Asymptotic compactness of global trajectories generated by the Navier–Stokes–Poisson equations of a compressible fluid

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Abstract. In this paper, we consider the global behavior of weak solutions of Navier–Stokes–Poisson equations in time in a bounded domain–arbitrary forces. After proving the existence of bounded absorbing sets, we also obtain the conclusion on asymptotic compactness of global trajectories generated by the Navier–Stokes–Poisson equations of a compressible fluid.

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1. Introduction

We consider the following Navier–Stokes–Poisson equation with the external force:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \quad (1.1)$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla \operatorname{div} \mathbf{u} + \nabla p(\varrho) + \varrho \nabla \Phi = \varrho \mathbf{f}, \quad (1.2)$$

$$\Delta \Phi = 4\pi g \left(\varrho - \frac{1}{|\Omega|} \int_{\Omega} \varrho dx \right), \quad (1.3)$$

in $\Omega \times R^+$, and with the boundary condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega \times R^+, \quad (1.4)$$

$$\frac{\partial \Phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times R^+, \quad (1.5)$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with $(0 < \theta < 1)$, ν the outer normal vector, the time $t \in R^+ = (0, \infty)$, $\varrho = \varrho(t, x)$ the density, $\mathbf{u} = \mathbf{u}(t, x) = (u^1(x, t), u^2(x, t), u^3(x, t))$ the velocity, $\Phi = \Phi(t, x)$ the Newtonian gravitational

potential, $\mathbf{f} = (f^1(x, t), f^2(x, t), f^3(x, t))$ the external force, $\mu > 0$ and λ the viscosity constants satisfying $\lambda + \mu \geq 0$, $a = e^S$ the constant determined by the entropy S , and $g > 0$ the gravitational constant. For simplicity, we consider only the isentropic case when

$$p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma > \frac{3}{2}, \quad (1.6)$$

where $\gamma > 1$ the adiabatic constant. Physically, this system describes the motion of compressible viscous isentropic gas flow under the self-gravitational force with the external force. Such a fluid may be formulated as the Euler–Poisson equation, and several mathematical studies have been done for this Euler–Poisson equation [1, 15, 20, 21], where the viscosity and the external force are neglected, the equation is considered in the whole space $\mathbb{R}^3$, and the solution admits to have the compact support. Then, the contact angle between fluid and vacuum is not zero in equilibrium state, and establishing the existence of the solution in an appropriate function space including the equilibrium state is not easy because of this, even locally in time with spherically symmetry in space.

Initial boundary value problems and the initial value problem for the compressible Navier–Stokes–Poisson equations (of heat-conducting flow) have also been studied by a great many authors. In a series of papers, Makino [22, 23, 25] and his co-workers studied such a problem of vacuum for spherically symmetric Navier–Stokes–Poisson equation (of a specified pair of the viscosity constants, and without the external force) with solid core. In Ducomet et al. [3, 4] proved the existence of global weak solutions of compressible barotropic self-gravitating fluids for the whole and exterior domain case. Recently, we [16] further studied the existence of complete bounded trajectories and attractors for compressible barotropic self-gravitating fluid occupying a bounded domain, where the finite energy solutions were based on the concept of [4] and our work extended earlier results by Feireisl [10]. In Ducomet and Feireisl [5] again researched the dynamics of gaseous stars occupying a bounded domain where the temperature effects are considered, and their work developed a mathematical theory based on the concept of variational (weak) solutions. In Kobayashi and Suzuki [17] studied the Navier–Stokes–Poisson equation (1.1–1.6) without the external force, on the fixed bounded domain Ω without radial symmetry or solid core, and show the existence of the weak solution in a reasonable function space including the equilibrium state, emphasizing that the vacuum region $\{x \in \bar{\Omega} | \varrho(x, t) = 0\}$ can exist inside this domain Ω although the equilibrium state is everywhere positive in this problem. Recently, Wang with Tan [26] research the propagation of density-oscillations in solutions to the compressible Navier–Stokes–Poisson system. And Zhang with Tan [27] proved the existence of solutions to the Navier–Stokes–Poisson equations of a two-dimensional compressible flow in Orlicz–Sobolev function spaces.

It also should be noted that there are other forms of compressible Navier–Stokes–Poisson system which are slightly different from the system of (1.1)–(1.6) and can be used to simulate, for instance in semiconductor devices, the transport

of charged particles under the electric field of electrostatic potential force rather than the Newtonian gravitational potential force. The reader can refer to for instance [6, 18], and references therein for details. In Donatelli [6], gave the existence of local and global weak solutions for the bounded domain case. And the authors [18] studied the optimal decay rate of the density and momentum of the compressible Navier–Stokes–Poisson system in $\mathbb{R}^3$.

In this paper, we shall deal with globally defined finite energy weak solutions of the problem (1.1)–(1.6) as in [17], i.e.,

Definition 1.1. We shall say that $(\varrho, \mathbf{u}, \Phi)$ is finite energy weak solution of the problem (1.1)–(1.6) on $R^+ \times \Omega$, where R^+ is an open interval, if

- $\varrho, \mathbf{u}$ belong to the classes

$$\begin{aligned} \varrho &\in L_{loc}^\infty(R^+; L^\gamma(\Omega)) \cap L_{loc}^{s(\gamma)}(R^+; L^{s(\gamma)}(\Omega)), \quad u^i \in L_{loc}^2(R^+; W_0^{1,2}(\Omega)), \\ s(\gamma) &= \frac{5\gamma - 3}{3}, \quad i = 1, 2, 3; \end{aligned} \quad (1.7)$$

- the energy E is locally integrable on R^+ and the energy inequality

$$\frac{d}{dt} E[\varrho, \mathbf{u}, \Phi](t) + \int_{\Omega} \mu |\nabla \mathbf{u}(t)|^2 dx + \int_{\Omega} (\lambda + \mu) |\operatorname{div} \mathbf{u}(t)|^2 dx \leq \int_{\Omega} \varrho(t) \mathbf{f}(t) \cdot \mathbf{u}(t) dx \quad (1.8)$$

is satisfied in $\mathcal{D}'(R^+)$, where the energy E is given by the formula

$$E[\varrho, \mathbf{u}, \Phi](t) = \int_{\Omega} \frac{1}{2} \varrho(t) |\mathbf{u}(t)|^2 dx + \frac{a}{\gamma - 1} \int_{\Omega} \varrho^\gamma(t) dx - \frac{1}{8\pi g} \int_{\Omega} |\nabla \Phi|^2 dx; \quad (1.9)$$

- the Eqs. (1.1), (1.2) are satisfied in $D'(R^+ \times \Omega)$; moreover, (1.1) holds in $D'(R^+ \times \mathbb{R}^3)$ provided $\varrho, \mathbf{u}$ were prolonged to be zero on $\mathbb{R}^3/\Omega$;
- for any $t \in R^+$, the Eq. (1.3) holds a.e for $x \in \Omega$ and (1.5) holds in the sense of traces;
- the Eq. (1.1) is satisfied in the sense of renormalized solutions, it means that

$$b(\varrho)_t + \operatorname{div}(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div} \mathbf{u} = 0 \quad (1.10)$$

holds in $D'(R^+ \times \Omega)$ for any b satisfying

$$b \in C^0([0, \infty]) \cap C^1((0, \infty)), \quad |b'(t)| \leq C t^{-\lambda_0}, \quad t \in (0, 1), \quad \lambda_0 < 1, \quad (1.11)$$

with growth conditions at infinity

$$|b'(t)| \leq C t^{\lambda_1}, \quad t \geq 1, \quad \text{where } C > 0, \quad -1 < \lambda_1 \leq \frac{s(\gamma)}{2} - 1. \quad (1.12)$$

The existence of above globally defined finite energy weak solutions of Navier–Stokes–Poisson equation (1.1)–(1.6) with proper initial condition, without the external force $\mathbf{f}$, was proved in [17] for $\Omega \subset \mathbb{R}^3$, where the b only satisfies

$$b \in C^1(R), \quad b'(z) = 0, \quad \text{if } |z| \text{ is large} \quad (1.13)$$

and especially the Φ , satisfying the bounder condition (1.5) and $\int_{\Omega} \Phi dx = 0$ for any $t \in R^+$, can be uniquely determined by ϱ . Referring to the proof of

Theorem 7.7 in [24], we can easily see there also exists globally finite energy weak solution defined as in Definition 1.1 for Navier–Stokes–Poisson equation (1.1)–(1.6).

From the definition of renormalized solutions, we can infer the mass m to be conserved (see [12, Proposition 2.1] for details), i.e.

$$m = \int_{\Omega} \varrho(x, t) dx \text{ is independent of } t \in R^+. \quad (1.14)$$

Now we state our main results of this paper, which are regarded as the generalization of Feireisl and Petzeltová [8–10] concerning the Navier–Stokes equation without the Poisson term.

Theorem 1.2. *Let*

$$\gamma > \frac{5}{3} \quad (1.15)$$

and let $\mathbf{f} = (f^1(t, x), f^2(t, x), f^3(t, x))$ be a sequence of bounded measurable function satisfying

$$\max_{i=1,2,3} \left\{ \operatorname{ess\,sup}_{t \in R^+, x \in \Omega} |f^i(t, x)| \right\} \leq K. \quad (1.16)$$

Then there exists a constant E_{∞} , depending solely on γ , K and on the total mass m , having the following property:

Given E_0 , there exists a time $T = T(E_0)$ such that

$$E(t) := E[\varrho, \mathbf{u}, \Phi](t) \leq E_{\infty} \quad \text{for a.e. } t > T, \quad (1.17)$$

provided

$$\operatorname{ess\,lim\,sup}_{t \rightarrow 0^+} E(t) \leq E_0 \quad (1.18)$$

and $\varrho, \mathbf{u}, \Phi$ is a finite energy weak solution of the problem (1.1)–(1.6).

The above Theorem 1.2 will play a crucial role in the proof of the following Theorem 1.3.

Theorem 1.3. *Let*

$$\gamma > \frac{5}{3}$$

and let $\mathbf{f}_n = (f_n^1(t, x), f_n^2(t, x), f_n^3(t, x))$ be a sequence of bounded measurable functions satisfying

$$\max_{i=1,2,3} \left\{ \operatorname{ess\,sup}_{t \in R^+, x \in \Omega} |f_n^i(t, x)| \right\} \leq K, \quad (1.19)$$

for all $n = 1, 2, \dots$. Let $\varrho_n, \mathbf{u}_n, \Phi_n$ be a sequence of finite energy weak solutions of the problem (1.1)–(1.6) on the time interval R^+ and such that

$$\int_{\Omega} \varrho_n(t, x) dx = m, \quad (1.20)$$

$$\operatorname{ess\,lim\,sup}_{t \rightarrow 0^+} E(t) \leq E_0, \quad (1.21)$$

independent of $n = 1, 2, \dots$

Then any sequence of times $t_n \rightarrow \infty$ contains a subsequence (not relabeled) such that

$$\varrho_n(t_n + t) \rightarrow \bar{\varrho}(t) \text{ strongly in } L^\alpha(\Omega) \text{ and weakly in } L^\gamma(\Omega) \text{ for any } 1 \leq \alpha < \gamma, \quad (1.22)$$

$$\nabla \Phi_n(t_n + t) \rightarrow \nabla \bar{\Phi}(t) \text{ strongly in } W^{1,\alpha}(\Omega), \quad 1 \leq \alpha < \gamma, \quad (1.23)$$

$$(\varrho_n \mathbf{u}_n)(t_n + t) \rightarrow (\bar{\varrho} \bar{\mathbf{u}})(t) \text{ weakly in } (L^1(\Omega))^3 \quad (1.24)$$

for any $t \in R$. Moreover,

$$\int_J \int_{\Omega} |\varrho_n(t_n + t, x) - \bar{\varrho}(t, x)|^\gamma dx dt \rightarrow 0, \quad (1.25)$$

$$\nabla \Phi_n(t_n + t, x) \rightarrow \nabla \bar{\Phi}(t, x) \text{ strongly in } W^{1,\gamma}(\Omega \times J), \quad (1.26)$$

$$\int_J \int_{\Omega} |\varrho_n(t_n + t, x) \mathbf{u}_n(t_n + t, x) - \bar{\varrho}(t, x) \bar{\mathbf{u}}(t, x)| dx dt \rightarrow 0 \quad (1.27)$$

for any bounded interval $J \subset R$. Here $\bar{\varrho}, \bar{\mathbf{u}}, \bar{\Phi}$ is a globally defined (i.e. on $I=R$) finite energy weak solution of the problem (1.1)–(1.6) with driving force $\bar{\mathbf{f}}$,

$$\bar{\mathbf{f}} = \lim_{n \rightarrow \infty} \mathbf{f}_n(t_n, \cdot) \text{ in the weak star topology of the space } L^\infty(R \times \Omega), \quad (1.28)$$

and such that

$$\operatorname{ess\,sup}_{t \in R} E[\bar{\varrho}, \bar{\mathbf{u}}, \bar{\Phi}](t) \leq E_\infty. \quad (1.29)$$

2. The proof of Theorem 1.2

We start to prove Theorem 1.2 by the following three steps.

2.1. Preliminaries

Lemma 2.1. Let $\varrho, \mathbf{u}$ be a renormalized solution of (1.1), i.e. the identity

$$b(\varrho)_t + \operatorname{div}(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\operatorname{div}\mathbf{u} = 0$$

holds in $\mathcal{D}(R^+ \times \mathbb{R}^3)$ for any functions b satisfying (1.11) and (1.12), then, the total mass is conserved, i.e. $\varrho \in C(R^+, L^\gamma_{\text{weak}}(\Omega))$ and (1.14) holds.

Moreover, denoting

$$S_\varepsilon[v] = \vartheta_\varepsilon * v \quad \text{where } \vartheta_\varepsilon = \vartheta_\varepsilon(x) \text{ is a regularizing sequence,} \quad (2.1)$$

we have

$$\partial_t S_\varepsilon[b(\varrho)] + \operatorname{div}(S_\varepsilon[b(\varrho)]\mathbf{u}) + S_\varepsilon[(b'(\varrho)\varrho - b(\varrho))\operatorname{div}\mathbf{u}] = r_\varepsilon \quad (2.2)$$

on $R^+ \times \mathbb{R}^3$, where

$$r_\varepsilon \rightarrow 0 \quad \text{in } L^2_{\text{loc}}(R^+; L^\alpha(\Omega)) \quad \text{for } \varepsilon \rightarrow 0 \quad \text{with } \alpha = \frac{2\beta}{\beta+2} \quad (2.3)$$

provided

$$b(\varrho) \quad \text{is in } L^\infty_{\text{loc}}(R^+, L^\beta(\Omega)), \quad \beta \geq 2. \quad (2.4)$$

Proof. Seeing [8, Lemma 2.1] or [24, Lemma 6.7-Lemma 6.9]. $\square$

Lemma 2.2. *Let ϱ , $\mathbf{u}$, Φ be a renormalized solution of (1.1)–(1.6). Under the assumptions of Theorem 1.2, we have, being redefined on a set of measure zero if necessary, the energy E has locally bounded variation on R^+ , and*

$$E(t+) = \lim_{s \rightarrow t+} E(s) \leq \lim_{s \rightarrow t-} E(s) = E(t-) \quad \text{for any } t \in R^+. \quad (2.5)$$

Moreover,

$$E(t_2-) \leq (1 + E(t_1+))e^{c(m,K)(t_2-t_1)} - 1 \quad \text{for all } 0 < t_1 < t_2, \quad (2.6)$$

where the $c(m, K)$ depends on γ , g , π , a , K , and m .

Proof. Considering (1.3) and (1.5), we can obtain

$$\left(\int_{\Omega} |\nabla \Phi|^2 dx \right)^{\frac{1}{2}} \leq c_0 \left(\int_{\Omega} \varrho^{\frac{6}{5}} dx \right)^{\frac{5}{6}}$$

by $L^{\frac{6}{5}}$ elliptic estimate (or see [24, Lemma 4.27, 3.17]) and Sobolev's inequality, where c_0 is a constant determined by Ω . Combining (1.14) with the interpolation inequality

$$\left(\int_{\Omega} \varrho^{\frac{6}{5}} dx \right)^{\frac{5}{6}} \leq \left(\int_{\Omega} \varrho dx \right)^{\frac{5\gamma-6}{6(\gamma-1)}} \left(\int_{\Omega} \varrho^\gamma dx \right)^{\frac{1}{6(\gamma-1)}},$$

we obtain

$$\int_{\Omega} |\nabla \Phi|^2 dx \leq c_0^2 m^{\frac{5\gamma-6}{3(\gamma-1)}} \left(\int_{\Omega} \varrho^\gamma dx \right)^{\frac{1}{3(\gamma-1)}}.$$

By Young inequality, we have

$$\frac{1}{8\pi g} \|\nabla \Phi\|_{L^2(\Omega)}^2 \leq c_{0,1} + \frac{a}{2(\gamma-1)} \|\varrho\|_{L^\gamma(\Omega)}^\gamma. \quad (2.7)$$

By virtue of (1.9), we obtain

$$\frac{a}{2(\gamma-1)} \|\varrho\|_{L^\gamma(\Omega)}^\gamma \leq E(t) + c_{0,1}. \quad (2.8)$$

We denote $E_1(t)$ satisfying

$$\frac{d}{dt} E_1(t) + \int_{\Omega} \mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 dx = \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} dx \quad \text{a.e for } t \in R^+. \quad (2.9)$$

Thus $E_2 := (E - E_1) \in L^1_{\text{loc}}(R^+)$. In view of (1.8), we have

$$\frac{d}{dt}E_2(t) \leq 0 \text{ in } \mathcal{D}'(R^+). \quad (2.10)$$

Above analysis tell us that E can be written as a sum of an absolutely function and a nonincreasing function, and consequently, E is a continuous except a countable set of points in which (2.5) holds.

By virtue of (1.16), the right-hand side of (1.8) may be estimated as follows:

$$\begin{aligned} \int_{\Omega} \varrho \mathbf{f} \cdot \mathbf{u} dx &\leq K \left(\int_{\Omega} \varrho dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \varrho |\mathbf{u}|^2 dx \right)^{\frac{1}{2}} \leq \sqrt{2m}K \left(1 + \int_{\Omega} \varrho |\mathbf{u}|^2 dx \right) \\ &\leq \sqrt{2m}K \left(1 + 2E(t) + \frac{1}{4\pi g} \|\Phi\|_{L^2(\Omega)}^2 \right) \\ &\leq \sqrt{2m}K(1 + 4c_{0,1} + 4E(t)) \leq \sqrt{2m}K \\ &\quad \max\{1 + 4c_{0,1}, 4\}(1 + E(t)) := c(m, K)(1 + E(t)), \end{aligned} \quad (2.11)$$

whence (2.6) is a straightforward consequence of the Gronwall lemma. $\square$

2.2. Decay estimates

Similarly to [8], we can get the following assertion, which plays a crucial role in the proof of Theorem 1.2.

Proposition 2.3. *Under the hypotheses of Theorem 1.2, there exists a constant L , depending solely on γ , K and m , enjoying the following property:*

If

$$E((T+1)-) > E(T+) - 1 \quad \text{for a certain } T \in R^+, \quad (2.12)$$

then

$$\sup_{t \in (T, T+1)} E(t+) \leq L. \quad (2.13)$$

The proof will be carried over by a series of auxiliary results:

Lemma 2.4. *Under the hypotheses of Theorem 1.2 and (2.12), there exists a constant $c_1 = c_1(K, m)$, such that*

$$\int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt \leq c_1 \left(1 + \int_T^{T+1} \|\varrho(t)\|_{L^{\frac{3}{2}}(\Omega)} dt \right). \quad (2.14)$$

Proof. Seeing [8, Lemma 3.1]. $\square$

Lemma 2.5. *Under the hypotheses of Theorem 1.2 and (2.12), there exists a constant $c_2 = c_2(K, m)$, such that*

$$E(t+) \leq c_2 \left(1 + \int_T^{T+1} \|\varrho(s)\|_{L^{\gamma}(\Omega)}^{\gamma} ds \right) \text{ for any } t \in [T, T+1]. \quad (2.15)$$

Proof. Integrating (2.6) for the choice $t_2 = T + 1$ with respect to t_1 , one gets

$$E((T + 1)-) \leq \int_T^{T+1} (1 + E(s))e^{\sqrt{2m}K(T+1-s)} ds \leq c_{2,1} \left(1 + \int_T^{T+1} E(s) ds \right).$$

From (2.12), we get

$$E(T+) \leq E((T + 1)-) + 1 \leq c_{2,2} \left(1 + \int_T^{T+1} E(s) ds \right). \quad (2.16)$$

Then, taking $t_1 = T$ in (2.6) and using (2.16), we obtain

$$E(t+) \leq E(t-) \leq c_{2,3} \left(1 + \int_T^{T+1} E(s) ds \right) \text{ for any } t \in [T, T + 1]. \quad (2.17)$$

Now, by virtue of the Hölder inequality and Lemma 2.2, we can get

$$\int_T^{T+1} \int_{\Omega} \varrho |\mathbf{u}|^2 dx dt \leq c_{2,4} \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\frac{3}{2}}(\Omega)} \left(1 + \int_T^{T+1} \|\varrho(t)\|_{L^{\frac{3}{2}}(\Omega)} dt \right). \quad (2.18)$$

Using the interpolation inequality,

$$\|\varrho\|_{L^{\frac{3}{2}}(\Omega)} \leq \|\varrho\|_{L^1(\Omega)}^{1-\eta} \|\varrho\|_{L^\gamma(\Omega)}^\eta \quad \text{for } \eta = \frac{\gamma}{3(\gamma-1)}. \quad (2.19)$$

and (2.8), we deduce

$$\int_T^{T+1} \int_{\Omega} \varrho |\mathbf{u}|^2 dx dt \leq c_{2,5}(m) \sup_{t \in [T, T+1]} (E(t) + 1)^{\frac{1}{3(\gamma-1)}} \left(1 + \int_T^{T+1} \|\varrho(t)\|_{L^\gamma(\Omega)}^{\frac{\gamma}{3(\gamma-1)}} dt \right)$$

and, consequently,

$$\begin{aligned} \sup_{t \in [T, T+1]} E(t+) &\leq c_{2,6}(m) \left[1 + \int_T^{T+1} \|\varrho(t)\|_{L^\gamma(\Omega)}^\gamma dt + \left(\sup_{t \in [T, T+1]} E(t)^{\frac{1}{3(\gamma-1)}} + 1 \right) \right. \\ &\quad \left. \times \left(1 + \int_T^{T+1} \|\varrho(t)\|_{L^\gamma(\Omega)}^{\frac{\gamma}{3(\gamma-1)}} dt \right) \right]. \end{aligned}$$

□

As, by virtue of (1.15), $\frac{1}{3(\gamma-1)} < \frac{1}{2}$, the desired conclusion follows.

Let us introduce the operators $\mathcal{B}_i$ considered by Bogovskii [2]:

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, and $p, r \in (1, \infty)$ given numbers. Then there exists a bounded linear operator $\mathcal{B}$,*

$$\begin{aligned} \mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3] : \left\{ f \in L^p(\Omega) \left| \int_{\Omega} f dx = 0 \right. \right\} &\mapsto [W_0^{1,p}(\Omega)]^3, \\ \|\mathcal{B}\{f\}\|_{W_0^{1,p}(\Omega)} &\leq c_3(p, \Omega) \|f\|_{L^p(\Omega)} \end{aligned} \quad (2.20)$$

such that $\mathbf{v} = \mathcal{B}\{f\}$ solves the problem

$$\operatorname{div} \mathbf{v} = f \quad \text{a.e. in } \Omega, \quad \mathbf{v}|_{\partial\Omega} = 0. \quad (2.21)$$

Moreover, if $f \in L^P(\Omega)$ can be written in the form

$$f = \operatorname{div} \mathbf{g} \quad \text{for a certain } \mathbf{g} \in [L^\gamma(\Omega)]^3, \mathbf{g} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

then

$$\|\mathcal{B}\{f\}\|_{L^\gamma(\Omega)} \leq c_4(p, \gamma, \Omega) \|\mathbf{g}\|_{L^\gamma(\Omega)}. \quad (2.22)$$

The proof follows the ideas of Galdi [2, 14, Chapter III, Theorem 3.3] and is given in [7, Proposition 2.1].

Now, consider the test functions of the form

$$\varphi_i(t, x) = \psi(t) \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\}, \quad i = 1, 2, 3,$$

where $0 \leq \psi \leq 1$, $\psi \in \mathcal{D}(T, T+1)$, S_ϵ are the smoothing operators considered in Lemma 2.1, and

$$b \in C^1(R), \quad b(z) = z^\theta \text{ for } z \geq 1 \text{ with } \theta = \min \left\{ \frac{1}{4}, \frac{2\gamma-3}{3\gamma} \right\}. \quad (2.23)$$

Taking φ_i as test functions for (1.2) and making use of Lemma 2.1, 2.6, we obtain, after a lengthy but straightforward computation, the following formula:

$$\begin{aligned} & a \int_T^{T+1} \int_\Omega \psi \varrho^\gamma S_\epsilon[b(\varrho)] dx dt \\ &= \int_T^{T+1} \psi \left(\int_\Omega a \varrho^\gamma dx \right) \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx dt + (\lambda + \mu) \int_T^{T+1} \int_\Omega \psi S_\epsilon[b(\varrho)] \operatorname{div} \mathbf{u} dx dt \\ &\quad - \int_T^{T+1} \int_\Omega \psi_t \varrho u^i \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\} dx dt \\ &\quad + \mu \int_T^{T+1} \int_\Omega \psi \partial_{x_j} u^i \partial_{x_j} \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\} dx dt \\ &\quad - \int_T^{T+1} \int_\Omega \psi \varrho u^i u^j \partial_{x_j} \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\} dx dt \\ &\quad + \int_T^{T+1} \int_\Omega \psi \varrho u^i \mathcal{B}_i \left\{ S_\epsilon[(b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u}] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[(b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u}] dx \right\} dx dt \\ &\quad + \int_T^{T+1} \int_\Omega \psi \varrho u^i \mathcal{B}_i \left\{ r_\epsilon - \frac{1}{|\Omega|} \int_\Omega r_\epsilon dx \right\} dx dt - \int_T^{T+1} \int_\Omega \psi \varrho u^i \mathcal{B}_i \{ \operatorname{div}(S_\epsilon[b(\varrho)]) \mathbf{u} \} dx dt \\ &\quad - \int_T^{T+1} \int_\Omega \psi \varrho f_i \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\} dx dt \\ &\quad + \int_T^{T+1} \int_\Omega \psi \varrho \partial_i \Phi \mathcal{B}_i \left\{ S_\epsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\epsilon[b(\varrho)] dx \right\} dx dt. \end{aligned} \quad (2.24)$$

At this stage, the ten integrals on the right-hand side of the above relation will be estimated in terms of the norms of ϱ , $\mathbf{u}$ and Φ . The main tools used is the Hölder inequality and the Sobolev embedding theorems together with the

estimates for $\mathcal{B}$ presented in Lemma 2.6. Skipping the technicalities we refer to [7, Section 3], [8, Section 3] and [17, Section 4] for details. One has

$$\left| \int_T^{T+1} \psi \left(\int_{\Omega} a \varrho^\gamma dx \right) \left(\frac{1}{|\Omega|} \int_{\Omega} S_\varepsilon[b(\varrho)] dx \right) dt \right| \leq c_5(m) \int_T^{T+1} \int_{\Omega} \varrho^\gamma dx dt, \quad (2.25)$$

$$\left| \int_T^{T+1} \int_{\Omega} \psi S_\varepsilon[b(\varrho)] \operatorname{div} \mathbf{u} dx dt \right| \leq c_6(m) \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)} dt, \quad (2.26)$$

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi_t \varrho u^i \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_{\Omega} S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \leq c_7(m) \\ & \int_T^{T+1} |\psi_t| \|\sqrt{\varrho} \mathbf{u}\|_{L^2(\Omega)} dt, \end{aligned} \quad (2.27)$$

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi \partial_{x_j} u^i \partial_{x_j} \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_{\Omega} S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \leq c_8(m) \\ & \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)} dt, \end{aligned} \quad (2.28)$$

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi \varrho u^i u^j \partial_{x_j} \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_{\Omega} S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \\ & \leq c_9(m) \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^\gamma(\Omega)} \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt, \end{aligned} \quad (2.29)$$

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi \varrho u^i \mathcal{B}_i \left\{ S_\varepsilon[(b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u}] - \frac{1}{|\Omega|} \int_{\Omega} S_\varepsilon \right. \right. \\ & \quad \left. \left. [(b(\varrho) - b'(\varrho)\varrho) \operatorname{div} \mathbf{u}] dx \right\} dx dt \right| \\ & \leq c_{10}(m) \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^\gamma(\Omega)} \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt, \end{aligned} \quad (2.30)$$

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi \varrho u^i \mathcal{B}_i \left\{ r_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} r_\varepsilon \right\} dx dt \right| \\ & \leq c_{11} \int_T^{T+1} \|\varrho\|_{L^\gamma(\Omega)} \|\mathbf{u}\|_{W^{1,2}(\Omega)} \|r_\varepsilon\|_{L^{q_2}(\Omega)} dt. \end{aligned}$$

Consequently, by virtue of (2.3),

$$\lim_{\varepsilon \rightarrow 0^+} \left| \int_T^{T+1} \int_{\Omega} \psi \varrho u^i \mathcal{B}_i \left\{ r_\varepsilon - \frac{1}{|\Omega|} \int_{\Omega} r_\varepsilon \right\} dx dt \right| = 0, \quad (2.31)$$

since, in view of (2.23),

$$b(\varrho) \text{ is in } L_{loc}^\infty(R^+, L^\beta(\Omega)) \text{ with } \beta = \max \left\{ 2, \frac{2\gamma}{2\gamma-3} \right\}, \quad q_2 = \frac{2\beta}{\beta+2}.$$

$$\left| \int_T^{T+1} \int_\Omega \psi \varrho u^i \mathcal{B}_i \{ \operatorname{div} (S_\varepsilon[b(\varrho)] \mathbf{u}) \} dx dt \right| \leq c_{12}(m) \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^\gamma(\Omega)} \\ \times \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt, \quad (2.32)$$

and

$$\left| \int_T^{T+1} \int_\Omega \psi \varrho f_i \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \leq c_{13}(K, m). \quad (2.33)$$

Next, we estimate the last integral term on the right-hand side of (2.24). Using Hölder inequality, Sobolev embedding theorems, Lemma 2.6 and elliptic estimate of (1.3), we obtain, if $\frac{5}{3} < \gamma \leq 3$,

$$\left| \int_T^{T+1} \int_\Omega \psi \varrho \partial_i \Phi \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \\ \leq \int_T^{T+1} \|\varrho\|_{L^\gamma(\Omega)} \|\partial_i \Phi\|_{L^{\frac{\gamma}{\gamma-1}}(\Omega)} \left\| \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon[b(\varrho)] dx \right\} \right\|_{L^\infty(\Omega)} dt \\ \leq c_{14,1}(m) \int_T^{T+1} \|\varrho\|_{L^\gamma(\Omega)} \|\nabla \Phi\|_{W^{1, \frac{3\gamma}{4\gamma-3}}(\Omega)} dt \\ \leq c_{14,2}(m) \int_T^{T+1} \|\varrho\|_{L^\gamma(\Omega)} \|\varrho\|_{L^{\frac{3\gamma}{4\gamma-3}}(\Omega)} dt. \quad (2.34)$$

Then, for $\frac{5}{3} < \gamma < 3$, using the interpolation inequality, we have

$$\|\varrho\|_{L^{\frac{3\gamma}{4\gamma-3}}(\Omega)} \leq \|\varrho\|_{L^1(\Omega)}^\theta \|\varrho\|_{L^\gamma(\Omega)}^{1-\theta}, \quad \theta = \frac{4\gamma-6}{3(\gamma-1)}.$$

Consequently, in view of (2.34), we conclude

$$\left| \int_T^{T+1} \int_\Omega \psi \varrho \partial_i \Phi \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \\ \leq c_{14,3}(m) \sup_{t \in [T, T+1]} \|\varrho\|_{L^\gamma(\Omega)}^{\frac{2\gamma}{3(\gamma-1)}} \leq c_{14}(m) \sup_{t \in [T, T+1]} \|\varrho\|_{L^\gamma(\Omega)}^\gamma. \quad (2.35)$$

And, for $\gamma = 3$, we have

$$\left| \int_T^{T+1} \int_\Omega \psi \varrho \partial_i \Phi \mathcal{B}_i \left\{ S_\varepsilon[b(\varrho)] - \frac{1}{|\Omega|} \int_\Omega S_\varepsilon[b(\varrho)] dx \right\} dx dt \right| \\ \leq c_{15}(m) \sup_{t \in [T, T+1]} \|\varrho\|_{L^\gamma(\Omega)}. \quad (2.36)$$

If $\gamma > 3$, the last integral term on the right-hand side of (2.24) can be estimated as following

$$\begin{aligned} & \left| \int_T^{T+1} \int_{\Omega} \psi \varrho \partial_i \Phi \mathcal{B}_i \left\{ S_{\varepsilon}[b(\varrho)] - \frac{1}{|\Omega|} \int_{\Omega} S_{\varepsilon}[b(\varrho)] dx \right\} dx dt \right| \\ & \leq \int_T^{T+1} \|\nabla \Phi\|_{L^{\infty}(\Omega)} \left\| \mathcal{B}_i \left\{ S_{\varepsilon}[b(\varrho)] - \frac{1}{|\Omega|} \int_{\Omega} S_{\varepsilon}[b(\varrho)] dx \right\} \right\|_{L^{\infty}(\Omega)} \\ & \quad \times \int_{\Omega} \varrho dx dt \leq c_{16}(m) \sup_{t \in [T, T+1]} \|\varrho\|_{L^{\gamma}(\Omega)}. \end{aligned} \quad (2.37)$$

Approximating the characteristic function of the interval $[T, T+1]$ by a sequence ψ_{ε} and letting $\varepsilon \rightarrow 0$ in (2.25)–(2.37), we conclude

$$\begin{aligned} & \int_T^{T+1} \int_{\Omega} \varrho^{\gamma+\theta} dx dt \leq c_{17}(K, m) \left[1 + \sup_{t \in [T, T+1]} \|\sqrt{\varrho} \mathbf{u}(t)\|_{L^2(\Omega)} \right. \\ & \quad \left. + \left(1 + \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\gamma}(\Omega)} \right) \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt \right]. \end{aligned} \quad (2.38)$$

On the other hand, interpolating the spaces L^1 , $L^{\gamma+\theta}$, one has

$$\int_T^{T+1} \|\varrho\|_{L^{\gamma}(\Omega)}^{\gamma} dt \leq c_{18}(m) \left[\int_T^{T+1} \int_{\Omega} \varrho^{\gamma+\theta} dx dt \right]^{\frac{\gamma-1}{\gamma+\theta-1}}. \quad (2.39)$$

Moreover, by virtue of Lemma 2.4,

$$\begin{aligned} & \left| \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\gamma}(\Omega)} \int_T^{T+1} \|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega)}^2 dt \right|^{\frac{\gamma-1}{\gamma+\theta-1}} \\ & \leq c_{19}(K, m) \left[1 + \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\gamma}(\Omega)} \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\frac{3}{2}}(\Omega)} \right]^{\frac{\gamma-1}{\gamma+\theta-1}} \\ & \leq c_{20}(K, m) \left[1 + \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^{\gamma}(\Omega)} \right]^{\frac{4\gamma-3}{3(\gamma+\theta-1)}}. \end{aligned} \quad (2.40)$$

Finally, by virtue of (1.9), (2.7) and (2.8), we obtain

$$\begin{aligned} & \operatorname{ess\,sup}_{t \in [T, T+1]} \|\sqrt{\varrho} \mathbf{u}(t)\|_{L^2(\Omega)} \\ & \leq \sup_{t \in [T, T+1]} \sqrt{2E(t+) + \frac{1}{4\pi g} \|\nabla \Phi\|_{L^2(\Omega)}^2} \leq \sup_{t \in [T, T+1]} 2\sqrt{E(t+) + c_{2,5}(m)}. \end{aligned} \quad (2.41)$$

Combining Lemma 2.5 with estimates (2.38)–(2.41), we infer

$$\begin{aligned} & \sup_{t \in [T, T+1]} E(t+) \\ & \leq c_{21}(K, m) \left(1 + \sup_{t \in [T, T+1]} 2\sqrt{E(t+) + c_{2,5}(m)} + \sup_{t \in [T, T+1]} \|\varrho(t)\|_{L^\gamma(\Omega)}^{\frac{4\gamma-3}{3(\gamma+\theta-1)}} \right). \end{aligned} \quad (2.42)$$

By virtue of the hypothesis (1.15), we have

$$\frac{4\gamma-3}{3(\gamma+\theta-1)} < \gamma,$$

provided θ is as in (2.23). Consequently, (2.42) implies the existence of the constant L having the property claimed in Proposition 2.3.

2.3. Conclusion of the proof of Theorem 1.2

To begin, observe there exists $T = T(E_0, m) > 0$ such that $E(T-) > E((T-1)+) - 1$. Indeed if it was not the case, then, by virtue of (2.5) and (1.18), when the t is enough large, we infer the energy would become negative and the absolute value of energy E can be arbitrary large. This contradict (2.8). Thus $E(t_0) \leq L$ for a certain $t_0 < T$, where L is the constant from Proposition 2.3.

Next, we claim that for any integer $n \geq 0$ we have that

$$E((t_0 + n)+) \leq L, \quad (2.43)$$

By induction, assume $E((t_0 + n)+) \leq L$. By Proposition 2.3 and (2.5), either

$$\sup_{t \in (t_0 + n, t_0 + n+1)} E(t+) \leq L,$$

and, consequently, $E((t_0 + n + 1)-) \leq L$, or

$$E((t_0 + n + 1)+) \leq E((t_0 + n + 1)-) \leq E((t_0 + n)+) - 1 \leq L - 1.$$

Finally, by virtue of Lemma 2.2 and (2.43), we can take

$$E_\infty = (1 + L)e^{c(m, K)} - 1.$$

Theorem 1.2 has been proved.

3. The proof of Theorem 1.3

The rest of the paper will be devoted to the proof of Theorem 1.3. Similarly to [9], the main idea is also based on careful analysis of the “defect measure”

$$v = \overline{\log(\varrho)}\varrho - \log(\bar{\varrho})\bar{\varrho}. \quad (3.1)$$

Here, and in what follows, the symbol $\bar{\varrho}$ denotes a weak limit of the sequence of time shifts

$$\varrho_{n, t_n}(t, x) = \begin{cases} \varrho_n(t_n + t, x), & \text{for } t > -t_n, \\ 0 & \text{if } t \leq -t_n. \end{cases}$$

While $\overline{b(\varrho)}$ stands for a weak limit of a composition $b(\varrho_{n, t_n})$.

We shall show that the function

$$D(t) = \int_{\Omega} v(t, x) dx \quad t \in R$$

is continuous, nonnegative and uniformly bounded on R , and satisfies

$$D'(t) + \Psi(D(t)) \leq 0 \quad \text{in } \mathcal{D}'(R),$$

where Ψ is convex, strictly positive for arguments, and $\Psi(0) = 0$. Consequently, $D \equiv 0$ which implies strong convergence of the density claimed in (1.22).

The core of this section is thus the proof of the existence of the function Ψ which is dependent of the initial date and yields uniform decay of oscillations in the density component of the finite energy solutions.

3.1. Basic estimates and convergence of the sequence of time shifts

We start with a straightforward consequence of Theorem 1.2:

Proposition 3.1. *Under the hypotheses of Theorem 1.3, there exists a time $T = T(E_0)$ a constant L depending on K and m such that*

$$\sup_{t \geq T} \|\varrho_n(t)\|_{L^\gamma(\Omega)} \leq L, \quad (3.2)$$

$$\text{ess sup}_{t \geq T} \|\sqrt{\varrho_n(t)} \mathbf{u}_n(t)\|_{L^2(\Omega)} \leq L, \quad (3.3)$$

$$\sup_{t \geq T} \|\nabla \Phi_n(t)\|_{W^{1,\gamma}(\Omega)} \leq L \quad (3.4)$$

and

$$\int_J \int_{\Omega} |\varrho_n|^{\gamma+\theta} dx dt \leq L, \quad \theta = \frac{2}{3}\gamma - 1, \quad (3.5)$$

$$\int_J \|\mathbf{u}_n(t)\|_{W_0^{1,2}(\Omega)}^2 dt \leq L, \quad (3.6)$$

for any time interval $J \subset (T, \infty)$, $|J| \leq 1$ independent of $n = 1, 2, \dots$

Proof. By the virtue of (2.41) and (2.8), the estimates (3.2), (3.3) and (3.6) are a straightforward consequence of Theorem 1.2 and the energy inequality (1.8), and the estimates (3.4) can be obtained by elliptic estimate (see [24, Lemma 4.27]). The bound (3.5) can be proved in exactly the same way as the estimate (2.38) (or see [24, Lemma 7.52]). One has only to observe that, assuming (1.15) and knowing the energy E is bounded by E_∞ .

Now, similarly as above, we define

$$\mathbf{u}_{n,t_n}(t) = \mathbf{u}_n(t_n + t), \quad \mathbf{f}_{n,t_n}(t) = \mathbf{f}_n(t_n + t) \quad \text{and} \quad \Phi_{n,t_n} = \Phi_n(t_n + t).$$

In view of the estimates in Proposition 3.1, since $(\varrho_{n,t_n}, \mathbf{u}_{n,t_n})$ can be zero outside Ω , so we can pass to subsequences (not relabeled) such that

$$\begin{aligned} \varrho_{n,t_n} &\rightarrow \bar{\varrho} \quad \text{weakly star in } L^\infty(R; L^\gamma(\mathbb{R}^3)), \\ \varrho &\geq 0 \quad \text{a.e. in } R \times \mathbb{R}^3, \quad \varrho = 0 \text{ in } \mathbb{R}^3/\Omega, \end{aligned} \quad (3.7)$$

$$\begin{aligned} \mathbf{u}_{n,t_n} &\rightarrow \bar{\mathbf{u}} \quad \text{weakly in } L^2(J; (W_0^{1,2}(\Omega))^3) \\ &\text{and in } L^2(J, (W^{1,2}(\mathbb{R}^3))^3), \quad \mathbf{u} = \mathbf{0} \text{ in } (\mathbb{R}^3/\Omega) \times J \end{aligned} \quad (3.8)$$

for any bounded interval $J \subset R$, where $\bar{\mathbf{u}}$ is a function belonging to $L^2_{\text{loc}}(R; (W_0^{1,2}(\Omega))^3)$ satisfying

$$\int_J \|\bar{\mathbf{u}}\|_{W_0^{1,2}(\Omega)}^2 \leq L \quad \text{for any } J \subset R, |J| \leq 1, \quad (3.9)$$

and

$$\begin{aligned} \mathbf{f}_{n,t_n} &\rightarrow \bar{\mathbf{f}} \quad \text{weakly in } L^\infty(R; \Omega), \\ \varrho_{n,t_n} \mathbf{f}_{n,t_n} &\rightarrow \bar{\varrho} \bar{\mathbf{f}} \quad \text{weakly star in } L^\infty(R; L^\gamma(\Omega)). \end{aligned}$$

Moreover, making use of the estimates (3.2), (3.6), the fact that ϱ_{n,t_n} solve (1.1), and the Arzela-Ascoli theorem, we deduce

$$\varrho_{n,t_n} \rightarrow \bar{\varrho} \quad \text{in } C(J; L_{\text{weak}}^\gamma(\Omega)) \quad \text{for any compact interval } J \subset R \quad (3.10)$$

and

$$\bar{\varrho} \in BC(R; L_{\text{weak}}^\gamma(\Omega)) \quad (3.11)$$

(see [11, Section 3] for details).

Now, the relation (3.10) implies strong convergence of ϱ_{n,t_n} in $C(J; W^{-1,2}(\Omega))$, which, combined with (3.8), yields

$$\varrho_{n,t_n} \mathbf{u}_{n,t_n} \rightarrow \bar{\varrho} \bar{\mathbf{u}} \quad \text{weakly in } L^2(J; (L^{\frac{6\gamma}{6+\gamma}}(\mathbb{R}^3))^3) \quad \text{for any compact interval } J \subset R. \quad (3.12)$$

Consequently we obtain

$$\partial_t \bar{\varrho} + \operatorname{div}(\bar{\varrho} \bar{\mathbf{u}}) = 0 \quad \text{in } \mathcal{D}'(R \times \mathbb{R}^3). \quad (3.13)$$

Similarly as above, one can estimate the time derivative of $\varrho_{n,t_n} \mathbf{u}_{n,t_n}$ by means of the Eq. (1.2), which, together with the estimates (3.2)–(3.4), yields

$$\varrho_{n,t_n} \mathbf{u}_{n,t_n} \rightarrow \bar{\varrho} \bar{\mathbf{u}} \quad \text{in } C(J; L_{\text{weak}}^{\frac{2\gamma}{\gamma+1}}(\Omega)) \quad (3.14)$$

(see [24, Section 7.10] for details), and

$$\bar{\varrho} \bar{\mathbf{u}} \in BC(R; L_{\text{weak}}^{\frac{2\gamma}{1+\gamma}}(\Omega)). \quad (3.15)$$

Now, the relation (3.14) implies strong convergence of $\varrho_{n,t_n} \mathbf{u}_{n,t_n}$ in the space $C(J; W^{-1,2}(\Omega))$ and we infer, exactly as when proving (3.12), that

$$\varrho_{n,t_n} u_{n,t_n}^i u_{n,t_n}^j \rightarrow \bar{\varrho} \bar{u}^i \bar{u}^j \quad \text{weakly in } L^2(J; L^{\frac{6\gamma}{3+4\gamma}}(\Omega)), \quad i, j = 1, 2, 3 \quad (3.16)$$

for any bounded $J \subset R$. By virtue of (3.5), we can obtain

$$\varrho_{n,t_n} \rightarrow \bar{\varrho} \quad \text{weakly in } L^{\gamma+\theta}(J \times \mathbb{R}^3) \quad (3.17)$$

and

$$p(\varrho_{n,t_n}) \rightarrow \overline{p(\bar{\varrho})} \quad \text{weakly in } L^{\frac{\gamma+\theta}{\gamma}}(J \times \Omega) \quad (3.18)$$

for any bounded interval $J \subset R$.

Finally, by virtue of (1.3), (1.5), (3.4), (3.9) and (3.10), we can infer

$$\nabla \Phi_{n,t_n} \rightarrow \nabla \bar{\Phi} \quad \text{weakly} - * \text{ in } L^\infty(R^+; W^{1,\gamma}(\Omega)), \quad (3.19)$$

$$\Delta \Phi_{n,t_n} \rightarrow \Delta \bar{\Phi} \text{ in } C(J; L_{\text{weak}}^\gamma(\Omega)), \quad (3.20)$$

for any bounded interval $J \subset R$, where $\bar{\Phi}$ satisfy

$$\Delta \bar{\Phi} = 4\pi g \left(\bar{\varrho} - \frac{1}{|\Omega|} \int_\Omega \bar{\varrho} \right), \quad \text{a.e in } \Omega \quad \text{for any } t \in R^+, \quad (3.21)$$

and

$$\left. \frac{\partial \bar{\Phi}}{\partial \nu} \right|_{\partial \Omega} = 0 \text{ holds in the sense of traces for any } t \in R^+.$$

Furthermore, in the case of $\gamma \geq 3$, by virtue of (3.2), (3.4), embedding theorem and Hölder inequality, we have

$$\varrho_{n,t_n} \nabla \Phi_{n,t_n} \rightarrow \overline{\varrho \nabla \bar{\Phi}} \text{ in } L^\infty(R^+; L^p(\Omega)) \text{ for } 1 < p < \gamma. \quad (3.22)$$

The other case of $3 > \gamma > \frac{5}{3}$, we obtain

$$\varrho_{n,t_n} \nabla \Phi_{n,t_n} \rightarrow \overline{\varrho \nabla \bar{\Phi}} \text{ in } L^\infty(R^+; L^p(\Omega)) \text{ for } 1 < p < \frac{3\gamma}{6-\gamma}. \quad (3.23)$$

Consequently we get

$$\begin{aligned} & \partial_t(\bar{\varrho} \bar{\mathbf{u}}) + \operatorname{div}(\bar{\varrho} \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}) - \mu \Delta \bar{\mathbf{u}} - (\lambda + \mu) \nabla(\operatorname{div} \bar{\mathbf{u}}) + \overline{\nabla p(\bar{\varrho})} + \overline{\varrho \nabla \bar{\Phi}} \\ & = \overline{\varrho \mathbf{f}} \text{ in } \mathcal{D}'(R \times \Omega), \end{aligned} \quad (3.24)$$

□

3.2. Some results about the density and the effective viscous flux

By the definition of renormalized solutions, the function ϱ_{n,t_n} satisfy

$$\partial_t b(\varrho_{n,t_n}) + \operatorname{div}(b(\varrho_{n,t_n}) \mathbf{u}_{n,t_n}) + (b'(\varrho_{n,t_n}) \varrho_{n,t_n} - b(\varrho_{n,t_n})) \operatorname{div} \mathbf{u}_{n,t_n} = 0 \quad (3.25)$$

holds in $\mathcal{D}'((-t_n, \infty) \times \mathbb{R}^3)$ for any b satisfying (1.11) and (1.12). Moreover,

$$\varrho_{n,t_n} \in C(J; L^\alpha(\Omega)) \text{ for any compact } J \subset (-t_n, \infty). \quad (3.26)$$

Analogously the same holds also for $\bar{\varrho}$, i.e.,

$$\partial_t b(\bar{\varrho}) + \operatorname{div}(b(\bar{\varrho}) \bar{\mathbf{u}}) + (b'(\bar{\varrho}) \bar{\varrho} - b(\bar{\varrho})) \operatorname{div} \bar{\mathbf{u}} = 0 \text{ in } \mathcal{D}'(R \times \mathbb{R}^3), \quad (3.27)$$

$$\bar{\varrho} \in BC(R; L^\alpha(\Omega)) \quad 1 \leq \alpha < \gamma, \quad (3.28)$$

in particular,

$$\log(\bar{\varrho})\bar{\varrho} \in BC(R; L^\alpha(\Omega)) \quad \text{for } 1 \leq \alpha < \gamma \quad (3.29)$$

(see [24, Lemma 7.57] for details).

At this stage, we introduce functions

$$M_k(z) = \begin{cases} z \log(z) & \text{for } 0 \leq z < k, \\ k \log(k) + (1 + \log(k))(z - k) & \text{for } z \geq k \end{cases} \quad (3.30)$$

and

$$T_k(z) = \min\{z, k\}, \quad z \geq 0$$

for any $k \geq 1$.

Clearly, $M_k(z)$ can be written on the form:

$$M_k(z) = (\log k + 1)z + l_k(z), \quad l_k(z) = z(\log z - \log k) 1_{\{z \leq k\}} - z 1_{\{z \leq k\}} - k 1_{\{z > k\}}.$$

We observe that $l_k(z)$ satisfies (1.11) and (1.12) with any $\lambda_1 > -1$, so we deduce from (3.25) and (3.27) that

$$\begin{aligned} \partial_t M_k(\varrho_{n,t_n}) + \operatorname{div}(M_k(\varrho_{n,t_n})\mathbf{u}_{n,t_n}) + T_k(\varrho_{n,t_n})\operatorname{div}\mathbf{u}_{n,t_n} \\ = 0 \text{ in } \mathcal{D}'((-t_n, \infty) \times \mathbb{R}^3) \end{aligned} \quad (3.31)$$

and

$$\partial_t M_k(\bar{\varrho}) + \operatorname{div}(M_k(\bar{\varrho})\bar{\mathbf{u}}) + T_k(\bar{\varrho})\operatorname{div}\bar{\mathbf{u}} = 0 \text{ in } \mathcal{D}'(\mathbb{R} \times \mathbb{R}^3) \quad (3.32)$$

Now, similar as in the proof of (3.10), we have

$$M_k(\varrho_{n,t_n}) \rightarrow \overline{M_k(\varrho)} \quad \text{in } C(J; L_{\text{weak}}^\gamma(\Omega)), \quad (3.33)$$

where

$$\overline{M_k(\varrho)} \in BC(R; L_{\text{weak}}^\alpha(\Omega)), \quad 1 \leq \alpha < \gamma, \quad (3.34)$$

and the bound in (3.34) depends solely on α , in particular, it is independent of k .

Lemma 3.2. *Under the hypotheses of Theorem 1.3, we have*

$$\sup_{t > T-t_n} \sup_n \int_{\Omega} \varrho_{n,t_n}(t) \log(\varrho_{n,t_n}(t)) - M_k(\varrho_{n,t_n}(t)) dx \leq r_1(k)$$

where

$$r_1(k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Proof. See reference [9, Lemma 3.1]. □

Corollary 3.3. *Under the hypotheses of Theorem 1.3, we have*

$$\varrho_{n,t_n} \log(\varrho_{n,t_n}) \rightarrow \overline{\varrho \log(\varrho)} \quad \text{in } C(J; L_{\text{weak}}^\alpha(\Omega))$$

for any compact interval $J \subset \mathbb{R}$ and any fixed $1 \leq \alpha < \gamma$. Moreover,

$$\sup_{t \in \mathbb{R}} \int_{\Omega} \overline{\varrho \log(\varrho)}(t) - \overline{M_k(\varrho)}(t) dx \leq r_1(k)$$

where

$$r_1(k) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Finally, we can use (3.25) to get

$$\begin{aligned} & \partial_t T_k(\varrho_{n,t_n}) + \operatorname{div}(T_k(\varrho_{n,t_n})\mathbf{u}_{n,t_n}) \\ & + k \operatorname{sgn}^+(\varrho_{n,t_n} - k) \operatorname{div} \mathbf{u}_{n,t_n} = 0 \quad \text{in } \mathcal{D}'((-t_n, \infty) \times \mathbb{R}^3). \end{aligned} \quad (3.35)$$

Similarly as in (3.9)–(3.11) we can pass to the limit for $n \rightarrow \infty$ to obtain

$$\begin{aligned} T_k(\varrho_{n,t_n}) & \rightarrow \overline{T_k(\varrho)} \quad \text{in } C(J; L^\beta(\Omega)) \quad \text{for any compact } J \subset R, 1 \leq \beta < \infty, \\ \overline{T_k(\varrho)} & \in BC(R; L^\alpha(\Omega)) \quad 1 \leq \alpha < \gamma \quad \text{independent of } k, \\ k \operatorname{sgn}^+(\varrho_{n,t_n} - k) \operatorname{div} \mathbf{u}_{n,t_n} & \rightarrow \chi_k \quad \text{weakly in } L^2(J \times \Omega) \\ & \text{for any bounded } J \subset R, \end{aligned} \quad (3.37)$$

and

$$\partial_t \overline{T_k(\varrho)} + \operatorname{div}(\overline{T_k(\varrho)} \bar{\mathbf{u}}) + \chi_k = 0 \quad \text{in } \mathcal{D}'(R \times \mathbb{R}^3) \quad (3.38)$$

where T_k are the cut-off functions defined at the beginning of the section.

Next, we shall investigate the properties of the quantity

$$p(\varrho) - (\lambda + 2\mu) \operatorname{div} \mathbf{u}$$

called usually the effective viscous flux. It turns out that it is “more regular” than its components, in particular, it exhibits certain weak continuity. This is the crucial property used in the proof of existence of weak solutions as presented in Lions [19]

Lemma 3.4. *Under the hypotheses of Theorem 1.3, we have*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_J \int_\Omega (p(\varrho_{n,t_n}) - (\lambda + 2\mu) \operatorname{div} \mathbf{u}_{n,t_n}) T_k(\varrho_{n,t_n}) dx dt \\ & = \int_J \int_\Omega (\overline{p(\varrho)} - (\lambda + 2\mu) \operatorname{div} \bar{\mathbf{u}}) \overline{T_k(\varrho)} dx dt \end{aligned}$$

for any bounded interval $J \subset R$ and all $k = 1, 2, \dots$

Proof. Consider the operators

$$\mathcal{A}_j[v] = \Delta^{-1} \partial_{x_j}(v), \quad j = 1, 2, 3,$$

more specifically,

$$\mathcal{A}_j[\xi] = \mathcal{F}^{-1} \left\{ \frac{-i\xi_j}{|\xi|^2} \mathcal{F}\{v\}(\xi) \right\}, \quad j = 1, 2, 3,$$

where $\mathcal{F}$ denotes the Fourier transform. By means of the Mikhlin multiplier theorem, we have

$$\|\partial_{x_i} \mathcal{A}_j[v]\|_{L^p(\Omega)} \leq c(p) \|v\|_{L^p(\mathbb{R}^3)}, \quad \text{for any } 1 < p < \infty$$

and

$$\|\mathcal{A}_i[v]\|_{L^q(\Omega)} \leq c(q, r)\|v\|_{L^r(\mathbb{R}^3)},$$

where $r \leq q \leq \frac{3\gamma}{3-\gamma}$ if $1 < r < 3$, q arbitrary finite if $r = 3$, $q = \infty$ for $r > 3$.

Now, we use the quantities

$$\varphi_i(x, t) = \psi(t)\phi(x)\mathcal{A}_i[T_k(\varrho_{n,t_n})], \quad \psi \in \mathcal{D}(Z), \quad \phi \in \mathcal{D}(\Omega), \quad i = 1, 2, 3$$

as test functions for (1.2) (as always, ϱ_{n,t_n} is prolonged by zero outside Ω):

$$\begin{aligned} & \int_J \int_{\Omega} \phi \psi [p(\varrho_{n,t_n}) - (\lambda + 2\mu) \operatorname{div} \mathbf{u}_{n,t_n}] T_k(\varrho_{n,t_n}) dx dt \\ &= \mu \int_J \int_{\Omega} \psi \left\{ \nabla \phi \cdot \nabla u_{n,t_n}^i \mathcal{A}_i[T_k(\varrho_{n,t_n})] - u_{n,t_n}^i \partial_{x_j} \psi \partial_{x_j} \mathcal{A}_i[T_k(\varrho_{n,t_n})] \right. \\ & \quad \left. + (\mathbf{u}_{n,t_n} \cdot \nabla \phi) T_k(\varrho_{n,t_n}) \right\} dx dt \\ & \quad + \int_J \int_{\Omega} \psi [(\lambda + \mu) \operatorname{div} \mathbf{u}_{n,t_n} - p(\varrho_{n,t_n})] \partial_{x_i} \phi \mathcal{A}_i[T_k(\varrho_{n,t_n})] dx dt \\ & \quad - \int_J \int_{\Omega} \phi \varrho_{n,t_n} u_{n,t_n}^i \left\{ \partial_t \psi \mathcal{A}_i[T_k(\varrho_{n,t_n})] + k \psi \mathcal{A}_i[\operatorname{sgn}^+(\varrho_{n,t_n} - k) \operatorname{div} \mathbf{u}_{n,t_n}] \right\} dx dt \\ & \quad - \int_J \int_{\Omega} \psi \left\{ \varrho_{n,t_n} u_{n,t_n}^i (u_{n,t_n}^j \partial_{x_j} \phi) \mathcal{A}_i[T_k(\varrho_{n,t_n})] + \phi \varrho_{n,t_n} f_{n,t_n}^i \mathcal{A}_i[T_k(\varrho_{n,t_n})] \right\} dx dt \\ & \quad + \int_J \int_{\Omega} \psi u_{n,t_n}^i \left\{ T_k(\varrho_{n,t_n}) \mathcal{R}_{i,j}[\phi \varrho_{n,t_n} u_{n,t_n}^j] - \phi \varrho_{n,t_n} u_{n,t_n}^j \mathcal{R}_{i,j}[T_k(\varrho_{n,t_n})] \right\} dx dt \\ & \quad + \int_J \int_{\Omega} \psi \phi \varrho_{n,t_n} \partial_{x_i} \Phi_{n,t_n} \mathcal{A}_i[T_k(\varrho_{n,t_n})] dx dt = \sum_{i=1}^6 J_{i,n}. \end{aligned} \quad (3.39)$$

where the operators $\mathcal{R}_{i,j}$ are defined as

$$\mathcal{R}_{i,j}[v] = \mathcal{F}^{-1} \left\{ \frac{\xi_i \xi_j}{|\xi|^2} \mathcal{F}\{v\}(\xi) \right\}.$$

Here, we have made use of (3.35).

Analogously, we can repeat the above arguments considering the Eqs. (3.24), (3.38) and the test functions

$$\varphi_i(x, t) = \psi \phi \mathcal{A}_i[\overline{T_k(\varrho)}], \quad i = 1, 2, 3$$

to deduce

$$\begin{aligned} & \int_J \int_{\Omega} \phi \psi [\overline{p(\varrho)} - (\lambda + 2\mu) \operatorname{div} \bar{\mathbf{u}}] \overline{T_k(\varrho)} dx dt \\ &= \mu \int_J \int_{\Omega} \psi \left\{ \nabla \phi \cdot \nabla \bar{u}^i \mathcal{A}_i[\overline{T_k(\varrho)}] - \bar{u}^i \partial_{x_j} \psi \partial_{x_j} \mathcal{A}_i[\overline{T_k(\varrho)}] + (\bar{\mathbf{u}} \cdot \nabla \phi) \overline{T_k(\varrho)} \right\} dx dt \\ & \quad + \int_J \int_{\Omega} \psi [(\lambda + \mu) \operatorname{div} \bar{\mathbf{u}} - \overline{p(\varrho)}] \partial_{x_i} \phi \mathcal{A}_i[\overline{T_k(\varrho)}] dx dt \\ & \quad - \int_J \int_{\Omega} \phi \bar{\varrho} \bar{u}^i \left\{ \partial_t \psi \mathcal{A}_i[\overline{T_k(\varrho)}] + k \psi \mathcal{A}_i[\chi_k] \right\} dx dt \end{aligned}$$

$$\begin{aligned}
& - \int_J \int_{\Omega} \psi \{ \bar{\varrho} \bar{u}^i (u_{n,t_n}^j \partial_{x_j} \phi) \mathcal{A}_i[\overline{T_k(\varrho)}] + \phi \bar{\varrho} \bar{f}^i \mathcal{A}[\overline{T_k(\varrho)}] \} dx dt \\
& + \int_J \int_{\Omega} \psi \bar{u}^i \left\{ \overline{T_k(\varrho)} \mathcal{R}_{i,j}[\phi \bar{\varrho} \bar{u}^j] - \phi \bar{\varrho} \bar{u}^j \mathcal{R}_{i,j}[\overline{T_k(\varrho)}] \right\} dx dt \\
& + \int_J \int_{\Omega} \psi \phi \bar{\varrho} \partial_{x_i} \Phi \mathcal{A}_i[\overline{T_k(\varrho)}] dx dt = \sum_{i=1}^6 J_i.
\end{aligned} \tag{3.40}$$

From (3.36), we have

$$\mathcal{A}_i[T_k(\varrho)] \rightarrow \mathcal{A}_i[\overline{T_k(\varrho)}] \quad \text{in } C(J \times \bar{\Omega})$$

as $n \rightarrow \infty$. Therefore, $J_{2,n} \rightarrow J_2$ by (3.8) and (3.18). Next, (3.36) implies

$$\partial_j \mathcal{A}_i[T_k(\varrho_{n,t_n})] \rightarrow \partial_j \mathcal{A}_i[\overline{T_k(\varrho)}] \quad \text{in } C(J; W^{-1,2}(\Omega))$$

and therefore, $J_{1,n} \rightarrow J_1$ by (3.8) and (3.36). The first term of $J_{3,n}$ converges to that of J_3 by (3.14). We have, on the other hand, (3.37), and therefore,

$$\mathcal{A}_i[\text{sgn}^+(\varrho_{n,t_n} - k) \text{div} \mathbf{u}_{n,t_n}] \rightarrow \mathcal{A}_i[\chi_k] \quad \text{in } L^2(J \times \Omega)$$

by the compact injection $W^{1,2} \hookrightarrow L^2(\Omega)$ and the continuity of $\mathcal{A}_i: L^2(\Omega) \rightarrow W^{1,2}(\Omega)$. Therefore, the second term of $J_{3,n}$ converges to that of J_3 by (3.8). Next, the convergence $J_{4,n} \rightarrow J_4$ follows from (3.16). We have (3.14), (3.36), and therefore,

$$\begin{aligned}
& T_k(\varrho_{n,t_n}) \mathcal{R}_{i,j}[\phi \varrho_{n,t_n} u_{n,t_n}^i] - \phi \varrho_{n,t_n} u_{n,t_n}^i \mathcal{R}_{ij}[T_k(\varrho_{n,t_n})] \\
& \rightarrow \overline{T_k(\varrho)} \mathcal{R}_{i,j}[\phi \varrho u^i] - \phi \varrho u^i \mathcal{R}_{ij}[\overline{T_k(\varrho)}]
\end{aligned}$$

by the following Lemma 3.5 (see [17, Lemma 3.4]) in the case of $\frac{1}{p} + \frac{\gamma+1}{2\gamma} = \frac{1}{r}$ with $\gamma > \frac{6}{5}$, i.e., $p > \frac{6\gamma}{2\gamma-3}$. $\square$

Lemma 3.5. *If $p, q, r > 1$ are in $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $v_n \rightarrow v$ and $w_n \rightarrow w$ weakly in $L^p(\mathbb{R}^3)$ and $L^q(\mathbb{R}^3)$, respectively, then $v_n \mathcal{R}_{i,j}[w_n] - w_n \mathcal{R}_{i,j}[v_n] \rightarrow v \mathcal{R}_{i,j}[w] - w \mathcal{R}_{i,j}[v]$ weakly in $L^r(\mathbb{R}^3)$.*

This implies $J_{5,n} \rightarrow J_5$ by (3.8). Finally, we have (3.22) if $\gamma \geq 3$ and (3.23) for $1 < p < \frac{3\gamma}{6-\gamma}$ if $1 < \gamma < 3$, respectively, which implies $J_{6,n} \rightarrow J_6$. Thus, we obtain the conclusion of Lemma 3.4.

3.3. Compactness of the density and momenta

The aim of the present section is to prove the compactness of the density and momenta. These conclusion can be easily obtained by the same argument in Sections 5–7 of [9]. Here we only sketch the spirit of proof for reader's convenience. We refer to [9, Section 5–7] for details.

Firstly, let us fix $\tau_1 < \tau_2$ arbitrary. By virtue of (3.26) and (3.31), we deduce

$$\begin{aligned} & \int_{\Omega} M_k(\varrho_{n,t_n})(\tau_2) dx - \int_{\Omega} M_k(\varrho_{n,t_n})(\tau_1) dx + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\lambda + 2\mu} p(\varrho_{n,t_n}) T_k(\varrho_{n,t_n}) dx dt \\ &= \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\lambda + 2\mu} p(\varrho_{n,t_n}) T_k(\varrho_{n,t_n}) - \operatorname{div} \mathbf{u}_{n,t_n} T_k(\varrho_{n,t_n}) dx dt. \end{aligned} \quad (3.41)$$

Now, applying the same treatment to the Eq. (3.32) and making use of (3.28), we obtain

$$\begin{aligned} & \int_{\Omega} M_k(\bar{\varrho})(\tau_2) dx - \int_{\Omega} M_k(\bar{\varrho})(\tau_1) dx + \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\lambda + 2\mu} \overline{p(\varrho)} \overline{T_k(\varrho)} dx dt \\ &= \int_{\tau_1}^{\tau_2} \int_{\Omega} \frac{1}{\lambda + 2\mu} \overline{p(\varrho)} \overline{T_k(\varrho)} - \operatorname{div} \bar{\mathbf{u}} \overline{T_k(\varrho)} dx dt. \end{aligned} \quad (3.42)$$

Taking the difference of (3.41), (3.42) and making use of Lemma 3.4 together (3.33), we get

$$\begin{aligned} & \int_{\Omega} (\overline{M_k(\varrho)} - M_k(\bar{\varrho}))(\tau_2) dx - \int_{\Omega} (\overline{M_k(\varrho)} - M_k(\bar{\varrho}))(\tau_1) dx \\ &+ \frac{1}{\lambda + 2\mu} \limsup_{n \rightarrow \infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} p(\varrho_{n,t_n}) T_k(\varrho_{n,t_n}) - \overline{p(\varrho)} \overline{T_k(\varrho)} dx dt \\ &\leq \int_{\tau_1}^{\tau_2} \int_{\Omega} (\overline{T_k(\varrho)} - T_k(\bar{\varrho})) \operatorname{div} \bar{\mathbf{u}} dx dt. \end{aligned} \quad (3.43)$$

At this stage, we shall need the following three crucial lemmas:

Lemma 3.6. *There exists a constant $d > 0$, such that*

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} p(\varrho_{n,t_n}) T_k(\varrho_{n,t_n}) - \overline{p(\varrho)} \overline{T_k(\varrho)} dx dt + r_2(k)(\tau_2 - \tau_1) \\ &\geq d \int_{\tau_1}^{\tau_2} \Psi \left(\int_{\Omega} \overline{M_k(\varrho)} - M_k(\bar{\varrho}) dx \right) dt, \end{aligned}$$

where

$$r_2(k) \rightarrow 0 \quad \text{for } k \rightarrow \infty,$$

and Ψ is the convex function from the following Lemma 3.7.

Lemma 3.7. *Fix $\eta \in (\frac{1}{\gamma+1}, 1)$ and consider the function Ψ (depending on η) determined by the relation*

$$\Psi(z^\eta + z^{\frac{1}{\eta}}) = z^{\eta+1} \quad \text{for all } z \geq 0.$$

Then Ψ is convex, strictly increasing for $z \geq 0$, $\Psi(0)=0$.

For the proof of above two lemmas, we refer to [9, Lemma 6.1, Lemma 5.3] for details.

Lemma 3.8. *Under the assumptions of Theorem 1.3, we have*

$$\sup_{k>1} \limsup_{n \rightarrow \infty} \int_{\tau_1}^{\tau_2} \int_{\Omega} |T_k(\varrho_{n,t_n}) - T_k(\bar{\varrho})|^{\gamma+1} dx dt \leq L(\tau_1, \tau_2).$$

Proof. See [24, Lemma 7.56]. $\square$

Now, using the conclusion of Lemma 3.6 together with Corollary 3.3, one can pass to the limit in (3.43) for $k \rightarrow \infty$ to obtain

$$\begin{aligned} & \int_{\Omega} v(\tau_2, x) dx - \int_{\Omega} v(\tau_1, x) dx + \frac{d}{\lambda + 2\mu} \int_{\tau_1}^{\tau_2} \Psi \left(\int_{\Omega} v(t, x) dx \right) dt \\ & \leq \limsup_{k \rightarrow \infty} \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} (\overline{T_k(\varrho)} - T_k(\bar{\varrho})) \operatorname{div} \bar{\mathbf{u}} dx dt \right|, \end{aligned} \quad (3.44)$$

where the “defect measure” v is given by (3.1). The term on the right-hand side may be treated as

$$\begin{aligned} & \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} (\overline{T_k(\varrho)} - T_k(\bar{\varrho})) \operatorname{div} \bar{\mathbf{u}} dx dt \right| \\ & \leq \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} (\overline{T_k(\varrho)} - T_k(\bar{\varrho}))^2 dx dt \right|^{\frac{1}{2}} \left| \int_{\tau_1}^{\tau_2} \int_{\Omega} |\operatorname{div} \bar{\mathbf{u}}|^2 dx dt \right|^{\frac{1}{2}} \\ & \leq \left(\int_{\tau_1}^{\tau_2} \int_{\Omega} |\overline{T_k(\varrho)} - T_k(\bar{\varrho})| dx dt \right)^{\frac{\gamma-1}{2\gamma}} \left(\int_{\tau_1}^{\tau_2} \int_{\Omega} |\overline{T_k(\varrho)} - T_k(\bar{\varrho})|^{\gamma+1} dx dt \right)^{\frac{1}{2\gamma}} \\ & \quad \times \left(\int_{\tau_1}^{\tau_2} \int_{\Omega} |\operatorname{div} \bar{\mathbf{u}}|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

By virtue of (3.9), Lemma 3.8 and the following inequality

$$\begin{aligned} & \left(\int_{\tau_1}^{\tau_2} \int_{\Omega} |\overline{T_k(\varrho)} - T_k(\bar{\varrho})|^p dx dt \right)^{\frac{1}{p}} \leq L(\tau_1, \tau_2) k^{(\frac{1}{\gamma+\theta(\gamma)} - \frac{1}{p})(\gamma+\theta(\gamma))}, \\ & k > 0, \quad 1 \leq p < \gamma + \theta(\gamma), \end{aligned}$$

we have proved the term on the right-hand side of (3.44) tends to zero for $k \rightarrow \infty$.

It follows from the relations (3.29), (3.34) and Corollary 3.3 that the function

$$D(t) = \int_{\Omega} v(t, x) dx$$

is continuous and globally bounded on the whole real line $\mathbb{R}$. Moreover, by virtue of (3.44), the inequality

$$D(\tau_2) - D(\tau_1) + \frac{d}{\lambda + 2\mu} \int_{\tau_1}^{\tau_2} \Psi(D(t)) dt \leq 0$$

holds for any $\tau_1 < \tau_2$.

Consequently, $D \equiv 0$, in other words, we have proved (1.22). Moreover, combining this fact with (3.17), we get (1.23), (1.25) and (1.26). The strong convergence of the ϱ_{n,t_n} implies that

$$\overline{\varrho \mathbf{f}} = \bar{\varrho} \bar{\mathbf{f}}, \quad \overline{p(\varrho)} = p(\bar{\varrho}), \quad \text{and} \quad \overline{\varrho \Phi} = \bar{\varrho} \bar{\Phi}.$$

Thus the relation (3.24) being taken into account, $\bar{\varrho}$, $\bar{\mathbf{u}}$, $\bar{\Phi}$ is a globally defined weak solution (1.1)–(1.4).

Moreover, make use of (3.10), (3.16) and (1.23), we have

$$\begin{aligned} \int_I E[\bar{\varrho}, \bar{\mathbf{u}}, \bar{\Phi}](t) dt &= \int_I \left(\frac{1}{2} \|\sqrt{\bar{\varrho}} \bar{\mathbf{u}}\|_2^2 + \frac{a}{\gamma - 1} \|\bar{\varrho}\|_\gamma^\gamma - \frac{1}{8\pi g} \|\nabla \bar{\Phi}\|_2^2 \right) dt \\ &\leq \liminf_{n \rightarrow \infty} \int_I \left(\frac{1}{2} \|\sqrt{\varrho_{n,t_n}} \mathbf{u}_{n,t_n}\|_2^2 + \frac{a}{\gamma - 1} \|\varrho_{n,t_n}\|_\gamma^\gamma \right. \\ &\quad \left. - \frac{1}{8\pi g} \|\nabla \Phi_{n,t_n}\|_2^2 \right) dt \leq |I| E_\infty \end{aligned}$$

for arbitrary I , which yields (1.29).

To complete the proof of Theorem 1.3, we have only to show compactness of the momenta, i.e., the relation (1.27). Now, we can write

$$\varrho_{n,t_n} \mathbf{u}_{n,t_n} = (\varrho_{n,t_n})^{\frac{1}{2}} (\varrho_{n,t_n})^{\frac{1}{2}} \mathbf{u}_{n,t_n},$$

then, by virtue of (1.25),

$$(\varrho_{n,t_n})^{\frac{1}{2}} \rightarrow (\bar{\varrho})^{\frac{1}{2}} \quad \text{strongly in } L^2(J \times \Omega) \quad (3.45)$$

and, in view of (3.3),

$$(\varrho_{n,t_n})^{\frac{1}{2}} \mathbf{u}_{n,t_n} \rightarrow (\bar{\varrho})^{\frac{1}{2}} \bar{\mathbf{u}} \quad \text{weakly in } [L^2(J \times \Omega)]^3 \quad (3.46)$$

for any bounded $J \subset \mathbb{R}$. Moreover, making use of (3.16), we infer

$$\begin{aligned} &\|(\varrho_{n,t_n})^{\frac{1}{2}} \mathbf{u}_{n,t_n}\|_{L^2(J \times \Omega)}^2 \\ &= \int_J \int_\Omega \varrho_{n,t_n} |\mathbf{u}_{n,t_n}|^2 dx dt \rightarrow \int_J \int_\Omega \bar{\varrho} |\bar{\mathbf{u}}|^2 dx dt = \|(\bar{\varrho})^{\frac{1}{2}} \bar{\mathbf{u}}\|_{L^2(J \times \Omega)}^2. \end{aligned}$$

In other words, the convergence in (3.46) is strong and (1.27) follow. Then

$$\varrho_{n,t_n} \mathbf{f}_{n,t_n} \cdot \mathbf{u}_{n,t_n} \rightarrow \bar{\varrho} \bar{\mathbf{f}} \cdot \bar{\mathbf{u}} \quad \text{in } \mathcal{D}(J \times \Omega)$$

and consequently, the energy inequality (1.8) holds for $\bar{\varrho}$, $\bar{\mathbf{u}}$, $\bar{\Phi}$.

Since (1.24) follows immediately from (3.14). Theorem 1.3 has been proved.

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