{p-1}} dx. \end{aligned}$$

We let $\varepsilon \rightarrow 0$ to get $0 \leq \int_{\Omega} L(u, v) dx \leq (\lambda - \beta) \int_{\Omega} m |u|^p dx$. If $\int_{\Omega} m |u|^p dx > 0$ (resp. $\int_{\Omega} m |u|^p dx < 0$) and $\beta \geq \lambda$ (resp. $\beta \leq \lambda$) then $L(u, v) = 0$. Hence $u = cv$ for some $c > 0$ and therefore $\alpha = \beta$. If $\int_{\Omega} m |u|^p dx = 0$ then $L(u, v) = 0$ and we conclude again that $u = cv$ and $\alpha = \beta$. $\square$

Next we consider an arbitrary weight m . The approach used now is the one introduced in [7, 8]. We consider for any fixed $\lambda \in \mathbb{R}$ the new eigenvalue problem of parameter $\mu(\lambda)$:

$$\begin{cases} -\Delta_p u + (V - \lambda m) |u|^{p-2} u = \mu(\lambda) |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (9)$$

and we set

$$\mu(\lambda) \stackrel{\text{def}}{=} \lambda_1(V - \lambda m, 1). \quad (10)$$

From Theorem 1 we know that $\mu(\lambda) > -\infty$ is the unique principal eigenvalue of problem (9). Let us denote here by φ_{λ} the L^p -normalized positive eigenfunction associated to $\mu(\lambda)$. Our purpose is to find a reasonable assumption on V and m so that there exists at least one $\lambda \in \mathbb{R}$ such that $\mu(\lambda) = 0$. Indeed, it is straightforward proving the following:

Lemma 5. $\mu(\lambda) = 0$ if and only if λ is a principal eigenvalue of (P) . In that case the associated eigenfunctions are multiples of φ_{λ} .

We start by giving some properties of the function $\mu(\lambda)$. Properties (i), (ii) and (iii) are proved in [8] (for bounded V, m) but we include their proof for the sake of completeness. Property (iv) generalizes for indefinite weights m the value μ^* of Theorem 3.1 of [8]. See also Remark 10.

Proposition 6.

(i) $\mu : \mathbb{R} \rightarrow \mathbb{R}$ is concave and differentiable with

$$\mu'(\lambda) = - \int_{\Omega} m \varphi_{\lambda}^p dx \quad \text{for all } \lambda \in \mathbb{R}; \quad (11)$$

(ii) if $|\Omega_+| > 0$ (resp. $|\Omega_-| > 0$) then $\lim_{\lambda \rightarrow +\infty} \mu(\lambda) = -\infty$ (resp. $\lim_{\lambda \rightarrow -\infty} \mu(\lambda) = -\infty$);

- (iii) if $|\Omega_-| = 0$ then μ is strictly decreasing;
- (iv) $\sup_{\lambda \in \mathbb{R}} \mu(\lambda) = \alpha(V, m)$ where

$$\alpha(V, m) \stackrel{\text{def}}{=} \inf \left\{ E_V(u); \|u\|_p = 1, \int_{\Omega} m |u|^p dx = 0 \right\}. \quad (12)$$

The infimum above is finite if and only if $|\Omega_+| < |\Omega|$. In this case $\alpha(V, m)$ is achieved at some $0 \leq \xi_0 \in W_0^{1,p}(\Omega)$.

Proof.

- (i) The concavity of μ follows from the concavity of $E_{V-\lambda m}$ as a function of λ . In particular μ is continuous. Now let $\lambda_n \rightarrow \lambda$ and $\varphi_n, \varphi_\lambda$ be the L^p -normalized positive eigenfunctions related to λ_n, λ respectively. Since the potentials $V - \lambda_n m$ are uniformly bounded in $L^r(\Omega)$, we apply Lemma 2 with $\omega \equiv 1$ to get

$$\|\varphi_n\|_p^p \leq C_1 E_{V-\lambda_n m}(\varphi_n) + C_2 = C_1 \mu(\lambda_n) + C_2.$$

So we conclude that (φ_n) is a bounded sequence in $W_0^{1,p}(\Omega)$. Hence there exists φ_0 such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi_0$ in $W_0^{1,p}(\Omega)$, strongly in $L^{pr'}(\Omega)$. Then $\|\varphi_0\|_p = 1$ and from

$$\mu(\lambda) = \lim_{n \rightarrow +\infty} \mu(\lambda_n) \geq E_{V-\lambda m}(\varphi_0) \geq \mu(\lambda)$$

we infer that $\varphi_0 = \varphi_\lambda$. Furthermore

$$\begin{aligned} \mu(\lambda_n) &= E_{V-\lambda_n m}(\varphi_n) = E_{V-\lambda m}(\varphi_n) + (\lambda - \lambda_n) \int_{\Omega} m |\varphi_n|^p dx \\ &\geq \mu(\lambda) + (\lambda - \lambda_n) \int_{\Omega} m |\varphi_n|^p dx \end{aligned}$$

and replacing λ (resp. φ) by λ_n (resp. φ_n) in this inequality we have, for $\lambda_n > \lambda$,

$$- \int_{\Omega} m |\varphi_n|^p dx \leq \frac{\mu(\lambda_n) - \mu(\lambda)}{\lambda_n - \lambda} \leq - \int_{\Omega} m |\varphi_\lambda|^p dx.$$

Passing to the limit we get (11). A similar argument holds if $\lambda_n < \lambda$.

- (ii) Since $|\Omega_+| > 0$ there exists a function $\xi \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} m |\xi|^p dx > 0$ and $\int_{\Omega} |\xi|^p dx = 1$. Then, for all $\lambda > 0$

$$\mu(\lambda) \leq \int_{\Omega} (|\nabla \xi|^p + (V - \lambda m) |\xi|^p) dx \longrightarrow -\infty \text{ as } \lambda \rightarrow +\infty.$$

A similar proof holds if $|\Omega_-| > 0$.

- (iii) This result is clear from the fact that $\int_{\Omega} m |\varphi_\lambda|^p dx > 0$ for any $\lambda \in \mathbb{R}$. Indeed, if $\lambda_1 < \lambda_2$ then $\mu(\lambda_1) = E_V(\varphi_{\lambda_1}) - \lambda_1 \int_{\Omega} m |\varphi_{\lambda_1}|^p dx > E_V(\varphi_{\lambda_1}) - \lambda_2 \int_{\Omega} m |\varphi_{\lambda_1}|^p dx \geq \mu(\lambda_2)$.
- (iv) Let us prove that $\sup_{\lambda \in \mathbb{R}} \mu(\lambda) = \alpha(V, m)$. One can see easily that

$$\alpha(V, m) \geq \mu(\lambda)$$

for all $\lambda \in \mathbb{R}$ and that $\alpha(V, m) = +\infty$ if and only if $m > 0$ a.e on Ω . We distinguish two cases:

- (a) $|\Omega_-| = 0$. In this case we know that $\mu(\lambda)$ is strictly decreasing so that

$$\sup \mu(\lambda) = \lim_{\lambda \rightarrow -\infty} \mu(\lambda). \quad (13)$$

Let $\lambda_n \rightarrow -\infty$ when $n \rightarrow \infty$ and let φ_n be the associated L^p -normalized eigenfunction. From lemma 2 we get

$$\mu(\lambda_n) = E_{V-\lambda_n m}(\varphi_n) \geq E_V(\varphi_n) \geq C_1^{-1}(\|\varphi_n\|^p - C_2)$$

for all $\lambda_n \leq 0$. Thus the sequence (φ_n) is bounded in $W_0^{1,p}(\Omega)$. Then there exists $\varphi \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $\varphi_n \rightharpoonup \varphi$ in $W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. One has $\|\varphi\|_p = 1$ and

$$\alpha(V, m) \geq \lim_{n \rightarrow \infty} \mu(\lambda_n) \geq E_V(\varphi) - \lim_{n \rightarrow \infty} \lambda_n \int_{\Omega} m |\varphi_n|^p dx. \quad (14)$$

If $m > 0$ a.e on Ω then, from (14), $\lim_{\lambda \rightarrow -\infty} \mu(\lambda) = +\infty = \alpha(V, m)$. Otherwise $\alpha(V, m) < +\infty$ so that μ is bounded from above and we conclude from (14) that $\int_{\Omega} m |\varphi|^p dx = 0$. Therefore φ is admissible in the definition of $\alpha(V, m)$ and, again from (14), we get

$$\alpha(V, m) \geq \lim_{n \rightarrow \infty} \mu(\lambda_n) \geq E_V(\varphi) \geq \alpha(V, m)$$

and the result follows from (13).

- (b) $|\Omega_-| > 0$. In this case it follows from (i) and (ii) that μ is bounded from above. Then $\sup_{\lambda \in \mathbb{R}} \mu(\lambda)$ is achieved at some λ_0 that satisfies $0 = \mu'(\lambda_0) = -\int_{\Omega} m |\varphi_{\lambda_0}|^p dx$. We conclude that φ_{λ_0} is admissible in the definition of $\alpha(V, m)$ so that

$$\alpha(V, m) \leq E_V(\varphi_{\lambda_0}) = \mu(\lambda_0) = \sup_{\lambda \in \mathbb{R}} \mu(\lambda).$$

Then the conclusion follows. Finally, let us show that $\alpha(V, m)$ is achieved whenever it is finite. Take a minimizing sequence (u_n) for $\alpha(V, m)$ and apply Lemma 2 with $\omega \equiv 1$ to get

$$\|u_n\|^p \leq C_1 E_V(u_n) + C_2.$$

Since the right hand side goes to $C_1 \alpha(V, m) + C_2$ we conclude that (u_n) is bounded. By standard arguments one can show that (u_n) converges weakly to some u_0 that realizes $\alpha(V, m)$. $\square$

We state now the main result of this section:

Theorem 7. *Assume that V and m satisfy (H1).*

- (i) *If $|\Omega_-| = 0$ then there exists a principal eigenvalue of (P) if and only if $\alpha(V, m) > 0$. In this case the principal eigenvalue is unique and it is characterized by*

$$\lambda_1(V, m) = \min_M E_V. \quad (15)$$

(ii) If $|\Omega_-| > 0$ then there exists a principal eigenvalue of (P) if and only if $\alpha(V, m) \geq 0$. More precisely:

(a) If $\alpha(V, m) > 0$ then (P) admits exactly two principal eigenvalues $\lambda_{-1}(V, m) < \lambda_1(V, m)$, with $\lambda_1(V, m)$ characterized as in (15) and

$$\lambda_{-1}(V, m) = -\min_{M^-} E_V \quad (16)$$

where $M^- \stackrel{\text{def}}{=} \{u \in W_0^{1,p}(\Omega); \int_{\Omega} m|u|^p dx = -1\}$.

(b) If $\alpha(V, m) = 0$ then (P) has an unique principal eigenvalue $\lambda_1(V, m)$ given by

$$\lambda_1(V, m) = \inf_M E_V = -\inf_{M^-} E_V.$$

These infima are not achieved. Moreover any function $u \neq 0$ in $W_0^{1,p}(\Omega)$ satisfying

$$E_V(u) = \int_{\Omega} m|u|^p dx = 0 \quad (17)$$

is an eigenfunction associated to $\lambda_1(V, m)$.

Proof.

- (i) From (iii) and (iv) of Proposition 6 it is clear that $\alpha(V, m) > 0$ is a necessary and sufficient condition for the curve $\mu(\lambda)$ to vanish at some unique value λ_1 . It follows that $E_V(u) \geq \lambda_1 \int_{\Omega} m|u|^p dx$ for every $u \neq 0$. Moreover, equality holds for $u = \varphi_{\lambda_1}$. From formula (11) we know that $\int_{\Omega} m|\varphi_{\lambda_1}|^p dx > 0$ and consequently (15) holds.
- (ii) By Proposition 6 (ii), $\mu(\lambda)$ is a concave differentiable function going to $-\infty$ when $\lambda \rightarrow \pm\infty$. Therefore $0 = \alpha(V, m) = \sup_{\lambda \in \mathbb{R}} \mu(\lambda) = \mu(\tilde{\lambda})$ for some $\tilde{\lambda} \in \mathbb{R}$ providing a principal eigenvalue of Problem (P). Conversely, if Problem (P) has a principal eigenvalue, say λ , then $0 = \mu(\lambda) \leq \alpha(V, m)$.
- (a) If $\alpha(V, m) > 0$ then $\mu(\lambda)$ vanishes twice, let's say in $\lambda_{-1}(V, m) < \lambda_1(V, m)$. Let us show that $\mu' > 0$ in $\lambda_{-1}(V, m)$. Indeed, assume that $\mu'(\lambda_{-1}(V, m)) = -\int_{\Omega} m|\varphi_{-1}|^p dx = 0$, where φ_{-1} is the L^p -normalized positive eigenfunction associated to $\lambda_{-1}(V, m)$. Then φ_{-1} is an admissible function in the definition of $\alpha(V, m)$ and we have $\alpha(V, m) \leq E_V(\varphi_{-1}) = \mu(\lambda_{-1}(V, m)) = 0$, a contradiction. Thus we have shown that $\int_{\Omega} m|\varphi_{-1}|^p dx < 0$. After a renormalisation we can assume that $\int_{\Omega} m|\varphi_{-1}|^p dx = -1$. Then formula (16) is directly deduced from the definition of $\mu(\lambda)$. A similar argument holds for $\lambda_1(V, m)$ and we omit it.
- (b) If $\alpha(V, m) = 0$ then $\mu(\lambda_0) = 0$ at some λ_0 providing a principal eigenvalue of (P). By Lemma 4 this point λ_0 is unique. Hence $\mu'(\lambda_0) = -\int_{\Omega} m|\varphi_{\lambda_0}|^p dx = 0$. Let us prove that $\lambda_0 = \inf_M E_V$. Take $u \in M$

and assume that $u \geq 0$ by replacing u by $|u|$ if necessary. Picone's identity applied to $u_T \stackrel{\text{def}}{=} \min\{u, T\}$ and $\varphi_{\lambda_0} + \varepsilon$ yields

$$\begin{aligned} 0 &\leq \int_{\Omega} L(u_T, \varphi_{\lambda_0} + \varepsilon) dx = \int_{\Omega} R(u_T, \varphi_{\lambda_0} + \varepsilon) dx \\ &= \int_{\Omega} \left[|\nabla u_T|^p - |\nabla \varphi_{\lambda_0}|^{p-2} \nabla \left(\frac{u_T^p}{(\varphi_{\lambda_0} + \varepsilon)^{p-1}} \right) \nabla \varphi_{\lambda_0} \right] dx \\ &= \int_{\Omega} |\nabla u_T|^p dx - \lambda_0 \int_{\Omega} m \left(\frac{u_T^p}{(\varphi_{\lambda_0} + \varepsilon)^{p-1}} \right) \varphi_{\lambda_0}^{p-1} dx \\ &\quad + \int_{\Omega} V \left(\frac{u_T^p}{(\varphi_{\lambda_0} + \varepsilon)^{p-1}} \right) \varphi_{\lambda_0}^{p-1} dx. \end{aligned}$$

Now we let $\varepsilon \rightarrow 0$ and $T \rightarrow \infty$ to get $E_V(u) \geq \lambda_0$. Consider the sequence $u_n \in M$ given by

$$u_n = \frac{\varphi_{\lambda_0} + \frac{\psi}{n}}{\left(\int_{\Omega} m |\varphi_{\lambda_0} + \frac{\psi}{n}|^p dx \right)^{\frac{1}{p}}}, \quad (18)$$

for some fixed $0 < \psi \in C^\infty(\Omega)$ such that $\int_{\Omega} m \psi^p dx > 0$ and $\int_{\Omega} m \varphi_{\lambda_0}^{p-1} \psi dx > 0$. One can easily prove that $\int_{\Omega} m |\varphi_{\lambda_0} + \frac{\psi}{n}|^p dx > 0$ for n large enough. Moreover, for such n we can find $0 < t_n, s_n < \frac{1}{n}$ such that

$$E_V(\varphi_{\lambda_0} + \frac{\psi}{n}) = \frac{1}{n} \langle E'_V(\varphi_{\lambda_0} + t_n \psi), \psi \rangle$$

and

$$\int_{\Omega} m |\varphi_{\lambda_0} + \frac{\psi}{n}|^p dx = \frac{p}{n} \int_{\Omega} m |\varphi_{\lambda_0} + s_n \psi|^{p-1} \psi dx.$$

Hence $E_V(u_n) \rightarrow \lambda_0$. Since φ_{λ_0} satisfies (17) it is clear that λ_0 is not achieved. By repeating the above argument for $u \in M^-$ and $\psi < 0$ such that $\int_{\Omega} m \psi^p dx < 0$ and $\int_{\Omega} m \varphi_{\lambda_0}^{p-1} \psi dx < 0$ we can show that $\lambda_0 = \inf_{M^-} E_V$. Finally for any L^p -normalized function u satisfying $E_V(u) = \int_{\Omega} m |u|^p dx = 0$ one has

$$\sup_{\lambda \in \mathbb{R}} \mu(\lambda) = 0 = E_V(u) = E_{V-\lambda_0 m}(u) \geq \mu(\lambda_0) = 0,$$

so u achieves $\mu(\lambda_0)$. Then $u = c\varphi_{\lambda_0}$ for some constant c and the result follows. We put $\lambda_1(V, m) = \lambda_0$. $\square$

Remark 8. As a consequence of Theorem 7 (ii) (b) we obtain, when $|\Omega_-| > 0$ and $\alpha(V, m) = 0$, that the function ξ_0 of Proposition 6 (iv) is an eigenfunction associated to $\lambda_1(V, m)$. Therefore ξ_0 is unique.

Remark 9. By Lemma 4 all the principal eigenvalues of (P) are simple and there are no other principal eigenvalues. Moreover, using similar arguments to those of [11] one can prove that the principal eigenvalues of Theorem 7 are isolated in the spectrum.

Remark 10. Notice that the condition $\alpha(V, m) > 0$ in Theorem 7 (ii) is equivalent to $\mu(\lambda_0) > 0$ for some λ_0 . In this sense, (ii) corresponds to Theorem 4.2 of [8] whereas the case $\alpha(V, m) = 0$ in Theorem 7 (ii) (b) is equivalent to $\mu(\lambda_0) = 0, \mu(\lambda) < 0$ for all λ , as stated in Theorem 4.2 of [8].

It is not difficult to construct an example of V and m such that $\alpha(V, m) = 0$:

Example 10.1. Let Ω be a bounded domain in $\mathbb{R}^N$ and $B_0 \subset \Omega$ be an open subset such that $\|\varphi_1\|_{2, B_0}^2 = \frac{1}{2}$, where φ_1 is the first eigenfunction of $(-\Delta, H_0^1(\Omega))$. Set

$$m = m_\varepsilon = \begin{cases} -\varepsilon, & \text{in } \Omega \setminus B_0 \\ \varepsilon, & \text{in } B_0 \end{cases} \quad (19)$$

and

$$V = V_\varepsilon = \begin{cases} -\lambda_1 - \frac{1}{2\varepsilon}, & \text{in } \Omega \setminus B_0 \\ -\lambda_1 + \frac{1}{2\varepsilon}, & \text{in } B_0 \end{cases} \quad (20)$$

where λ_1 is the first eigenvalue of $(-\Delta, H_0^1(\Omega))$. We have $\int_\Omega m \varphi_1^2 dx = 0$. If u is such that $\int_\Omega m u^2 dx = 1$ (resp. 0) then

$$\|u\|_{2, B_0}^2 - \|u\|_{2, \Omega \setminus B_0}^2 = \frac{1}{\varepsilon} \text{ (resp. 0)}$$

and

$$E_V(u) = \|u\|^2 - (\lambda_1 + \frac{1}{2\varepsilon})\|u\|_{2, \Omega \setminus B_0}^2 - (\lambda_1 - \frac{1}{2\varepsilon})\|u\|_{2, B_0}^2 \quad (21)$$

$$> \frac{1}{2\varepsilon}(\|u\|_{2, B_0}^2 - \|u\|_{2, \Omega \setminus B_0}^2) = \frac{1}{2\varepsilon^2} \text{ (resp. } \geq 0) \quad (22)$$

Therefore E_V is non-negative and vanishes only for multiples of φ_1 . Hence φ_1 achieves $\alpha(V, m) = 0$ and $\lambda_1(V, m) = \frac{1}{2\varepsilon^2}$.

When $m \geq 0$ and $\text{supp } m \neq \Omega$ one may ask whether or not $\alpha(V, m)$ is the first eigenvalue of $-\Delta_p + V$ on $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus \text{supp}(m)$. For establishing this equality, we require Ω_0 to be p -stable in the capacity sense, i.e., $\forall u \in W^{1,p}(\mathbb{R}^N)$ there holds:

$$u = 0 \text{ quasi-everywhere on } \mathbb{R}^N \setminus \overline{\Omega_0} \implies u = 0 \text{ quasi-everywhere on } \mathbb{R}^N \setminus \Omega_0.$$

For other characterizations and examples of p -stable sets, we refer to [6, 16].

Proposition 11. *Let $m \geq 0$ satisfy (H1) and $\Omega_0 \stackrel{\text{def}}{=} \Omega \setminus \text{supp}(m)$. If Ω_0 is a non-empty p -stable set then $\alpha(V, m) = \lambda_1(V, 1, \Omega_0)$, i.e., $\alpha(V, m)$ is the first eigenvalue of the equation*

$$-\Delta_p u + V|u|^{p-2}u = \lambda|u|^{p-2}u \quad \text{in } \Omega_0; \quad u = 0 \quad \text{on } \partial\Omega_0. \quad (23)$$

Proof. Let $0 < \omega_0 \in W_0^{1,p}(\Omega_0)$ be the first eigenfunction of $(-\Delta_p + V, \Omega_0)$ L^p -normalized. If one extends ω_0 by 0 over Ω one gets a function admissible for $\alpha(V, m)$. Thus

$$\alpha(V, m) \leq E_V(\omega_0) = \lambda_1(V, 1, \Omega_0).$$

For the reverse inequality, let $u \geq 0$ be an admissible function for $\alpha(V, m)$ and u^* be its *quasi-continuous* representative extended by zero to $\mathbb{R}^N$. We claim that $u^* = 0$ quasi-everywhere on $\mathbb{R}^N \setminus \overline{\Omega}_0$ so that $u^* \in W_0^{1,p}(\Omega_0)$. Indeed, assume by contradiction that $u^* > 0$ on $A \subset \mathbb{R}^N \setminus \overline{\Omega}_0$ with $\text{cap}(A) > 0$. We choose A_0 such that $\text{cap}(A_0) < \text{cap}(A)$ and u^* is continuous in $\Omega \setminus A_0$. Then $A \not\subset A_0$ so that u^* is continuous at some $y \in A$. Hence $u^* > 0$ in $B(y, r) \subset \mathbb{R}^N \setminus \overline{\Omega}_0$ for some $r > 0$ and consequently $\int_{\Omega} m|u^*|^p dx > 0$, a contradiction. $\square$

The next proposition shows that the existence of principal eigenvalues of (P) is equivalent to E_V to be bounded from below on M .

Proposition 12. *If $\alpha(V, m) < 0$ or $\alpha(V, m) = 0$ when $m \geq 0$ and Ω_0 is p -stable then E_V is unbounded from below on M .*

Proof. Assume that $\alpha(V, m) < 0$ and let $u_0 \geq 0$ realize $\alpha(V, m)$, i.e., $E_V(u_0) < 0$ and $\int_{\Omega} m|u_0|^p dx = 0$. If $m \geq 0$ then $u_0 = 0$ on Ω_+ . Let $0 \leq w \in W_0^{1,p}(\Omega)$ be such that $w > 0$ on Ω_+ and $\int_{\Omega} m|w|^p dx = 1$. Then $\int_{\Omega} m|w + tu_0|^p dx = \int_{\Omega_+} m|w|^p dx = 1$ and $E_V(w + tu_0) = E_V(w) + t^p E_V(u_0) \rightarrow -\infty$ when $t \rightarrow \infty$.

If $m^- \neq 0$ then $|(\Omega \setminus \Omega_+) \cap \Omega_{0,+}| > 0$, where $\Omega_{0,+} \stackrel{\text{def}}{=} \{x \in \Omega; u_0(x) > 0\}$, otherwise $\int_{\Omega} m|u_0|^p dx < 0$. We pick $0 \leq w \in W_0^{1,p}(\Omega)$ such that $w > 0$ on $\Omega_+ \cap \Omega_{0,+}$. Then, for each n , there are $0 < s_n, t_n < \frac{1}{n}$ such that

$$\int_{\Omega} m|u_0 + \frac{w}{n}|^p dx = \frac{p}{n} \int_{\Omega_+} m|u_0 + s_n w|^{p-2} (u_0 + s_n w) w dx > 0$$

and

$$E_V\left(u_0 + \frac{w}{n}\right) = E_V(u_0) + \frac{1}{n} \langle E'_V(u_0 + t_n w), w \rangle.$$

Hence

$$\frac{E_V(u_0 + \frac{w}{n})}{\int_{\Omega} m|u_0 + \frac{w}{n}|^p dx} \rightarrow -\infty.$$

Now let us treat the case when $\alpha(V, m) = 0$, $m \geq 0$ and Ω_0 is p -stable. Assume by contradiction that $\lambda_1 \stackrel{\text{def}}{=} \inf_M E_V > -\infty$. We prove that E_V satisfies the $(PSC)_{\lambda_1}$ condition on M , which implies that λ_1 is achieved (notice that conditions $(PSC)_{\lambda_1}$ and $(PS)_{\lambda_1}$ are equivalent, as stated in [5], Prop. 4.3). Using that $E_V(u) = E_V(|u|)$ for all $u \in M$ and the Harnack inequality one would get that λ_1 is a principal eigenvalue, a contradiction with Theorem 7 (ii)(b).

Let (u_n) be a $(PSC)_{\lambda_1}$ sequence for $\tilde{E}_V$, the restriction of E_V to M . For some sequence $\varepsilon_n \rightarrow 0$ there holds

$$(PSC1) \quad E_V(u_n) \longrightarrow \lambda_1$$

$$(PSC2) \quad |\langle E'_V(u_n), \xi \rangle| \leq \frac{\varepsilon_n}{1 + \|u_n\|} \|\xi\| \quad \text{for all } \xi \in T_{u_n} M.$$

Let us assume that (u_n) is unbounded and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, there is some v_0 such that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \rightarrow v_0$ in $L^{pr'}(\Omega)$. For

any $v, w \in W_0^{1,p}(\Omega)$, let $\tau(v, w) = \int_{\Omega} m|v|^{p-2}vw \, dx$. We choose $\xi = (v_n - v_0) - \tau(u_n, v_n - v_0)u_n$ in $(PSC2)$ and divide it by $\|u_n\|^{p-1}$ to obtain

$$\begin{aligned} & |\langle E'_V(v_n), v_n - v_0 \rangle - \tau(v_n, v_n - v_0)E_V(u_n)| \\ & \leq \frac{\varepsilon_n \|u_n\|}{1 + \|u_n\|} \left\| \frac{v_n - v_0}{\|u_n\|^p} - \tau(v_n, v_n - v_0)v_n \right\|. \end{aligned}$$

Let us recall that the (S^+) property of the p -Laplacian claims that $v_n \rightarrow v_0$ if $v_n \rightharpoonup v_0$ and $\langle E'_V(v_n), v_n - v_0 \rangle \rightarrow 0$. Thus by letting $n \rightarrow \infty$ above we get $v_n \rightarrow v_0$ in $W_0^{1,p}(\Omega)$. Furthermore, when replacing $v_n - v_0$ by any $w \in W_0^{1,p}(\Omega)$ and letting $n \rightarrow \infty$ we find that v_0 realizes $\alpha(V, m)$ and that λ_1 is an eigenvalue for (P) (therefore, a principal eigenvalue). This is impossible by Theorem 7 (ii)(b). Therefore u_n is bounded and we can repeat the argument applied to v_n to show that u_n is, up to a subsequence, convergent. Hence the $(PSC)_{\lambda_1}$ condition is proved. $\square$

3. Non-principal eigenvalues

Now we turn to the search of a first non-principal eigenvalue of (P). More precisely, we look for the first eigenvalue to the right of $\lambda_1(V, m)$ when $\alpha(V, m) \geq 0$. A natural candidate for this eigenvalue is the minimax value obtained by the Mountain-Pass theorem applied to a family of paths depending on the principal eigenvalue of (P). Such approach has been widely used so far. For instance, in [4] and [5] the authors characterized the second eigenvalue of $-\Delta_p$ under Dirichlet and Neumann boundary conditions. We consider two cases, namely: $\alpha(V, m) > 0$ and $\alpha(V, m) = 0$. The first one stands for a standard case, whereas the second one presents some difficulties related to the geometry of E_V .

Let us set $\tilde{E}_V$ to be the restriction of E_V to M . We denote by $\varphi_1 \in M$ the positive eigenfunction associated to $\lambda_1(V, m)$ when $\alpha(V, m) > 0$ and by φ_0 the L^p -normalized positive eigenfunction associated to $\lambda_1(V, m)$ and satisfying $\int_{\Omega} m|\varphi_0|^p \, dx = 0$ when $\alpha(V, m) = 0$.

3.1. The case $\alpha(V, m) > 0$

Our aim is to prove that the minimax value

$$\lambda_2(V, m) = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} E_V(\gamma(t)), \quad (24)$$

where $\Gamma = \{\gamma \in \mathcal{C}([0,1], M); \gamma(0) = -\varphi_1, \gamma(1) = \varphi_1\}$, provides a first non-principal eigenvalue of (P). One can easily construct paths belonging to Γ to show that $\lambda_2(V, m)$ is finite. Next we collect some results in order to show that $\tilde{E}_V$ has the expected mountain-pass geometry.

Proposition 13. *Assume that $\alpha(V, m) > 0$. Then the functional $\tilde{E}_V$ satisfies the (PS) condition.*

Proof. Let (u_n) be a Palais–Smale sequence in M for $\tilde{E}_V$, i.e., there exists $\varepsilon_n \rightarrow 0$ such that

$$(PS1) \quad E_V(u_n) \longrightarrow c$$

$$(PS2) \quad \left| \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \xi \, dx + \int_{\Omega} V |u_n|^{p-2} u_n \xi \, dx \right| \leq \varepsilon_n \|\xi\|$$

for all $\xi \in T_{u_n} M$.

For any $v \in W_0^{1,p}(\Omega)$ we consider its projection on $T_{u_n} M$:

$$v - \left(\int_{\Omega} m |u_n|^{p-2} u_n v \, dx \right) u_n \, dx.$$

After replacing it in (PS2) we get

$$\begin{aligned} & \left| \langle E'_V(u_n), v \rangle - E_V(u_n) \int_{\Omega} m |u_n|^{p-2} u_n v \, dx \right| \\ & \leq \varepsilon_n \left\| v - \left(\int_{\Omega} m |u_n|^{p-2} u_n v \, dx \right) u_n \right\| \end{aligned}$$

for all $v \in W_0^{1,p}(\Omega)$. If u_n is bounded then there exists $u_0 \in W_0^{1,p}(\Omega)$ such that, up to a subsequence, $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, strongly in $L^{pr'}(\Omega)$. Choosing $v = u_n - u_0$ in the above inequality and passing to the limit we obtain

$$\lim_{n \rightarrow +\infty} \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u_0) \, dx = 0.$$

The (S^+) property of the p -Laplacian then implies that u_n converges strongly in $W_0^{1,p}(\Omega)$ to u_0 . It remains to prove that the (PS) sequence is bounded. If $m > 0$ the result comes from (PS1) and Lemma 2. Otherwise we apply Lemma 14 below. $\square$

Lemma 14. Assume that $\alpha(V, m) > 0$ with $V, m \in L^r(\Omega)$ satisfying (H1). If $V_n \rightharpoonup V, m_n \rightharpoonup m$ in $L^r(\Omega)$ and u_n is a sequence such that $E_{V_n}(u_n)$ and $\int_{\Omega} m_n |u_n|^p \, dx$ are bounded then u_n is bounded.

Proof. Assume by contradiction that u_n is unbounded. Consider $v_n \stackrel{\text{def}}{=} \frac{u_n}{\|u_n\|}$ which converges weakly to some $v_0 \in W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. Thus, passing to the limit, we get $\int_{\Omega} m |v_0|^p \, dx = 0$ and

$$\lim_{n \rightarrow +\infty} E_{V_n}(v_n) = \lim_{n \rightarrow +\infty} \frac{E_{V_n}(u_n)}{\|u_n\|^p} = 0.$$

Hence

$$1 + \int_{\Omega} V |v_0|^p \, dx = 1 + \lim_{n \rightarrow +\infty} \int_{\Omega} V_n |v_n|^p \, dx = \lim_{n \rightarrow +\infty} E_{V_n}(v_n) = 0.$$

We deduce in particular that $v_0 \not\equiv 0$. Then $\frac{v_0}{\|v_0\|_p}$ is admissible in the definition of $\alpha(V, m)$ so that $\alpha(V, m) \leq E_V\left(\frac{v_0}{\|v_0\|_p}\right) \leq 0$, a contradiction. $\square$

Proposition 15. $\lambda_2(V, m)$ is the first non-principal eigenvalue of problem (P), in the sense that there is no eigenvalue of (P) in $]\lambda_1(V, m), \lambda_2(V, m)[$.

Proof. It is standard to see that $\pm\varphi_1$ are strict local minima of $\tilde{E}_V$. Furthermore we can prove as in Lemma 6 of [4] that, for every $\varepsilon > 0$,

$$\lambda_1(V, m) < \inf\{E_V(u); \quad u \in M \cap \partial B(\varphi_{\pm 1}, \varepsilon)\}.$$

As a consequence $\lambda_2(V, m) > \lambda_1(V, m)$. By the version of the Mountain pass theorem on C^1 -manifolds of [4] we can deduce that the right hand side in (24) is a critical value of $\tilde{E}_V$, so that $\lambda_2(V, m)$ is an eigenvalue of (P). Moreover, by Lemma 4 we get that $\lambda_2(V, m)$ is not principal. Let us prove that there is no eigenvalue of (P) in $]\lambda_1(V, m), \lambda_2(V, m)[$. Let us assume by contradiction that (λ, u) solves (P) with $\lambda \in]\lambda_1(V, m), \lambda_2(V, m)[$. Then u changes sign. Let us show that $\int_{\Omega} m|u^{\pm}|^p dx > 0$. Indeed, if for instance $\int_{\Omega} m|u^+|^p dx = 0$ (similar proof for u^-) then $\alpha(V, m) \leq E_V(\frac{u^+}{\|u^+\|_p}) = \lambda \int_{\Omega} m(\frac{u^+}{\|u^+\|_p})^p dx = 0$, a contradiction. If $\int_{\Omega} m|u^+|^p dx < 0$ then

$$-\lambda = E_V\left(\frac{u^+}{(-\int_{\Omega} m|u^-|^p dx)^{1/p}}\right) \geq -\lambda_{-1}(V, m) > -\lambda_1(V, m),$$

a contradiction. It follows that

$$\int_{\Omega} m|u^+ - tu^-|^p dx = \int_{\Omega} m|u^+|^p dx + t^p \int_{\Omega} m|u^-|^p dx > 0,$$

so we can set

$$\gamma_1(t) \stackrel{\text{def}}{=} \frac{u^+ - tu^-}{\left(\int_{\Omega} m|u^+ - tu^-|^p dx\right)^{\frac{1}{p}}},$$

which is a path in M going from $\frac{u^+}{\left(\int_{\Omega} m|u^+|^p dx\right)^{\frac{1}{p}}}$ to u and such that $E_V(\gamma_1(t)) = \lambda$.

In a similar way, we set

$$\gamma_2(t) \stackrel{\text{def}}{=} \frac{(1-t)u^+ - u^-}{\left(\int_{\Omega} m|(1-t)u^+ - u^-|^p dx\right)^{\frac{1}{p}}}$$

in order to get a path in M going from u to $\frac{-u^-}{\left(\int_{\Omega} m|u^-|^p dx\right)^{\frac{1}{p}}}$ and such that $E_V(\gamma_2(t)) = \lambda$. Hence $\gamma_1 \cup \gamma_2$ provides a path which contradicts the definition of $\lambda_2(V, m)$ in (24). $\square$

Remark 16. If one replaces the function u by φ_2 , an eigenfunction related to $\lambda_2(V, m)$, in the proof of Proposition 15 then the path $\gamma_1 \cup \gamma_2$ is a minimizing path for $\lambda_2(V, m)$.

The next proposition shows that one can enlarge the family of paths in (24) and keep the same minimax level:

Proposition 17. $\lambda_2(V, m) = \inf_{\gamma \in \Gamma_0} \max_{t \in [0, 1]} E_V(\gamma(t))$ where

$$\Gamma_0 \stackrel{\text{def}}{=} \{\gamma \in C([0, 1], M); \quad \gamma(0) \geq 0, \gamma(1) \leq 0\}. \quad (25)$$

Proof. We argue as in [4]. Assume by contradiction that there exists $\gamma \in \Gamma_0$ such that $\max_\gamma E_V < \lambda_2(V, m)$. We will construct a path in Γ which also stays strictly below $\lambda_2(V, m)$. This will contradict (24). Consider $\mathcal{A} \stackrel{\text{def}}{=} \{u \in M; E_V(u) < \lambda_2(V, m)\}$. By Lemma 14 of [4], $\mathcal{A}$ possesses at most two path components (because φ_1 and $-\varphi_1$ are the only critical points of $\tilde{E}_V$ in $\mathcal{A}$). If φ_1 and $-\varphi_1$ lie in the same component then the conclusion follows. If not, $\gamma(0)$ can be connected by a path to either φ_1 or $-\varphi_1$. Since $E_V(u) = E_V(\pm|u|)$ for all $u \in M$, we can always find a path from $\gamma(0)$ to φ_1 in $\mathcal{A}$ by taking absolute value. By a similar argument we prove that $\gamma(1)$ can be joined to $-\varphi_1$ in $\mathcal{A}$. $\square$

3.2. The singular case $\alpha(V, m) = 0$

We assume now that $|\Omega_-| > 0$. When $\alpha(V, m) = 0$ it has been proved previously that the unique principal eigenvalue $\lambda_1(V, m)$ of (P) is simple and has a positive L^p -normalized eigenfunction φ_0 satisfying (17). Therefore the mountain-pass procedure of minimizing E_V over paths going from φ_0 to $-\varphi_0$ is no longer suitable since $\varphi_0 \notin M$. We adopt the approach of minimizing E_V over free endpoints paths in M going from a positive function to a negative one and staying at an energy level above $\lambda_1(V, m)$. This approach has been introduced in [5] for a Neumann problem having some similar features to (P).

Let us introduce

$$\lambda_2(V, m) \stackrel{\text{def}}{=} \inf_{\gamma \in \Gamma_0} \max_{t \in [0, 1]} E_V(\gamma(t)),$$

where Γ_0 is defined in (25) above. We intend to prove that $\lambda_2(V, m)$ is the first non-principal eigenvalue of (P). To this aim we show that

$$\lambda_2(V, m) = \bar{\lambda}_2(V, m) \stackrel{\text{def}}{=} \inf_{\gamma \in \bar{\Gamma}_0} \max_{t \in [0, 1]} E_V(\gamma(t)), \quad (26)$$

where $\bar{\Gamma}_0 \stackrel{\text{def}}{=} \{\gamma \in C([0, 1], M); \gamma(0) = u_1, \gamma(1) = -u_1\}$ for every $0 \leq u_1 \in M$ such that $E_V(u_1) < \lambda_2(V, m)$.

We start by proving the following result:

Proposition 18. $\lambda_2(V, m) > \lambda_1(V, m)$.

Proof. One has trivially $\lambda_2(V, m) \geq \lambda_1(V, m)$. Let us assume by contradiction that equality holds. Then there exists $\gamma_k \in \Gamma_0$ such that $\max_{t \in [0, 1]} E_V(\gamma_k(t)) \rightarrow \lambda_1(V, m)$ when $k \rightarrow \infty$. For every k one can find $t_k \in [0, 1]$ satisfying

$$\int_{\Omega} m |\gamma_k(t_k)^+|^p dx = \int_{\Omega} m |\gamma_k(t_k)^-|^p dx = \frac{1}{2}. \quad (27)$$

We set $u_k = \gamma_k(t_k)$. Notice, from (27), that $2^{\frac{1}{p}} u_k^{\pm} \in M$ so that $E_V(u_k^{\pm}) \geq \frac{1}{2} \lambda_1(V, m)$. Hence

$$\begin{aligned} \frac{1}{2} \lambda_1(V, m) &\leq E_V(u_k^{\pm}) = E_V(u_k) - E_V(u_k^{\mp}) \\ &\leq \max_{t \in [0, 1]} E_V(\gamma_k(t)) - \frac{\lambda_1(V, m)}{2} \rightarrow \frac{\lambda_1(V, m)}{2} \end{aligned}$$

so that

$$\lim_{k \rightarrow +\infty} E_V(u_k^{\pm}) = \frac{\lambda_1(V, m)}{2}. \quad (28)$$

Let us show that the sequence (u_k) is bounded in $W_0^{1,p}(\Omega)$. Assume by contradiction that the sequence (u_k) is unbounded and set $v_k = \frac{u_k}{\|u_k\|}$, which, up to a subsequence, converges weakly to some $v_0 \in W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. From

$$1 = \|v_k\|^p = E_V(v_k) - \int_{\Omega} V |v_k|^p dx$$

we deduce that $v_0 \neq 0$. Since $\int_{\Omega} m |u_k|^p dx = 1$ it follows, dividing by $\|u_k\|^p$ and passing to the limit, that $\int_{\Omega} m |v_0|^p dx = 0$. Then

$$0 = \alpha(V, m) \leq E_v \left(\frac{v_0}{\|v_0\|_p} \right) \leq \frac{1}{\|v_0\|^p} \lim_{k \rightarrow +\infty} E_V(v_k) \leq \lim_{k \rightarrow +\infty} \frac{\lambda_1(V, m)}{\|u_k\|^p} = 0.$$

Hence $E_V(v_0) = 0$ and by Theorem 7 (ii) (b) and Remark 8, v_0 is, up to a multiplicative constant, the unique constant-sign solution of (P) associated to $\lambda_1(V, m)$. As a consequence the sequence u_k^+ or u_k^- (let's say u_k^-) converges in measure to 0. By (27), (28) and Lemma 20 below we conclude that (u_k^-) is bounded. Then, up to a subsequence, (u_k^-) converges weakly to some u_0 that satisfies $E_V(u_0) = \frac{1}{2} \lambda_1(V, m)$ and $\int_{\Omega} m |u_0|^p dx = \frac{1}{2}$. Hence $2^{\frac{1}{p}} u_0 \in M$ realizes $\lambda_1(V, m)$. Thus $u_0 > 0$, a contradiction with the fact that u_k^- converges to 0 in measure. Therefore the sequence u_k is bounded and as above (passing to the limit for a convergent subsequence) we prove that the value $\lambda_1(V, m)$ is achieved at some point of M , a contradiction. $\square$

When considering the sequence u_n given in (18) we get:

Corollary 19. *There exists $u_1 \in M$ such that $u_1 \geq 0$ and $E_V(u_1) < \lambda_2(V, m)$.*

Lemma 20. *Let m satisfy (H1) and assume that $|\Omega_-| > 0$ and $\alpha(V, m) = 0$. If $u_n \geq 0$ is a sequence in M such that $|\{x \in \Omega; u_n(x) > 0\}| \rightarrow 0$ and $E_V(u_n)$ is bounded then (u_n) is bounded.*

Proof. The proof is similar to the proof of Lemma 14. Assume by contradiction that u_n is unbounded. Consider $v_n \stackrel{\text{def}}{=} \frac{u_n}{\|u_n\|}$, which converges weakly to some $v_0 \in W_0^{1,p}(\Omega)$ and strongly in $L^{pr'}(\Omega)$. Thus, passing to the limit, we get $\int_{\Omega} m |v_0|^p dx = 0$ and

$$\lim_{n \rightarrow +\infty} E_V(v_n) = \lim_{n \rightarrow +\infty} \frac{E_V(u_n)}{\|u_n\|^p} = 0.$$

Moreover

$$1 + \int_{\Omega} V |v_0|^p dx = 1 + \lim_{n \rightarrow +\infty} \int_{\Omega} V |v_n|^p dx = \lim_{n \rightarrow +\infty} E_V(v_n) = 0.$$

We deduce in particular that $v_0 \neq 0$. Hence $\frac{v_0}{\|v_0\|_p}$ is admissible in the definition of $\alpha(V, m)$ so $\alpha(V, m) \leq E_V(\frac{v_0}{\|v_0\|_p}) \leq 0$. By Proposition 6 (iv) (b) and Remark 8, $v_0 = c\varphi_0 > 0$ on Ω for some $c > 0$, in contradiction with $|\{x \in \Omega; v_n(x) > 0\}| = |\{x \in \Omega; u_n(x) > 0\}| \rightarrow 0$. $\square$

Since $\lambda_1(V, m)$ is not achieved, the Palais–Smale condition (PS) and the Palais–Smale–Cerami condition (PSC) (see for instance [5]) for $\tilde{E}_V$ fails at the level $\lambda_1(V, m)$. However, the $(PSC)_c$ condition holds at any upper level c :

Proposition 21. *The functional E_V satisfies the $(PSC)_c$ condition on M for every $c > \lambda_1(V, m)$.*

Proof. Let $c > \lambda_1(V, m)$ and (u_n) be a $(PSC)_c$ sequence in M for $\tilde{E}_V$, i.e., there exists $\varepsilon_n \rightarrow 0$ such that

$$\begin{aligned} (PSC1) \quad & E_V(u_n) \rightarrow c \\ (PSC2) \quad & |\langle E'_V(u_n), \xi \rangle| \leq \frac{\varepsilon_n}{1 + \|u_n\|} \|\xi\| \quad \text{for all } \xi \in T_{u_n}M. \end{aligned}$$

Let us assume that (u_n) is unbounded and set $v_n = \frac{u_n}{\|u_n\|}$. Up to a subsequence, there is some v_0 such that $v_n \rightharpoonup v_0$ in $W_0^{1,p}(\Omega)$ and $v_n \rightarrow v_0$ in $L^{pr'}(\Omega)$. For any $v, w \in W_0^{1,p}(\Omega)$, let $\tau(v, w) = \int_{\Omega} m |v|^{p-2} v w dx$. We choose $\xi = (v_n - v_0) - \tau(u_n, v_n - v_0)u_n$ in $(PSC2)$ and divide by $\|u_n\|^{p-1}$ to obtain

$$\begin{aligned} & |\langle E'_V(v_n), v_n - v_0 \rangle - \tau(v_n, v_n - v_0) E_V(u_n)| \\ & \leq \frac{\varepsilon_n \|u_n\|}{1 + \|u_n\|} \left\| \frac{v_n - v_0}{\|u_n\|^p} - \tau(v_n, v_n - v_0) v_n \right\|. \end{aligned} \quad (29)$$

By letting $n \rightarrow \infty$ and using the (S^+) property of the p -Laplacian we get that $v_n \rightarrow v_0$ in $W_0^{1,p}(\Omega)$. Hence $E_V(v_0) = 0$ and $\int_{\Omega} m |v_0|^p dx = 0$ so that $v_0 = d\varphi_0$ for some constant $d \neq 0$ (since $\|v_n\| = 1$ for every n). Furthermore, if we replace $v_n - v_0$ by any $w \in W_0^{1,p}(\Omega)$ and let $n \rightarrow \infty$ we see that c is an eigenvalue for (P) whose associated eigenfunction v_0 is sign-constant. This is impossible by Lemma 4. Therefore u_n is bounded and convergent up to a subsequence. $\square$

Theorem 22. *For every $0 \leq u_1 \in M$ such that $E_V(u_1) < \lambda_2(V, m)$ we have:*

- (i) $\bar{\lambda}_2(V, m)$ defined in (26) is an eigenvalue of (P).
- (ii) $\bar{\lambda}_2(V, m) = \lambda_2(V, m)$.
- (iii) $\lambda_2(V, m)$ is the first non-principal eigenvalue of (P), i.e., there is no eigenvalue of (P) in $]\lambda_1(V, m), \lambda_2(V, m)[$.

Proof.

- (i) It is a consequence of Theorem 4.1 of [5].
- (ii) Clearly $\bar{\lambda}_2(V, m) \geq \lambda_2(V, m)$. To show the reverse inequality let $\epsilon > 0$ be small enough and $\gamma \in \Gamma$ such that

$$\max_{\gamma} E_V \leq \lambda_2(V, m) + \epsilon.$$

Put $u_0 = \gamma(0)$. We claim that either there exists a path $\tilde{\gamma}$ in M from u_0 to u_1 such that E_V stays below the level $\lambda_2(V, m) + \epsilon$ on $\tilde{\gamma}$ or u_1 and $-u_1$ can be connected by a path staying also below the level $\lambda_2(V, m) + \epsilon$. In both cases the conclusion is that $\bar{\lambda}_2(V, m) \leq \lambda_2(V, m) + \epsilon$ and the conclusion follows (since a similar argument holds for $\gamma(1)$ and $-u_1$). For that purpose we perturb a little bit V in order to have the Palais–Smale condition satisfied everywhere. From Lemma 23 below, $\alpha(t) \stackrel{\text{def}}{=} \alpha(V + t, m)$ is differentiable and $\alpha'(0) = \int_{\Omega} |\varphi_0|^p dx > 0$ where φ_0 is the positive L^p -normalized eigenfunction associated to $\lambda_1(V, m)$ satisfying (17). Therefore $\alpha(V + t, m) > 0$ for $t > 0$ small enough. We also choose $t > 0$ small enough to have $\max\{E_{V+t}(u_0), E_{V+t}(u_1)\} < \lambda_2(V, m) + \epsilon$. Let us consider the open set $\mathcal{O} \stackrel{\text{def}}{=} \{u \in M; E_{V+t}(u) < \lambda_2(V, m) + \epsilon\}$. Observe that, if $t > 0$ is small enough, then $\lambda_2(V, m) < \lambda_2(V + t, m)$. This comes from the following observation: if γ is a path that realizes $\lambda_2(V + t, m)$ (cf. Remark 16) then

$$\lambda_2(V, m) \leq \max_{\gamma} E_V < \max_{\gamma} E_{V+t} = \lambda_2(V + t, m).$$

It follows from Lemma 14 of [4] that $\mathcal{O}$ has at most two arcwise connected components (because $\varphi_1(V + t, m)$ and $-\varphi_1(V + t, m)$ are the only critical points of $\tilde{E}_{V+t}$ in $\mathcal{O}$). If u_1 and $-u_1$ lie in the same component then it comes that $\bar{\lambda}_2(V, m) \leq \lambda_2(V, m) + \epsilon$. Otherwise u_0 can be connected by a path to either u_1 or $-u_1$. But since $E_{V+t}(u) = E_{V+t}(\pm|u|)$ for every $u \in M$, we can always find a path from u_0 to u_1 by taking absolute value.

- (iii) The proof of Proposition 15 remains valid in this case. Observe that any eigenfunction u associated to an eigenvalue different than $\lambda_1(V, m)$ satisfies $\int_{\Omega} m|u|^{\pm p} dx \neq 0$ by Theorem 7 (ii) (b). $\square$

Lemma 23. *Let $V, \tilde{V} \in L^r(\Omega)$ with r as in (H1). Then the mapping $\alpha(t) \stackrel{\text{def}}{=} \alpha(V + t\tilde{V}, m)$ is concave, differentiable at $t = 0$ and there exists u_0 that realizes $\alpha(V, m)$ and satisfies $\alpha'(0) = \int_{\Omega} \tilde{V}|u_0|^p dx$.*

Proof. The proof is similar to the one in Proposition 6 (i), so we omit it. $\square$

Remark 24. A sequence of eigenvalues of (P) by the Ljusternik–Schnirelmann principle can be obtained when either $\alpha(V, m) > 0$ or $\alpha(V, m) = 0$ and $|\Omega_-| > 0$. To this aim, one may apply, for instance, an equivariant version of a deformation theorem suitable for the (PSC) condition (see [9] or [10]).

4. Continuity properties of λ_1 and λ_2

We are now concerned with the behavior of $\lambda_1(V, m)$ and $\lambda_2(V, m)$ with respect to V and m . By the dependence of these values on $\alpha(V, m)$ one is naturally led to analyze the behavior of this last parameter. Continuity properties of $\lambda_1(V, m)$ and $\lambda_2(V, m)$ are useful when dealing with optimization problems with respect to V or m .

Proposition 25. *Let $V, m \in L^r(\Omega)$ satisfy (H1) and assume that $V_k \rightharpoonup V, m_k \rightharpoonup m$ in $L^r(\Omega)$.*

- (i) *If $|\Omega_-| > 0$ then $\alpha(V_k, m_k) \rightarrow \alpha(V, m)$.*
- (ii) *If $|\Omega_-| = 0$ then $\alpha(V_k, m) \rightarrow \alpha(V, m)$ and $\liminf \alpha(V_k, m_k) \geq \alpha(V, m)$.*

Proof.

- (i) Since $m_k \rightharpoonup m$ in $L^r(\Omega)$ there holds $\int_{\Omega_{\pm}} m_k dx \rightarrow \int_{\Omega_{\pm}} m dx \gtrless 0$ so that, for k large, $\Omega_{k, \pm} \stackrel{\text{def}}{=} \{x \in \Omega; m_k(x) \gtrless 0\}$ have positive measure. Hence there are λ, λ_k and u, u_k L^p -normalized such that

$$\alpha(V, m) = \mu_1(\lambda) = E_V(u), \quad \int_{\Omega} m|u|^p dx = 0$$

and

$$\alpha(V_k, m_k) = \mu_1(\lambda_k) = E_{V_k}(u_k), \quad \int_{\Omega} m_k|u_k|^p dx = 0.$$

By Lemma 2 one has $\|u_k\|^p \leq C(\alpha(V_k, m_k) + 1)$ so that $\alpha(V_k, m_k) = \mu_1(\lambda_k)$ is bounded from below. We claim that λ_k is bounded. Indeed, assume by contradiction that, up to a subsequence, $\lambda_k \rightarrow \infty$. Take w such that $\|w\|_p = 1$ and $\int_{\Omega} m|w|^p dx > 0$. Then $\alpha(V_k, m_k) \leq E_V(w) - \lambda_k \int_{\Omega} m_k|w|^p dx \rightarrow -\infty$ and we reach a contradiction. A similar argument prevents from $\lambda_k \rightarrow -\infty$. From the last inequality one can still infer that $\alpha(V_k, m_k)$ is bounded from above. Thus u_k is bounded and, up to a subsequence, $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_k \rightarrow u$ in $L^{pr'}(\Omega)$. As a consequence $E_V(u) \leq \liminf E_{V_k}(u_k)$. Therefore $\alpha(V, m) \leq \liminf \alpha(V_k, m_k)$. On the other hand,

$$\begin{aligned} \alpha(V_k, m_k) &= \mu_1(\lambda_k) \leq E_{V_k - \lambda_k m_k}(u) = \alpha(V, m) \\ &\quad + \int_{\Omega} (V_k - V)|u|^p dx - \lambda_k \int_{\Omega} m_k|u|^p dx \end{aligned}$$

and consequently $\limsup \alpha(V_k, m_k) \leq \alpha(V, m)$.

- (ii) Let $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be such that $\|w\|_p = 1$ and $\int_{\Omega} m|w|^p dx = 0$. Then

$$\alpha(V_k, m) \leq E_{V_k}(w) \leq \|w\|^p + \|V_k\|_r \|w\|_\infty |\Omega|^{\frac{1}{r'}} \leq \|w\|^p + M \|w\|_\infty |\Omega|^{\frac{1}{r'}}, \quad (30)$$

where M is such that $\|V_k\|_r \leq M$ for all k . Let $\xi_k, \xi \in W_0^{1,p}(\Omega)$ realize $\alpha(V_k, m)$, $\alpha(V, m)$ respectively. From Lemma 2, we have

$$\|\xi_k\|^p \leq C_1 E_V(\xi_k) + C_2 = C_1 \alpha(V_k, m) + C_2, \quad (31)$$

so the sequence ξ_k is bounded. Now, from (12), one has

$$\int_{\Omega} (V - V_k) |\xi|^p dx \leq \alpha(V, m) - \alpha(V_k, m) \leq \int_{\Omega} (V - V_k) |\xi_k|^p dx,$$

and, as ξ_k is bounded in $L^{pr'}(\Omega)$, we conclude that $\alpha(V_k, m) \rightarrow \alpha(V, m)$.

From Lemma 2 we know that $\alpha(V_k, m_k)$ is bounded from below. If $\alpha(V_k, m_k) \rightarrow \infty$ there is nothing to prove. Otherwise, up to a subsequence, there is u such that $u_k \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and $u_k \rightarrow u$ in $L^{pr'}(\Omega)$, where u_k realizes $\alpha(V_k, m_k)$. Since $\int_{\Omega} m|u|^p dx = \lim \int_{\Omega} m_k|u_k|^p dx = 0$ we have $\alpha(V, m) \leq E_V(u) \leq \liminf \alpha(V_k, m_k)$. $\square$

Proposition 26. *Assume that $V_k \rightharpoonup V$ and $m_k \rightharpoonup m$ in $L^r(\Omega)$ with r as in (H1), and $\alpha(V, m) \geq 0$. If $\alpha(V, m) = 0$ assume also that $\alpha(V_k, m_k) \geq 0$ for every $k \in \mathbb{N}$ and $|\Omega_-| > 0$. Then, up to a subsequence,*

- (i) $\lambda_1(V_k, m_k) \rightarrow \lambda_1(V, m)$
- (ii) $\lambda_2(V_k, m_k) \rightarrow \lambda_2(V, m)$.

Proof.

- (i) If $\alpha(V, m) > 0$ then, up to a subsequence, $\alpha(V_k, m_k) > 0$ for k large enough by Proposition 25. Let $\varphi_k = \varphi_{1,k}$ and φ_1 realize $\lambda_1(V_k, m_k)$, $\lambda_1(V, m)$ respectively and $w \in M \cap L^\infty(\Omega)$. Since $\int_{\Omega} m_k|w|^p dx \rightarrow 1$ one has

$$|\lambda_1(V_k, m_k)| \leq \frac{|E_{V_k}(w)|}{\int_{\Omega} m_k|w|^p dx} \leq \frac{\|w\|^p + \|V_k\|_r \|w\|_{\infty} |\Omega|^{\frac{1}{r}}}{\int_{\Omega} m_k|w|^p dx}$$

so that $\lambda_1(V_k, m_k)$ is bounded. We infer from Lemma 14 that φ_k is bounded. Moreover, there holds

$$\lambda_1(V, m) - \lambda_1(V_k, m_k) \int_{\Omega} m_k |\varphi|^p dx \geq \int_{\Omega} (V - V_k) |\varphi|^p dx$$

and

$$\int_{\Omega} (V - V_k) |\varphi_k|^p dx \geq \lambda_1(V, m) \int_{\Omega} m |\varphi_k|^p dx - \lambda_1(V_k, m_k).$$

Hence

$$\begin{aligned} & \frac{\int_{\Omega} (V - V_k) |\varphi_k|^p dx + \lambda_1(V_k, m_k) \int_{\Omega} m |\varphi_k|^p dx}{\int_{\Omega} m |\varphi_k|^p dx} - \lambda_1(V_k, m_k) \int_{\Omega} m_k |\varphi|^p dx \\ & \geq \lambda_1(V, m) - \lambda_1(V_k, m_k) \int_{\Omega} m_k |\varphi|^p dx \geq \int_{\Omega} (V - V_k) |\varphi|^p dx \end{aligned} \quad (32)$$

so that $\lambda_1(V_k, m_k) - \lambda_1(V, m) \rightarrow 0$.

If $\alpha(V, m) = 0$ with $\alpha(V_k, m) \geq 0$ for all k then we can assume, up to a subsequence, that $\alpha(V_k, m_k) > 0$ for all k or $\alpha(V_k, m_k) = 0$ for all k . In the

first case consider, for any $\epsilon > 0$, a function u_ϵ given by (18) and such that $E_V(u_\epsilon) \leq \lambda_1(V, m) + \epsilon$. Then

$$\begin{aligned} \int_{\Omega} (V - V_k) |u_\epsilon|^p dx - \epsilon &\leq \lambda_1(V, m) - \lambda_1(V_k, m_k) \int_{\Omega} m_k |u_\epsilon|^p dx \\ &\leq \int_{\Omega} (V - V_k) |\varphi_k|^p dx \end{aligned}$$

and the conclusion follows as in (32). We assume now that $\alpha(V_k, m_k) = 0$ for every k . Let $\xi_k, \xi \geq 0$ realize respectively $\alpha(V_k, m_k), \alpha(V, m)$ (they are unique by Remark 8). One can easily show that $\xi_k \rightarrow \xi$. We choose $v \in W_0^{1,p}(\Omega)$ such that $\int_{\Omega} m \xi^{p-1} v dx > 0$ (this is possible thanks to the strict positivity of ξ). Therefore $\int_{\Omega} m_k \xi_k^{p-1} v dx > 0$ for k large, so that

$$\lambda_1(V_k, m_k) = \frac{\langle E'_{V_k}(\xi_k), v \rangle}{p \int_{\Omega} m_k \xi_k^{p-1} v dx} \rightarrow \frac{\langle E'_V(\xi), v \rangle}{p \int_{\Omega} m \xi^{p-1} v dx} = \lambda_1(V, m).$$

(ii) We first prove the upper semicontinuity. Let $\epsilon > 0$ and take $\gamma \in \Gamma$ such that

$$\max_t E_V(\gamma(t)) < \lambda_2(V, m) + \epsilon.$$

By continuity we deduce that, for k sufficiently large,

$$\max_t \frac{E_{V_k}(\gamma(t))}{\int_{\Omega} m_k |\gamma(t)|^p} < \lambda_2(V, m) + \epsilon. \quad (33)$$

By Proposition 17 there holds $\lambda_2(V_k, m_k) \leq \max_t \frac{E_{V_k}(\gamma(t))}{\int_{\Omega} m_k |\gamma(t)|^p}$, and consequently

$$\limsup_{k \rightarrow \infty} \lambda_2(V_k, m_k) \leq \lambda_2(V, m) + \epsilon.$$

Since ϵ is arbitrary, the upper semicontinuity follows. To prove the lower semicontinuity, suppose by contradiction that, for a subsequence, $\lambda_2(V_k, m_k) \rightarrow c_0$ with $c_0 < \lambda_2(V, m)$. Assume first that $\alpha(V, m) > 0$. Let $\phi_k \in M$ be an eigenfunction associated to $\lambda_2(V_k, m_k)$. Then, by Lemma 14, we deduce that the sequence ϕ_k is bounded so that, up to a subsequence, $\phi_k \rightharpoonup \phi_0$ in $W_0^{1,p}(\Omega)$ and $\phi_k \rightarrow \phi_0$ in $L^{pr'}(\Omega)$ for some ϕ_0 . From

$$-\Delta_p \phi_k = (\lambda_2(V_k, m_k) m_k - V_k) |\phi_k|^{p-2} \phi_k$$

we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} |\nabla \phi_k|^{p-2} \nabla \phi_k \nabla (\phi_k - \phi_0) dx = 0,$$

and by the (S^+) property of Δ_p we infer that $\varphi_k \rightarrow \varphi_0$ in $W_0^{1,p}(\Omega)$. Thus ϕ_0 is a solution of (P) for $\lambda = c_0$. Hence $c_0 = \lambda_1(V, m)$ and we can assume that $\phi_0 > 0$ in Ω . Since $\phi_k \rightarrow \phi_0$ in $L^{pr'}(\Omega)$ we find that $|\{\phi_k < 0\}| \rightarrow 0$. On the other hand, one can adapt Theorem 3.2 from [11] to get the following estimate:

$$|\{\phi_k < 0\}| \geq [C \|\lambda_2(V_k, m_k) m_k - V_k\|_r]^{-\sigma},$$

where C and σ are constants depending only on p and N . By the boundedness of $\lambda_2(V_k, m_k)$, V_k and m_k we reach a contradiction. Therefore $c_0 = \lambda_2(V, m)$. If $\alpha(V, m) = 0$ then $\psi_k \stackrel{\text{def}}{=} \frac{\phi_k}{\|\phi_k\|}$ is, up to a subsequence, weakly convergent to some ψ_0 . From

$$E_V(\psi_k) = E_{V_k}(\psi_k) + \int_{\Omega} (V - V_k) |\psi_k|^p dx$$

we get $E_V(\psi_k) \rightarrow 0$. Hence, up to a multiplicative constant, ψ_0 realizes $\alpha(V, m)$, i.e., ψ_0 is a constant-sign solution of (P). Therefore $|\{\phi_k < 0\}| \rightarrow 0$ and we reach a contradiction as above. $\square$

References

- [1] Allegretto, W., Huang, Y.-X.: A Picone's identity for the p -Laplacian and applications. *NonLinear Anal. TMA* **32**(7), 819–830 (1998)
- [2] Anane, A.: Etude des valeurs propres et de la résonance pour l'opérateur p -laplacien. Thèse de Doctorat, Université Libre de Bruxelles (1987)
- [3] Anane, A., Tsouli, N.: On the second eigenvalue of the p -Laplacian. In: Benkirane, A., Gossez, J.-P. (eds.) *Nonlinear PDE*. Pitman Research Notes in Mathematics **343**, 1–9 (1996)
- [4] Arias, M., Campos, J., Cuesta, M., Gossez, J.-P.: Asymmetric elliptic problems with indefinite weights. *Ann. Inst. H. Poincaré An. Non Li.* **19**, 581–616 (2002)
- [5] Arias, M., Campos, J., Cuesta, M., Gossez, J.-P.: An asymmetric Neumann problem with weights. *Ann. Inst. H. Poincaré An. Non Li.* **25**(2), 267–280 (2008)
- [6] Bucur, D., Buttazzo, G.: Variational methods in shape optimization problems. In: *Progress in Nonlinear Differential Equations and their Applications*, 65. Birkhäuser Boston, Inc., Boston (2005)
- [7] Binding, P.A., Huang, Y.X.: The principal eigencurve for the p -Laplacian. *Differ. Integral Equ.* **8**(2), 405–414 (1995)
- [8] Binding, P.A., Huang, Y.X.: Existence and nonexistence of positive eigenfunctions for the p -Laplacian. *Proc. Am. Math. Soc.* **123**(6), 1833–1838 (1995)
- [9] Corvellec, J.-N.: Quantitative deformation theorems and critical point theory. *Pac. J. Math.* **187**(2), 263–279 (1999)
- [10] Corvellec, J.-N., Degiovanni, M., Marzocchi, M.: Deformation properties for continuous functionals and critical point theory. *Topol. Methods Nonlinear Anal.* **1**(1), 151–171 (1993)
- [11] Cuesta, M.: Eigenvalue problems for the p -Laplacian with indefinite weight. *Elec. J. Differ. Equ.* 1–9 (2001)

- [12] del Pezzo, L.M., Fernandez Bonder, J.: An optimization problem for the first eigenvalue of the p -Laplacian plus a potential. *Commun. Partial Differ. Equ.* **5**(4), 675–690 (2006)
- [13] Fleckinger, J., Hernandez, J., de Thélin, F.: Existence of multiple principal eigenvalues for some indefinite linear eigenvalue problems. *Bolletino U.M.I* (8) **7-B**, 159–188 (2004)
- [14] Garcia Azorero, J.P., Peral Alonso, I.: Existence and nonuniqueness for the p -Laplacian nonlinear eigenvalues. *Commun. Partial Differ. Equ.* **12**(12), 1389–1400 (1987)
- [15] Godoy, T., Gossez, J.-P., Paczka, S.: A minimax formula for the principal eigenvalues of Dirichlet problems and its applications. 2006 International Conference in honor of Jacqueline Fleckinger. *Elect. J. Differ. Equ. Conf.* **16**, 137–154 (2007)
- [16] Henrot, A., Pierre, M.: “Variation et optimisation de formes. Une analyse géométrique” Collection: Mathématiques et Applications, vol. 48. Springer, New York (2005)
- [17] Ladyzhenskaya, O., Uraltseva, N.: “Equations aux Dérivées Partielles du Type Elliptique du Second Ordre”, Ed. Masson (France) (1975)
- [18] Lions, J.-L.: “Équations différentielles opérationnelles et problèmes aux limites”, Die Grundlehren der mathematischen Wissenschaften, Bd. 111. Springer-Verlag, Berlin-Gttingen-Heidelberg (1961)
- [19] Lopez-Gomez, J.: The maximum principle and the existence of principal eigenvalues for some linear weighted eigenvalue problems. *J. Differ. Equ.* **127**, 263–294 (1996)
- [20] Otani, M., Teshima, T.: On the first eigenvalue of some quasilinear elliptic equations. *Proc. Jpn. Acad.* **64**, 8–10 (1988)
- [21] Serrin, J.: Local behavior of solutions of quasilinear equations. *Acta Math.* **111**, 247–302 (1962)
- [22] Ziemer, W.P.: Weakly Differentiable Functions. GTM 120, Springer-Verlag (1989)

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