On a fourth-order quasilinear elliptic equation of concave–convex type

Ederson Moreira dos Santos

Abstract. We consider a fourth-order quasilinear equation depending on a nonnegative parameter λ and with subcritical or critical growth. Such equation is equivalent to a Hamiltonian system and the main goal of this work is to prove the existence of at least two positive and infinitely many solutions for such equation when the parameter λ is positive and small enough.

Mathematics Subject Classification (2000). Primary 35J40; Secondary 35J55.

Keywords. Fourth-order equation, Hamiltonian system, Variational methods.

1. Introduction and statement of the results

In this work we consider a fourth-order quasilinear equation of concave–convex type under Navier’s boundary conditions. Such equation is equivalent to a Hamiltonian system, which is a perturbation of a well known 2×2 -system under Dirichlet boundary conditions. The equation treated here depends on a parameter $\lambda \geq 0$ and we are concerned with the existence of infinitely many solutions and at least two positive solutions. In order to obtain some of these results we need to impose that the parameter λ is positive and sufficiently small.

The basic hypotheses used in showing the multiplicity results are superlinearity, subcriticality or criticality of the problem made in a generalized sense. Our approach is variational and due to the form of the corresponding functional, it is based on a detailed study of the Nehari’s manifold together with Ekeland’s Variational Principle [15], a Riesz’s representation theorem for some Sobolev spaces [12], the Mountain Pass Theorem [2], the Fountain and Dual Fountain Theorems [4, 6, 23], some arguments based in the Krasnoselskii genus theory found in [17] and compactness arguments, either derived from the subcriticality nature

of the problem, or from the Concentration-Compactness Principle [20] employed in the case when the equation has a critical exponential growth.

This work is motivated by the earlier works [5–10, 12, 14, 17, 22] and the purpose of this paper is to extend some of the results in those papers to problem (P_λ) presented below.

In this paper we consider

$$(P_\lambda) \quad \begin{cases} -\Delta \left((-\Delta u)^{\frac{1}{p}} \right) = u^q + \lambda g(x) u^r & \text{in } \Omega, \\ u, -\Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

on a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with smooth boundary, where for short we write $u^q = |u|^{q-1}u$, $u^r = |u|^{r-1}u$ and $(-\Delta u)^{\frac{1}{p}} = |-\Delta u|^{\frac{1}{p}-1}(-\Delta u)$, $\lambda \geq 0$ is a parameter and $g \in C^1(\bar{\Omega})$ is positive inside Ω .

We denote by E and F the Sobolev spaces $W^{2, \frac{p+1}{p}}(\Omega) \cap W_0^{1, \frac{p+1}{p}}(\Omega)$ and $W^{2, \frac{q+1}{q}}(\Omega) \cap W_0^{1, \frac{q+1}{q}}(\Omega)$ respectively and by E' the dual space of E . We consider E endowed with the norm (see Lemma 9.17 of [18])

$$\|u\| = \left(\int |\Delta u|^{\frac{p+1}{p}} dx \right)^{\frac{p}{p+1}}, \quad u \in E.$$

We say that $u \in E$ is a weak solution of (P_λ) provided

$$\int |\Delta u|^{\frac{1}{p}-1} \Delta u \Delta v dx = \int |u|^{q-1} u v dx + \lambda \int g(x) |u|^{r-1} u v dx, \quad \forall v \in E.$$

At first sight equation (P_λ) may seem to be uncommon, but our interest on studying (P_λ) comes from its equivalence (see Theorem 1.1) with the Hamiltonian system

$$(S_\lambda) \quad \begin{cases} -\Delta u = v^p & \text{in } \Omega, \\ -\Delta v = u^q + \lambda g(x) u^r & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

and the operator $-\Delta((-\Delta \cdot)^{\frac{1}{p}})$ is a variation of the biharmonic operator as well as the p -Laplacian is a variation of the Laplacian.

We study (P_λ) under the following hypotheses:

- (H1) $0 < r < \frac{1}{p} < q$;
- (H2) $\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{N}$;
- (H3) $g \in C^1(\bar{\Omega})$ and $g > 0$ in Ω .

Hypothesis (H1) provides the concave–convex behavior for (P_λ) . Note, when $p = 1$, (P_λ) becomes an equation involving the biharmonic operator and in this case $u \mapsto u^r$ is in fact a concave function while $u \mapsto u^q$ is convex on $[0, +\infty)$.

Hypothesis (H2) determines the critical or subcritical growth of (P_λ) , which occurs with the identity or strict inequality respectively.

With relation to (H3), we assume $g \in C^1(\overline{\Omega})$ in order to get the equivalence between (P_λ) and (S_λ) . The assumption $g > 0$ in Ω is employed to get the existence of infinitely many solutions for (P_λ) either by means of the Dual Fountain Theorem or by some properties of the genus theory. But we emphasize that the condition $g \geq 0$ in Ω and not identically zero is sufficient to provide our results concerning the existence of positive solutions in the subcritical case.

We would like to say that our results concerning the case with $\lambda = 0$ and with subcritical growth, extend some of the results in [5] and [8], since those papers do not deal with $p \geq \frac{N+4}{N-4}$ or $q \geq \frac{N+4}{N-4}$ for $N \geq 5$ (see for instance Eq. (3.2) in [5]).

From the above hypotheses, the weak solutions of (P_λ) are precisely the critical points of the functional

$$I_\lambda(u) = \frac{p}{p+1} \int |\Delta u|^{\frac{p+1}{p}} dx - \frac{1}{q+1} \int |u|^{q+1} dx - \frac{\lambda}{r+1} \int g(x)|u|^{r+1} dx, \quad (1.1)$$

which is well defined on E and of class C^1 .

Following the same ideas of the proofs of Theorem 1.1 in [12] and Theorem 1.1 in [14], we present the equivalence between (P_λ) and (S_λ) .

Theorem 1.1. *Suppose (H1)–(H3) and let $u \in E$ be a weak solution of (P_λ) and denote $v = (-\Delta u)^{\frac{1}{p}}$. If the strict inequality holds at (H2) then (u, v) is a classical solution of (S_λ) and (u, v) lies in $C^{2,\alpha}(\overline{\Omega}) \times C^{2,\alpha}(\overline{\Omega})$, with $\alpha = \min\{p, q\}$ when $p < 1$ or $q < 1$ and α is any number in $(0, 1)$ when $p \geq 1$ and $q \geq 1$. If the identity holds at (H2) then $(u, v) \in E \times F$ is a strong solution of (S_λ) .*

In view of the equivalence between (P_λ) and (S_λ) , we introduce the following definition.

Definition 1.2. By a nonnegative weak solution u of (P_λ) we mean that $u \in E$ is a critical point of I_λ and satisfies $u, -\Delta u \geq 0$ a.e. in Ω .

For the subcritical case, combining Theorem 1.1 with the classical strong maximum principle applied to both equations of (S_λ) , we obtain that a nontrivial nonnegative weak solutions u of (P_λ) satisfies $u(x), -\Delta u(x) > 0$ for every $x \in \Omega$.

Here $|\cdot|_{q+1}$ and $|\cdot|_{r+1}$ denotes the $L^{q+1}(\Omega)$ -norm and $L^{r+1}(\Omega)$ -norm respectively, φ_1 the positive first eigenfunction satisfying $\int \varphi_1 dx = 1$ and λ_1 the first eigenvalue of $(-\Delta, H_0^1(\Omega))$.

In order to make clear that the results in this paper extend some of the results in [12] and [14], we introduce the following notations:

$$|g|_{*,r+1} := \sup_{v \in E, \|v\|=1} \int g(x) |v|^{r+1} dx, \quad K := \left(\frac{1-pr}{p(q-r)} \right)^{\frac{1-pr}{pq-1}} \frac{pq-1}{p(q-r)},$$

$$R := \inf_{u \in E, u \neq 0} \frac{\|u\|^{\frac{p+1}{p}}}{|u|_{q+1}^{\frac{p+1}{p}}}$$

and for the critical case, that is, when the identity holds at (H2), R is the best constant for the noncompact Sobolev embedding of E into $L^{q+1}(\Omega)$, and in this case we denote R by S . Also:

$$\begin{aligned}\mathcal{N}_\lambda &= \{u \in E : \langle I'_\lambda(u), u \rangle = 0\}, \\ \mathcal{N}_\lambda^+ &= \{u \in \mathcal{N}_\lambda : (1 - pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} > 0\}, \\ \mathcal{N}_{\lambda,0} &= \{u \in \mathcal{N}_\lambda : (1 - pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} = 0\}, \\ \mathcal{N}_\lambda^- &= \{u \in \mathcal{N}_\lambda : (1 - pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} < 0\}.\end{aligned}$$

We will prove that I_λ achieves its minimum value on $\mathcal{N}_\lambda$ and also in parts of $\mathcal{N}_\lambda$, provided $\lambda > 0$ is small enough. For that matter, we add to the above set of hypotheses the following condition:

$$(H4) \quad 0 < \lambda < \lambda^* := \frac{KR^{\frac{p(q+1)(1-pr)}{(p+1)(pq-1)}}}{|g|_{*,r+1}},$$

and we obtain our nontrivial nonnegative weak solutions for (P_λ) investigating the minimization problems

$$c_{\lambda,0} := \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) \quad \text{and} \quad c_{\lambda,1} := \inf_{u \in \mathcal{N}_\lambda^-} I_\lambda(u).$$

At this point we state our results concerning (P_λ) in the subcritical case.

Theorem 1.3. *Suppose (H1), the strict inequality at (H2) and (H3).*

- (I) *For every $\lambda \geq 0$ there exists a sequence (u_k) of weak solutions of (P_λ) such that $I_\lambda(u_k) \rightarrow +\infty$.*
- (II) *For every $\lambda > 0$ there exists a sequence $(\bar{u}_k)$ of weak solutions of (P_λ) such that $I_\lambda(\bar{u}_k) < 0$ and $I_\lambda(\bar{u}_k) \rightarrow 0$.*
- (III) *If (H4) is also satisfied then there exist at least two nontrivial nonnegative weak solutions $u_{\lambda,0}$ and $u_{\lambda,1}$ of (P_λ) with: $u_{\lambda,0}$ is a local minimum for I_λ , $u_{\lambda,0} \in \mathcal{N}_\lambda^+$, $u_{\lambda,1} \in \mathcal{N}_\lambda^-$, $I_\lambda(u_{\lambda,0}) = c_{\lambda,0}$ and $I_\lambda(u_{\lambda,1}) = c_{\lambda,1}$.*
- (IV) *If $\lambda = \lambda^*$ then there exists at least one nontrivial nonnegative weak solution $u_{\lambda,0}$ of (P_λ) with: $u_{\lambda,0} \in \mathcal{N}_{\lambda,0} \cup \mathcal{N}_\lambda^+$ and $I_\lambda(u_{\lambda,0}) = c_{\lambda,0}$.*
- (V) *If $p, q \geq 1$ then there exists $\lambda^\sharp > 0$ such that (P_λ) has no nontrivial nonnegative weak solution for $\lambda > \lambda^\sharp$.*

Before we state our results for the critical case, we add another hypothesis which was introduced in [14]:

$$(H5) \quad \begin{cases} \text{If } N = 3, \dots, 6 & \text{then } p < \frac{N}{N-2}; \\ \text{If } N = 7, \dots, 12 & \text{then } \frac{1}{2} < p < \frac{N}{N-2}; \\ \text{If } N \geq 13 & \text{then } \frac{N+6}{2N-6} \leq p < \frac{N}{N-2}. \end{cases}$$

Now we state our results concerning (P_λ) with critical exponential growth.

Theorem 1.4. *Suppose (H1), the identity at (H2) and (H3).*

- (a) *There exists $\lambda_* \in (0, \lambda^*)$ such that if $\lambda \in (0, \lambda_*)$ then there exists a sequence $(\bar{u}_k)$ of weak solutions of (P_λ) such that $I_\lambda(\bar{u}_k) < 0$ and $I_\lambda(\bar{u}_k) \rightarrow 0$.*
- (b) *If (H_4) is satisfied there also exists at least one nontrivial nonnegative weak solution $u_{\lambda,0}$ for (P_λ) with: $u_{\lambda,0}$ is a local minimum for I_λ , $u_{\lambda,0} \in \mathcal{N}_\lambda^+$ and $I_\lambda(u_{\lambda,0}) = c_{\lambda,0}$.*
- (c) *If $\lambda = \lambda^*$ then there exists at least one nontrivial nonnegative weak solution $u_{\lambda,0}$ of (P_λ) with: $u_{\lambda,0} \in \mathcal{N}_{\lambda,0} \cup \mathcal{N}_\lambda^+$ and $I_\lambda(u_{\lambda,0}) = c_{\lambda,0}$.*
- (d) *There exists $\lambda^{**} \in (0, \lambda^*]$ such that, if $\lambda \in (0, \lambda^{**})$, (H_5) and $g(x) \geq \alpha > 0 \ \forall x \in \Omega$ are also satisfied (α stands for a constant), then there exists a second nontrivial nonnegative weak solution $u_{\lambda,1}$ for (P_λ) with: $u_{\lambda,1} \in \mathcal{N}_\lambda^-$ and $I_\lambda(u_{\lambda,1}) = c_{\lambda,1}$.*
- (e) *If $p, q \geq 1$ then there exists $\lambda^\sharp > 0$ such that (P_λ) has no nontrivial nonnegative weak solution for $\lambda > \lambda^\sharp$.*

This paper is organized as follows. In Sect. 2, we show some results that are applied to deal with (P_λ) with both subcritical or critical growth. In Sect. 3 we present the proof for our results concerning the subcritical case. In Sect. 4 we treat the critical problem. Finally, at Sect. 5 we discuss deeply about our solutions as well as about some open problems.

2. Preliminary results

The first part of this section provides the tools employed to construct nontrivial nonnegative weak solutions for (P_λ) . In the second part we are concerned with the compactness properties of I_λ . In the third part, we prove some lemmas that help us with the existence of infinitely many solutions for (P_λ) . Also, throughout this text we assume that hypotheses (H1)–(H3) are satisfied.

2.1. Minimization problems

In order to accomplish the proof of our results concerning the existence of multiple nontrivial nonnegative weak solutions for (P_λ) with $\lambda > 0$ sufficiently small, we employ the arguments of minimization on the Nehari manifold $\mathcal{N}_\lambda$ found in [12].

The idea of minimize I_λ on $\mathcal{N}_\lambda$ under (H4) provides a way to find local minimums for I_λ , because the points on $\mathcal{N}_\lambda$ where I_λ achieves $c_{\lambda,0}$ are all local minimums for I_λ when it is considered defined on the whole space E ; see Lemma 2.12. By means of this argument we obtain our first nontrivial nonnegative weak solution.

Following the approach adopted in [12], the necessary condition to a point $u \in E$ to be a local minimum of I_λ leads us to consider the splitting of $\mathcal{N}_\lambda$ into $\mathcal{N}_\lambda^+$, $\mathcal{N}_{\lambda,0}$ and $\mathcal{N}_\lambda^-$.

When (H4) is satisfied, Lemma 2.12 also guarantees that the points on $\mathcal{N}_\lambda$ such that I_λ assumes the value $c_{\lambda,0}$ should lie in $\mathcal{N}_\lambda^+$ and $\mathcal{N}_{\lambda,0} = \{0\}$ by (iv) of Lemma 2.5. So, we also investigate the minimization problem $c_{\lambda,1}$, mainly

motivated by Lemma 2.13. With this procedure we obtain our second nontrivial nonnegative weak solution.

After these comments, we start with a lemma which justifies our method to find nonnegative weak solutions, since it consists basically on minimizations of I_λ on $\mathcal{N}_\lambda$ or in parts of $\mathcal{N}_\lambda$.

Lemma 2.1. *I_λ is bounded from below on $\mathcal{N}_\lambda$.*

Proof. If $u \in \mathcal{N}_\lambda$ then $\|u\|^{\frac{p+1}{p}} - |u|_{q+1}^{q+1} - \lambda \int g(x)|u|^{r+1}dx = 0$. So

$$\begin{aligned} I_\lambda(u) &= \frac{p}{p+1} \int |\Delta u|^{\frac{p+1}{p}} dx - \frac{1}{q+1} \int |u|^{q+1} dx - \frac{\lambda}{r+1} \int g(x)|u|^{r+1} dx \\ &= \left(\frac{p}{p+1} - \frac{1}{q+1} \right) \int |\Delta u|^{\frac{p+1}{p}} dx + \left(\frac{1}{q+1} - \frac{1}{r+1} \right) \lambda \int g(x)|u|^{r+1} dx \\ &\geq \frac{pq-1}{(p+1)(q+1)} \|u\|^{\frac{p+1}{p}} - \frac{q-r}{(q+1)(r+1)} \lambda |g|_{*,r+1} \|u\|^{r+1} \\ &\geq -\frac{(1-pr)(q-r)}{(p+1)(q+1)(r+1)} \left(\frac{p(q-r)}{pq-1} \right)^{\frac{p(r+1)}{1-pr}} \left(\lambda |g|_{*,r+1} \right)^{\frac{p+1}{1-pr}}. \end{aligned}$$

□

Given $u \in E$, $u \neq 0$, define $i : [0, +\infty) \rightarrow \mathbb{R}$ by $i(t) := t^{\frac{1}{p}-r} \|u\|^{\frac{p+1}{p}} - t^{q-r} |u|_{q+1}^{q+1}$, which is motivated by the expression

$$\langle I'_\lambda(tu), tu \rangle = t^{r+1} \left(t^{\frac{1}{p}-r} \|u\|^{\frac{p+1}{p}} - t^{q-r} |u|_{q+1}^{q+1} - \lambda \int g(x)|u|^{r+1}dx \right).$$

So, for $t > 0$, $tu \in \mathcal{N}_\lambda$ if and only if $i(t) = \lambda \int g(x)|u|^{r+1}dx$. On the other hand, (H1) implies that, for each $u \in E \setminus \{0\}$, $i(t)$ is bounded from above, $i(t) \rightarrow -\infty$ as $t \rightarrow +\infty$ and $i(t)$ attains its maximum value at

$$t_{\max} := \left(\frac{1-pr}{p(q-r)} \frac{\|u\|^{\frac{p+1}{p}}}{|u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}}.$$

The next lemma clarifies the purpose of assumption (H4).

Lemma 2.2. *If (H4) is satisfied then for each $u \in E \setminus \{0\}$, the set $\{tu : t > 0\}$ intersects $\mathcal{N}_\lambda$ exactly twice. More specifically, there exists a unique $t^+ = t^+(u) > 0$ such that $t^+u \in \mathcal{N}_\lambda^-$ and a unique $t^- = t^-(u) > 0$ such that $t^-u \in \mathcal{N}_\lambda^+$. Moreover, $t^- < t_{\max} < t^+$, $I_\lambda(t^-u) = \min_{t \in [0, t^+]} I_\lambda(tu)$ and $I_\lambda(t^+u) = \max_{t \geq t^-} I_\lambda(tu)$.*

Proof. Let $u \in E \setminus \{0\}$ and consider $j : [0, +\infty) \rightarrow \mathbb{R}$ given by $j(t) = I_\lambda(tu)$. Then $j \in C([0, +\infty), \mathbb{R}) \cap C^1((0, +\infty), \mathbb{R})$ and

$$j'(t) = t^r \left(i(t) - \lambda \int g(x)|u|^{r+1}dx \right).$$

On the other hand, by (H4):

$$\begin{aligned} i(t_{\max}) - \lambda \int g(x) |u|^{r+1} dx &= K \frac{\|u\|^{\frac{(q-r)(p+1)}{pq-1}}}{|u|_{q+1}^{\frac{(q+1)(1-pr)}{pq-1}}} - \|u\|^{r+1} \lambda \int g(x) \left| \frac{u}{\|u\|} \right|^{r+1} dx \\ &\geq K \frac{\|u\|^{\frac{(q-r)(p+1)}{pq-1}}}{|u|_{q+1}^{\frac{(q+1)(1-pr)}{pq-1}}} R^{\frac{p}{p+1} \frac{(q+1)(1-pr)}{pq-1}} - \lambda |g|_{*,r+1} \|u\|^{r+1} \\ &= \left(K R^{\frac{p}{p+1} \frac{(q+1)(1-pr)}{pq-1}} - \lambda |g|_{*,r+1} \right) \|u\|^{r+1} > 0 \end{aligned}$$

and so, there exist t^-, t^+ satisfying $0 < t^- < t_{\max} < t^+$ such that

$$j'(t) < 0 \text{ in } (0, t^-) \cup (t^+, +\infty); \quad j'(t) > 0 \text{ in } (t^-, t^+) \quad \text{and} \quad j'(t^+) = j'(t^-) = 0.$$

In this way $\{tu : t > 0\} \cap \mathcal{N}_\lambda = \{t^-u, t^+u\}$, $I_\lambda(t^-u) = \min_{t \in [0, t^+]} I_\lambda(tu)$ and $I_\lambda(t^+u) = \max_{t \geq t^-} I_\lambda(tu)$. From $i'(t^-) > 0$, $i'(t^+) < 0$ one concludes that $t^-u \in \mathcal{N}_\lambda^+$ and $t^+u \in \mathcal{N}_\lambda^-$. $\square$

Remark 2.3. If $\lambda = \lambda^*$ then there exists $t_0 = t_0(u) > 0$ such that $t_0u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda$, and $I_\lambda(t_0u) = \min\{I_\lambda(tu) : t > 0, tu \in \mathcal{N}_\lambda\}$.

The next technical lemma will help us to construct sequences that will converge to nontrivial nonnegative weak solutions of (P $_\lambda$).

Lemma 2.4. *If (H4) is satisfied then the infimum*

$$\mu_{\lambda,0} = \inf_{u \in E, |u|_{q+1}=1} \left(K \|u\|^{\frac{(q-r)(p+1)}{pq-1}} - \lambda \int g(x) |u|^{r+1} dx \right)$$

is positive.

Proof. Under (H4) one can write $\lambda = \lambda^* - \tau$ with $\tau > 0$. Then, for each $u \in E$ such that $|u|_{q+1} = 1$,

$$\begin{aligned} K \|u\|^{\frac{(q-r)(p+1)}{pq-1}} - \lambda \int g(x) |u|^{r+1} dx &\geq K \|u\|^{\frac{(q-r)(p+1)}{pq-1}} - \lambda \|u\|^{r+1} |g|_{*,r+1} \\ &= \|u\|^{r+1} \left(K \|u\|^{\frac{(q+1)(1-pr)}{pq-1}} - \lambda^* |g|_{*,r+1} \right) + \|u\|^{r+1} \tau |g|_{*,r+1} \\ &\geq (\lambda^* - \lambda) |g|_{*,r+1} R^{\frac{p(r+1)}{p+1}}. \end{aligned}$$

Therefore, one concludes that $\mu_{\lambda,0} \geq (\lambda^* - \lambda) |g|_{*,r+1} R^{\frac{p(r+1)}{p+1}}$. $\square$

Next, for $u \in E \setminus \{0\}$ set

$$\psi(u) = K \frac{\|u\|^{\frac{(q-r)(p+1)}{pq-1}}}{|u|_{q+1}^{\frac{(q+1)(1-pr)}{pq-1}}} - \lambda \int g(x) |u|^{r+1} dx.$$

For $t > 0$ and $|u|_{q+1} = 1$, we have

$$\psi(tu) = t^{r+1} \left(K \|u\|^{\frac{(q-r)(p+1)}{pq-1}} - \lambda \int g(x) |u|^{r+1} dx \right).$$

So, from Lemma 2.4, given $\gamma > 0$

$$\inf_{u \in E, |u|_{q+1} \geq \gamma} \psi(u) = \inf_{u \in E, |u|_{q+1} \geq \gamma} \psi \left(|u|_{q+1} \frac{u}{|u|_{q+1}} \right) \geq \gamma^{r+1} \mu_{\lambda,0}. \quad (2.1)$$

In particular, if (H4) is verified then the infimum given at (2.1) is bounded away from zero.

Suppose (H4) satisfied. From the continuity of $t_{\max} : E \setminus \{0\} \rightarrow \mathbb{R}$ and the uniqueness property of t^+ , given in Lemma 2.2, we show that $t^+ : E \setminus \{0\} \rightarrow \mathbb{R}$ is continuous. So, the function

$$\Phi : E \setminus \{0\} \rightarrow \mathbb{R}, \quad \Phi(u) = t^+ \left(\frac{u}{\|u\|} \right) - \|u\|$$

is continuous. In particular,

$$\mathcal{N}_{\lambda}^{-} = \Phi^{-1}(0) = \left\{ u \in E \setminus \{0\} : t^+ \left(\frac{u}{\|u\|} \right) = \|u\| \right\}$$

is a closed set of $E \setminus \{0\}$, and from (iii) of Lemma 2.5 we will conclude that it is also a closed set of E . On the other hand

$$\begin{aligned} U_2 &= \Phi^{-1}((-\infty, 0)) = \left\{ u \in E \setminus \{0\} : \|u\| > t^+ \left(\frac{u}{\|u\|} \right) \right\} \\ U_1^* &= \Phi^{-1}((0, +\infty)) = \left\{ u \in E \setminus \{0\} : \|u\| < t^+ \left(\frac{u}{\|u\|} \right) \right\} \end{aligned}$$

are open sets of $E \setminus \{0\}$ and therefore of E . From (iii) of Lemma 2.5 we will conclude that $U_1 = U_1^* \cup \{0\}$ is an open set of E .

Now we present a lemma which shows some of the properties of the sets $\mathcal{N}_{\lambda}$, $\mathcal{N}_{\lambda}^+$, $\mathcal{N}_{\lambda}^-$ and $\mathcal{N}_{\lambda,0}$.

Lemma 2.5. *For any $\lambda > 0$:*

- (i) $\mathcal{N}_{\lambda}^+ \cup \mathcal{N}_{\lambda,0}$ is bounded, more specifically, for any $u \in \mathcal{N}_{\lambda}^+ \cup \mathcal{N}_{\lambda,0}$

$$\|u\| < \left(\frac{(q-r)(p+1)}{(r+1)(pq-1)} \lambda |g|_{*,r+1} \right)^{\frac{p}{1-pr}}. \quad (2.2)$$

- (ii) $\mathcal{N}_{\lambda}^-$ is unbounded.
 (iii) For any $u \in \mathcal{N}_{\lambda}^-$

$$\|u\| > \left(\frac{1-pr}{p(q-r)} R^{\frac{p(q+1)}{p+1}} \right)^{\frac{p}{pq-1}}.$$

(note the bound does not depend on λ)

If (H4) is satisfied then:

- (iv) $\mathcal{N}_{\lambda,0} = \{0\}$.
 (v) $\mathcal{N}_{\lambda}^{-}$ disconnects E in the sense that, given $u \in U_1$, $v \in U_2$ and $h : [0, 1] \rightarrow E$ continuous such that $h(0) = u$ and $h(1) = v$, the image of h intersects $\mathcal{N}_{\lambda}^{-}$. Furthermore, $\mathcal{N}_{\lambda}^{+} \subset U_1$.
 (vi) Let $S_E = \{u \in E : \|u\| = 1\}$. Then, $T^{+} : S_E \rightarrow \mathcal{N}_{\lambda}^{-}$ defined by $T^{+}(u) = t^{+}(u)u$ is a homeomorphism.

Proof. (i) If $u \in (\mathcal{N}_{\lambda}^{+} \cup \mathcal{N}_{\lambda,0}) \setminus \{0\}$ then

$$\|u\|^{\frac{p+1}{p}} - |u|_{q+1}^{q+1} - \lambda \int g(x) |u|^{r+1} dx = 0 \text{ and } (1-pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} \geq 0.$$

Since $pq > 1$ by (H1), one has

$$\begin{aligned} I_{\lambda}(u) &= -\frac{1}{(p+1)(r+1)} \left((1-pr)\|u\|^{\frac{p+1}{p}} - \frac{(q-r)(p+1)}{q+1} |u|_{q+1}^{q+1} \right) \\ &< -\frac{1}{(p+1)(r+1)} \left((1-pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} \right) \leq 0, \end{aligned}$$

and so, using the fact that $u \in \mathcal{N}_{\lambda}$,

$$0 > I_{\lambda}(u) = \frac{pq-1}{(p+1)(q+1)} \|u\|^{\frac{p+1}{p}} - \frac{q-r}{q+1} \lambda \int g(x) |u|^{r+1} dx,$$

which implies

$$\begin{aligned} \frac{pq-1}{(p+1)(q+1)} \|u\|^{\frac{p+1}{p}} &< \frac{q-r}{(r+1)(q+1)} \lambda \int g(x) |u|^{r+1} dx \\ &\leq \frac{q-r}{(r+1)(q+1)} \lambda |g|_{*,r+1} \|u\|^{r+1}, \end{aligned}$$

that is,

$$\|u\| < \left(\frac{(q-r)(p+1)}{(r+1)(pq-1)} \lambda |g|_{*,r+1} \right)^{\frac{p}{1-pr}}.$$

(ii) Let

$$\lambda_0 = \inf_{u \in E, \|u\|=1} |u|_{q+1}.$$

Then $\lambda_0 = 0$, since otherwise $\|u\| \leq \frac{1}{\lambda_0} |u|_{q+1}$ for every $u \in E$, which would imply E would be a closed subspace of $L^{q+1}(\Omega)$, which is certainly false, once $C_0^{\infty}(\Omega) \subset E \subset L^{q+1}(\Omega)$, $C_0^{\infty}(\Omega)$ is dense in $L^{q+1}(\Omega)$ and $E \neq L^{q+1}(\Omega)$.

If $u \in E$ and $\|u\| = 1$ then from Lemma 2.2

$$\|t^{+}(u)u\| = t^{+}(u) > \left(\frac{1-pr}{p(q-r)|u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}}.$$

The fact that $\lambda_0 = 0$ and the last inequality imply that $\mathcal{N}_{\lambda}^{-}$ is unbounded.

(iii) Given $u \in E \setminus \{0\}$, by Lemma 2.2 there exists a unique $t^+(u) > 0$ such that $t^+(u)u \in \mathcal{N}_\lambda^-$ and

$$t^+ > \left(\frac{1 - pr}{p(q - r)} \frac{\|u\|^{\frac{p+1}{p}}}{|u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}} := t_{\max}.$$

Note that

$$\frac{1 - pr}{p(q - r)} \frac{\|u\|^{\frac{p+1}{p}}}{|u|_{q+1}^{q+1}} = \frac{1 - pr}{p(q - r)} \frac{1}{\|u\|^{\frac{pq-1}{p}}} \frac{\|u\|^{q+1}}{|u|_{q+1}^{q+1}} \geq \frac{1 - pr}{p(q - r)} R^{\frac{p(q+1)}{p+1}} \frac{1}{\|u\|^{\frac{pq-1}{p}}}.$$

From the last inequality one gets

$$t_{\max} \geq \left(\frac{1 - pr}{p(q - r)} R^{\frac{p(q+1)}{p+1}} \right)^{\frac{p}{pq-1}} \frac{1}{\|u\|}, \quad \text{that is, } t_{\max}\|u\| \geq \left(\frac{1 - pr}{p(q - r)} R^{\frac{p(q+1)}{p+1}} \right)^{\frac{p}{pq-1}}.$$

In this way

$$\|t^+(u)u\| > t_{\max}\|u\| \geq \left(\frac{1 - pr}{p(q - r)} R^{\frac{p(q+1)}{p+1}} \right)^{\frac{p}{pq-1}}.$$

If $u \in \mathcal{N}_\lambda^-$ then $t^+(u) = 1$. This remark together with the last inequality prove the claim.

(iv) It follows directly from Lemma 2.2.

(v) and (vi) If $u \in \mathcal{N}_\lambda^+$ then $(1 - pr)\|u\|^{\frac{p+1}{p}} - p(q - r)|u|_{q+1}^{q+1} > 0$, and so

$$t^+ \left(\frac{u}{\|u\|} \right) > t_{\max} \left(\frac{u}{\|u\|} \right) = \left(\frac{1 - pr}{p(q - r)} \frac{\|u\|^{q+1}}{|u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}} > \left(\frac{\|u\|^{q+1}}{\|u\|^{\frac{pq-1}{p}}} \right)^{\frac{p}{pq-1}} = \|u\|,$$

that is, $u \in U_1$. Therefore, $\mathcal{N}_\lambda^+ \subset U_1$.

Now, let $u \in U_1$, $v \in U_2$ and $h : [0, 1] \rightarrow E$ continuous such that $h(0) = u$ and $h(1) = v$.

If $h(t) \neq 0$ for every $t \in [0, 1]$, then $\Phi \circ h : [0, 1] \rightarrow \mathbb{R}$ is continuous and satisfies $\Phi \circ h(0) = \Phi(u) > 0$, $\Phi \circ h(1) = \Phi(v) < 0$. So, there exists $t^* \in (0, 1)$ such that $\Phi \circ h(t^*) = 0$, that is, $h(t^*) \in \mathcal{N}_\lambda^-$.

If $h(t) = 0$ for some $t \in [0, 1]$ define $A = h^{-1}(0)$. A is a compact set of $[0, 1]$ and $\alpha = \sup A < 1$ because $h(1) \neq 0$. So, $h : [\alpha, 1] \rightarrow E$ satisfies $h(\alpha) = 0$ and $h(t) \neq 0$ for $t \in (\alpha, 1]$. Since $0 \in U_1$ and U_1 is open, there exists $\delta > 0$ such that $\alpha + \delta < 1$, $h(\alpha + \delta) \in U_1$. So, the function $h : [\alpha + \delta, 1] \rightarrow \mathbb{R}$ has the same properties of the function of the last paragraph. Therefore, one concludes that there exists $t^* \in (0, 1)$ such that $h(t^*) \in \mathcal{N}_\lambda^-$.

To finish, the function $T^+ : S_E \rightarrow \mathcal{N}_\lambda^-$, given by $T^+(u) = t^+(u)u$, is continuous because t^+ is continuous. The continuity of T^{-1} follows from the fact that $T^{-1}(w) = \frac{w}{\|w\|}$. $\square$

Remark 2.6. It follows from the proof of item (i) of Lemma 2.5 that, if $u \in \mathcal{N}_\lambda$ and $I_\lambda(u) \leq 0$ then the inequality (2.2) holds true.

The following result will help us to guarantee that certain minimizing sequences of I_λ on parts of $\mathcal{N}_\lambda$ are (PS)-sequences and also to prove that a certain critical point of I_λ is a local minimum for it.

Lemma 2.7. *Suppose (H4) satisfied. Given $u \in \mathcal{N}_\lambda \setminus \{0\}$ there exists $\delta > 0$ and a differentiable positive function $t : \{w \in E : \|w\| < \delta\} \rightarrow \mathbb{R}$, satisfying $t(0) = 1$, $t(w)(u - w) \in \mathcal{N}_\lambda$ for every $w \in E$ with $\|w\| < \delta$, and*

$$\begin{aligned} & \langle t'(0), v \rangle \\ &= \frac{(p+1) \int |\Delta u|^{\frac{1}{p}-1} \Delta u \Delta v dx - p(q+1) \int |u|^{q-1} u v dx - p(r+1) \lambda \int g(x) |u|^{r-1} u v dx}{(1-pr) \|u\|^{\frac{p+1}{p}} - p(q-r) |u|_{q+1}^{q+1}}. \end{aligned}$$

Proof. Define $F : (0, +\infty) \times E \rightarrow \mathbb{R}$ by

$$F(t, w) = pt^{\frac{1}{p}} \|u - w\|^{\frac{p+1}{p}} - pt^q \|u - w\|_{q+1}^{q+1} - p\lambda t^r \int g(x) |u - w|^{r+1} dx.$$

So, $F \in C^1((0, +\infty) \times E, \mathbb{R})$, $F(1, 0) = 0$ and $F_t(1, 0) = \|u\|^{\frac{p+1}{p}} - pq|u|_{q+1}^{q+1} - \lambda pr \int g(x) |u|^{r+1} dx = (1-pr) \|u\|^{\frac{p+1}{p}} - p(q-r) |u|_{q+1}^{q+1} \neq 0$ by (iv) of Lemma 2.5. Then, by the Implicit Function Theorem, there exists $\delta > 0$, $t : \{w \in E : \|w\| < \delta\} \rightarrow \mathbb{R}$ of class C^1 such that

- $F(t(w), w) = 0$ for every $w \in E$, $\|w\| < \delta$,
- $t(0) = 1$,
- $t(w) > 0$ for every $w \in E$ with $\|w\| < \delta$.

Then,

$$\begin{aligned} 0 &= t(w)F(t(w), w) \\ &= p \left(\|t(w)(u - w)\|^{\frac{p+1}{p}} - |t(w)(u - w)|_{q+1}^{q+1} - \lambda \int g(x) |t(w)(u - w)|^{r+1} dx \right) \end{aligned}$$

that is, $t(w)(u - w) \in \mathcal{N}_\lambda$ for every $w \in E$ with $\|w\| < \delta$. If one differentiates the identity $F(t(w), w) = 0$ with relation to w , one gets

$$0 = \langle F_t(t(w), w)t'(w) + F_w(t(w), w), v \rangle, \text{ for every } v \in E.$$

Putting $w = 0$ in the above identity

$$0 = \langle F_t(1, 0)t'(0) + F_w(1, 0), v \rangle = F_t(1, 0)\langle t'(0), v \rangle + \langle F_w(1, 0), v \rangle$$

and from the last identity

$$\begin{aligned} \langle t'(0), v \rangle &= -\frac{\langle F_w(1, 0), v \rangle}{F_t(1, 0)} \\ &= \frac{(p+1) \int |\Delta u|^{\frac{1}{p}-1} \Delta u \Delta v dx - p(q+1) \int |u|^{q-1} u v dx - p(r+1) \lambda \int g(x) |u|^{r-1} u v dx}{(1-pr) \|u\|^{\frac{p+1}{p}} - p(q-r) |u|_{q+1}^{q+1}}. \end{aligned}$$

□

The proof of Lemma 2.1 gives us a lower bound for $c_{\lambda,0}$. Now we present an upper bound for it. We present a negative upper bound for $c_{\lambda,0}$ and the fact that $c_{\lambda,0}$ is negative is very important in our calculations.

Fix $u_* \in E$, a solution of

$$\begin{cases} -\Delta((-\Delta u)^{\frac{1}{p}}) = g(x)u^r & \text{in } \Omega, \\ u, -\Delta u \geq 0 & \text{in } \Omega, \\ u, -\Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

which is obtained by a minimization of

$$J(u) = \frac{p}{p+1} \int |\Delta u|^{\frac{p+1}{p}} dx - \frac{1}{r+1} \int g(x) |u|^{r+1} dx$$

on E . The argument employed to prove that the minimum value of J is attained at an element $u_* \in E$ satisfying $u_*, -\Delta u_* \geq 0$ a.e. in Ω is similar to that employed in the proof of Theorem 1.4 in [12].

If (H4) is satisfied then from Lemma 2.2, there exists a unique $t_* = t^-(u_*) > 0$ such that $t_* u_* \in \mathcal{N}_\lambda^+$. Then, using respectively the fact that $t_* u_* \in \mathcal{N}_\lambda$, and $t_* u_* \in \mathcal{N}_\lambda^+$ we have

$$\begin{aligned} I_\lambda(t_* u_*) &= -\frac{1-pr}{(p+1)(r+1)} \|t_* u_*\|^{\frac{p+1}{p}} + \frac{q-r}{(r+1)(q+1)} |t_* u_*|_{q+1}^{q+1} \\ &< -\frac{1-pr}{(p+1)(r+1)} \|t_* u_*\|^{\frac{p+1}{p}} + \frac{1-pr}{p} \frac{1}{(r+1)(q+1)} \|t_* u_*\|^{\frac{p+1}{p}} \\ &= -\frac{(1-pr)(pq-1)}{p(p+1)(q+1)(r+1)} t_*^{\frac{p+1}{p}} \|u_*\|^{\frac{p+1}{p}}. \end{aligned}$$

We emphasize t_* depends on λ . From the last inequality and Lemma 2.1 we have

$$\begin{aligned} c_{\lambda,0} &\geq -\frac{(1-pr)(q-r)}{(p+1)(q+1)(r+1)} \left(\frac{p(q-r)}{pq-1} \right)^{\frac{p(r+1)}{1-pr}} (\lambda |g|_{*,r+1})^{\frac{p+1}{1-pr}}, \\ c_{\lambda,0} &< -\frac{(1-pr)(pq-1)}{p(p+1)(q+1)(r+1)} t_*^{\frac{p+1}{p}} \|u_*\|^{\frac{p+1}{p}}. \end{aligned} \quad (2.3)$$

It worths to note that for $\lambda = \lambda^*$, we take $t_* = t_0(u_*)$, given by Remark 2.3 to guarantee that

$$\begin{aligned} c_{\lambda,0} &\geq -\frac{(1-pr)(q-r)}{(p+1)(q+1)(r+1)} \left(\frac{p(q-r)}{pq-1} \right)^{\frac{p(r+1)}{1-pr}} \left(\lambda |g|_{*,r+1} \right)^{\frac{p+1}{1-pr}}, \\ c_{\lambda,0} &\leq -\frac{(1-pr)(pq-1)}{p(p+1)(q+1)(r+1)} t_*^{\frac{p+1}{p}} \|u_*\|^{\frac{p+1}{p}}. \end{aligned} \quad (2.4)$$

Next we employ the Ekeland's Variational Principle to show that (H4) guarantee the existence of a minimizing sequence of $c_{\lambda,0}$ and a minimizing sequence of $c_{\lambda,1}$ with a very special property.

Lemma 2.8. *Suppose (H4) satisfied. Then there exists a minimizing sequence of I_λ on $\mathcal{N}_\lambda$ which is also a (PS)-sequence for I_λ .*

Proof. It is clear that $\mathcal{N}_\lambda$ is a closed set of E and therefore a complete metric space. Since $c_{\lambda,0} > -\infty$, by Lemma 2.1, the Ekeland's Variational Principle guarantees the existence of a sequence (u_n) in $\mathcal{N}_\lambda$ satisfying:

$$I_\lambda(u_n) < c_{\lambda,0} + \frac{1}{n}, \quad I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n}\|u_n - w\|, \quad \forall n \in \mathbb{N}, \quad \forall w \in \mathcal{N}_\lambda. \quad (2.5)$$

Since $u_n \in \mathcal{N}_\lambda$, by (2.3)

$$\begin{aligned} I_\lambda(u_n) &= \frac{pq-1}{(p+1)(q+1)}\|u_n\|^{\frac{p+1}{p}} - \frac{q-r}{(q+1)(r+1)}\lambda \int g(x)|u_n|^{r+1}dx < c_{\lambda,0} + \frac{1}{n} \\ &< -\frac{(1-pr)(pq-1)}{p(p+1)(q+1)(r+1)}t_*^{\frac{p+1}{p}}\|u_*\|^{\frac{p+1}{p}} \end{aligned} \quad (2.6)$$

for n sufficiently large.

From (2.6) one obtains

$$(q-r)\lambda|g|_{*,r+1}\|u_n\|^{r+1} \geq (q-r)\lambda \int g(x)|u_n|^{r+1}dx > \frac{(1-pr)(pq-1)}{p(p+1)}t_*^{\frac{p+1}{p}}\|u_*\|^{\frac{p+1}{p}}$$

and so

$$\|u_n\|^{r+1} > \frac{(1-pr)(pq-1)}{p(q-r)(p+1)} \frac{t_*^{\frac{p+1}{p}}}{\lambda} \frac{\|u_*\|^{\frac{p+1}{p}}}{|g|_{*,r+1}}. \quad (2.7)$$

The next step is to prove that there exists a subsequence of (u_n) , also denoted by (u_n) , such that $I'_\lambda(u_n) \rightarrow 0$ in E' . By contradiction suppose the reverse. One can suppose $I'_\lambda(u_n) \neq 0$ for every $n \in \mathbb{N}$. Let $T: E \rightarrow E'$ be the homeomorphism given by Lemma 3.2 in [12], and $U_n = T^{-1}(I'_\lambda(u_n))$. From definition of T one has $\langle T(U_n), U_n \rangle = \|U_n\|^{\frac{p+1}{p}}$ and $\|u\| = |T(u)|_*^p$ for every $u \in E$. So, one concludes that

$$\langle T(U_n), U_n \rangle = \|U_n\|^{\frac{p+1}{p}} = |I'_\lambda(u_n)|_*^{p+1}$$

and

$$\left\langle I'_\lambda(u_n), \frac{U_n}{\|U_n\|} \right\rangle = \frac{|I'_\lambda(u_n)|_*^{p+1}}{|I'_\lambda(u_n)|_*^p} = |I'_\lambda(u_n)|_*. \quad (2.8)$$

Fix $n \in \mathbb{N}$ for a while. Making $u = u_n$ in Lemma 2.7, one knows that for $w = \delta \frac{U_n}{\|U_n\|}$, with $0 < \delta < \delta_n$, δ_n given in Lemma 2.7, and $\tau_n(\delta) := t_n(\delta \frac{U_n}{\|U_n\|})$

$$w_{\delta,n} = \tau_n(\delta) \left[u_n - \delta \frac{U_n}{\|U_n\|} \right] \in \mathcal{N}_\lambda, \quad \text{for every } 0 < \delta < \delta_n.$$

Note here $\tau_n(\delta)$ is a function defined on $(0, \delta_n)$ and $t_n(\delta \frac{U_n}{\|U_n\|})$ is the function of Lemma 2.7.

From (2.5) and the fact that $w_{\delta,n} \in \mathcal{N}_\lambda$ one gets

$$\begin{aligned} \frac{1}{n} \|u_n - w_{\delta,n}\| &\geq I_\lambda(u_n) - I_\lambda(w_{\delta,n}) - \langle I'_\lambda(w_{\delta,n}), u_n \rangle + \langle I'_\lambda(w_{\delta,n}), u_n \rangle \\ &= I_\lambda(u_n) - I_\lambda(w_{\delta,n}) - \langle I'_\lambda(w_{\delta,n}), u_n \rangle \\ &\quad + (1 - \tau_n(\delta)) \langle I'_\lambda(w_{\delta,n}), u_n \rangle + \delta \tau_n(\delta) \left\langle I'_\lambda(w_{\delta,n}), \frac{U_n}{\|U_n\|} \right\rangle. \end{aligned} \quad (2.9)$$

Assume (see Lemma 2.9)

$$I_\lambda(u_n) - I_\lambda(w_{\delta,n}) - \langle I'_\lambda(w_{\delta,n}), u_n \rangle = o(\delta). \quad (2.10)$$

Then,

$$\frac{1}{n} \|w_{\delta,n} - u_n\| \geq (1 - \tau_n(\delta)) \langle I'_\lambda(w_{\delta,n}), u_n \rangle + \delta \tau_n(\delta) \left\langle I'_\lambda(w_{\delta,n}), \frac{U_n}{\|U_n\|} \right\rangle + o(\delta). \quad (2.11)$$

But $w_{\delta,n} - u_n = (\tau_n(\delta) - 1)u_n - \delta \tau_n(\delta) \frac{U_n}{\|U_n\|}$ and from the notation $\tau_n(\delta) = t_n(\delta \frac{U_n}{\|U_n\|})$ one has $\tau'_n(0) = \langle t'_n(0), \frac{U_n}{\|U_n\|} \rangle$. So dividing (2.11) by $\delta > 0$ and taking the limit when $\delta \rightarrow 0$, from (2.8) and (2.11) one reaches at

$$\begin{aligned} \frac{1}{n} (1 + |\tau'_n(0)| \|u_n\|) &\geq \frac{1}{n} \left\| \tau'_n(0) u_n - \frac{U_n}{\|U_n\|} \right\| = \frac{1}{n} \left\| \lim_{\delta \rightarrow 0} \frac{w_{\delta,n} - u_n}{\delta} \right\| \\ &\geq -\tau'_n(0) \langle I'_\lambda(u_n), u_n \rangle + \langle I'_\lambda(u_n), \frac{U_n}{\|U_n\|} \rangle = |I'_\lambda(u_n)|_*, \end{aligned}$$

that is,

$$|I'_\lambda(u_n)|_* \leq \frac{1}{n} (1 + |\tau'_n(0)| \|u_n\|) \leq \frac{C}{n} (1 + |\tau'_n(0)|) \quad (2.12)$$

since (u_n) is bounded. In virtue of (2.12) the next step is to show that $|\tau'_n(0)|$ is uniformly bounded with relation to n . One has from Lemma 2.7 that

$$|\tau'_n(0)| = \left| \langle t'_n(0), \frac{U_n}{\|U_n\|} \rangle \right| \leq \frac{C_1}{\left| (1 - pr) \|u_n\|^{\frac{p+1}{p}} - p(q-r) |u_n|_{q+1}^{q+1} \right|}$$

because (u_n) is bounded and $\|\frac{U_n}{\|U_n\|}\| = 1$. So, one needs to prove that $|(1 - pr) \|u_n\|^{\frac{p+1}{p}} - p(q-r) |u_n|_{q+1}^{q+1}|$ is bounded away from zero. By contradiction, suppose that there exists a subsequence of (u_n) , also denoted by (u_n) , such that

$$(1 - pr) \|u_n\|^{\frac{p+1}{p}} - p(q-r) |u_n|_{q+1}^{q+1} = o(1). \quad (2.13)$$

From (2.13) and (2.7)

$$|u_n|_{q+1}^{q+1} = \frac{1 - pr}{p(q-r)} \|u_n\|^{\frac{p+1}{p}} + o(1) \geq C,$$

that is, $|u_n|_{q+1} \geq \gamma$ for some $\gamma > 0$ and

$$\begin{aligned} \frac{pq-1}{1-pr} \left(|u_n|_{q+1}^{q+1} \right)^{\frac{p(q-r)}{pq-1}} &= \frac{pq-1}{1-pr} \left(\frac{1-pr}{p(q-r)} \|u_n\|^{\frac{p+1}{p}} \right)^{\frac{p(q-r)}{pq-1}} + o(1) \\ &= K \|u_n\|^{\frac{(q-r)(p+1)}{pq-1}} + o(1). \end{aligned} \quad (2.14)$$

From (2.13), the definition of K and the fact that $u_n \in \mathcal{N}_\lambda$

$$\begin{aligned} \lambda \int g(x) |u_n|^{r+1} dx &= \|u_n\|^{\frac{p+1}{p}} - |u_n|_{q+1}^{q+1} = \left(\frac{p(q-r)}{1-pr} - 1 \right) |u_n|_{q+1}^{q+1} + o(1) \\ &= \frac{pq-1}{1-pr} |u_n|_{q+1}^{q+1} + o(1). \end{aligned} \quad (2.15)$$

So, from (2.1), (2.14) and (2.15) with $\alpha = \frac{(q+1)(1-pr)}{pq-1}$

$$\begin{aligned} 0 &< \mu_{\lambda,0} \gamma^{r+1+\alpha} \leq |u_n|_{q+1}^\alpha \psi(u_n) \\ &= |u_n|_{q+1}^\alpha \left(K \frac{\|u_n\|^{\frac{(q-r)(p+1)}{pq-1}}}{|u_n|_{q+1}^\alpha} - \frac{pq-1}{1-pr} |u_n|_{q+1}^{q+1} \right) + o(1) \\ &= K \|u_n\|^{\frac{(q-r)(p+1)}{pq-1}} - \frac{pq-1}{1-pr} |u_n|_{q+1}^{\frac{p(q-r)(q+1)}{pq-1}} + o(1) = o(1) \end{aligned}$$

which is clearly impossible.

Now, to conclude that $I'_\lambda(u_n) \rightarrow 0$ one just need to prove (2.10). It is going to be proved below in the form of a lemma. $\square$

Let $s > 1$ be a real number. Then, by the mean value theorem for derivatives, for all $t, h \in \mathbb{R}$, there exists $0 < \theta < 1$ such that

$$|t|^s + (s-1)|t+h|^s - s|t+h|^{s-2}(t+h)t = s \left(|t+h|^{s-2}(t+h) - |t+\theta h|^{s-2}(t+\theta h) \right) h. \quad (2.16)$$

We employ (2.16) to prove (2.10).

Lemma 2.9. (2.10) holds true.

Proof. Let $\kappa_{\delta,n} = I_\lambda(u_n) - I_\lambda(w_{\delta,n}) - \langle I'_\lambda(w_{\delta,n}), u_n \rangle$. By the fact that $w_{\delta,n} \in \mathcal{N}_\lambda$, one has

$$\begin{aligned} \kappa_{\delta,n} &= \frac{p}{p+1} \|u_n\|^{\frac{p+1}{p}} - \frac{1}{q+1} |u_n|_{q+1}^{q+1} - \frac{\lambda}{r+1} \int g(x) |u_n|^{r+1} dx - \frac{p}{p+1} \|w_{\delta,n}\|^{\frac{p+1}{p}} \\ &\quad + \frac{1}{q+1} |w_{\delta,n}|_{q+1}^{q+1} + \frac{\lambda}{r+1} \int g(x) |w_{\delta,n}|^{r+1} dx - \int |\Delta w_{\delta,n}|^{\frac{1}{p}-1} \Delta w_{\delta,n} \Delta u_n dx \\ &\quad + \int |w_{\delta,n}|^{q-1} w_{\delta,n} u_n dx + \lambda \int g(x) |w_{\delta,n}|^{r-1} w_{\delta,n} u_n dx \\ &\quad + \|w_{\delta,n}\|^{\frac{p+1}{p}} - |w_{\delta,n}|_{q+1}^{q+1} - \lambda \int g(x) |w_{\delta,n}|^{r+1} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{p}{p+1} \left(\|u_n\|^{\frac{p+1}{p}} + \frac{1}{p} \|w_{\delta,n}\|^{\frac{p+1}{p}} - \frac{p+1}{p} \int |\Delta w_{\delta,n}|^{\frac{1}{p}-1} \Delta w_{\delta,n} \Delta u_n dx \right) \\
&\quad - \frac{1}{q+1} \left(|u_n|_{q+1}^{q+1} + q |w_{\delta,n}|_{q+1}^{q+1} - (q+1) \int |w_{\delta,n}|^{q-1} w_{\delta,n} u_n dx \right) \\
&\quad - \frac{\lambda}{r+1} \left(\int g(x) \left(|u_n|^{r+1} + r |w_{\delta,n}|^{r+1} - (r+1) |w_{\delta,n}|^{r-1} w_{\delta,n} u_n \right) dx \right).
\end{aligned}$$

Let $h_{\delta,n} = w_{\delta,n} - u_n$. From the last identity and from (2.16) one gets, for some $0 < \theta, \sigma, \tau < 1$,

$$\begin{aligned}
\kappa_{\delta,n} &= \int \left(|\Delta w_{\delta,n}|^{\frac{1}{p}-1} \Delta w_{\delta,n} - |\Delta u_n + \theta \Delta h_{\delta,n}|^{\frac{1}{p}-1} (\Delta u_n + \theta \Delta h_{\delta,n}) \right) \Delta h_{\delta,n} dx \\
&\quad - \int \left(|w_{\delta,n}|^{q-1} w_{\delta,n} - |u_n + \sigma h_{\delta,n}|^{q-1} (u_n + \sigma h_{\delta,n}) \right) h_{\delta,n} dx \\
&\quad - \lambda \int g(x) \left(|w_{\delta,n}|^{r-1} w_{\delta,n} - |u_n + \tau h_{\delta,n}|^{r-1} (u_n + \tau h_{\delta,n}) \right) h_{\delta,n} dx.
\end{aligned}$$

Dividing the last identity by $\delta > 0$:

$$\begin{aligned}
\frac{\kappa_{\delta,n}}{\delta} &= \int \left(|\Delta w_{\delta,n}|^{\frac{1}{p}-1} \Delta w_{\delta,n} - |\Delta u_n + \theta \Delta h_{\delta,n}|^{\frac{1}{p}-1} (\Delta u_n + \theta \Delta h_{\delta,n}) \right) \frac{\Delta h_{\delta,n}}{\delta} dx \\
&\quad - \int \left(|w_{\delta,n}|^{q-1} w_{\delta,n} - |u_n + \sigma h_{\delta,n}|^{q-1} (u_n + \sigma h_{\delta,n}) \right) \frac{h_{\delta,n}}{\delta} dx \\
&\quad - \lambda \int g(x) \left(|w_{\delta,n}|^{r-1} w_{\delta,n} - |u_n + \tau h_{\delta,n}|^{r-1} (u_n + \tau h_{\delta,n}) \right) \frac{h_{\delta,n}}{\delta} dx.
\end{aligned}$$

So, from the Lebesgue's Dominated Convergence Theorem one obtains that $\kappa_{\delta,n} = o(\delta)$ just as desired. $\square$

Lemma 2.10. *If (H₄) holds true then there exists a minimizing sequence of I_λ on $\mathcal{N}_\lambda^-$ which is also a (PS)-sequence for I_λ .*

Proof. By (vi) of Lemma 2.5 one has that $\mathcal{N}_\lambda^-$ is a closed set of E and therefore a complete metric space. By Ekeland's Variational Principle there exists a sequence (u_n) in $\mathcal{N}_\lambda^-$ satisfying:

$$I_\lambda(u_n) < c_{\lambda,1} + \frac{1}{n}, \quad I_\lambda(w) \geq I_\lambda(u_n) - \frac{1}{n} \|u_n - w\|, \quad \forall n \in \mathbb{N}, \quad \forall w \in \mathcal{N}_\lambda^-. \quad (2.17)$$

By the estimate obtained in the proof of Lemma 2.1 involving $\|u\|$ to the powers $\frac{p+1}{p}$ and $r+1$, one concludes from (H1) that (u_n) is bounded and from (iii) of Lemma 2.5 one has that (u_n) is bounded away from zero. The rest of the proof is just a copy of the proof of Lemma 2.8, going from (2.7). $\square$

Remark 2.11. It follows from the proof of Lemma 2.2 that under (H4)

$$c_{\lambda,0} = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda^+} I_\lambda(u).$$

Another important fact is that

$$c_{\lambda,0} = \inf_{u \in \mathcal{N}_\lambda} I_\lambda(u) = \inf_{u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_{\lambda,0}} I_\lambda(u)$$

when $\lambda = \lambda^*$.

Proof. Let $\lambda = \lambda^*$ and $(u_n) \subset \mathcal{N}_\lambda$ such that $I_\lambda(u_n) \rightarrow c_0$ as $n \rightarrow \infty$. For each $n \in \mathbb{N}$ let $w_n = t_0(u_n)u_n \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_{\lambda,0}$, t_0 given by Remark 2.3. Since $I_\lambda(w_n) \leq I_\lambda(u_n)$ the desired inequality is obtained. $\square$

The next two lemmas present important properties of the points that realizes the minimization problems involving $c_{\lambda,0}$ or $c_{\lambda,1}$.

Lemma 2.12. *Under (H4), if $u \in \mathcal{N}_\lambda$ is such that $I_\lambda(u) = c_{\lambda,0}$ then $u \in \mathcal{N}_\lambda^+$ and u is a local minimum for I_λ when it is considered defined on the whole space E .*

Proof. Let $u \in \mathcal{N}_\lambda$ such that $I_\lambda(u) = c_{\lambda,0}$. From the existence of $t^+(u)$ e $t^-(u)$, given by Lemma 2.2 and from $I_\lambda(t^-(u)u) < I_\lambda(t^+(u)u)$ one gets $u \in \mathcal{N}_\lambda^+$.

Since $u \in \mathcal{N}_\lambda^+$, $t^- = 1$. Then from the inequality involving t^- and $t_{\max}$ given at Lemma 2.2,

$$1 < \left(\frac{1 - pr}{p(q-r)} \frac{\|u\|^{\frac{p+1}{p}}}{|u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}}.$$

So, for certain $\delta > 0$ sufficiently small

$$1 < \left(\frac{1 - pr}{p(q-r)} \frac{\|w - u\|^{\frac{p+1}{p}}}{|w - u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}}, \quad \forall w \in E, \|w\| < \delta. \quad (2.18)$$

By Lemma 2.7, for $\delta > 0$ even small if necessary, let $t : \{w \in E : \|w\| < \delta\} \rightarrow \mathbb{R}$ such that $t(w)(u - w) \in \mathcal{N}_\lambda$ for every $w \in E$ with $\|w\| < \delta$.

Once $t(w) \rightarrow 1$ as $\|w\| \rightarrow 0$, one can assume that

$$t(w) < \left(\frac{1 - pr}{p(q-r)} \frac{\|w - u\|^{\frac{p+1}{p}}}{|w - u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}}, \quad \forall w \in E, \|w\| < \delta.$$

The last inequality combined with Lemma 2.2 imply that $t(w)(u - w) \in \mathcal{N}_\lambda^+$. Once more by Lemma 2.2 and from $I_\lambda(u) = c_{\lambda,0}$,

$$\begin{aligned} I_\lambda(t(u - w)) &\geq I_\lambda(t(w)(u - w)) \\ &\geq I_\lambda(u), \quad \forall t \in \left[0, \left(\frac{1 - pr}{p(q-r)} \frac{\|w - u\|^{\frac{p+1}{p}}}{|w - u|_{q+1}^{q+1}} \right)^{\frac{p}{pq-1}} \right]. \end{aligned}$$

Therefore, from (2.18) one concludes that $I_\lambda(u - w) \geq I_\lambda(u)$ for every $w \in E$ with $\|w\| < \delta$, that is, u is a local minimum for I_λ . $\square$

Lemma 2.13. *Let λ be any real number. If $u \in \mathcal{N}_\lambda^-$ is such that $I_\lambda(u) = c_{\lambda,1}$ then u is a critical point of I_λ .*

Proof. Consider the open set $A = \{u \in E : (1 - pr)\|u\|^{\frac{p+1}{p}} - p(q-r)|u|_{q+1}^{q+1} < 0\}$ and $G : A \rightarrow \mathbb{R}$ defined by

$$G(u) = \langle I'_\lambda(u), u \rangle = \|u\|^{\frac{p+1}{p}} - |u|_{q+1}^{q+1} - \lambda \int g(x)|u|^{r+1} dx.$$

So, $G \in C^1(A, \mathbb{R})$, $\mathcal{N}_\lambda^- = G^{-1}(\{0\})$ and for each $u \in \mathcal{N}_\lambda^-$

$$\begin{aligned} \langle G'(u), u \rangle &= \frac{p+1}{p} \|u\|^{\frac{p+1}{p}} - (q+1)|u|_{q+1}^{q+1} - \lambda(r+1) \int g(x)|u|^{r+1} dx \\ &= \frac{1-pr}{p} \|u\|^{\frac{p+1}{p}} - (q-r)|u|_{q+1}^{q+1} < 0. \end{aligned} \quad (2.19)$$

If $u \in \mathcal{N}_\lambda^-$ is such that $I_\lambda(u) = c_{\lambda,1}$ then the Lagrange's multiplier theorem guarantees the existence of $\theta \in \mathbb{R}$ such that

$$\langle I'_\lambda(u), v \rangle = \theta \langle G'(u), v \rangle, \quad \forall v \in E,$$

in particular

$$0 = \langle I'_\lambda(u), u \rangle = \theta \langle G'(u), u \rangle.$$

It follows from the last inequality and from (2.19) that $\theta = 0$ and therefore that u is a critical point of I_λ . $\square$

2.2. The Palais-Smale condition

Here we prove some compactness properties of I_λ .

As we said before, we obtain nontrivial nonnegative weak solutions for (P_λ) investigating the minimization problems which define $c_{\lambda,0}$ and $c_{\lambda,1}$. We have constructed (PS)-sequences associated to such problems, the next step is to prove that I_λ satisfies the (PS)-condition at such levels.

To obtain the results concerning the existence of infinitely many solutions in the subcritical case, we apply the Fountain and the Dual Fountain Theorems. For that matter, it is important to know if E has a Schauder basis. In 1932, Banach proposed in p. 111 of his book [3] the following classical problem in Functional Analysis: Does every separable Banach space have a Schauder basis? This question remained without answer until 1973 when Enflo [11] gave an example of a separable Banach space which fails to have a basis. On the other hand, it was proved in [16] that for any integer $k \geq 0$ and $1 < p < +\infty$, the Sobolev spaces $W^{k,p}(\Omega)$ and $W_0^{k,p}(\Omega)$ have Schauder bases. But E does not belong to any of the above two classes of Sobolev spaces. Fortunately, from Theorem 9.15 and Lemma 9.17 in [18], we know that $-\Delta : E \rightarrow L^{\frac{p+1}{p}}(\Omega)$ is an isometric isomorphism. So, from the existence of a Schauder basis for $L^{\frac{p+1}{p}}(\Omega) = W^{0,\frac{p+1}{p}}(\Omega)$, we conclude that E also has a Schauder basis. Also concerning the application of the Fountain and Dual Fountain Theorems, we need to know if such Schauder basis is a shrinking basis. And once more the answer is positive, since E is a reflexive space (see Theorem 3.2.13 in [1] or Theorem 1 in [19]).

For the sake of completeness we introduce the following definition (Definition 3.2.5 in [1]).

Definition 2.14. Let $(e_k)_{k=1}^\infty$ be a Schauder basis for a Banach space X with associated biorthogonal functionals $(e_k^*)_{k=1}^\infty$, that is, $e_j^*(e_k) = \delta_{jk}$ for all $j, k \in \mathbb{N}$. We say that (e_k) is shrinking if $(e_k^*)_{k=1}^\infty$ is a Schauder basis for X' , the dual space of X .

In this way, let us denote by (e_k) a Schauder basis for E with associated biorthogonal functionals (e_k^*) . Since E is a reflexive Banach space such basis is a shrinking basis.

For each $n \geq 1$, following the notation from Sect. 3.2 in [23], we call

$$Y_n := [e_j]_{j=1}^n, \quad Z_n := \overline{[e_j]_{j \geq n}}.$$

Lemma 2.15. Let λ and β be real numbers. Every $(PS)_\beta^*$ -sequence with relation to (Y_n) of I_λ is bounded.

Proof. Let (u_{n_j}) be a $(PS)_\beta^*$ -sequence with relation to (Y_n) for I_λ , that is,

$$n_j \rightarrow +\infty, \quad u_{n_j} \in Y_{n_j}, \quad I_\lambda(u_{n_j}) \rightarrow \beta, \quad I'_\lambda|_{Y_{n_j}}(u_{n_j}) \rightarrow 0.$$

In particular there exist $C > 0$ and a sequence of positive real numbers $\sigma_{n_j} \rightarrow 0$, such that for every n_j and for every $w \in Y_{n_j}$

$$|I_\lambda(u_{n_j})| \leq C \quad \text{and} \quad |\langle I'_\lambda(u_{n_j}), w \rangle| \leq \sigma_{n_j} \|w\|.$$

Since

$$(q+1)I_\lambda(u_{n_j}) - \langle I'_\lambda(u_{n_j}), u_{n_j} \rangle = \frac{pq-1}{p+1} \|u_{n_j}\|^{\frac{p+1}{p}} - \frac{q}{r+1} \lambda \int g(x) |u_{n_j}|^{r+1} dx$$

one gets

$$\begin{aligned} \frac{pq-1}{p+1} \|u_{n_j}\|^{\frac{p+1}{p}} &= (q+1)I_\lambda(u_{n_j}) - \langle I'_\lambda(u_{n_j}), u_{n_j} \rangle + \frac{q}{r+1} \lambda \int g(x) |u_{n_j}|^{r+1} dx \\ &\leq (q+1)C + \sigma_{n_j} \|u_{n_j}\| + |\lambda| \frac{q}{r+1} |g|_{*,r+1} \|u_{n_j}\|^{r+1}, \end{aligned}$$

which implies that (u_{n_j}) is bounded, since $\frac{p+1}{p} > r+1$ by (H1). $\square$

Proposition 2.16. If the strict inequality holds at (H2) then for every $\lambda \in \mathbb{R}$, the functional I_λ satisfies the $(PS)^*$ -condition with relation to (Y_n) . In particular, I_λ satisfies the (PS) -condition (see Remark 3.19 in [23]).

Proof. Let (u_{n_j}) be a $(PS)_\beta^*$ -sequence with relation to (Y_n) for I_λ . By Lemma 2.15, (u_{n_j}) is bounded. Since E is a reflexive Banach space one can suppose that $u_{n_j} \rightharpoonup u$ for some $u \in E$.

Writing $I'_\lambda = T + K_\lambda$, with $T, K_\lambda : E \rightarrow E'$ where

$$\langle K_\lambda(v), w \rangle = - \int |v|^{q-1} v w dx - \lambda \int g(x) |v|^{r-1} v w dx, \quad \forall v, w \in E.$$

It follows from (H1), (H3) and the strict inequality at (H2) that K_λ maps weakly convergent sequences into strongly convergent sequences. On the other hand, the operator T has been studied at Lemma 3.2 in [12].

In particular,

$$\begin{aligned}\langle T(u_{n_j}) - T(u), u_{n_j} - u \rangle &= \langle I'_\lambda(u_{n_j}) - I'_\lambda(u), u_{n_j} - u \rangle - \langle K_\lambda(u_{n_j}) - K_\lambda(u), u_{n_j} - u \rangle \\ &= \langle I'_\lambda(u_{n_j}) - I'_\lambda(u), u_{n_j} - u \rangle + o(1) = \langle I'_\lambda(u_{n_j}), u \rangle + o(1).\end{aligned}$$

Since (e_n) is a Schauder basis for E and from the definition of (Y_n) , there exists a sequence $(\bar{u}_{n_j})$ such that for each n_j , $\bar{u}_{n_j} \in Y_{n_j}$ and $\bar{u}_{n_j} \rightarrow u$. As T and K_λ maps bounded sets into bounded sets, the sequence $(I'_\lambda(u_{n_j}))$ is bounded in E' and then,

$$\begin{aligned}\langle T(u_{n_j}) - T(u), u_{n_j} - u \rangle &= \langle I'_\lambda(u_{n_j}), u \rangle + o(1) \\ &= \langle I'_\lambda(u_{n_j}), u - \bar{u}_{n_j} \rangle + \langle I'_\lambda(u_{n_j}), \bar{u}_{n_j} \rangle + o(1) \\ &= o(1).\end{aligned}\tag{2.20}$$

On the other hand, it follows from the inequality (2.2) in [21] and the definition of T that for each $p > 0$ there exists $C_p > 0$ such that for all $v, w \in E$

$$\langle T(v) - T(w), v - w \rangle \geq \begin{cases} C_p \|v - w\|^{\frac{p+1}{p}} & \text{if } 0 < p \leq 1, \\ C_p \int \frac{|\Delta v - \Delta w|^2}{(|\Delta v| + |\Delta w|)^{1-\frac{1}{p}}} dx & \text{if } p > 1. \end{cases}\tag{2.21}$$

In the case that $0 < p \leq 1$, it follows directly from (2.20) and (2.21) that $u_{n_j} \rightarrow u$ in E .

In the case that $p > 1$, by Hölder's inequality, for all $v, w \in E$

$$\begin{aligned}\|v - w\|^{\frac{p+1}{p}} &= \int \left(\frac{|\Delta v - \Delta w|^{\frac{p+1}{p}}}{(|\Delta v| + |\Delta w|)^{\frac{\frac{p+1}{p} \cdot p - 1}{2}}} (|\Delta v| + |\Delta w|)^{\frac{\frac{p+1}{p} \cdot p - 1}{2}} \right) dx \\ &\leq \left(\int \frac{|\Delta v - \Delta w|^2}{(|\Delta v| + |\Delta w|)^{1-\frac{1}{p}}} dx \right)^{\frac{p+1}{2p}} \left(\int (|\Delta v| + |\Delta w|)^{\frac{p+1}{p}} dx \right)^{\frac{p-1}{2p}}.\end{aligned}\tag{2.22}$$

So, from (2.20)–(2.22) one obtains that $u_{n_j} \rightarrow u$ in E .

Now, for each $n \in \mathbb{N}$, let $P_n : E \rightarrow Y_n$ the projector of E onto Y_n . Since (e_n) is a shrinking basis of E , one has that for every $\varphi \in E'$, $\varphi \circ P_n \rightarrow \varphi$ in E' , which implies that

$$\left| \varphi|_{Y_n} \right|_* \rightarrow |\varphi|_*.$$

In this way,

$$\begin{aligned} |I'_\lambda(u)|_* &= \left| I'_\lambda(u) \right|_{Y_{n_j}}|_* + o(1) \leq \left| (I'_\lambda(u) - I'_\lambda(u_{n_j})) \right|_{Y_{n_j}}|_* + \left| I'_\lambda(u_{n_j}) \right|_{Y_{n_j}}|_* + o(1) \\ &\leq \left| I'_\lambda(u) - I'_\lambda(u_{n_j}) \right|_* + \left| I'_\lambda(u_{n_j}) \right|_{Y_{n_j}}|_* + o(1) = o(1), \end{aligned}$$

that is, u is a critical point of I_λ . $\square$

In the critical case, when the identity holds at (H2), we are not able to prove that the functional I_λ satisfies the $(PS)^*_\beta$ -condition with relation to (Y_n) for any $\beta \in \mathbb{R}$. Instead of that, we employ the Concentration-Compactness Principle [20] and a Riesz's representation theorem for some Sobolev spaces on Ω [12] to locate the levels where I_λ satisfies the (PS)-condition.

The next lemma deals with the compactness property of I_λ when (P_λ) is a critical problem. In this case, the constant R defined in Sect. 1 becomes S , the best constant for the noncompact Sobolev embedding of E into $L^{q+1}(\Omega)$.

Proposition 2.17. *If the identity holds at (H2) then for every $\lambda \in \mathbb{R}$, the functional I_λ satisfies the $(PS)_\beta$ -condition for every*

$$\beta < c_{\lambda,0} + \frac{2}{N} S^{\frac{pN}{2(p+1)}}.$$

Proof. See the proof of Proposition 3.2 in [14]. $\square$

Remark 2.18. We emphasize that the arguments of Sect. 3 in [14] cannot be adapted to obtain the $(PS)^*_\beta$ -condition for I_λ with relation to (Y_n) , because the claim (3.1) in [14] may not be true when (u_n) is only a $(PS)^*_\beta$ -sequence.

2.3. Checking the structure to get infinitely many solutions

Here we prove some results that are employed to get the existence of infinitely many solutions for (P_λ) .

In order to apply the Fountain and the Dual Fountain Theorem we need to prove:

Lemma 2.19. *If $\frac{1}{p+1} + \frac{1}{s+1} > 1 - \frac{2}{N}$ then*

$$\beta_n := \sup_{u \in Z_n, \|u\|=1} |u|_{s+1} \rightarrow 0, \text{ as } n \rightarrow +\infty.$$

Proof. It is clear that $0 < \beta_{n+1} \leq \beta_n$, so $\beta_n \rightarrow \beta \geq 0$, $n \rightarrow +\infty$. For every $n \geq 1$, there exists $u_n \in Z_n$ such that $\|u_n\| = 1$ and $|u_n|_{s+1} \geq \beta_n/2$. Since (e_k) is a shrinking basis, one has that $u_n \rightharpoonup 0$ in E . From the compact Sobolev embedding of E into $L^{s+1}(\Omega)$, $u_n \rightarrow 0$ in $L^{s+1}(\Omega)$, which proves that $\beta = 0$. $\square$

As one can see at part (a) of the statement of Theorem 1.4, we intend to obtain a sequence $(\bar{u}_k)$ of weak solutions of (P_λ) with negative energy. Once we need to verify the (PS)-condition to such levels, by means of Lemma 2.16 we prove:

Lemma 2.20. *If the identity holds at (H2) then*

$$c_{\lambda,0} + \frac{2}{N} S^{\frac{pN}{2(p+1)}} \geq 0 \text{ for every } \lambda \in (0, \alpha\lambda^*], \quad (2.23)$$

$$\text{with } \alpha = \alpha(p, r, N) = \left(\frac{p(r+1)}{1-pr} \right)^{\frac{1-pr}{p+1}} \left(\frac{p(q-r)}{1-pr} \right)^{\frac{1-pr}{pq-1}}.$$

Proof. It follows from a straightforward calculation employing the lower bound of $c_{\lambda,0}$ given at Lemma 2.1. $\square$

3. The proof of Theorem 1.3

3.1. Item (I)

Here we employ the Fountain Theorem as stated at Theorem 3.6 in [23]. We have that $I_\lambda \in C^1(E, \mathbb{R})$ is an even functional and we need to verify the conditions (A1)–(A4) of Theorem 3.6 in [23].

It is clear that $G := \mathbb{Z}/2$ acts isometrically on $E = \overline{\bigoplus_{j=1}^\infty [e_j]}$, $[e_j] \simeq \mathbb{R}$ for each $j \in \mathbb{N}$ and the antipodal action of G on $\mathbb{R}$ is admissible by the Borsuk-Ulam Theorem. In this way (A1) is verified.

Also,

$$I_\lambda(u) \leq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{1}{q+1} |u|_{q+1}^{q+1}.$$

Since on the finite-dimensional space Y_n all norms are equivalent, from $q > \frac{1}{p}$ at (H1), for each $n \in \mathbb{N}$ there exists $\rho_n > 0$ large enough such that

$$a_n := \max_{u \in Y_n, \|u\|=\rho_n} I_\lambda(u) < 0,$$

which verifies (A2).

On the other hand, there exists $C > 0$ such that for all $x \in \Omega$, $t \in \mathbb{R}$

$$\frac{1}{q+1} |t|^{q+1} + \frac{\lambda}{r+1} g(x) |t|^{r+1} \leq C(1 + |t|^{q+1}).$$

Define

$$\beta_n := \sup_{u \in Z_n, \|u\|=1} |u|_{q+1}$$

so that, on Z_n , one has

$$I_\lambda(u) \geq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - C |u|_{q+1}^{q+1} - C |\Omega| \geq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - C \beta_n^{q+1} \|u\|^{q+1} - C |\Omega|.$$

Choosing $r_n := (C \beta_n^{q+1} (q+1))^{-\frac{p}{pq-1}}$, one obtains, if $u \in Z_n$ and $\|u\| = r_n$ then

$$I_\lambda(u) \geq \left(\frac{p}{p+1} - C \beta_n^{q+1} \right) (C \beta_n^{q+1} (q+1))^{-\frac{p+1}{pq-1}} \frac{1 + C \beta_n^{q+1} (q+1)}{C \beta_n^{q+1} (q+1)} - C |\Omega|.$$

By Lemma 2.19, $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. So (A3) is proved.

Finally, (A4) is verified because I_λ satisfies the (PS) condition by Lemma 2.16.

In this way, the conclusion follows from Theorem 3.6 in [23].

3.2. Item (II)

Here we employ the Dual Fountain Theorem as stated at Theorem 3.18 in [23]. As before (A1) is verified, $I_\lambda \in C^1(E, \mathbb{R})$ is an even functional. Now we verify the conditions (B1)–(B4) of Theorem 3.18 in [23].

In order to verify (B1), we define

$$\beta_n = \sup_{u \in Z_n, \|u\|=1} |u|_{r+1}.$$

On the other hand, there exists $C > 0$ such that

$$\|u\| \leq C \Rightarrow \frac{R^{\frac{p(q+1)}{p+1}}}{q+1} \|u\|^{q+1} \leq \frac{p}{2(p+1)} \|u\|^{\frac{p+1}{p}}.$$

So, if $u \in Z_n$ and $\|u\| \leq C$ then

$$\begin{aligned} I_\lambda(u) &\geq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{R^{\frac{p(q+1)}{p+1}}}{q+1} \|u\|^{q+1} - \frac{\lambda}{r+1} |g|_\infty |u|_{r+1}^{r+1} \\ &\geq \frac{p}{2(p+1)} \|u\|^{\frac{p+1}{p}} - \frac{\lambda}{r+1} |g|_\infty \beta_n^{r+1} \|u\|^{r+1}. \end{aligned} \quad (3.1)$$

If one chooses $\rho_n := (4^{\frac{p+1}{p}} \frac{\lambda}{r+1} |g|_\infty \beta_n^{r+1})^{\frac{p}{1-pr}}$ one gets that

$$a_n := \inf_{u \in Z_n, \|u\|=\rho_n} I_\lambda(u) \geq 0,$$

which proves (B1).

With relation to (B2), one has

$$I_\lambda(u) \leq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{\lambda}{r+1} \int g(x) |u|^{r+1} dx.$$

From (H3),

$$u \mapsto \left(\int g(x) |u|^{r+1} dx \right)^{\frac{1}{r+1}}$$

defines a norm on Y_n . From the equivalence between any two norms on Y_n and from (H1), one finds $0 < r_n < \rho_n$ such that

$$b_n := \max_{u \in Y_n, \|u\|=r_n} I_\lambda(u) < 0,$$

which verifies (B2).

Since $\beta_n \rightarrow 0$ by Lemma 2.19, one obtain by (3.1) that

$$d_n := \inf_{u \in Z_n, \|u\| \leq \rho_n} I_\lambda(u) \rightarrow 0, \quad n \rightarrow +\infty,$$

which verifies (B3).

Finally, by Lemma 2.16 one knows that I_λ satisfies the $(PS)^*$ -condition with relation to (Y_n) , which verifies (B4).

3.3. Item (III)

The proof of this item is based on Lemmas 2.8, 2.10, 2.12, 2.13, 2.16 and it is similar to the proofs of Theorems 1.4, 1.5 in [12].

3.4. Item (IV)

To prove this item, we employ an approximation argument identical to that employed in the proof of Theorem 1.6 in [12].

3.5. Item (V)

Let

$$\lambda^\sharp := \max \left\{ 1, \left(\frac{C}{\int g(x) u_*^r \varphi_1 dx} \right)^{\frac{1}{1+pr}} \right\},$$

where C stands for the maximum value of the function $h : [0, t) \rightarrow \mathbb{R}$ defined by $h(t) = \lambda_1^{\frac{p+1}{p}} t^{\frac{1}{p}} - t^q$ and (u_*, v_*) is the unique solution of

$$\begin{cases} -\Delta u_* = v_*^p & \text{in } \Omega, \\ -\Delta v_* = g(x) u_*^r & \text{in } \Omega, \\ u_*, v_* > 0 & \text{in } \Omega, \\ u_*, v_* = 0 & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

The hypothesis $pr < 1$ is crucial for the existence and uniqueness of a (classical) solution for (3.2). For the uniqueness see Theorem 5.1 in [13]. For the existence, it is obtained minimizing

$$h(u) = \frac{p}{p+1} \int |\Delta u|^{\frac{p+1}{p}} dx - \frac{1}{r+1} \int g(x) |u|^{r+1} dx, \quad u \in E$$

on E and applying a regularity argument similar to statement of Theorem 1.1.

The first step is the following: if $\lambda \geq 1$, then any nontrivial nonnegative solution of (P_λ) satisfies

$$u \geq \lambda^p u_* \text{ and } v \geq \lambda v_* \text{ in } \Omega. \quad (3.3)$$

In view of Theorem 1.1, let

$$S = \left\{ s > 0 : u > s^{\frac{r+1}{r}} \lambda^p u_*, v > s^{\frac{p+1}{p}} \lambda v_* \right\}$$

and

$$s^* = \sup S.$$

To prove (3.3), it is enough to show that $s^* \geq 1$. It can be proved by contradiction, following the arguments employed in the proof of Theorem 5.1 in [13].

For that, it is important to see that if $s \in S \cap (0, 1)$, then

$$\begin{cases} -\Delta(u - s^{\frac{r+1}{r}} \lambda^p u_*) = v^p - s^{\frac{r+1}{r}} \lambda^p v_*^p > s^{p+1} \lambda^p v_*^p - s^{\frac{r+1}{r}} \lambda^p v_*^p \\ \quad = s^{p+1} \lambda^p v_*^p (1 - s^{\frac{1}{r}-p}) > 0, \\ -\Delta(v - s^{\frac{p+1}{p}} \lambda v_*) = u^q + \lambda g(x) u^r - s^{\frac{p+1}{p}} \lambda g(x) u_*^r > \lambda g(x) (s^{r+1} \lambda^{pr} u_*^r - s^{\frac{p+1}{p}} u_*^r) \\ \quad = \lambda g(x) s^{r+1} u_*^r (\lambda^{pr} - s^{\frac{1}{p}-r}) > 0, \end{cases}$$

in Ω (remember $\lambda \geq 1$).

Suppose $\lambda \geq 1$. If (P_λ) has a nontrivial nonnegative weak solution, then taking $v = \varphi_1$ at the definition of weak solution for (P_λ) one has:

$$\lambda_1 \int (-\Delta u)^{\frac{1}{p}} \varphi_1 dx = \int u^q \varphi_1 dx + \lambda \int g(x) u^r \varphi_1 dx.$$

Since $p, q \geq 1$ and $pq > 1$, $\lambda \geq 1$, by the means of the Jensen's inequality and from (3.3), one has that

$$\lambda_1^{\frac{p+1}{p}} \left(\int u \varphi_1 dx \right)^{\frac{1}{p}} \geq \left(\int u \varphi_1 dx \right)^q + \lambda^{1+pr} \int g(x) u_*^r \varphi_1 dx.$$

Then

$$\lambda^{1+pr} \leq \frac{\lambda_1^{\frac{p+1}{p}} \left(\int u \varphi_1 dx \right)^{\frac{1}{p}} - \left(\int u \varphi_1 dx \right)^q}{\int g(x) u_*^r \varphi_1 dx} \leq \frac{C}{\int g(x) u_*^r \varphi_1 dx}.$$

From the last inequality, one gets the desired estimate for λ .

4. The proof of Theorem 1.4

When the identity holds at (H2), we are not able to prove that I_λ satisfies the $(PS)_\beta^*$ -condition for any $\beta \in \mathbb{R}$, therefore we cannot apply the Dual Fountain Theorem to get the existence of infinitely many weak solutions for (P_λ) with negative energy. Also, since I_λ does not satisfy the $(PS)_\beta$ -condition for large values of β , we cannot apply the Fountain Theorem to obtain the existence of a sequence of weak solutions of (P_λ) on which the energy goes to infinity. In this way, our results concerning the existence of infinitely many solutions for (P_λ) with critical exponential growth are obtained by means of the genus theory, adapting some arguments contained in [17].

4.1. Item (a)

From definition of I_λ one has

$$\begin{aligned} I_\lambda(u) &= \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{1}{q+1} |u|_{q+1}^{q+1} - \frac{\lambda}{r+1} \int g(x) |u|^{r+1} dx \\ &\geq \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{1}{q+1} \frac{1}{S^{\frac{p(q+1)}{p+1}}} \|u\|^{q+1} - \frac{\lambda}{r+1} |g|_{*,r+1} \|u\|^{r+1}. \end{aligned}$$

If one defines

$$h_\lambda(t) = \frac{p}{p+1} t^{\frac{p+1}{p}} - \frac{1}{q+1} \frac{1}{S^{\frac{p(q+1)}{p+1}}} t^{q+1} - \frac{\lambda}{r+1} |g|_{*,r+1} t^{r+1}, \quad t \geq 0$$

then

$$I_\lambda(u) \geq h_\lambda(\|u\|), \quad \forall u \in E. \quad (4.1)$$

From (H1) and from the analysis of h'_λ one sees that $h'_\lambda \leq 0$ for every $t > 0$ if $\lambda \geq \lambda^*$ and h'_λ has exactly two zeros in $(0, +\infty)$ if $0 < \lambda < \lambda^*$, the smaller one is a local minimum and the bigger one is a local maximum for h_λ .

On the other hand, the maximum value of h_λ is positive if and only if

$$0 < \lambda < \left(\frac{pN}{p(N-2)-2} \right)^{\frac{p(q-r)}{pq-1}} \frac{r+1}{q+1} \lambda^*.$$

Combining this with the analysis of h'_λ made above one sees that necessarily

$$\left(\frac{pN}{p(N-2)-2} \right)^{\frac{p(q-r)}{pq-1}} \frac{r+1}{q+1} < 1.$$

Let

$$\lambda_* := \min \left\{ \alpha(p, r, N) \lambda^*, \left(\frac{pN}{p(N-2)-2} \right)^{\frac{p(q-r)}{pq-1}} \frac{r+1}{q+1} \lambda^* \right\}.$$

From Lemma 2.20 and from the above discussion it has been proved that:

Lemma 4.1. *If $\lambda \in (0, \lambda_*)$ then I_λ satisfies $(PS)_\beta$ for every $c \leq 0$ and the maximum value of h_λ is positive.*

For every $\lambda \in (0, \lambda_*)$ the behavior of h_λ is the same as that given at Fig. 4.1 in [17].

Let $0 < R_{0,\lambda} < R_{1,\lambda}$ the two positive zeros of h_λ and consider $\tau_\lambda : [0, +\infty) \rightarrow [0, 1]$ nonincreasing and C^∞ such that

$$\tau_\lambda \equiv 1 \text{ on } [0, R_{0,\lambda}) \text{ and } \tau_\lambda \equiv 0 \text{ on } [R_{1,\lambda}, +\infty).$$

Then define $\varphi_\lambda(u) = \tau_\lambda(\|u\|)$ and

$$J_\lambda(u) = \frac{p}{p+1} \|u\|^{\frac{p+1}{p}} - \frac{1}{q+1} |u|_{q+1}^{q+1} \varphi_\lambda(u) - \frac{\lambda}{r+1} \int g(x) |u|^{r+1} dx.$$

As in (4.1) one has $J_\lambda(u) \geq \bar{h}_\lambda(\|u\|)$ for every $u \in E$, where

$$h_\lambda(t) := \frac{p}{p+1} t^{\frac{p+1}{p}} - \frac{1}{q+1} \frac{1}{S^{\frac{p(q+1)}{p+1}}} t^{q+1} \tau_\lambda(t) - \frac{\lambda}{r+1} |g|_{*,r+1} t^{r+1}, \quad t \geq 0$$

has the same behavior as $\bar{h}$ in Fig. 4.2 in [17].

Similar to Lemma 4.2 in [17], by means of Lemma 4.1 one easily proves:

Lemma 4.2. *If $\lambda \in (0, \lambda_*)$ then:*

- $J_\lambda \in C^1(E, \mathbb{R})$.

- If $J_\lambda(u) \leq 0$ then $\|u\| < R_{0,\lambda}$ and $J_\lambda(v) = I_\lambda(v)$ for every v in a small neighborhood of u .
- J_λ verifies $(PS)_\beta$ for every $\beta < 0$.

Let Σ be the class of all closed and symmetric with relation to the origin subsets of $E \setminus \{0\}$ and for each $A \in \Sigma$ denote by $\gamma(A)$ the genus of A .

Lemma 4.3. *Let $\lambda > 0$. Given $n \in \mathbb{N}$, there exists $\epsilon = \epsilon(n) > 0$ such that*

$$\gamma(\{u \in E : J_\lambda(u) \leq -\epsilon\}) \geq n.$$

Proof. See the proof of Lemma 4.3 in [17]. Here the assumption (H3) plays an important rule. $\square$

Lemma 4.4. *Suppose $\lambda \in (0, \lambda_*)$. Let $\Sigma_k := \{A \in \Sigma : \gamma(A) \geq k\}$, $K_c := \{u \in E : J'_\lambda(u) = 0, J_\lambda(u) = c\}$ and*

$$c_k := \inf_{A \in \Sigma_k} \sup_{u \in A} J_\lambda(u).$$

Then (c_k) is a nondecreasing sequence of negative real numbers and $c_k \rightarrow 0$ as $k \rightarrow +\infty$. Furthermore, if $c = c_k = c_{k+1} = \dots = c_{k+r}$ then $\gamma(K_c) \geq r + 1$.

Proof. First of all, as in Lemma 4.4 in [17], one obtains that (c_k) is a nondecreasing sequence of negative real numbers. To see that $c_k \rightarrow 0$ as $k \rightarrow +\infty$, let $A \in \Sigma_k$.

Firstly, if $\gamma(A) = k + s$ for some $s \geq 0$ then from item (8) in Proposition 4.1 in [17] one has that $A \cap Z_{k+r+2} \neq \emptyset$. Since $J_\lambda(u) \geq I_\lambda(u)$ for every $u \in E$, one relates such fact with property (B3) in Theorem 3.18 in [23], which is satisfied by I_λ and was proved at Sect. 3.2 in this paper.

Secondly, if $\gamma(A) = +\infty$ then by item (7) in Proposition 4.1 in [17] A is not compact. Here one has two cases. The first, if A is unbounded then one has $\sup_{u \in A} J_\lambda(u) = +\infty$. The second case, if A is bounded and noncompact then $C \cap Z_n \neq \emptyset$ for every $n \in \mathbb{N}$ and once more one relate such fact with property (B3) in Theorem 3.18 in [23].

From both cases one concludes that $c_k \rightarrow 0$ as $k \rightarrow +\infty$.

The rest of the claims are proved as in Lemma 4.4 in [17], by means of a classical deformation lemma as that stated in Lemma 3.1 in [23]. $\square$

Therefore, if $\lambda \in (0, \lambda_*)$ then from Lemmas 4.2 and 4.4, (c_k) is a sequence of negative critical values of I_λ and $c_k \rightarrow 0$ as $k \rightarrow +\infty$.

4.2. Items (b)–(d)

To obtain the existence of multiple nontrivial nonnegative weak solutions for (P_λ) in the critical case, our procedure is similar to that employed in the subcritical problem. First we obtain a local minimum for I_λ minimizing I_λ over $\mathcal{N}_\lambda$. In order to get the existence of a second nontrivial nonnegative weak solution we need to compare $c_{\lambda,1}$ with a Mountain Pass level of I_λ . With the purpose of proving

that such Mountain Pass level is below $c_{\lambda,0} + \frac{2}{N} S^{\frac{pN}{2(p+1)}}$ we need to employ the restrictions given by (H5). To be able to proceed as in the proof of Theorem 1.5 in [14], we employ $g(x) > \alpha > 0$ for all $x \in \Omega$. The last hypothesis is employed to construct a regular solution for (P_λ) , with λ small, by means of the method of subsolution and supersolution (see Remark 4.1 in [13]).

The proof of these items are based on the results obtained at Sects. 2.1 and 2.2 and they are similar to the proofs of Theorems 1.2, 1.3 and 1.5 in [14].

4.3. Item (e)

The proof of this item is based on the Jensen's inequality and is similar to the proof of item (V) of Theorem 1.3.

5. Final remarks

Under hypotheses (H1), the strict inequality at (H2), (H3) or (H1), the identity at (H2), (H3), (H5), one can also obtain a critical value of I_λ , for small values of $\lambda > 0$, by means of the Mountain Pass Theorem. For instance see Theorem 1.8 in [12] and Theorem 1.6 in [14].

For small values of $\lambda > 0$, the sequence of critical points of I_λ , given by the Fountain Theorem at item (I) of Theorem 1.3, lies in $\mathcal{N}_\lambda^-$. It is a consequence of Lemma 2.5.

For $\lambda > 0$ sufficiently small, the sequence of critical points of I_λ , given by the Dual Fountain Theorem at item (II) of Theorem 1.3, lies in $\mathcal{N}_\lambda^+$. Again it is a consequence of Lemma 2.5, but now combined with Remark 2.6.

By the same reasons presented at the last paragraph, the sequence of critical points of I_λ , given by item (a) in Theorem 1.4 by means of genus theory, lies in $\mathcal{N}_\lambda^+$ provided $\lambda > 0$ is sufficiently small.

It was proved at Theorem 1.10 in [12] that (S_λ) with $\lambda = 0$ has a pair (u, v) of classical solution with $u, v > 0$ in Ω provided the pair of positive powers (p, q) lies between the critical hyperbola and the hyperbola $pq > 1$, that is, provided

$$1 > \frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{N}$$

is satisfied. If we proceed as there, we are able to obtain the same result, by taking the limit on $(u_{\lambda,1})$ as $\lambda \rightarrow 0$, with $u_{\lambda,1}$ given at (III) of Theorem 1.3.

It is an open question to obtain a nonexistence result of nontrivial weak solutions for (P_λ) for large values of λ , eliminating the restriction $p, q \geq 1$ at item (V) in Theorem 1.3 and at item (e) of Theorem 1.4.

It is an open problem to prove the existence of two nontrivial nonnegative weak solutions for (P_λ) over all the critical hyperbola and with $\lambda > 0$ sufficiently small, that is, to eliminate the restriction (H5) at item (d) of Theorem 1.4.

Acknowledgments

The author would like to thank Prof. D.G. de Figueiredo, Prof. Jorge Mujica from IMECC-UNICAMP and Prof. Pedro Ubilla from USACH for their helpful suggestions and comments.

References

- [1] Albiac, F., Kalton, N.J.: Topics in Banach Space Theory. Springer, Heidelberg (2006)
- [2] Ambrosetti, A., Rabinowitz, P.H.: Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14**, 349–381 (1973)
- [3] Banach, S.: Théorie des opérations linéaires. Reprint of the 1932 original. Édition Jacques Gabay, Sceaux (1993)
- [4] Bartsch, T.: Infinitely many solutions of a symmetric Dirichlet problem. *Nonlinear Anal. TMA* **20**, 1205–1216 (1993)
- [5] Bartsch, T., de Figueiredo, D.G.: Infinitely many solutions of nonlinear elliptic systems. Topics in nonlinear analysis. In: *Progr. Nonlinear Differential Equations Appl.*, vol. 35, pp. 51–67. Birkhäuser, Basel (1999)
- [6] Bartsch, T., Willem, M.: On an elliptic equation with concave and convex nonlinearities. *Proc. Am. Math. Soc.* **123**, 3555–3561 (1995)
- [7] Bernis, F., Garcia Azorero, J., Peral Alonso, I.: Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order. *Adv. Differ. Equ.* **1**(2), 219–240 (1996)
- [8] de Figueiredo, D.G., Felmer, P.: On superquadratic elliptic systems. *Trans. Am. Math. Soc.* **343**(1), 99–116 (1994)
- [9] de Figueiredo, D.G., Ruf, B.: Elliptic systems with nonlinearities of arbitrary growth. *Mediterr. J. Math.* **1**, 417–431 (2004)
- [10] do Ó, J.M., Ubilla, P.: A multiplicity result for a class of superquadratic Hamiltonian systems. *Electron. J. Differ. Equ.* **15**, 1–14 (2003)
- [11] Enflo, P.: A counterexample to the approximation problem in Banach spaces. *Acta Math.* **130**, 309–317 (1973)
- [12] dos Santos, E.M.: Multiplicity of solutions for a fourth-order quasilinear nonhomogeneous equation. *J. Math. Anal. Appl.* **342**, 277–297 (2008)
- [13] dos Santos, E.M.: On the existence of positive solutions for a nonhomogeneous elliptic system. *Portugaliae Mathematica* (2009, to appear)

- [14] dos Santos, E.M.: Positive solutions for a fourth-order quasilinear equation with critical Sobolev exponent. *Commun. Contemp. Math.* (2009, to appear)
- [15] Ekeland, I.: On the variational principle. *J. Math. Anal. Appl.* **47**, 324–353 (1974)
- [16] Fučík, S., John, O., Nečas, J.: On the existence of Schauder bases in Sobolev spaces. *Comment. Math. Univ. Carolinae* **13**, 163–175 (1972)
- [17] Garcia Azorero, J., Peral Alonso, I.: Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term. *Trans. Am. Math. Soc.* **323**(2), 877–895 (1991)
- [18] Gilbarg, D., Trudinger, N.S.: *Elliptic partial differential equations of second order*. Springer, Heidelberg (1977)
- [19] James, R.C.: Bases and reflexivity of Banach spaces. *Ann. Math.* **52**(2), 518–527 (1950)
- [20] Lions, P.L.: The Concentration-Compactness Principle in the Calculus of Variations (The limit case, Part I.). *Rev. Mat. Iberoamericana* **1**(1), 145–201 (1985)
- [21] Simon, J.: Régularité de la solution d’une equation non lineaire dans $\mathbb{R}^N$. *Lecture Notes in Math.*, vol. 665. Springer, Heidelberg (1978)
- [22] Tarantello, G.: On nonhomogeneous elliptic equations involving critical Sobolev exponent. *Ann. Inst. Henri Poincaré* **9**(3), 281–304 (1992)
- [23] Willem, M.: *Minimax Theorems*. Birkhäuser, Boston (1996)

E. M. dos Santos
IMECC-UNICAMP,
Cidade Universitária,
Rua Sérgio Buarque de Holanda 651,
Barão Geraldo, Caixa Postal 6065,
Campinas, SP 13083-859,
Brazil,
e-mail: ederson@ime.unicamp.br

Received: 18 March 2008.

Accepted: 23 September 2008.