Classical solutions of many-dimensional elliptic–parabolic free boundary problems

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Abstract. We consider a free boundary problem modeling fluid flow in a partially saturated porous media. In that context an unknown function represents the pressure and satisfies an elliptic equation in the saturated domain and a quasilinear parabolic equation in the unsaturated domain. The principal part of this work is the investigation of the smoothness properties of an unknown (free) boundary between domains of ellipticity and parabolicity.

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1. Statement of the problem and the main result

Let $\Omega \subset R^n$ be a doubly-connected domain such that $\partial\Omega = \Gamma^+ \cup \Gamma^-$, $\Gamma^+ \cap \Gamma^- = \emptyset$, $\Omega_T = \Omega \times (0, T)$, $\Gamma_T^\pm = \Gamma^\pm \times [0, T]$, the function $c(\sigma)$, $\sigma \in R^1$, be continuous piecewise smooth such that $c(\sigma) = 0$ for $\sigma \leq 0$ and $c'(\sigma) > 0$ for $\sigma \geq 0$.

Consider the problem

$$\frac{\partial}{\partial t} c(u) - \Delta u = 0 \quad \text{in } \Omega_T, \quad (1.1)$$

$$c(u(y, 0)) = c(u_0(y)) \quad \text{in } \Omega, \quad (1.2)$$

$$u(y, t) = g(y, t) \quad \text{on } \Gamma_T^\pm, \quad (1.3)$$

where $u_0(y)$, $g(y, t)$ are given functions. This problem arises in the theory of partially saturated flows in porous media where $c(u)$ models the concentration of the fluid.

The level set $\{u = 0\}$ splits the domain Ω on two regions in which Eq. (1.1) is respectively parabolic and elliptic. Our main goal to describe the regularity properties of the set $\{u = 0\}$ which specifies the unknown (free) boundary in this problem.

In the one dimensional case a complete analysis of the existence, uniqueness, and regularity of weak and viscosity solutions can be found in the works [1, 6, 8–10, 12, 13, 15, 17–19] (see also the references therein). The best result with respect to the smoothness of a free boundary in this case is contained in [6] where it was proved that the curve $\{u = 0\}$ is from the C^1 class. The paper [7] deals with the case of two spatial variables and it was shown that the free boundary is from the C class under some assumptions of monotonicity type on the data. The properties of weak solutions in the many-dimensional case are investigated in the works [1, 8].

In this paper we shall establish that under some assumptions about the smoothness of the initial data and the boundary conditions the nonlinear free boundary problem of the form (1.1)–(1.3) with $\Omega \subset R^n$, $n \geq 2$, has a classical solution locally in time and a free boundary can be represented as a certain function that has spatial and time derivatives from Hölder classes. Our approach resembles the method used in [3–5] to solve the Verigin and the Stefan or the Hele–Shaw problems.

For each moment t we denote by $\Gamma(t) \subset \Omega$ a desired free boundary, $\Gamma(t) \cap \Gamma^\pm = \emptyset$. Let $\Omega^\pm(t)$ be the subdomains of Ω such that $\partial\Omega^\pm(t) = \Gamma(t) \cup \Gamma^\pm$. For the sake of simplicity we suppose

$$c(u) = \begin{cases} 0, & u \leq 0, \\ u, & u \geq 0. \end{cases} \quad (1.4)$$

From the proof of the solvability of the free boundary problem one can see that it is only important for us that $c'(u) > 0$ for $u \geq 0$. Since we consider Problem (1.1)–(1.3) as a free boundary problem we assume that parallel with u is continuous on $\Gamma(t)$ the flux $(\nabla u, \nu(t))$ is continuous on $\Gamma(t)$ as well (cf. [13]), where $\nu(t)$ is the unit normal to $\Gamma(t)$ directed into $\Omega^+(t)$.

We shall study the free boundary problem for the system

$$L_0^+ u^+(y, t) \equiv \frac{\partial u^+(y, t)}{\partial t} - \Delta_y u^+(y, t) = 0 \quad \text{in } \Omega^+(t), \quad t \in (0, T), \quad (1.5)$$

$$L_0^- u^-(y, t) \equiv -\Delta_y u^-(y, t) = 0 \quad \text{in } \Omega^-(t), \quad t \in (0, T), \quad (1.6)$$

$$u^+(y, t) = u^-(y, t) = 0, \quad \text{on } \Gamma(t), \quad t \in (0, T), \quad (1.7)$$

$$\frac{\partial u^+(y, t)}{\partial \nu(t)} = \frac{\partial u^-(y, t)}{\partial \nu(t)}, \quad \text{on } \Gamma(t), \quad t \in (0, T), \quad (1.8)$$

$$u^\pm(y, t) = g^\pm(y, t), \quad \text{on } \Gamma_T^\pm, \quad (1.9)$$

$$u^\pm(y, 0) = u_0^\pm(y), \quad \text{on } \overline{\Omega^\pm}, \quad (1.10)$$

$$\Gamma \equiv \Gamma(0), \quad \Omega^\pm = \Omega^\pm(0) \text{ are given.} \quad (1.11)$$

We assume that $\Gamma, \Gamma^\pm \in C^{4+\alpha}$, $g^\pm(y, 0) \in C^{4+\alpha}(\Gamma^\pm)$ and the function $u_0^- \in C^{4+\alpha}(\overline{\Omega^-})$ and satisfies the conditions

$$\Delta_y u_0^-(y) = 0 \text{ in } \Omega^-(0), \quad u_0^- = g^-(y, 0) \quad \text{on } \Gamma^-, \quad u_0^- = 0 \quad \text{on } \Gamma. \quad (1.12)$$

We will also assume that

$$u_0^+ = 0 \quad \text{on } \Gamma, \quad \frac{\partial u_0^+}{\partial \nu(0)} = \frac{\partial u_0^-}{\partial \nu(0)} \geq \gamma > 0, \quad \pm g^\pm(y, 0) > 0, \quad \pm u_0^\pm > 0 \quad \text{in } \Omega^\pm(0), \quad (1.13)$$

where γ is some number, $\Delta_y = \nabla_y^2$, $\nabla_y = (\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \dots, \frac{\partial}{\partial y_n})$.

The proposed method of the investigation consists of three steps:

- (i) Reducing of the free boundary problem to a problem in a fixed domain and next to a nonlinear functional equation $\varphi = K(\varphi)$ in some Banach space where vector φ represents the unknown free boundary and the functions $u^\pm$;
- (ii) Solving a model problem and a linear system in the fixed domain. This is the principal part of the paper;
- (iii) Proving that K is a contraction, so that there exists a unique fixed point.

We will use the Banach space $C^{l, l/2}(\overline{\Omega_T})$ of functions $u(x, t)$ with the norm ($l > 0$, l is not integer, $l = [l] + \{l\}$)

$$|u|_{\Omega_T}^{(l)} = \sum_{j=0}^{[l]} \sum_{2r+s=j} |D_t^r D_x^s u|^{(0)} + \sum_{2r+s=[l]} \langle D_t^r D_x^s u \rangle_{x, \Omega_T}^{(l-[l])} + \sum_{0 < l-2r-s < 2} \langle D_t^r D_x^s u \rangle_{t, \Omega_T}^{(\frac{l-2r-s}{2})}$$

where $|D_t^r D_x^s u|^{(0)} = \max_{\Omega_T} |D_t^r D_x^s u|$, $\langle u \rangle_{x, \Omega_T}^{(\alpha)}$ and $\langle u \rangle_{t, \Omega_T}^{(\alpha)}$ are Hölder's constants of a function $u(x, t)$ in x and t respectively. The Hölder space $C^l(\Omega)$ of functions $u(x)$ is defined in a similar way.

Introduce the seminorm [16]

$$[u]_{\Omega_T}^{(\alpha, \beta)} = \sup_{(x, t), (y, \tau) \in \overline{\Omega_T}} \frac{|u(x, t) - u(y, t) - u(x, \tau) + u(y, \tau)|}{|x - y|^\alpha |t - \tau|^\beta}, \quad \alpha, \beta \in (0, 1),$$

and the spaces $S^{i+\alpha}(\overline{\Omega_T})$, $i = 0, 1, 2$, $E^{2+\alpha}(\overline{\Omega_T})$ with the norms

$$\begin{aligned} |u|_{S^\alpha(\overline{\Omega_T})} &\equiv |u|_{\Omega_T}^{(\alpha)} + [u]_{\Omega_T}^{(\alpha, \alpha/2)}, \\ |u|_{S^{1+\alpha}(\overline{\Omega_T})} &\equiv |u|_{\Omega_T}^{(1+\alpha)} + [u]_{\Omega_T}^{(\alpha, \frac{1+\alpha}{2})} + [u_x]_{\Omega_T}^{(\alpha, \alpha/2)}, \\ |u|_{S^{2+\alpha}(\overline{\Omega_T})} &\equiv |u|_{\Omega_T}^{(2+\alpha)} + [u_x]_{\Omega_T}^{(\alpha, \frac{1+\alpha}{2})} + [u_{xx}]_{\Omega_T}^{(\alpha, \alpha/2)} + [u_t]_{\Omega_T}^{(\alpha, \alpha/2)}, \\ |u|_{E^{2+\alpha}(\overline{\Omega_T})} &\equiv |u|_{\Omega_T}^{(1+\alpha)} + |u_x|_{\Omega_T}^{(1+\alpha)} + [u_x]_{\Omega_T}^{(\alpha, \frac{1+\alpha}{2})} + [u_{xx}]_{\Omega_T}^{(\alpha, \alpha/2)}. \end{aligned}$$

By using a local parametrization, we can define surfaces of the classes $C^{l, l/2}$ and $S^{i+\alpha}$ and corresponding spaces of functions on such surfaces.

Let us formulate now the main result.

Theorem 1.1. *Let the surfaces Γ , $\Gamma^\pm \in C^{4+\alpha}$, the functions $u_0^\pm(y) \in C^{4+\alpha}(\overline{\Omega^\pm})$, $g^\pm(y, \tau) \in C^{4+\alpha}(\Gamma_T^\pm)$, and the first order consistency conditions be fulfilled. Then*

there exists a unique solution of Problem (1.5)–(1.13) for some $T > 0$ such that $u^+ \in S^{2+\alpha}(\cup_{t \in [0, T]} \Omega^+(t))$, $u^- \in E^{2+\alpha}(\cup_{t \in [0, T]} \Omega^-(t))$, $\cup_{t \in [0, T]} \Gamma(t) \in S^{2+\alpha}$.

2. The nonlinear functional equation and linearization

2.1. Reduction of the free boundary problem (1.5)–(1.13) to a problem in a fixed domain

We now construct the diffeomorphic map $\Omega^\pm(t) \rightarrow \Omega^\pm$, $\Gamma(t) \rightarrow \Gamma$ that will lead us to a problem in the fixed domain. Let $\omega = (\omega_1, \omega_2, \dots, \omega_{n-1})$ be the internal parameter on Γ . We write Γ in the form $y = \bar{R}(\omega)$ ($\bar{R}(\omega) \subset \Omega$) and denote by $\bar{n}(\omega)$ the unite normal to Γ directed into Ω^+ . Let $\gamma_0 > 0$ be such that surfaces $\{y = \bar{R}(\omega) + \eta \bar{n}(\omega), \quad |\eta| < 2\gamma_0\}$ have no self-intersections and intersections with $\Gamma^\pm$. On the set

$$N = \{y \in R^n : \text{dist}(y, \Gamma) < 3/2\gamma_0\}$$

we introduce the local coordinates (ω, η) by

$$y(\omega, \eta) = \bar{R}(\omega) + \eta \bar{n}(\omega), \quad \bar{R}(\omega) \subset \Gamma.$$

We represent $\Gamma(t)$ in the form

$$\Gamma(t) = \{(y, t) : y = \bar{R}(\omega) + \bar{n}(\omega) \rho(\omega, t)\},$$

where $\rho(\omega, t)$ is unknown and it is assumed to satisfy the conditions

$$|\rho(\omega, t)| \leq c_0 \gamma_0, \quad \rho(\omega, 0) = 0, \quad 0 < c_0 < 1. \quad (2.1)$$

Let $\chi(\lambda) \in C_0^\infty$ be such that $\chi(\lambda) = 1$ if $|\lambda| < \gamma_0/4$, $\chi(\lambda) = 0$ if $|\lambda| > 3\gamma_0/4$, $|\chi^{(k)}(\lambda)| \leq c_1/\gamma_0^k$, $k = 1, 2, 3$. We choose c_0 in (2.1) such that $c_0 < 1/(2c_1)$, then $1 + \chi'(\lambda)\rho(\omega, t) > 1/2$. Following Hanzawa [11] (see also [5] for details), we introduce the map $e_\rho : R_x^n \times [0, T] \rightarrow R_y^n \times [0, T]$ by

$$\begin{aligned} y &= x \quad \text{if } \text{dist}(x, \Gamma) > 3\gamma_0/4, \\ y &= x_0(\omega) + [\lambda(x) + \chi(\lambda(x))\rho(\omega, t)]\bar{n}(\omega(x)) \quad \text{if } \text{dist}(x, \Gamma) \leq 3\gamma_0/4, \\ t &= t, \end{aligned} \quad (2.2)$$

where $\omega = \omega(x)$ and $\lambda = \lambda(x)$ are the local coordinates of $x \in N$, $x_0(\omega)$ is the projection of x on Γ . In the strip N we have

$$y = x + \chi(\lambda(x))\rho(\omega, t)\bar{n}(\omega(x)). \quad (2.3)$$

Under the condition $c_0 < 1/(2c_1)$ the map e_ρ is a homeomorphism and

$$e_\rho(\omega, \lambda, t) = (\omega, \lambda(x) + \chi(\lambda(x))\rho(\omega, t), t),$$

i.e., $\omega(x) = \omega(y)$, $\eta(y) = \lambda(x) + \chi(\lambda(x))\rho(\omega, t)$. Note that in the local variables the surface $\Gamma(t)$ is given by

$$\Phi_\rho(y, t) = \eta(y) - \rho(\omega, t) = 0.$$

By e_ρ the domains $\Omega^\pm$ are mapped onto $\Omega^\pm(t)$ for each t and System (1.5)–(1.13) is transformed into another system in the cylinders $\Omega_T^\pm$ for the functions $v^\pm(x, t) = u^\pm(e_\rho(x, t)) = u^\pm(y, t) \circ e_\rho(x, t)$, $(x, t) \in \Omega_T^\pm$, and $\rho(\omega, t)$.

Denote by $\nabla_\rho = E_\rho(x, t)\nabla$, where $\nabla \equiv \nabla_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ and $E_\rho(x, t)$ is the inverse matrix to the Jacobi matrix of the transform $e_\rho: x \rightarrow y$ so that $\nabla_y = \nabla_\rho$, $\Delta_y = \nabla_\rho^2$. Since $\rho(\omega, 0) = 0$ we have $E_\rho(x, 0) = I$. In view of (2.3)

$$h_\rho(x, t) \equiv \frac{\partial x}{\partial t}(y, t) \big|_{(y, t) = e_\rho(x, t)} = -\bar{n}(\omega) \chi(\lambda) \frac{\partial \rho(\omega, t)}{\partial t}, \quad (2.4)$$

hence,

$$\frac{\partial u^+}{\partial t} = \frac{\partial v^+}{\partial t} + (h_\rho, \nabla_x) v^+.$$

Now we are ready to rewrite System (1.5)–(1.13) in the new variables:

$$L_\rho^+ v^+(x, t) = \frac{\partial v^+}{\partial t} + (h_\rho, \nabla_x) v^+ - \nabla_\rho^2 v^+ = 0 \quad \text{in } \Omega_T^+, \quad (2.5)$$

$$L_\rho^+ v^-(x, t) = -\nabla_\rho^2 v^- = 0 \quad \text{in } \Omega_T^-, \quad (2.6)$$

$$v^+(x, t) = v^-(x, t) = 0, \quad \text{on } \Gamma_T, \quad (2.7)$$

$$\frac{\partial v^+(x, t)}{\partial n} - \frac{\partial v^-(x, t)}{\partial n} = 0, \quad \text{on } \Gamma_T, \quad (2.8)$$

$$v^\pm(x, 0) = v_0^\pm(x), \quad \text{on } \Omega^\pm, \quad \rho(\omega, 0) = 0, \quad (2.9)$$

$$v^\pm(x, t) = g^\pm(x, t) \quad \text{on } \Gamma_T^\pm \quad (2.10)$$

$$\Delta_x v_0^- = 0, \quad v_0^- = g^-(x, 0) \quad \text{on } \Gamma^-, \quad v_0^- = 0 \quad \text{on } \Gamma, \quad (2.11)$$

$$v_0^+ = 0 \quad \text{on } \Gamma, \quad \frac{\partial v_0^+}{\partial n} = \frac{\partial v_0^-}{\partial n} \geq \gamma > 0, \quad \pm g(x, 0) > 0. \quad (2.12)$$

System (2.5)–(2.10) can be written briefly in the form

$$\mathbb{A}(\psi) = F, \quad \psi = (v^+, v^-, \rho), \quad (2.13)$$

where the nonlinear operator $\mathbb{A}$ is defined by the left-hand sides of (2.5)–(2.10) and $F(x, t)$ is defined by the initial data and the boundary conditions on $\Gamma_T^\pm$.

2.2. Linearization of the problem $\mathbb{A}(\psi) = F$

From the definition of h_ρ in (2.4) it follows

$$(h_\rho, \nabla_x) v^+ \big|_{t=0} = - \frac{\partial v_0^+}{\partial n} \frac{\partial \rho(\omega, t)}{\partial t} \big|_{t=0}.$$

By Eqs. (2.7) and (1.13) we get from (2.5)

$$\rho_1(\omega) \equiv \frac{\partial \rho(\omega, t)}{\partial t} \big|_{t=0} = -\Delta v_0^+ / (\partial v_0^+ / \partial n), \quad v_0^+ = u_0^+.$$

From (1.5) we can calculate

$$u_1^+(y) \equiv \frac{\partial u^+(y, t)}{\partial t} \big|_{t=0} = \Delta_y u_0^+.$$

We extend the functions $u_0^\pm$ from $\Omega^\pm$ such that the extended functions again are in $C^{4+\alpha}$ and then construct the functions $w^\pm(y, t)$ on Ω_T and $\sigma(\omega, t)$ on Γ_T with the properties (see [14], Chapter 4)

$$\begin{aligned} w^\pm(y, 0) &= u_0^\pm(y), \quad \frac{\partial w^+(y, t)}{\partial t} \Big|_{t=0} = u_1^+(y), \\ \sigma(y, 0) &= 0, \quad \frac{\partial \sigma(\omega, t)}{\partial t} \Big|_{t=0} = \rho_1(\omega). \end{aligned} \quad (2.14)$$

Moreover,

$$|w^\pm|_{C^{4+\alpha, \frac{4+\alpha}{2}}(\overline{\Omega}_T)} + |\sigma|_{C^{4+\alpha, \frac{4+\alpha}{2}}(\Gamma_T)} \leq C \left(|u_0^+|_{\Omega^+}^{(4+\alpha)} + |u_0^-|_{\Omega^-}^{(4+\alpha)} \right). \quad (2.15)$$

Here and forthcoming we denote by C various constants that are defined only by the given data.

Let $e_\sigma(x, t)$ be defined as in (2.2) with $\rho(\omega, t)$ changed by $\sigma(\omega, t)$. We consider now e_σ as the map $R_T^n \rightarrow R_T^n$. One can check that for the function $w^\pm \circ e_\sigma$

$$\lim_{\varepsilon \rightarrow 0} \frac{w^\pm \circ e_{\sigma+\varepsilon\delta} - w^\pm \circ e_\sigma}{\varepsilon} = \left(\frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi(\lambda(x)) \delta(\omega(x), t). \quad (2.16)$$

Introduce the new dependent variables

$$\begin{aligned} \delta(\omega, t) &\equiv \rho(\omega, t) - \sigma(\omega, t), \theta^\pm(x, t) \equiv v^\pm(x, t) - w^\pm \circ e_\sigma - \left(\frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \\ &\quad \times \chi(\lambda) \delta(\omega, t), \end{aligned} \quad (2.17)$$

and remark that for any function f

$$(L_0^\pm f) \circ e_\rho = L_\rho(f \circ e_\rho),$$

where the operators $L_0^\pm$ are defined in (1.5) and (1.6).

The next step is to single out the linear part of the nonlinear operator A from (2.13). After simple identical transforms we get from System (2.5)–(2.10) a new system

$$\begin{aligned} L_0^\pm \theta^\pm(x, t) &= \left\{ (L_0^\pm - L_\sigma^\pm) \theta^\pm(x, t) - (L_0^\pm w^\pm) \circ e_{\sigma+\delta} \right\} \\ &\quad + \left\{ L_\sigma^\pm \left(w^\pm \circ e_{\sigma+\delta} - w^\pm \circ e_\sigma - \left(\frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi(\lambda) \delta \right) \right. \\ &\quad \left. - (L_{\sigma+\delta}^\pm - L_\sigma^\pm) \left(\theta^\pm + \left(\frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \chi \delta \right) + (L_{\sigma+\delta}^\pm - L_\sigma^\pm) \right. \\ &\quad \left. \times (w^\pm \circ e_{\sigma+\delta} - w^\pm \circ e_\sigma) \right\} \\ &\equiv F_1^\pm(x, t; \theta^\pm, \delta) + F_2^\pm(x, t; \theta^\pm, \delta), \quad \text{in } \Omega_T^\pm, \end{aligned} \quad (2.18)$$

$$\theta^\pm(x, t) + \left(\frac{\partial w^\pm}{\partial \lambda} \circ e_\sigma \right) \delta = -w^\pm \circ e_\sigma \equiv F_4^\pm(x, t), \quad \text{on } \Gamma_T, \quad (2.19)$$

$$\begin{aligned} \frac{\partial \theta^+}{\partial n}(x, t) - \frac{\partial \theta^-}{\partial n}(x, t) &= \left(\frac{\partial}{\partial n} w^- \circ e_\sigma - \frac{\partial}{\partial n} w^+ \circ e_\sigma \right) \\ &+ \left(\frac{\partial}{\partial n} \left(\frac{\partial w^-}{\partial \lambda} \circ e_\sigma \right) - \frac{\partial}{\partial n} \left(\frac{\partial w^+}{\partial \lambda} \circ e_\sigma \right) \right) \delta \equiv F_5(x, t; \delta), \quad \text{on } \Gamma_T, \end{aligned} \quad (2.20)$$

$$\theta^\pm(x, t) = g^\pm(x, t) - w^\pm \circ e_\sigma \equiv F_3^\pm(x, t), \quad \text{on } \Gamma_T^\pm. \quad (2.21)$$

We recall that $\chi(\lambda) = 1$ near Γ , and e_ρ and e_σ are the identical map out of some vicinity of Γ , and in view of the definition of the coordinate λ we have $\partial u^\pm / \partial \lambda = \partial u^\pm / \partial n$ as $x \in \Gamma$.

If D is some Banach space of functions $u(x, t)$, we shall denote by D_0 the subspace of D of functions that are equal to zero at $t = 0$ together with all admissible time derivatives. From our construction it follows that we are looking for a solution to (2.18)–(2.21) in classes

$$\theta^+(x, t) \in S_0^{2+\alpha}(\overline{\Omega}_T^+), \quad \theta^-(x, t) \in E_0^{2+\alpha}(\overline{\Omega}_T^-), \quad \delta(\omega, t) \in S_0^{2+\alpha}(\Gamma_T). \quad (2.22)$$

It is important now that for such classes with nought the next relations, analogous to ones in the space $C_0^{l, l/2}$ (see [14, Ch.4]), take place

$$\begin{aligned} |u|_{C_0^{l', l'/2}(\overline{\Omega}_T)} &\leq CT^{\frac{l-l'}{2}} |u|_{C_0^{l, l/2}(\overline{\Omega}_T)}, \quad |u|_{S_0^{l'}(\overline{\Omega}_T)} \leq CT^{\frac{l-l'}{2}} |u|_{S_0^l(\overline{\Omega}_T)}, \quad l' < l, \\ |u|_{E_0^{2+\alpha'}(\overline{\Omega}_T)} &\leq CT^{\frac{\alpha-\alpha'}{2}} |u|_{E_0^{2+\alpha}(\overline{\Omega}_T)}, \quad \alpha' < \alpha, \end{aligned} \quad (2.23)$$

$$|uv|_{C_0^{l, l/2}(\overline{\Omega}_T)} \leq CT^{\frac{l-[l]}{2}} |u|_{C_0^{l, l/2}(\overline{\Omega}_T)} |v|_{C_0^{l, l/2}(\overline{\Omega}_T)}.$$

Introduce the Banach space $\mathcal{H} \equiv S_0^{2+\alpha}(\overline{\Omega}_T^+) \times E_0^{2+\alpha}(\overline{\Omega}_T^-) \times S_0^{2+\alpha}(\Gamma_T)$ of vector functions $\psi = (\theta^+, \theta^-, \delta)$ with the norm

$$\|\psi\| \equiv |\theta^+|_{S^{2+\alpha}(\overline{\Omega}_T^+)} + |\theta^-|_{E^{2+\alpha}(\overline{\Omega}_T^-)} + |\delta|_{S^{2+\alpha}(\Gamma_T)}. \quad (2.24)$$

One can see that the right-hand sides of Eqs. (2.18)–(2.21) are equal to zero at $t = 0$ provided $\psi \in \mathcal{H}$ and, by using inequalities (2.24), the next estimates are valid for $\psi, \psi_1, \psi_2 \in \mathcal{H}$ and small T :

$$\begin{aligned} |F_1^\pm(x, t; \psi)|_{S^\alpha(\overline{\Omega}_T^\pm)} &\leq C(\|\psi\|)(1 + \|\psi\|)T^{\alpha/2}, \quad |F_2^\pm(x, t; \psi)|_{S^\alpha(\overline{\Omega}_T^\pm)} \\ &\leq C(\|\psi\|)\|\psi\|^2, \end{aligned} \quad (2.25)$$

$$|F_1^\pm(x, t; \psi_2) - F_1^\pm(x, t; \psi_1)|_{S^\alpha(\overline{\Omega}_T^\pm)} \leq C(\|\psi_1\|, \|\psi_2\|)T^{\alpha/2}\|\psi_2 - \psi_1\|;$$

$$|F_2^\pm(x, t; \psi_2) - F_2^\pm(x, t; \psi_1)|_{S^\alpha(\overline{\Omega}_T^\pm)} \leq C(\|\psi_1\|, \|\psi_2\|)(\|\psi_1\| + \|\psi_2\|)\|\psi_2 - \psi_1\|; \quad (2.26)$$

$$\left| F_3^\pm(x, t) \right|_{S^{2+\alpha}(\Gamma_T^\pm)} + \left| F_4^\pm(x, t) \right|_{S^{2+\alpha}(\Gamma_T)} \leq CT^{\alpha/2}; \quad (2.27)$$

$$\left| F_5^\pm(x, t; \psi) \right|_{S^{1+\alpha}(\overline{\Omega_T^\pm})} \leq C(\|\psi\|)(1 + \|\psi\|)T^{\alpha/2},$$

$$\left| F_5^\pm(x, t; \psi_2) - F_5^\pm(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^\pm})} \leq C(\|\psi_1\|, \|\psi_2\|)T^{\alpha/2} \|\psi_2 - \psi_1\|, \quad (2.28)$$

where here and below $C(\|\psi\|)$, $C(\|\psi_1\|, \|\psi_2\|)$ are bounded for bounded $\|\psi\|$, $\|\psi_1\|$, $\|\psi_2\|$. The estimates (2.26)–(2.28) are the consequence of that fact that the functions F_i consist of terms either more smooth that is required to be evaluated in a corresponding norms or terms that have “quadratic” nonlinearity with respect to the functions $\theta_1^\pm$, $\theta_2^\pm$, and δ .

In the following estimates we take advantage of the special form of the functions F_1 and F_2 (see [16])

$$F_i^-(x, t; \psi) = \sum_{k,m} F_i^{(k,m)}(x, t; \psi) \frac{\partial G_i^{(k,m)}(x, t; \psi)}{\partial x_m}, \quad i = 1, 2, \quad (2.29)$$

where the functions $F_1^{(k,m)}(x, t; \psi)$ or $G_1^{(k,m)}(x, t; \psi)$ are more regular than functions from $S^{1+\alpha}(\overline{\Omega_T^-})$ and the functions $F_2^{(k,m)}(x, t; \psi)$, $G_2^{(k,m)}(x, t; \psi)$ depend on θ^- , δ “quadratically”. For instance, the first term $(L_0^- - L_\sigma^-)\theta^-$ in (2.18) can be represented in the form

$$(L_0^- - L_\sigma^-)\theta^- = (\nabla_\sigma^2 - \nabla^2) \theta^- = \sum_{i,j=1}^n (a_{ij}(\omega, \sigma, \nabla\sigma) - \delta_{ij}) \frac{\partial}{\partial x_i} \left(b_{ij}(\omega, \sigma, \nabla\sigma) \frac{\partial \theta^-}{\partial x_j} \right), \quad (2.30)$$

where δ_{ij} is Kronecker’s symbol, $a_{ij}(\omega, \sigma, \nabla\sigma)$, $b_{ij}(\omega, \sigma, \nabla\sigma)$ are given smooth functions of ω , σ , $\nabla\sigma$, $a_{ij} \equiv \delta_{ij}$ at $t = 0$. Really, simple calculations give

$$\nabla_\sigma^2 \theta^- = \sum_{i,j=1}^n a_{i,j}(\omega, \sigma, \sigma_\omega) \frac{\partial^2 \theta^-}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(\omega, \sigma, \sigma_{\omega\omega}) \frac{\partial \theta^-}{\partial x_i},$$

from there the equality (2.30) follows. In a similar way Eq. (2.29) is obtained. These representations give the possibility to get the next estimates which will be applied below in the proof of Lemma 3.1:

$$\left| F_1^{(k,m)} \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_1^{(k,m)} \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \leq C(\|\psi\|)(1 + \|\psi\|)T^{\alpha/2}, \quad (2.31)$$

$$\begin{aligned} & \left| F_1^{(k,m)}(x, t; \psi_2) - F_1^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_1^{(k,m)}(x, t; \psi_2) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \\ & + \left| F_1^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_1^{(k,m)}(x, t; \psi_2) - G_1^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \\ & \leq C(\|\psi_1\|, \|\psi_2\|)T^{\alpha/2} \|\psi_2 - \psi_1\|, \end{aligned} \quad (2.32)$$

$$\left| F_2^{(k,m)} \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_2^{(k,m)} \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \leq C(\|\psi\|) \|\psi\|^2, \quad (2.33)$$

$$\left| F_2^{(k,m)}(x, t; \psi_2) - F_2^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_2^{(k,m)}(x, t; \psi_2) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})}$$

$$\begin{aligned}
& + \left| F_2^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \left| G_2^{(k,m)}(x, t; \psi_2) - G_2^{(k,m)}(x, t; \psi_1) \right|_{S^{1+\alpha}(\overline{\Omega_T^-})} \\
& \leq C(\|\psi_1\|, \|\psi_2\|)(\|\psi_1\| + \|\psi_2\|) \|\psi_2 - \psi_1\|. \quad (2.34)
\end{aligned}$$

As to the inequalities (2.31)–(2.34), they follow from the structure of corresponding expressions and can be checked directly for each term in the definitions of F_1^- and F_2^- in (2.18).

3. The model problem

First we consider an auxiliary problem. Let $z = (z_1, z_2, \dots, z_n) \in R_\pm^n = \{z : \pm z_n > 0\}$, $z' = (z_1, z_2, \dots, z_{n-1})$, $R_{\pm T}^n = R_\pm^n \times [0, T]$, $R_T^{n-1} = R^{n-1} \times [0, T]$. Let $u(z, t)$ be a solution of the Dirichlet problem

$$\begin{aligned}
\varepsilon \frac{\partial u}{\partial t}(z, t) - \Delta u &= f_1(z, t) \frac{\partial f_2(z, t)}{\partial z_i} \equiv f(z, t), \quad i = \overline{1, n}, \quad \text{in } R_{+T}^n, \\
u(z', 0, t) &= F(z', t), \quad \text{on } R_T^{n-1}, \quad u(z, t) \in S_0^{2+\alpha}(\overline{R_{+T}^n}), \quad (3.1)
\end{aligned}$$

where $f_i \in S^{1+\alpha}(\overline{R_{+T}^n})$, $i = 1, 2$, and at least one of them belongs to the space $S_0^{1+\alpha}(\overline{R_{+T}^n})$, $F \in S_0^{2+\alpha}(\overline{R_T^{n-1}})$, and at least one of f_i and also F either are finite in z or are rapidly decreasing at infinity.

Lemma 3.1. *Problem (3.1) has a unique solution bounded at infinity and*

$$|u|_{E^{2+\alpha}(\overline{R_{+T}^n})} \leq C(|f_1|_{S^{1+\alpha}(\overline{R_{+T}^n})} |f_2|_{S^{1+\alpha}(\overline{R_{+T}^n})} + |F|_{S^{2+\alpha}(\overline{R_T^{n-1}})}), \quad (3.2)$$

$$|u|_{S^{2+\alpha}(\overline{R_{+T}^n})} \leq C(\varepsilon)(|f|_{S^\alpha(\overline{R_{+T}^n})} + |F|_{S^{2+\alpha}(\overline{R_T^{n-1}})}), \quad (3.3)$$

where the constant C in (3.2) does not depend on ε .

The proof of this lemma will be given in Appendix.

Now we formulate the principal model problem such that nonlinear Problem (2.18)–(2.21) will inherit the main feature of this model problem. We wish to find unknown functions $u^+(z, t)$ on R_{+T}^n , $u^-(z, t)$ on R_{-T}^n , and $\rho(z', t)$ on R_T^{n-1} by the conditions

$$\frac{\partial u^+}{\partial t} - \Delta u^+(z, t) = F_1^+(z, t), \quad (z, t) \in R_{+T}^n, \quad (3.4)$$

$$\varepsilon \frac{\partial u^-}{\partial t} - \Delta u^-(z, t) = F_{21}^-(z, t) \frac{\partial F_{22}^-(z, t)}{\partial z_k} \equiv F_2^-(z, t), \quad k = \overline{1, n}, \quad (z, t) \in R_{-T}^n, \quad (3.5)$$

$$u^\pm(z', 0, t) + A\rho(z', t) = F_3^\pm(z', t), \quad z' \in R^{n-1}, \quad (3.6)$$

$$\frac{\partial u^+}{\partial z_n}(z', 0, t) - \frac{\partial u^-}{\partial z_n}(z', 0, t) = F_4(z', t), \quad z' \in R^{n-1}, \quad (3.7)$$

$$u^\pm \in S_0^{2+\alpha}(\overline{R_{\pm T}^n}), \quad \rho(z', t) \in S_0^{2+\alpha}(\overline{R_T^{n-1}}), \quad (3.8)$$

where $A = \text{const.} > 0$, $F_1^+ \in S_0^\alpha(\overline{R_{+T}^n})$, $F_{2i}^- \in S^{1+\alpha}(\overline{R_{-T}^n})$, $i = 1, 2$, and at least one of the two functions F_{2i}^- belongs to the space $S_0^{1+\alpha}(\overline{R_{-T}^n})$, $F_3^\pm \in S_0^{2+\alpha}(\overline{R_T^{n-1}})$, $F_4 \in S_0^{1+\alpha}(\overline{R_T^{n-1}})$, and the functions from the right-hand sides of (3.4)–(3.7) are finite or are rapidly decreasing at infinity.

Lemma 3.2. *For sufficiently small ε , $\varepsilon \leq \varepsilon_0$, there exists a unique solution to Eqs. (3.4)–(3.7) and*

$$\begin{aligned} & |u^+|_{S^{2+\alpha}(\overline{R_{+T}^n})} + |u^+|_{E^{2+\alpha}(\overline{R_{-T}^n})} + |\rho|_{S^{2+\alpha}(\overline{R_T^{n-1}})} \\ & \leq C \left(|F_1^+|_{S^\alpha(\overline{R_{+T}^n})} + |F_{21}^-|_{S^{1+\alpha}(\overline{R_{-T}^n})} |F_{22}^-|_{S^{1+\alpha}(\overline{R_{-T}^n})} + |F_3^+|_{S^{2+\alpha}(\overline{R_T^{n-1}})} \right. \\ & \quad \left. + |F_3^-|_{S^{2+\alpha}(\overline{R_T^{n-1}})} + |F_4|_{S^{1+\alpha}(\overline{R_T^{n-1}})} \right) \equiv CM(T), \end{aligned} \quad (3.9)$$

$$|u^-|_{S^{2+\alpha}(\overline{R_{-T}^n})} \leq C(\varepsilon)M(T), \quad (3.10)$$

where the constant C in (3.9) does not depend on ε and k .

Proof. The proof for $A \geq \gamma > 0$ is similar to the case $A = 1$. By using the standard parabolic theory and Lemma 3.1 we can assume that

$$F_1^+ \equiv 0, \quad F_{2i}^- \equiv 0, \quad i = 1, 2, \quad F_3 \equiv 0, \quad F_4 \equiv f(z', t).$$

Next, we can extend all given functions in (3.4)–(3.7) as finite functions in the domain $t \geq 0$ and by zero in $t < 0$ such that the extended functions remained in the same classes. Introduce the Fourier and Laplace transform as

$$\tilde{g}(\lambda, z_n, p) \equiv \frac{1}{(2\pi)^{n/2}} \int_0^\infty e^{-pt} dt \int_{R^{n-1}} e^{(-iz' \cdot \lambda)} g(z', t) dz', \quad \lambda \in R^{n-1}, \quad \text{Re } p > 0. \quad (3.11)$$

□

Then we get from (3.4)–(3.7) by applying transform (3.11)

$$-\frac{d^2}{dz_n^2} \widetilde{u^+} + (p + \lambda^2) \widetilde{u^+} = 0, \quad z_n > 0, \quad (3.12)$$

$$-\frac{d^2}{dz_n^2} \widetilde{u^-} + (\varepsilon p + \lambda^2) \widetilde{u^-} = 0, \quad z_n < 0, \quad (3.13)$$

$$\widetilde{u^\pm}(\lambda, 0, p) + A\tilde{\rho}(\lambda, p) = 0, \quad z_n = 0, \quad (3.14)$$

$$\left. \frac{d\widetilde{u^+}}{dz_n} \right|_{z_n=0} - \left. \frac{d\widetilde{u^-}}{dz_n} \right|_{z_n=0} = \tilde{f}(\lambda, p), \quad z_n = 0, \quad (3.15)$$

$$|\widetilde{u^\pm}| \leq C, \quad |z_n| \rightarrow \infty. \quad (3.16)$$

Simple calculations give

$$\begin{aligned}\widetilde{u}^+(\lambda, z_n, p) &= -\widetilde{\rho}(\lambda, p)e^{-z_n\sqrt{p+\lambda^2}}, \quad z_n \geq 0, \\ \widetilde{u}^-(\lambda, z_n, p) &= -\widetilde{\rho}(\lambda, p)e^{z_n\sqrt{\varepsilon p+\lambda^2}}, \quad z_n \leq 0, \\ \widetilde{\rho}(\lambda, p) &= \frac{1}{\sqrt{p+\lambda^2} + \sqrt{\varepsilon p+\lambda^2}} \widetilde{f}(\lambda, p) \equiv \widetilde{G}_\varepsilon(\lambda, p) \widetilde{f}(\lambda, p)\end{aligned}\quad (3.17)$$

and it is easy to see the denominator of the last equation does not vanish for $\operatorname{Re}(p) > 0$,

$$\left| \sqrt{p+\lambda^2} + \sqrt{\varepsilon p+\lambda^2} \right| > 0.$$

Let $G_\varepsilon(z', t)$ be the inverse Fourier-Laplace transformation of $\widetilde{G}_{\varepsilon c}(\lambda, p)$ then

$$\rho(z', t) = \int_0^t d\tau \int_{R^{n-1}} G_\varepsilon(z' - \xi, t - \tau) f(\xi, \tau) d\xi = G_\varepsilon * f \equiv M_\varepsilon f, \quad (3.18)$$

so that $\widetilde{G}_\varepsilon(\lambda, p)$ is the symbol of the operator M_ε . Remark that by virtue of the smoothness properties of the function $f(x, t)$ and its behavior at infinity all above performed operations are justified.

We now consider the limit case $\varepsilon = 0$ and the operator M_0 , moreover, let $M_{0,a}$, $a \geq 1$, be the operator with the symbol

$$\widetilde{g}_a(\lambda, p) = \frac{1}{\sqrt{p+a^2\lambda^2} + a|\lambda|}, \quad (3.19)$$

that is

$$M_{0,a}f \equiv \int_0^t d\tau \int_{R^{n-1}} g_a(z' - \xi, t - \tau) f(\xi, \tau) d\xi = g_a * f, \quad (3.20)$$

where $g_a(z', t)$ is the inverse Fourier-Laplace preimage of $\widetilde{g}_a(\lambda, p)$ in (3.19).

Lemma 3.3. *The operator $M_{0,a}$ in (3.20) is a bounded operator from $S^{i+\alpha}(\overline{R_T^{n-1}})$ into $S^{i+1+\alpha}(\overline{R_T^{n-1}})$, i.e.,*

$$|M_{0,a}f|_{S^{i+1+\alpha}(\overline{R_T^{n-1}})} \leq C |f|_{S^{i+\alpha}(\overline{R_T^{n-1}})}, \quad (3.21)$$

where the constant C does not depend on $a \geq 1$.

Proof of Lemma 3.3. The proof is based on utilizing of the explicit representation of the kernel $g_a(z', t)$ in (3.20) and on estimates analogous to [14, Ch.4], as it will be done in the proof of Lemma 3.1. For convenience and simplicity let us redenote the variable z' by x only for the time of this proof.

The preimage of the function $\widetilde{g}_a(\lambda, p)$ in (3.19) is

$$\begin{aligned}g_a(x, t) &= Ca^{-(n-1)}t^{-\frac{n}{2}} \int_1^\infty \omega^{-\frac{n+2}{2}} \left[(n-1) - 2\frac{x^2}{4a^2t\omega} \right] e^{-\frac{x^2}{4a^2t\omega}} d\omega \\ &\quad + Ca^{-(n-1)}t^{-\frac{n}{2}} e^{-\frac{x^2}{4a^2t}} \equiv g_1(x, t) + g_2(x, t)\end{aligned}\quad (3.22)$$

that can be obtained by using the representation $\operatorname{Erfc}(y) = -\int_1^\infty \frac{d}{d\alpha} \operatorname{Erfc}(\alpha y) d\alpha$ and ([2], 5.3.(4)). In accordance with (3.22) we have

$$M_{0,a}f(x, t) = g_1 * f + g_2 * f \equiv \rho_1(x, t) + \rho_2(x, t).$$

The properties of the function $\rho_2(x, t)$, in particular, the estimates of the form (3.21) for $\rho_2(x, t)$ can be obtained similarly as in [14, Ch.4], so that we consider more carefully the function $\rho_1(x, t)$. For the kernel $g_1(x, t)$ the next estimates

$$|D_t^r D_x^s g_1(x, t)| \leq C_{rs} a^{-(n-1)-|s|t-\frac{n}{2}-r-\frac{|s|}{2}} \int_1^\infty \omega^{-\frac{n+2+|s|}{2}} e^{-\frac{x^2}{4a^2t\omega}} d\omega \quad (3.23)$$

are valid.

In order to demonstrate the method of obtaining of the estimate (3.21) we consider only the part of one

$$[\rho_{1x_i x_j}]^{(\alpha, \frac{\alpha}{2})} \leq C |f|_{S^{1+\alpha}(\overline{R_T^{n-1}})}. \quad (3.24)$$

For $h > 0$ we define

$$s(x, t) \equiv \frac{\rho_1(x, t) - \rho_1(x, t-h)}{h^{\alpha/2}}, \quad \varphi(x, t) \equiv \frac{f(x, t) - f(x, t-h)}{h^{\alpha/2}}.$$

Since the kernel $g_{1x_i}(x, t)$ is odd we have the next representation

$$s_{x_i x_j}(x, t) = \int_0^t d\tau \int_{R^{n-1}} g_{1x_i}(x - \xi, t - \tau) [\varphi_{\xi_j}(\xi, \tau) - \varphi_{x_j}(x, \tau)] d\xi.$$

Consider the difference $(x, \bar{x} \in R^{n-1}, |x - \bar{x}| = \delta)$

$$\begin{aligned} & s_{x_i x_j}(x, t) - s_{\bar{x}_i \bar{x}_j}(\bar{x}, t) \\ &= \int_0^t d\tau \int_{|x-\xi| \leq 2\delta} g_{1x_i}(x - \xi, t - \tau) [\varphi_{\xi_j}(\xi, \tau) - \varphi_{x_j}(x, \tau)] d\xi \\ &\quad - \int_0^t d\tau \int_{|\bar{x}-\xi| \leq 2\delta} g_{1\bar{x}_i}(\bar{x} - \xi, t - \tau) [\varphi_{\xi_j}(\xi, \tau) - \varphi_{\bar{x}_j}(\bar{x}, \tau)] d\xi \\ &\quad + \int_0^t d\tau [\varphi_{\bar{x}_j}(\bar{x}, \tau) - \varphi_{x_j}(x, \tau)] \int_{|x-\xi| > 2\delta} g_{1x_i}(x - \xi, t - \tau) d\xi \\ &\quad + \int_0^t d\tau \int_{|x-\xi| > 2\delta} [g_{1x_i}(x - \xi, t - \tau) - g_{1\bar{x}_i}(\bar{x} - \xi, t - \tau)] [\varphi_{\xi_j}(\xi, \tau) - \varphi_{\bar{x}_j}(\bar{x}, \tau)] d\xi \\ &\equiv I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where $I_3 = 0$ in view of the oddness of the function $g_{1x_i}(x, t)$. For I_1 by virtue of (3.23) and $n \geq 2$ we have the estimate

$$|I_1| \leq C [f_{x_j}]^{(\alpha, \frac{\alpha}{2})} a^{-(n-1)-1} \int_1^\infty \omega^{-\frac{n+3}{2}} d\omega \int_{|x-\xi| \leq 2\delta} |x - \xi|^\alpha d\xi \int_0^t (t - \tau)^{-\frac{n}{2}-\frac{1}{2}} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)\omega}} d\tau$$

and the change of the variable $(t - \tau) = \eta \frac{(x - \xi)^2}{a^2 \omega}$ gives

$$\begin{aligned} |I_1| &\leq C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} a^{-1} \int_1^\infty \omega^{-2} d\omega \int_{|x - \xi| \leq 2\delta} |x - \xi|^{-(n-1)+\alpha} d\xi \int_0^{\frac{ta^2\omega}{(x-\xi)^2}} \eta^{-\frac{n+1}{2}} e^{-\frac{1}{4\eta}} d\eta \\ &\leq C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} |x - \bar{x}|^\alpha. \end{aligned} \quad (3.25)$$

The integral I_2 is evaluated analogously taking into account that $\{|x - \xi| < 2|x - \bar{x}|\} \subset \{|\bar{x} - \xi| < 3|x - \bar{x}|\}$ and by expanding of the domain of integration.

To estimate I_4 we apply the mean value theorem to the difference $g_{1x_i}(x - \xi, t - \tau) - g_{1\bar{x}_i}(\bar{x} - \xi, t - \tau)$. Let $x_\theta \in [x, \bar{x}]$. On the set $\{|x - \xi| > 2|x - \bar{x}|\}$ the values $|\bar{x} - \xi|$ and $|x_\theta - \xi|$ are equivalent and $\{|x - \xi| > 2|x - \bar{x}|\} \subset \{|\bar{x} - \xi| > |x - \bar{x}|\}$. In addition we use the estimate (3.23) for $|D_x^2 g_1|$ that leads to the inequality

$$\begin{aligned} |I_4| &\leq C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} |x - \bar{x}| \\ &\quad \times a^{-(n-1)-2} \int_1^\infty \omega^{-\frac{n+4}{2}} d\omega \int_{|x - \xi| > 2\delta} |\bar{x} - \xi|^\alpha d\xi \int_0^t (t - \tau)^{-\frac{n}{2}-1} e^{-\frac{(x-\xi)^2}{4a^2(t-\tau)\omega}} d\tau \\ &\leq C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} |x - \bar{x}| a^{-1} \int_1^\infty \omega^{-2} d\omega \int_{|\bar{x} - \xi| > \delta} |\bar{x} - \xi|^{-(n-1)-1+\alpha} d\xi \int_0^{\frac{ta^2\omega}{(\bar{x}-\xi)^2}} \eta^{-\frac{n+2}{2}} e^{-\frac{1}{4\eta}} d\eta \\ &\leq C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} |x - \bar{x}| \delta^{-1+\alpha} = C[f_{x_j}]^{(\alpha, \frac{\alpha}{2})} |x - \bar{x}|^\alpha, \end{aligned}$$

where we used the same change of the variable as in (3.25). Collecting now the estimates for I_k , $k = \overline{1, 4}$, we get

$$\langle s_{x_i x_j} \rangle_x^{(\alpha)} \leq C |f|_{S^{1+\alpha}(\overline{R_T^{n-1}})}$$

and, consequently, by the definition of $s(x, t)$, we obtain (3.24).

This completes the proof of Lemma 3.3. $\square$

Corollary 3.1. *The operator M_ε defined in (3.18) is a bounded operator from $S^{1+\alpha}(R_T^{n-1})$ into $S^{2+\alpha}(R_T^{n-1})$ for sufficiently small ε , i.e.,*

$$|M_\varepsilon f|_{S^{2+\alpha}(R_T^{n-1})} \leq C |f|_{S^{1+\alpha}(R_T^{n-1})}. \quad (3.26)$$

Proof of Corollary 3.1. Denote by m_ε the operator with the symbol

$$\widetilde{m}_\varepsilon = \sqrt{p + \lambda^2} + \sqrt{\varepsilon p + \lambda^2}, \quad \widetilde{m}_0 = \widetilde{m}_\varepsilon|_{\varepsilon=0},$$

such that $M_\varepsilon = m_\varepsilon^{-1}$, $M_0 = m_0^{-1}$. The operator $m_\varepsilon - m_0$ has the symbol

$$\begin{aligned} \widetilde{m}_\varepsilon - \widetilde{m}_0 &= \sqrt{\varepsilon p + \lambda^2} - |\lambda| = \frac{\varepsilon p}{\sqrt{\varepsilon p + \lambda^2} + |\lambda|} \\ &= \varepsilon^{1/2} \frac{1}{\sqrt{p + \varepsilon^{-1} \lambda^2} + \varepsilon^{-1/2} |\lambda|} p \equiv \varepsilon^{1/2} \widetilde{m_{0, \varepsilon^{-1/2}}} \cdot p, \end{aligned}$$

One can see that the operator with the symbol p maps $S^{2+\alpha}(\overline{R_T^{n-1}})$ into $S^\alpha(\overline{R_T^{n-1}})$. Due to Lemma 3.3 the operator with the symbol $\widetilde{m_{0, \varepsilon^{-1/2}}}$ is uniformly in $\varepsilon^{-1/2} \geq 1$

bounded from $S^\alpha(\overline{R_T^{n-1}})$ into $S^{1+\alpha}(\overline{R_T^{n-1}})$, therefore

$$\|m_\varepsilon - m_0\|_{S^{2+\alpha}(\overline{R_T^{n-1}}) \rightarrow S^{1+\alpha}(\overline{R_T^{n-1}})} \leq C\varepsilon^{1/2}.$$

Since by Lemma 3.3, the operator m_0 has the bounded inverse operator $M_0 = m_0^{-1}$ from $S^{1+\alpha}(\overline{R_T^{n-1}})$ to $S^{2+\alpha}(\overline{R_T^{n-1}})$, it follows that for small ε operator m_ε also has the bounded inverse operator from $S^{1+\alpha}(\overline{R_T^{n-1}})$ into $S^{2+\alpha}(\overline{R_T^{n-1}})$ and

$$\|M_\varepsilon\|_{S^{1+\alpha}(\overline{R_T^{n-1}}) \rightarrow S^{2+\alpha}(\overline{R_T^{n-1}})} = \|m_\varepsilon^{-1}\|_{S^{1+\alpha}(\overline{R_T^{n-1}}) \rightarrow S^{2+\alpha}(\overline{R_T^{n-1}})} \leq C, \quad \varepsilon \leq \varepsilon_0,$$

as was to be proved in (3.26).

Now we can finish the proof of Lemma 3.2. From Corollary 3.1 it follows that the function $\rho(z', t)$ from (3.18) satisfies the inequality

$$|\rho|_{S^{2+\alpha}(\overline{R_T^{n-1}})} \leq C|f|_{S^{1+\alpha}(\overline{R_T^{n-1}})}, \quad \varepsilon \leq \varepsilon_0,$$

where the constant C does not depend on ε .

Given function $\rho(z', t)$ we find the function $u^\pm(z', 0, t)$ from (3.6) and use the standard procedure to construct $u^\pm(z', z_n, t)$. The estimates (3.9), (3.10) follow from the parabolic theory and Lemma 3.1. It completes the proof of Lemma 3.2.

4. Linear problem in an arbitrary domain

Consider the problem to determine unknown functions $u^\pm(x, t)$ on $\Omega_T^\pm$ and $\rho(x, t)$ on Γ_T

$$L_0^+ u^+(x, t) \equiv \frac{\partial u^+}{\partial t} - \Delta u^+ = F_1^+(x, t), \quad (x, t) \in \Omega_T^+, \quad (4.1)$$

$$L_\varepsilon^- u^-(x, t) \equiv \varepsilon \frac{\partial u^-}{\partial t} - \Delta u^- = F_{21}^-(x, t) \frac{\partial F_{22}^-(x, t)}{\partial x_i} \equiv F_2^-(x, t), \quad (x, t) \in \Omega_T^-, \quad (4.2)$$

$$u^\pm(x, t) + A(x, t)\rho(x, t) = F_3^\pm(x, t), \quad (x, t) \in \Gamma_T, \quad (4.3)$$

$$\frac{\partial u^+}{\partial n} - \frac{\partial u^-}{\partial n} = F_4(x, t), \quad (x, t) \in \Gamma_T, \quad (4.4)$$

$$u^\pm(x, t) = F_5^\pm(x, t), \quad (x, t) \in \Gamma_T^\pm, \quad (4.5)$$

$$u^\pm \in S_0^{2+\alpha}(\overline{\Omega_T^\pm}), \quad \rho \in S_0^{2+\alpha}(\Gamma_T), \quad (4.6)$$

where $i = \overline{1, n}$, $A(x, t) \in C^{2+\alpha, \frac{2+\alpha}{2}}(\Gamma_T)$ is a given function, $A(x, t) \geq \nu > 0$, $F_1^+ \in S_0^\alpha(\Omega_T^+)$, $F_{21}^- \in S^{1+\alpha}(\overline{\Omega_T^-})$, $F_{22}^- \in S_0^{1+\alpha}(\overline{\Omega_T^-})$, $F_3^\pm \in S_0^{2+\alpha}(\Gamma_T)$, $F_4 \in S_0^{1+\alpha}(\Gamma_T)$, $F_5^\pm \in S_0^{2+\alpha}(\Gamma_T^\pm)$, $\varepsilon \leq \varepsilon_0$, ε_0 is from the Lemma 3.2. We shall write this problem in the form $L\psi = F$, where $\psi = (u^+, u^-, \rho)$, $F = \{F_i\}$.

Lemma 4.1. *For $\varepsilon \leq \varepsilon_0$ and for any $T > 0$ the problem (4.1)–(4.6) has unique solution from the pointed above spaces and the next estimates are valid ($\psi = (u^+, u^-, \rho)$)*

$$\begin{aligned} \|\psi\| &\equiv |u^+|_{S^{2+\alpha}(\overline{\Omega_T^+})} + |u^-|_{E^{2+\alpha}(\overline{\Omega_T^-})} + |\rho|_{S^{2+\alpha}(\Gamma_T)} \\ &\leq C_T \left\{ |F_1^+|_{S^\alpha(\overline{\Omega_T^+})} + |F_{21}^-|_{S^{1+\alpha}(\overline{\Omega_T^-})} |F_{22}^-|_{S^{1+\alpha}(\overline{\Omega_T^-})} + |F_3^+|_{S^{2+\alpha}(\Gamma_T)} \right. \\ &\quad \left. + |F_3^-|_{S^{2+\alpha}(\Gamma_T)} + |F_4|_{S^{1+\alpha}(\Gamma_T)} + |F_5^+|_{S^{2+\alpha}(\Gamma_T^+)} + |F_5^-|_{S^{2+\alpha}(\Gamma_T^-)} \right\} \equiv C_T M(T), \end{aligned} \quad (4.7)$$

$$|u^+|_{S^{2+\alpha}(\overline{\Omega_T^+})} + |u^-|_{S^{2+\alpha}(\overline{\Omega_T^-})} + |\rho|_{S^{2+\alpha}(\Gamma_T)} \leq C_T(\varepsilon)M(T), \quad (4.8)$$

where C_T in (4.7) does not depend on ε .

Proof. Solvability of the problem (4.1)–(4.6) and the estimate (4.8) for $\varepsilon > 0$ can be proved by the classical method of construction of a regularizing operator (an operator that is close to the inverse operator) in the same way, as it was done in [14, Ch.4] for a boundary value problem for a parabolic equation. In our case we will get the estimate (4.7) for solutions $\psi_\varepsilon = (u_\varepsilon^+, u_\varepsilon^-, \rho_\varepsilon)$ that is not depending on ε and will show that there exists some subsequence that goes to a limit as $\varepsilon \rightarrow 0$. $\square$

To derive the estimate (4.7) we apply the Schauder method. Below we use also apart from inequalities (2.24) the interpolation inequality in $C^l(\overline{\Omega})$ of the form

$$|u|_\Omega^{(l')} \leq \delta |u|_\Omega^{(l)} + C_\delta |u|_\Omega^{(0)}, \quad l' < l, \quad (4.9)$$

where δ is an arbitrary number.

Let us demonstrate the distinctive peculiarity of our case, that is the use of the space $E^{2+\alpha}$ for the unknown u^- , and connected with that the special form of the right-hand side of the Eq. (4.2), and derive the interior estimate for $u^-(x, t)$. Take a point x_0 , which lies strictly inside Ω^- , and let $B_r = B_r(x_0) \subset \Omega^-$ is the closed ball with the center at x_0 of the radius $r \leq r_0$. Let $\eta(x) \in C^\infty(R^n)$, $\eta(x) \equiv 1$ in $B_{r/2}(x_0)$, $\eta(x) \equiv 0$ outside $B_r(x_0)$, $|D^s \eta| \leq C(s)r_0^{-|s|}$. Denote $v^- = \eta(x)u^-$ and multiply Eq. (4.2) by $\eta(x)$, then

$$\begin{aligned} \varepsilon \frac{\partial v^-}{\partial t} - \Delta v^- &= (\eta(x)F_{21}^-) \frac{\partial F_{22}^-}{\partial x_i} - 2 \sum_j \eta_{x_j} \frac{\partial u^-}{\partial x_j} - (\Delta \eta)u^- \equiv \widetilde{F}_2 \\ &= \sum_{k,l,j} f_{1klj} \frac{\partial f_{2kl}}{\partial x_j}, \quad v^- = 0 \text{ on } \partial B_r(x_0), \quad v^- \in E_0^{2+\alpha}(R^n \times [0, T]). \end{aligned} \quad (4.10)$$

The right-hand side in (4.10) has the form we need. For example,

$$\eta_{x_j} \frac{\partial u^-}{\partial x_j} \equiv f_1 \frac{\partial f_2}{\partial x_j}, \quad f_1 \in S^{1+\alpha}(B_r \times [0, t]), f_2 \in S_0^{1+\alpha}(B_r \times [0, t]). \quad (4.11)$$

By virtue of the inequality (4.9) and the definition of the norm in the spaces $S^{1+\alpha}$, it is easy to check that in (4.11) $|f_1|_{S^{1+\alpha}} \leq C(r)$ and

$$\begin{aligned} |f_2|_{S^{1+\alpha}(B_r \times [0, T])} &\leq |u^-|_{S^{1+\alpha}(B_r \times [0, T])} \leq \delta |\nabla u^-|_{S^{1+\alpha}(B_r \times [0, T])} + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})} \\ &\leq \delta |u^-|_{E^{2+\alpha}(\overline{\Omega_T^-})} + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}. \end{aligned} \quad (4.12)$$

To get (4.12) we consider, for example, $[f_2]^{(\alpha, \frac{1+\alpha}{2})} = [u^-]^{(\alpha, \frac{1+\alpha}{2})}$. Let $g_h(x, t) = [u^-(x, t) - u^-(x, t-h)]/h^{\frac{1+\alpha}{2}}$. Then

$$[f_2]^{(\alpha, \frac{1+\alpha}{2})} = [u^-]^{(\alpha, \frac{1+\alpha}{2})} = \sup_{h>0} \langle g_h \rangle_x^{(\alpha)}.$$

Due to (4.9) $\langle g_h \rangle_x^{(\alpha)} \leq \delta |\nabla g_h|^{(\alpha)} + C_\delta |g_h|^{(0)}$, i.e., taking sup in h ,

$$[f_2]^{(\alpha, \frac{1+\alpha}{2})} = [u^-]^{(\alpha, \frac{1+\alpha}{2})} \leq \delta [\nabla u^-]^{(\alpha, \frac{1+\alpha}{2})} + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}.$$

Similar estimates lead to (4.12). Next we apply the same approach as in proof of Lemma 3.1 and get the inequality

$$\begin{aligned} |u^-|_{E^{2+\alpha}(B_{r/2} \times [0, T])} &\leq CM(T) + \delta |u^-|_{E^{2+\alpha}(\overline{\Omega_T^-})} + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})} \\ &\leq CM(T) + \delta \|\psi\| + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}. \end{aligned} \quad (4.13)$$

Analogous reasonings are applied for all others points $x_0 \in \overline{\Omega^\pm}$ by using properties of solutions to the model problems, which was considered above or are well known. For the points $x_0 \in \Gamma$ we use the model problem in Lemma 3.2. Let us remark also, that all the reasonings are carrying out for small enough T and the help of the inequalities (2.24) to estimate arising lower differential order terms. Taking now *sup* over all x_0 in the left-hand side of inequalities of the kind (4.13), we obtain with some $\beta > 0$

$$\|\psi\| \leq CM(T) + (\delta + CT^\beta) \|\psi\| + C_\delta \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})},$$

and, hence, for sufficiently small δ and T

$$\|\psi\| \leq CM(T) + C \langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}. \quad (4.14)$$

Note, since we consider T to be sufficiently small, we do not have in the right-hand side of (4.14) such terms as $\langle u^+ \rangle_t^{(\frac{1+\alpha}{2})}$ or $\langle \rho \rangle_t^{(\frac{1+\alpha}{2})}$ because, for example,

$$\langle u^+ \rangle_t^{(\frac{1+\alpha}{2})} \leq T^{\frac{1-\alpha}{2}} \left| \frac{\partial u^+}{\partial t} \right|_{\Omega_T^+}^{(0)} \leq T^{\frac{1-\alpha}{2}} \|\psi\|.$$

The characteristic peculiarity of our elliptic-parabolic case is that we can not do in the same way with $\langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}$. To estimate $\langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})}$ introduce for $h > 0$ the function $v(x, t) = [u^-(x, t) - u^-(x, t-h)]/h^{\frac{1+\alpha}{2}} \equiv \Delta_h u^-$, assuming that all

functions are zero in $t < 0$. It follows from the relations (4.2), (4.3), and (4.6) the function $v(x, t)$ satisfies the problem

$$\begin{aligned} \varepsilon \frac{\partial v}{\partial t} - \Delta v = & \left\{ [\Delta_h F_{21}(x, t)] \frac{\partial F_{22}(x, t)}{\partial x_i} - [\Delta_h F_{22}(x, t)] \frac{\partial F_{21}(x, t-h)}{\partial x_i} \right\} \\ & + \frac{\partial}{\partial x_i} [F_{21}(x, t-h) \Delta_h F_{22}(x, t)] \equiv \varphi_1(x, t) + \frac{\partial}{\partial x_i} \varphi_2(x, t), \quad (x, t) \in \Omega_T^-, \end{aligned} \quad (4.15)$$

$$v(x, t) = \Delta_h F_5^-(x, t) \equiv \varphi_3(x, t), \quad \text{on } \Gamma_T^-, \quad (4.16)$$

$$v(x, t) = \Delta_h F_3^- - A(x, t) \Delta_h \rho - \rho(x, t-h) \Delta_h A \equiv \varphi_4(x, t), \quad \text{on } \Gamma_T, \quad (4.17)$$

$$v(x, 0) = 0.$$

We represent $v(x, t)$ as the sum $v = v_1 + v_2$, where v_1 satisfies the relations (4.15)–(4.17) with $\varphi_1 \equiv \varphi_2 \equiv 0$ and v_2 satisfies (4.15)–(4.17) with $\varphi_3 \equiv \varphi_4 \equiv 0$. For the function v_1 by virtue of the maximum principle we have independently of $\varepsilon > 0$ the estimate

$$|v_1| \leq \langle F_5 \rangle_{t, \Gamma_T^+}^{(\frac{1+\alpha}{2})} + C \langle \rho \rangle_{t, \Gamma_T^+}^{(\frac{1+\alpha}{2})} + C |\rho|_{\Gamma_T^+}^{(0)} \leq CM(T) + CT^{\frac{1-\alpha}{2}} \|\psi\|. \quad (4.18)$$

On the other hand an estimate for $|v_2|_{\Omega_T^-}^{(0)}$ can be obtained from well known local estimates of the maximum of the modulus of the solution via integral norms of the solution and integral norms of the right-hand sides of the equation (recall that $v_2 = 0$ on $\partial\Omega$). Preliminary we change the variable $t = \varepsilon\tau$ and estimate $v(x, \varepsilon\tau)$ in $\Omega_{T/\varepsilon}$ locally over $\tau \in [\tau_0, \tau_0 + 1]$. We get by [14, Ch.3, Th.8.1]

$$\begin{aligned} & |v_2|_{\Omega^- \times [\tau_0, \tau_0+1]}^{(0)} \\ & \leq C \left(\|\varphi_1\|_{2, \Omega^- \times [\tau_0-1, \tau_0+2]} + \|\varphi_2\|_{2, \Omega^- \times [\tau_0-1, \tau_0+2]} + \|v_2\|_{2, \Omega^- \times [\tau_0-1, \tau_0+2]} \right), \end{aligned}$$

which gives after returning to the variable t

$$\begin{aligned} |v_2|_{\Omega^- \times [\varepsilon\tau_0, \varepsilon\tau_0+\varepsilon]}^{(0)} & \leq C \left(\left(\frac{1}{\varepsilon} \int_{\varepsilon\tau_0-\varepsilon}^{\varepsilon\tau_0+2\varepsilon} dt \int_{\Omega^-} \varphi_1^2 dx \right)^{\frac{1}{2}} + \left(\frac{1}{\varepsilon} \int_{\varepsilon\tau_0-\varepsilon}^{\varepsilon\tau_0+2\varepsilon} dt \int_{\Omega^-} \varphi_2^2 dx \right)^{\frac{1}{2}} \right. \\ & \quad \left. + \left(\frac{1}{\varepsilon} \int_{\varepsilon\tau_0-\varepsilon}^{\varepsilon\tau_0+2\varepsilon} dt \int_{\Omega^-} v_2^2 dx \right)^{\frac{1}{2}} \right) \\ & \leq C \left(\sup_{0 \leq t \leq T} \|\varphi_1(\cdot, t)\|_2 + \sup_{0 \leq t \leq T} \|\varphi_2(\cdot, t)\|_2 + \sup_{0 \leq t \leq T} \|v_2(\cdot, t)\|_2 \right). \end{aligned}$$

Taking here $\sup$ in $\tau_0 \in [0, T/\varepsilon]$ we go to the estimate independent of ε of the form

$$|v_2|_{\Omega_T^-}^{(0)} \leq CM(T) + C \sup_{0 \leq t \leq T} \|v_2(\cdot, t)\|_2, \quad (4.19)$$

where $\|\cdot\|_2$ is the norm in $L_2(\Omega^-)$.

To estimate independently of ε the norm $\sup_{0 \leq t \leq T} \|v_2(\cdot, t)\|_2$ we multiply the Eq. (4.15) by $2v_2(x, t)$ and integrate by parts over Ω^- . We arrive to

$$\varepsilon \frac{d}{dt} \int_{\Omega^-} v_2^2(x, t) dx + 2 \int_{\Omega^-} |\nabla v_2|^2 dx = 2 \int_{\Omega^-} \varphi_1 v_2 dx - 2 \int_{\Omega^-} \varphi_2 \frac{\partial v_2}{\partial x_i} dx. \quad (4.20)$$

Since $v_2 = 0$ on $\partial\Omega^-$, we can apply Poincaré's inequality, so that $\|v_2\|_{W_2^1(\Omega^-)} \leq C(\int_{\Omega^-} |\nabla v_2|^2 dx)^{1/2}$. Next we estimate each integral in the right-hand side of (4.20) by Young's inequality with δ and carry the terms $\delta \int_{\Omega^-} v_2^2(x, t) dx$ and $\delta \int_{\Omega^-} v_{2x_i}^2(x, t) dx$ to the left-hand side of (4.20) for sufficiently small δ , then

$$\begin{aligned} \varepsilon \frac{d}{dt} \int_{\Omega^-} v_2^2(x, t) dx + C \int_{\Omega^-} |\nabla v_2|^2 dx &\leq C \left(\int_{\Omega^-} \varphi_1^2 dx + \int_{\Omega^-} \varphi_2^2 dx \right) \\ &\leq CM^2(T) \equiv g_T. \end{aligned} \quad (4.21)$$

We denote $\int_{\Omega^-} v_2^2(x, t) dx \equiv y(t)$ and apply Poincaré's inequality once more, then it follows

$$\varepsilon y'(t) + Cy(t) \leq g_T, \quad y(0) = 0.$$

Hence,

$$y(t) \leq g_T \frac{1}{\varepsilon} \int_0^t e^{-\frac{C}{\varepsilon}(t-\tau)} d\tau = g_T \int_0^{t/\varepsilon} e^{-C\eta} d\eta \leq Cg_T.$$

It means that $\sup_{0 \leq t \leq T} \|v_2(\cdot, t)\|_2 \leq CM(T)$ independently of ε .

Thus, the last estimate and (4.19) provide $|v_2| \leq CM(T)$ that together with the estimate (4.18) gives

$$\langle u^- \rangle_{t, \Omega_T^-}^{(\frac{1+\alpha}{2})} \leq \sup_h |v|_{\Omega_T^-}^{(0)} \leq CT^{\frac{1-\alpha}{2}} \|\psi\| + CM(T). \quad (4.22)$$

Applying this estimate to (4.14) and choosing sufficiently small T_0 , we deduce the estimate (4.7) for $T \leq T_0$. Note that T_0 does not depend on the norms of the right-hand sides in (4.1)–(4.6), therefore we can repeat these estimates up to the time $T > 0$.

The solvability of the problem (4.1)–(4.6) for $\varepsilon = 0$ follows now from the estimate (4.7). From the sequence $(u_\varepsilon^+, u_\varepsilon^-, \rho_\varepsilon)$ we can extract a subsequence which converges correspondingly in the space $S^{2+\alpha'}(\overline{\Omega_T^+}) \times E^{2+\alpha'}(\overline{\Omega_T^-}) \times S^{2+\alpha'}(\overline{\Gamma_T})$ with $\alpha' < \alpha$. The limit vector function will satisfy the relations (4.1)–(4.6) and will belong to the space $S^{2+\alpha}(\overline{\Omega_T^+}) \times E^{2+\alpha}(\overline{\Omega_T^-}) \times S^{2+\alpha}(\overline{\Gamma_T})$.

This accomplish the proof of Lemma 4.1.

5. Original nonlinear problem: proof of Theorem 1.1

Set

$$\mathcal{B}_r = \{\varphi = (\theta^+, \theta^-, \delta) \in \mathcal{H}, \quad \|\varphi\| \leq r\}.$$

For any $\varphi \in \mathcal{B}_r$ we define the right-hand sides of Eqs. (2.18)–(2.21) so that we get the linear system of the form (4.1)–(4.6). Applying Lemma 4.1, we find an element $\bar{\varphi} \in \mathcal{H}$, i.e., $\bar{\varphi} = K(\varphi)$. By making use the estimates (2.26)–(2.28) and (4.7) for $\varepsilon = 0$, we deduce that for some $\beta > 0$

$$\|K(0)\| \leq CT^\beta, \quad \|K(\varphi_1) - K(\varphi_2)\| \leq C(T^\beta + \max_{i=1,2} \|\varphi_i\|) \|\varphi_1 - \varphi_2\|.$$

It means that for sufficiently small T and r the operator K is contractive and maps $\mathcal{B}_r$ into itself. Hence, there exists a unique solution to System (2.18)–(2.21) and, as a consequence, we get the proof of Theorem 1.1.

Appendix: Proof of Lemma 3.1

A solution of the problem (3.1) can be represented as the sum of the volume potential and the double layer potential

$$\begin{aligned} u(z, t) &= \int_0^t \int_{R_+^n} [\Gamma_\varepsilon(z' - \xi', z_n - \xi_n, t - \tau) - \Gamma_\varepsilon(z' - \xi', z_n + \xi_n, t - \tau)] f(\xi, \tau) d\xi d\tau \\ &\quad + \int_0^t \int_{R^{n-1}} G_\varepsilon(z' - \xi', z_n, t - \tau) F(\xi', \tau) d\xi' d\tau \equiv u_1(z, t) - u_2(z, t) + u_3(z, t), \end{aligned} \quad (\text{A.1})$$

where $\xi = (\xi_1, \dots, \xi_{n-1}, \xi_n) = (\xi', \xi_n) \in R^n$ and

$$\begin{aligned} \Gamma_\varepsilon(z, t) &= C\varepsilon^{\frac{n}{2}-1} t^{-\frac{n}{2}} e^{-\frac{z^2}{4t}\varepsilon}, \\ G_\varepsilon(z' - \xi', z_n, t) &= \frac{\partial}{\partial \xi_n} [\Gamma_\varepsilon(z' - \xi', z_n - \xi_n, t - \tau) - \Gamma_\varepsilon(z' - \xi', z_n + \xi_n, t - \tau)] \Big|_{\xi_n=0} \\ &= C\varepsilon^{\frac{n}{2}} t^{-\frac{n}{2}-1} e^{-\frac{(z' - \xi')^2 + z_n^2}{4t}\varepsilon}. \end{aligned}$$

The estimate (3.3) can be obtained for each potential $u_i(z, t)$, $i = 1, 2, 3$ separately by using the method from [14, Ch.4], but to prove (3.2) we exploit the special form of the function $f(z, t)$. As an example we derive the estimate of the term $[u_1 z]_{\overline{R_+^n}}^{(\alpha, \frac{1+\alpha}{2})}$ that is a part of the norm $|u_1|_{E^{2+\alpha}(\overline{R_+^n})}$. Only for the sake of brevity we assume now that $i \neq n$, then

$$u_{1z_j} = \int_0^t \int_{R_+^n} \Gamma_{\varepsilon z_j}(z - \xi, \tau) f_1(\xi, t - \tau) \frac{\partial}{\partial \xi_i} f_2(\xi, t - \tau) d\xi d\tau.$$

For any function $q(z, t)$ such that $q(z, t)$ is equal to zero as $t < 0$ denote

$$\triangle_h q \equiv \frac{q(z, t+h) - q(z, t)}{h^{\frac{1+\alpha}{2}}}.$$

Then

$$\begin{aligned}
 \Delta_h u_{1z_j} &= \int_0^{t+h} \int_{R_+^n} \Gamma_{\varepsilon z_j}(z - \xi, \tau) (\Delta_h f_1(\xi, t - \tau)) \frac{\partial}{\partial \xi_i} f_2(\xi, t - \tau) d\xi d\tau \\
 &\quad + \int_0^{t+h} \int_{R_+^n} \Gamma_{\varepsilon z_j}(z - \xi, \tau) \frac{\partial}{\partial \xi_i} [f_1(\xi, t + h - \tau) \Delta_h f_2(\xi, t - \tau)] d\xi d\tau \\
 &\quad - \int_0^{t+h} \int_{R_+^n} \Gamma_{\varepsilon z_j}(z - \xi, \tau) (\Delta_h f_2(\xi, t - \tau)) \frac{\partial}{\partial \xi_i} f_1(\xi, t + h - \tau) d\xi d\tau \\
 &\equiv v_1 + v_2 + v_3. \tag{A.2}
 \end{aligned}$$

To show the generic approach to estimating $\Delta_h u_{1z_j}$ we consider, for instance, the function $v_2(z, t)$. Denote by $g(z, t) \equiv f_1(z, t + h) \Delta_h f_2(z, t)$ and remark that the Hölder constant of the function $g(z, t)$ with respect to z satisfies the inequality

$$\langle g(z, t) \rangle_z^{(\alpha)} \leq C |f_1|_{S^{1+\alpha}(\overline{R_{+T}^n})} |f_2|_{S^{1+\alpha}(\overline{R_{+T}^n})}.$$

By integrating by parts, we transform the function $v_2(z, t)$ to the following form

$$\begin{aligned}
 v_2(z, t) &= \int_0^{t+h} \int_{R_+^n} \Gamma_{\varepsilon z_j}(z - \xi, \tau) \frac{\partial}{\partial \xi_i} [g(\xi, t - \tau) - g(z, t - \tau)] d\xi d\tau \\
 &= \int_0^{t+h} \int_{R_+^n} \Gamma_{\varepsilon \xi_j \xi_i}(z - \xi, \tau) [g(\xi, t - \tau) - g(z, t - \tau)] d\xi d\tau,
 \end{aligned}$$

and the last integral converges on account of $g(z, t)$ is the Hölder function. Let $z, \bar{z} \in \overline{R_+^n}$, $\delta = |z - \bar{z}|$, $K_{2\delta} = R_+^n \cap \{\xi : |z - \xi| \leq 2|z - \bar{z}| = 2\delta\}$. Let us represent the difference $v_2(z, t) - v_2(\bar{z}, t)$ as follows (cf. [14, Ch.4])

$$\begin{aligned}
 v_2(z, t) - v_2(\bar{z}, t) &= \int_0^{t+h} \int_{K_{2\delta}} \Gamma_{\varepsilon \xi_j \xi_i}(z - \xi, \tau) [g(\xi, t - \tau) - g(z, t - \tau)] d\xi d\tau \\
 &\quad - \int_0^{t+h} \int_{K_{2\delta}} \Gamma_{\varepsilon \xi_j \xi_i}(\bar{z} - \xi, \tau) [g(\xi, t - \tau) - g(\bar{z}, t - \tau)] d\xi d\tau \\
 &\quad + \int_0^{t+h} \int_{R_+^n \setminus K_{2\delta}} [\Gamma_{\varepsilon \xi_j \xi_i}(z - \xi, \tau) - \Gamma_{\varepsilon \xi_j \xi_i}(\bar{z} - \xi, \tau)] [g(\xi, t - \tau) - g(\bar{z}, t - \tau)] d\xi d\tau \\
 &\quad + \int_0^{t+h} [g(\bar{z}, t - \tau) - g(z, t - \tau)] d\tau \int_{R_+^n \setminus K_{2\delta}} \Gamma_{\varepsilon \xi_j \xi_i}(z - \xi, \tau) d\xi \equiv I_1 + I_2 + I_3 + I_4.
 \end{aligned}$$

We use the estimates of the fundamental solution $\Gamma_\varepsilon(z, t)$

$$|D_t^r D_z^s \Gamma_\varepsilon| \leq C_{rs} \varepsilon^{\frac{n}{2}-1+\frac{|s|}{2}} t^{-\frac{n}{2}-r-\frac{|s|}{2}} e^{-\gamma \frac{z^2}{t} \varepsilon}$$

to evaluate the terms I_k , $k = \overline{1, 4}$. For I_1 we have after the change of variable $\tau = \varepsilon(z - \xi)^2\omega$

$$\begin{aligned} |I_1| &\leq C \langle g \rangle_z^{(\alpha)} \varepsilon^{\frac{n}{2}} \int_{|z-\xi| < 2\delta} |z - \xi|^\alpha |z - \xi|^{-n} \varepsilon^{-\frac{n}{2}} d\xi \int_0^{\frac{t+h}{\varepsilon(z-\xi)^2}} \omega^{-\frac{n}{2}-1} e^{-\frac{\gamma}{\omega}} d\omega \\ &\leq C \langle g \rangle_z^{(\alpha)} \int_{|z-\xi| < 2\delta} |z - \xi|^{\alpha-n} d\xi = C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha. \end{aligned}$$

To estimate I_2 we take into account that $K_{2\delta} \subset \{\xi : |\bar{z} - \xi| \leq 3\delta\} \equiv \bar{K}_{3\delta}$ and expand the domain of integration to $\bar{K}_{3\delta}$ that gives

$$|I_2| \leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha.$$

The mean value theorem gives the estimate

$$\begin{aligned} |\Gamma_{\varepsilon\xi_j\xi_i}(z - \xi, \tau) - \Gamma_{\varepsilon\xi_j\xi_i}(\bar{z} - \xi, \tau)| &= |z - \bar{z}| |\nabla_z \Gamma_{\varepsilon\xi_j\xi_i}(z_\theta - \xi, \tau)| \\ &\leq C |z - \bar{z}| \varepsilon^{\frac{n}{2}-1+\frac{3}{2}} \tau^{-\frac{n}{2}-\frac{3}{2}} e^{-\gamma \frac{(z_\theta - \xi)^2}{\tau}} \varepsilon, \quad z_\theta \in [z, \bar{z}]. \end{aligned}$$

On the set $\{\xi : |z - \xi| > 2\delta\}$ quantities $|z - \xi|$, $|\bar{z} - \xi|$, and $|z_\theta - \xi|$ are equivalent, in particular, $1/2 \leq |z_\theta - \xi|/|\bar{z} - \xi| \leq 2$ (since $|\bar{z} - \xi| \geq |z - \bar{z}|$ and $|z_\theta - \xi| \geq |z - \bar{z}|$ on this set). With these remarks we can write

$$|I_3| \leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}| \int_{|\bar{z}-\xi| > \delta} \varepsilon^{\frac{n}{2}-1+\frac{3}{2}} |\bar{z} - \xi|^\alpha d\xi \int_0^{t+h} \tau^{-\frac{n}{2}-\frac{3}{2}} e^{-\gamma \frac{(\bar{z}-\xi)^2}{\tau}} \varepsilon d\tau,$$

and after changing of the variable $\tau = \varepsilon(\bar{z} - \xi)^2\omega$ we have

$$\begin{aligned} |I_3| &\leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}| \int_{|\bar{z}-\xi| > \delta} |\bar{z} - \xi|^{\alpha-n-1} d\xi \int_0^{\frac{t+h}{\varepsilon(\bar{z}-\xi)^2}} \omega^{-\frac{n}{2}-\frac{3}{2}} e^{-\frac{\gamma}{\omega}} d\omega \\ &\leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}| \delta^{\alpha-1} = C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha. \end{aligned}$$

As the preparatory step to consider I_4 we integrate by parts the internal integral in I_4 ,

$$I = \int_{R_+^n \setminus K_{2\delta}} \Gamma_{\varepsilon\xi_j\xi_i}(z - \xi, \tau) d\xi = \int_{\partial K_{2\delta}} \Gamma_{\varepsilon\xi_j}(z - \xi, \tau) \cos(\vec{n}, \xi_i) d\xi.$$

Since $\cos(\vec{n}, \xi_i) \neq 0$ only on $\partial K_{2\delta}$ where $|z - \xi| = 2\delta$ we get

$$|I| \leq C \delta^{n-1} \varepsilon^{\frac{n}{2}-1+\frac{1}{2}} \tau^{-\frac{n}{2}-\frac{1}{2}} e^{-\gamma \frac{\delta^2}{\tau}} \varepsilon,$$

and therefore

$$|I_4| \leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha \int_0^{t+h} \delta^{n-1} \varepsilon^{\frac{n}{2}-1+\frac{1}{2}} \tau^{-\frac{n}{2}-\frac{1}{2}} e^{-\gamma \frac{\delta^2}{\tau}} \varepsilon d\tau \leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha.$$

So, we proved that

$$|v_2(z, t) - v_2(\bar{z}, t)| \leq C \langle g \rangle_z^{(\alpha)} |z - \bar{z}|^\alpha \leq C \langle f_1 \rangle_z^{(\alpha)} [f_2]^{(\alpha, \frac{1+\alpha}{2})} |z - \bar{z}|^\alpha.$$

In a similar way we can treat the functions v_1 and v_2 from (A.2) and u_1 and u_2 from (A.1) so that

$$[u_{1z_j}]^{(\alpha, \frac{1+\alpha}{2})} = \sup_h \langle \Delta_h u_{1z_j}(z, t) \rangle_z^{(\alpha)} \leq C |f_1|_{S^{1+\alpha}(\overline{R_{T+}^n})} |f_2|_{S^{1+\alpha}(\overline{R_{T+}^n})}.$$

Other terms in the norm $|u|_{E^{2+\alpha}(\overline{R_{T+}^n})}$ can be considered similarly that gives (3.2).

It completes the proof of Lemma 3.1. $\square$

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