The initial boundary value problem for the nonlinear parabolic equation in unbounded domain

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Abstract. In this paper we consider the mixed problem for the equation $u_{tt} + A_1 u + A_2(u_t) + g(u_t) = f(x, t)$ in unbounded domain, where A_1 is a linear elliptic operator of the fourth order and A_2 is a nonlinear elliptic operator of the second order. Under natural assumptions on the equation coefficients and f we proof existence of a solution. This result contains, as a special case, some of known before theorems of existence. Essentially, in difference up to previous results we prove theorems of existence without the additional assumption on behavior of solution at infinity.

Mathematics Subject Classification (2000). Primary 35K55, Secondary 35K60.

Keywords. Initial boundary value problem, Nonlinear parabolic equation.

1. Introduction

Initial boundary value problems for nonlinear parabolic equations in unbounded domain were studied by many authors. Most of the known results about the existence and uniqueness require additional assumptions on the behavior at infinity by space variables x . So, such type results for boundary value problems of nonlinear parabolic equations of the fourth order with the second order time derivative in [18, 27] are received. In papers [8, 9, 17, 21, 24] similar results for the Cauchy problem of parabolic nonlinear equations was obtained. In [25] the authors have shown the uniqueness of a weak solution in the class of functions which do not grow faster than the function $e^{a|x|^\alpha}$ for $|x| \rightarrow \infty$.

The main goal of the given article is to present such conditions for initial boundary value problems of nonlinear parabolic equations with nonlinearities in unbounded domains by space variables, at which the existence of solution does not depend on its behaviour at $|x| \rightarrow +\infty$.

We consider an initial boundary value problem in the form which will allow to generalize above mentioned results about existence of solutions. Namely, we

consider the equation

$$\begin{aligned}
 A(u) \equiv & u_{tt}(x, t) + \sum_{i,j,k,l=1}^n (a_{ij}^{kl}(x) u_{x_i x_j}(x, t))_{x_k x_l} \\
 & - \sum_{i,j=1}^n (a_{ij}(x) u_{x_i}(x, t))_{x_j} - \sum_{i=1}^n (b_i(x) |u_{tx_i}(x, t)|^{p-2} u_{tx_i}(x, t))_{x_i} \\
 & + a_0(x) u(x, t) + c(x) |u_t(x, t)|^{q-2} u_t(x, t) = f(x, t)
 \end{aligned} \tag{1.1}$$

with the following boundary and initial conditions

$$u|_{S_T} = 0, \quad \frac{\partial u}{\partial \nu}|_{S_T} = 0, \tag{1.2}$$

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \tag{1.3}$$

where $S_T = \partial\Omega \times (0, T)$ and ν is an outward normal vector for S_T , $p > 2$, $q > 2$.

We assume that the coefficients of (1.1) satisfies the following conditions:

- (A₁) $a_{ij}^{kl} \in B^2(\Omega)$; $a_{ij}^{kl}(x) \stackrel{\text{a.e.}}{=} a_{kl}^{ij}(x)$, $(x \in \Omega)$, $i, j, k, l = 1, \dots, n$,
 $\sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) \xi_i \xi_j \xi_k \xi_l \stackrel{\text{a.e.}}{\geq} a_2 \sum_{i,j=1}^n \xi_i^2$, $(x \in \Omega)$, $\xi \in R^{n(n+1)/2}$ with a constant $a_2 > 0$;
- (A₂) $a_{ij} \in B^1(\Omega)$, $i, j = 1, \dots, n$, $a_{ij}(x) \stackrel{\text{a.e.}}{=} a_{ji}(x)$, $(x \in \Omega)$,
 $\sum_{i,j=1}^n a_{ij}(x) \xi_i \xi_j \stackrel{\text{a.e.}}{\geq} a_1 \sum_{i=1}^n \xi_i^2$, $(x \in \Omega)$, $\xi \in R^n$ with a constant $a_1 > 0$;
- (A₃) $a_0 \in L^\infty(\Omega)$, $a_0(x) \stackrel{\text{a.e.}}{\geq} a_0 \geq 0$, $(x \in \Omega)$ with a constant a_0 ;
- (B) $b_i \in B^1(\Omega)$, $b_i(x) \stackrel{\text{a.e.}}{\geq} b_0 > 0$, $(x \in \Omega)$ with a constant b_0 ;
- (C) $c \in L^\infty(\Omega)$, $c(x) \stackrel{\text{a.e.}}{\geq} c_0 > 0$, $(x \in \Omega)$ with a constant c_0 .

Denote

$$\begin{aligned}
 H_{0,1}^2(\Omega^R) &= \left\{ u : u \in H^2(\Omega^R); u|_{\partial\Omega \cap B_R} = 0, \frac{\partial u}{\partial \nu}|_{\partial\Omega \cap B_R} = 0 \right\}, \\
 W_{0,1}^{1,p}(\Omega^R) &= \{ u : u \in W^{1,p}(\Omega^R), u|_{\partial\Omega \cap B_R} = 0 \}, \\
 H_{0,\text{loc}}^k(\overline{\Omega}) &= \{ u \in H_{0,1}^k(\Omega^R), \forall_{R>1} \}, \quad (k = 1, 2), \\
 L_{\text{loc}}^2(\overline{\Omega}) &= \{ u \in L^2(\Omega^R), \forall_{R>1} \}, \quad W_{0,\text{loc}}^{1,p}(\overline{\Omega}) = \{ u : u \in W_{0,1}^{1,p}(\Omega^R), \forall_{R>1} \}, \\
 B^1(\Omega) &= \{ u : u, u_{x_i} \in L^\infty(\Omega), \quad i = 1, 2, \dots, n \}, \\
 B^2(\Omega) &= \{ u : u, u_{x_i}, u_{x_i x_j} \in L^\infty(\Omega), \quad i, j = 1, 2, \dots, n \}.
 \end{aligned}$$

In work is proved that if the above conditions are executed that there exist solutions without additional conditions at infinity by space variables.

Notice also that, similar problems is actual for another types equations, for example for the nonlinear parabolic equations with the first derivative with

respect to time considered in unbounded domain. So, in [5] it was showed that the uniqueness of Cauchy problem solution for the equation

$$u_t - \Delta^m(|u|^{p-2}u) + c|u|^{q-2}u = 0 \quad (1.4)$$

with $m = 1$, $c = 1$, $p = 2$, $q > 2$ does not depend on its behaviour for $|x| \rightarrow +\infty$. Similar results were obtained for $m \geq 1$, $p = 2$, $c > 0$, $q > 2$ in [2–4, 12, 13, 15, 16, 22, 23, 26]. In [11] it was showed that the equation

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \quad (1.5)$$

also possesses analogical property. In [10] it was showed that if $m = 1$, $c \geq 0$, $p > 2$ in (1.5), then the solution of Cauchy problem exists and it is unique in the class of functions which do not grow faster than $|x|^s$, as $|x| \rightarrow +\infty$, where s depends on p and n . A similar result was obtained in [1, 6, 7, 12, 19] for the degenerate quasi-linear parabolic equations of types (1.4), (1.5). We can observe, that our results is a partial analog of these researches for the case of equations with the second derivative by time.

2. The main results

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain and $\partial\Omega \in C^1$, $\Omega \cap K_R = \Omega^R$ be a regular domain for all $R > 1$, where for all $R > 1$, $K_R = \{x \in \mathbb{R}^n, |x| < R\}$ and $Q_T = \Omega \times (0, T)$, $T < +\infty$. In the domain Q_T we shall consider the problem (1.1)–(1.3).

In the sequel, we take the assumptions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, $(\mathbf{A}_3)$, $(\mathbf{B})$, $(\mathbf{C})$.

Under these assumptions we will obtain the existence of weak solution of problem (1.1)–(1.3).

Definition 2.1. We call a function u weak solution of the problem (1.1)–(1.3) if

$$\begin{aligned} u &\in L^2((0, T); H_{0,\text{loc}}^2(\overline{\Omega})), \\ u_t &\in L^p((0, T); W_{0,\text{loc}}^{1,p}(\overline{\Omega})) \cap L^q((0, T); L_{\text{loc}}^q(\overline{\Omega})) \cap C([0, T]; L_{\text{loc}}^2(\overline{\Omega})), \end{aligned}$$

and u satisfies the following integral equation

$$\begin{aligned} \int_{\Omega} u_t(x, \tau) v(x, \tau) dx + \int_{Q_{\tau}} \left[-u_t v_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j} v_{x_k x_l} \right. \\ \left. + \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + a_0(x) u v + \sum_{i=1}^n b_i(x) |u_{x_i t}|^{p-2} u_{x_i t} v_{x_i} \right. \\ \left. + c(x) |u|^{q-2} u_t v \right] dx dt = \int_{Q_{\tau}} f(x, t) v dx dt + \int_{\Omega} u_1(x) v(x, 0) dx \end{aligned}$$

$$\forall v \in L^2((0, T); H_{0,\text{loc}}^2(\overline{\Omega})) \cap L^q((0, T); L_{\text{loc}}^q(\overline{\Omega})) \cap L^p((0, T); W_{0,\text{loc}}^{1,p}(\overline{\Omega})),$$

$$v_t \in L^2((0, T); L_{\text{loc}}^2(\overline{\Omega})), \quad \forall \tau \in (0, T],$$

where $\text{supp } v$ is bounded. Moreover, u satisfies the condition $u|_{t=0} = u_0(x)$.

Let $V_{\text{loc}}(\overline{\Omega}) = \{v \in V(\Omega^R), \forall_{R>1}\}$, where

$$V(\Omega^R) = \left\{ v : v \in H_{0,1}^2(\Omega^R) \cap L^{2(q-1)}(\Omega^R), \right. \\ \left. v_{x_i} \in L^{2(p-1)}(\Omega^R), |v_{x_i}|^{\frac{p-2}{2}} v_{x_i x_i} \in L^2(\Omega^R) \right\} \quad \text{for } i = 1, \dots, n$$

and $H_0^2(\Omega^R) = \{u : u \in H^2(\Omega^R) : u|_{\partial\Omega^R} = 0, \frac{\partial u}{\partial \nu}|_{\partial\Omega^R} = 0\}$. Moreover, let $l \in \mathbb{R}$ be at least such, that $V(\Omega^R) \subset H^l(\Omega^R)$ and $l > \frac{n+4}{2}$.

First, we consider the following equation

$$A(u) \equiv f^R(x, t) \quad (2.1)$$

in the domain $Q_T^R = \Omega^R \times (0, T)$, $R > 1$. For this equation we put the following boundary and initial conditions

$$u|_{t=0} = u_0^R(x), \quad u_t|_{t=0} = u_1^R(x), \quad (2.2)$$

$$u|_{\partial\Omega^R} = \frac{\partial u}{\partial \nu}|_{\partial\Omega^R} = 0. \quad (2.3)$$

Definition 2.2. We call the function u a solution of the problem (2.1)–(2.3) if

$$u \in L^2((0, T); H_0^2(\Omega^R)), \quad u_{tt} \in L^2(Q_T^R), \\ u_t \in L^p((0, T); W_0^{1,p}(\Omega^R)) \cap L^q(Q_T^R) \cap L^2((0, T); H_0^2(\Omega^R))$$

and u satisfies the following integral equation

$$\int_{Q_\tau^R} \left[u_{tt}v + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j} v_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + a_0(x) uv \right. \\ \left. + \sum_{i=1}^n b_i(x) |u_{x_i t}|^{p-2} u_{x_i t} v_{x_i} + c(x) |u_t|^{q-2} u_t v \right] dx dt = \int_{Q_\tau^R} f^R(x, t) v dx dt$$

$\forall v \in L^2((0, T); H_0^2(\Omega^R)) \cap L^q(Q_T^R) \cap L^p((0, T); W_0^{1,p}(\Omega^R))$, $\forall \tau \in (0, T]$. Moreover, u satisfies the conditions (2.2).

Theorem 2.3. If the conditions $(\mathbf{A}_1) - (\mathbf{A}_3)$, $(\mathbf{B})$, $(\mathbf{C})$ hold and $f^R, f_t^R \in L^2(Q_T^R)$, $u_0^R \in H_0^2(\Omega^R) \cap H^4(\Omega^R)$, $u_1^R \in V(\Omega^R) \cap H_0^2(\Omega^R)$ then there exist a solution of the problem (2.1)–(2.3).

Proof. Consider the space $H_0^m(\Omega^R)$, where $m = \max\{4, l\}$ and $\{\phi_k(x)\}$ is the basis of this space and $\{\phi_k(x)\}$ is orthonormal basis in $L^2(\Omega^R)$. Next, consider the sequence of functions of the form

$$u^N(x, t) = \sum_{s=1}^N C_s^N(t) \phi_s(x)$$

for $N = 1, 2, \dots$, where the functions $C_1^N, \dots, C_N^N$ are the solution of the following Cauchy problem:

$$\begin{aligned} \int_{\Omega^R} \left[u_{tt}^N(x, t) \phi_s(x) + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^N(x, t) \phi_{s x_k x_l}(x) \right. \\ + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N(x, t) \phi_{s x_j}(x) + \sum_{i=1}^n b_i(x) |u_{t x_i}^N(x, t)|^{p-2} u_{t x_i}^N(x, t) \phi_{s x_i}(x) \\ + a_0(x) u^N(x, t) \phi_s(x) + c(x) |u_t^N(x, t)|^{q-2} u_t^N(x, t) \phi_s(x) \\ \left. - f(x, t) \phi_s(x) \right] dx = 0 \end{aligned} \quad (2.4)$$

with the conditions

$$C_s^N(0) = u_{0,s}^N, \quad C_{st}^N(0) = u_{1,s}^N, \quad s = 1, \dots, N, \quad (2.5)$$

and $\|u_0^N - u_0\|_{H_0^2(\Omega^R) \cap H^4(\Omega^R)} \rightarrow 0$, $\|u_1^N - u_1\|_{H_0^2(\Omega^R)} \rightarrow 0$, where $u_0^N(x) = \sum_{s=1}^N u_{0,s}^N \phi_s(x)$ and $u_1^N(x) = \sum_{s=1}^N u_{1,s}^N \phi_s(x)$.

From the Carathéodory theorem there exists the solution of problem (2.4)–(2.5), where C_s^N are continuous and C_{st}^N , $s = 1, \dots, N$ are absolutely continuous on the interval $[0, h_0]$. Taking into account the estimations obtained below we can state that $h_0 = T$.

Multiplying (2.4) by the functions $C_{st}^N(t) e^{-\eta t}$, $\eta > 0$ respectively, then summing by s from 1 to N and integrating with respect to t from 0 to τ , $\tau \in (0, T]$, we obtain

$$\begin{aligned} \int_{Q_\tau^R} \left[u_{tt}^N u_t^N e^{-\eta t} + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^N u_{t x_k x_l}^N e^{-\eta t} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{t x_j}^N e^{-\eta t} \right. \\ + \sum_{i=1}^n b_i(x) |u_{t x_i}^N|^{p-2} u_{t x_i}^N u_{t x_i}^N e^{-\eta t} + a_0(x) u^N u_t^N e^{-\eta t} \\ \left. + c(x) |u_t^N|^{q-2} u_t^N u_t^N e^{-\eta t} - f(x, t) u_t^N e^{-\eta t} \right] dx dt = 0. \end{aligned} \quad (2.6)$$

If we consider the respective components of the last equality we will have

$$\int_{Q_\tau^R} u_{tt}^N u_t^N e^{-\eta t} dx dt = \frac{1}{2} \int_{\Omega_\tau^R} |u_t^N|^2 e^{-\eta \tau} dx - \frac{1}{2} \int_{\Omega_0^R} |u_1^N|^2 dx + \frac{\eta}{2} \int_{Q_\tau^R} |u_t^N|^2 e^{-\eta t} dx dt,$$

where $\Omega_\tau^R = Q_\tau^R \cap \{t = \tau\}$. Next, from $(\mathbf{A}_1)$ we have

$$\begin{aligned} \int_{Q_\tau^R} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^N u_{t x_k x_l}^N e^{-\eta t} dx dt &\geq \frac{a_2}{2} \int_{\Omega_\tau^R} \sum_{i,j=1}^n |u_{x_i x_j}^N|^2 e^{-\eta \tau} dx \\ &\quad - \frac{1}{2} \int_{\Omega_0^R} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0 x_i x_j}^N u_{0 x_k x_l}^N dx + \frac{\eta a_2}{2} \int_{Q_\tau^R} \sum_{i,j=1}^n |u_{x_i x_j}^N|^2 e^{-\eta t} dx dt. \end{aligned}$$

From the assumption $(\mathbf{A}_2)$ and the initial condition we obtain

$$\begin{aligned} \int_{Q_\tau^R} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{t x_j}^N e^{-\eta t} dx dt &= \frac{1}{2} \int_{\Omega_\tau^R} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{x_j}^N e^{-\eta \tau} dx \\ &\quad - \frac{1}{2} \int_{\Omega_0^R} \sum_{i,j=1}^n a_{ij}(x) u_{0 x_i}^N u_{0 x_j}^N dx + \frac{\eta}{2} \int_{Q_\tau^R} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^N u_{x_j}^N e^{-\eta t} dx dt \\ &\geq \frac{a_1}{2} \int_{\Omega_\tau^R} \sum_{i=1}^n |u_{x_i}^N|^2 e^{-\eta \tau} dx + \frac{a_1 \eta}{2} \int_{Q_\tau^R} \sum_{i=1}^n |u_{x_i}^N|^2 e^{-\eta t} dx dt \\ &\quad - \frac{1}{2} \int_{\Omega_0^R} \sum_{i,j=1}^n a_{ij}(x) u_{0 x_i}^N u_{0 x_j}^N dx. \end{aligned}$$

Next, from the assumption $(\mathbf{B})$ we have

$$\int_{Q_\tau^R} \sum_{i=1}^n b_i(x) |u_{t x_i}^N|^{p-2} u_{t x_i}^N e^{-\eta t} dx dt \geq b_0 \int_{Q_\tau^R} \sum_{i=1}^n |u_{t x_i}^N|^p e^{-\eta t} dx dt.$$

From $(\mathbf{A}_3)$ we obtain

$$\begin{aligned} \int_{Q_\tau^R} a_0(x) u_t^N e^{-\eta t} dx dt &= \frac{1}{2} \int_{\Omega_\tau^R} a_0 |u^N|^2 e^{-\eta \tau} dx \\ &\quad - \frac{1}{2} \int_{\Omega_0^R} a_0(x) |u_0^N|^2 dx + \frac{\eta}{2} \int_{Q_\tau^R} a_0(x) |u^N|^2 e^{-\eta t} dx dt \geq -\frac{1}{2} \int_{\Omega_0^R} a_0(x) |u_0^N|^2 dx. \end{aligned}$$

From $(\mathbf{C})$ we have

$$\int_{Q_\tau^R} c(x) |u_t^N|^{q-2} u_t^N e^{-\eta t} dx dt \geq c_0 \int_{Q_\tau^R} |u_t^N|^q e^{-\eta t} dx dt.$$

Next,

$$\int_{Q_\tau^R} f(x, t) u_t^N e^{-\eta t} dx dt \leq \frac{1}{2} \int_{Q_\tau^R} |f(x, t)|^2 e^{-\eta t} dx dt + \frac{1}{2} \int_{Q_\tau^R} |u_t^N|^2 e^{-\eta t} dx dt.$$

Thus, from (2.6) we get the following inequality

$$\begin{aligned}
 & \int_{\Omega_\tau^R} \left[|u_t^N|^2 + a_2 \sum_{i,j=1}^n |u_{x_i x_j}^N|^2 + a_1 \sum_{i=1}^n |u_{x_i}^N|^2 \right] e^{-\eta\tau} dx + \int_{Q_\tau^R} \left[(\eta - 1) |u_t^N|^2 \right. \\
 & \quad \left. + c_0 |u_t^N|^q + \eta a_2 \sum_{i,j=1}^n |u_{x_i x_j}^N|^2 + \eta a_1 \sum_{i=1}^n |u_{x_i}^N|^2 + b_0 \sum_{i=1}^n |u_{tx_i}^N|^p \right] e^{-\eta t} dx dt \\
 & \leq \int_{\Omega_0^R} \left[|u_1^N|^2 + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0x_i x_j}^N u_{0x_k x_l}^N \right. \\
 & \quad \left. + \sum_{i,j=1}^n a_{ij}(x) u_{0x_i}^N u_{0x_j}^N + a_0(x) |u_0^N|^2 \right] dx + \int_{Q_\tau^R} |f(x, t)|^2 e^{-\eta t} dx dt. \quad (2.7)
 \end{aligned}$$

Due to the assumptions of Theorem 2.3 we can differentiate the system (2.4) with respect to t

$$\begin{aligned}
 & \int_{\Omega^R} \left[u_{ttt}^N(x, t) \phi_s(x) + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{tx_i x_j}^N(x, t) \phi_{sx_k x_l}(x) \right. \\
 & \quad + \sum_{i,j=1}^n a_{ij}(x) u_{tx_i}^N(x, t) \phi_{sx_j}(x) \\
 & \quad + \sum_{i=1}^n b_i(x) (|u_{tx_i}^N(x, t)|^{p-2} u_{tx_i}^N(x, t))_t \phi_{sx_i}(x) + a_0(x) u_t^N(x, t) \phi_s(x) \\
 & \quad \left. + c(x) (|u_t^N(x, t)|^{q-2} u_t^N(x, t))_t \phi_s(x) - f_t(x, t) \phi_s(x) \right] dx = 0. \quad (2.8)
 \end{aligned}$$

Multiplying (2.8) by the functions $C_{stt}^N(t)e^{-\eta t}$, respectively, then summing by s from 1 to N and integrating with respect to t from 0 to τ , $\tau \in (0, T]$, we obtain

$$\begin{aligned}
 & \int_{Q_\tau^R} \left[u_{ttt}^N u_{tt}^N e^{-\eta t} + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{tx_i x_j}^N u_{tx_k x_l}^N e^{-\eta t} \right. \\
 & \quad + \sum_{i,j=1}^n a_{ij}(x) u_{tx_i}^N u_{tx_j}^N e^{-\eta t} + \sum_{i=1}^n b_i(x) (|u_{tx_i}^N|^{p-2} u_{tx_i}^N)_t u_{tx_i}^N e^{-\eta t} \\
 & \quad \left. + a_0(x) u_t^N u_{tt}^N e^{-\eta t} + c(x) (|u_t^N|^{q-2} u_t^N)_t u_{tt}^N e^{-\eta t} - f_t(x, t) u_{tt}^N e^{-\eta t} \right] dx dt = 0. \quad (2.9)
 \end{aligned}$$

Observe, that for $t = 0$ multiplying (2.8) by $C_{stt}(0)$, by analogy to (2.9), we obtain

$$\begin{aligned} \int_{\Omega_0^R} & \left[|u_{tt}^N|^2(x, 0) + \sum_{i,j,k,l=1}^n \left(a_{ij}^{kl}(x) u_{x_i x_j}^N(x, 0) \right)_{x_k x_l} u_{tt}^N(x, 0) \right. \\ & - \sum_{i,j=1}^n \left(a_{ij}(x) u_{x_i}^N(x, 0) \right)_{x_j} u_{tt}^N(x, 0) \\ & - \sum_{i=1}^n \left(b_i(x) |u_{tx_i}^N(x, 0)|^{p-2} u_{tx_i}^N(x, 0) \right)_{x_i} u_{tt}^N(x, 0) \\ & + a_0(x) u^N(x, 0) u_{tt}^N(x, 0) \\ & \left. + c(x) |u_t^N(x, 0)|^{q-2} u_t^N(x, 0) u_{tt}^N(x, 0) - f(x, 0) u_{tt}^N(x, 0) \right] dx = 0. \quad (2.10) \end{aligned}$$

It is easy to prove that for all $u \in H_0^l(\Omega^R)$,

$$\int_{\Omega^R} (|u_{x_i}|^{p-2} u_{x_i x_i})^2 dx \leq K \left(\int_{\Omega^R} |u_{x_i x_i}|^2 dx \right)^{p-1},$$

where K depends on R . Then by conditions of Theorem 2.3

$$\begin{aligned} \int_{\Omega_0^R} & \left[\left| \sum_{i,j,k,l=1}^n \left(a_{ij}^{kl} u_{0x_i x_j}^N \right)_{x_k x_l} \right|^2 + \left| \sum_{i,j=1}^n \left(a_{ij} u_{0x_i}^N \right)_{x_j} \right|^2 \right. \\ & \left. + \left| \sum_{i=1}^n \left(b_i(x) |u_{1x_i}^N|^{p-2} u_{1x_i}^N \right)_{x_i} \right|^2 + |a_0 u_0^N|^2 + (c(x) |u_1^N|^{q-2} u_1^N) \right] dx \leq K_1 \end{aligned}$$

and from (2.10) it follows then

$$\int_{\Omega_0^R} |u_{tt}^N|^2 dx \leq M_1, \quad (2.11)$$

where M_1 does not depend on N . Returning to (2.9), we have from (2.11)

$$\begin{aligned} I_1 &:= \int_{Q_\tau^R} u_{ttt}^N u_{tt}^N e^{-\eta t} dx dt = \frac{1}{2} \int_{\Omega_\tau^R} |u_{tt}^N|^2 e^{-\eta \tau} dx - \frac{1}{2} \int_{\Omega_0^R} |u_{tt}^N|^2 dx \\ &+ \frac{\eta}{2} \int_{Q_\tau^R} |u_{tt}^N|^2 e^{-\eta t} dx dt \geq \frac{1}{2} \int_{\Omega_\tau^R} |u_{tt}^N|^2 e^{-\eta \tau} dx + \frac{\eta}{2} \int_{Q_\tau^R} |u_{tt}^N|^2 e^{-\eta t} dx dt - \frac{M_1}{2}. \end{aligned}$$

From the assumption $(\mathbf{A}_1)$, we have

$$\begin{aligned} I_2 &:= \int_{Q_\tau^R} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{tx_i x_j}^N u_{tx_k x_l}^N e^{-\eta t} dx dt \\ &\geq \frac{a_2}{2} \int_{\Omega_\tau^R} \sum_{i,j=1}^n |u_{tx_i x_j}^N|^2 e^{-\eta \tau} dx \\ &\quad - \frac{1}{2} \int_{\Omega_0^R} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{1x_i x_j}^N u_{1x_k x_l}^N dx + \frac{\eta a_2}{2} \int_{Q_\tau^R} \sum_{i,j=1}^n |u_{tx_i x_j}^N|^2 e^{-\eta t} dx dt. \end{aligned}$$

From the assumption $(\mathbf{A}_2)$ and the initial condition we obtain

$$\begin{aligned} I_3 &:= \int_{Q_\tau^R} \sum_{i,j=1}^n a_{ij}(x) u_{tx_i}^N u_{tx_j}^N e^{-\eta t} dx dt \geq \frac{a_1}{2} \int_{\Omega_\tau^R} \sum_{i=1}^n |u_{tx_i}^N|^2 e^{-\eta \tau} dx \\ &\quad + \frac{a_1 \eta}{2} \int_{Q_\tau^R} \sum_{i=1}^n |u_{tx_i}^N|^2 e^{-\eta t} dx dt - \frac{1}{2} \int_{\Omega_0^R} \sum_{i,j=1}^n a_{ij}(x) u_{1x_i}^N u_{1x_j}^N dx. \end{aligned}$$

Next, from the assumption $(\mathbf{B})$ we have

$$\begin{aligned} I_4 &:= \int_{Q_\tau^R} \sum_{i=1}^n b_i(x) (|u_{tx_i}^N|^{p-2} u_{tx_i}^N)_t u_{tt x_i}^N e^{-\eta t} dx dt \\ &\geq b_0(p-1) \int_{Q_\tau^R} \sum_{i=1}^n |u_{tx_i}^N|^p e^{-\eta t} dx dt, \\ I_5 &:= \int_{Q_\tau^R} a_0(x) u_t^N u_{tt}^N e^{-\eta t} dx dt \leq \frac{a_0^1}{2} \int_{Q_\tau^R} |u_t^N|^2 e^{-\eta t} dx dt + \frac{a_0^1}{2} \int_{Q_\tau^R} |u_{tt}^N|^2 e^{-\eta t} dx dt, \end{aligned}$$

where $a_0^1 = \text{ess sup}_\Omega a_0(x)$. From $(\mathbf{C})$ we have

$$\begin{aligned} I_6 &:= \int_{Q_\tau^R} c(x) (|u_t^N|^{q-2} u_t^N)_t u_{tt}^N e^{-\eta t} dx dt \\ &\geq c_0 \int_{Q_\tau^R} (q-1) |u_t^N|^{q-2} |u_{tt}^N|^2 e^{-\eta t} dx dt \geq 0. \\ I_7 &:= \int_{Q_\tau^R} f_t(x, t) u_{tt}^N e^{-\eta t} dx dt \leq \frac{1}{2} \int_{Q_\tau^R} |f_t(x, t)|^2 e^{-\eta t} dx dt + \frac{1}{2} \int_{Q_\tau^R} |u_{tt}^N|^2 e^{-\eta t} dx dt. \end{aligned}$$

Thus, from (2.9), taking into account the estimates of the integrals $I_1 - I_7$, we get the following inequality

$$\begin{aligned}
& \int_{\Omega_\tau^R} \left[|u_{tt}^N|^2 + a_2 \sum_{i,j=1}^n |u_{tx_i x_j}^N|^2 + a_1 \sum_{i=1}^n |u_{tx_i}^N|^2 \right] e^{-\eta\tau} dx + \int_{Q_\tau^R} \left[(\eta - a_0^1 - 1) |u_{tt}^N|^2 \right. \\
& + \eta a_1 \sum_{i=1}^n |u_{tx_i}^N|^2 + 2b_0(p-1) \sum_{i=1}^n |u_{tx_i}^N|^p + 2c_0(q-1) |u_t^N|^{q-2} |u_{tt}^N|^2 \\
& \left. + \eta a_2 \sum_{i,j=1}^n |u_{tx_i x_j}^N|^2 \right] e^{-\eta t} dx dt \leq \int_{Q_\tau^R} |f_t(x, t)|^2 e^{-\eta t} dx dt \\
& + \int_{\Omega_0^R} \left[\sum_{i,j=1}^n a_{ij}(x) u_{1x_i}^N u_{1x_j}^N + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{1x_i x_j}^N u_{1x_k x_l}^N \right] dx + M_1 \leq M_2.
\end{aligned} \tag{2.12}$$

From (2.7) and (2.12), choosing $\eta = a_0^1 + 2$, we obtain that $\{u^N\}$ is bounded in the space $L^\infty((0, T); H_0^2(\Omega^R))$ and $\{u_{tt}^N\}$ is bounded in $L^\infty((0, T); L^2(\Omega^R))$, and $\{u_t^N\}$ is bounded in $L^p((0, T); W_0^{1,p}(\Omega^R)) \cap L^q(Q_T^R) \cap L^\infty((0, T); H_0^2(\Omega^R))$. Hence, we can choose subsequences $\{u^{N_k}\}$, $\{u_t^{N_k}\}$, $\{u_{tt}^{N_k}\}$ such that $u^{N_k} \rightarrow u^{R*}$ -weakly in $L^\infty((0, T); H_0^2(\Omega^R))$, $u_t^{N_k} \rightarrow u_{tt}^R$ -weakly in $L^\infty((0, T); L^2(\Omega^R))$, $u_{tt}^{N_k} \rightarrow u_{tt}^R$ -weakly in $L^\infty((0, T); H_0^2(\Omega^R))$, $u_t^{N_k} \rightarrow u_t^R$ weakly in $L^p((0, T); W_0^{1,p}(\Omega^R)) \cap L^q(Q_T^R)$ as $N_k \rightarrow \infty$. Define the operator $B : L^p((0, T); W_0^{1,p}(\Omega^R)) \cap L^q(Q_T^R) \rightarrow L^{p'}((0, T); W_0^{-1,p}(\Omega^R)) + L^{q'}(Q_T^R)$ by the formula

$$\langle B(u), v \rangle = \int_{Q_T^R} \left[\sum_{i=1}^n b_i |u_{x_i}|^{p-2} u_{x_i} v_{x_i} + c |u|^{q-2} u \right] dx dt$$

for $u, v \in L^p((0, T); W_0^{1,p}(\Omega^R)) \cap L^q(Q_T^R)$. It is easy to see that B is monotonic, and $\{B(u_t^N)\}$ is bounded in the space $L^{p'}((0, T); W_0^{-1,p'}(\Omega^R)) + L^{q'}(Q_T^R)$. Hence, we can accept that $B(u_t^{N_k}) \rightarrow z$ weakly in $L^{p'}((0, T); W_0^{-1,p}(\Omega^R)) + L^{q'}(Q_T^R)$. Analogously as in [20, p.II, s.1,2] we can prove that $z = B(u_t^R)$. Since $u_{tt}^R \in L^\infty((0, T); L^2(\Omega^R))$, and $u_t \in L^\infty((0, T); H^2(\Omega^R))$, then $u_t \in C([0, T]; L^2(\Omega^R))$. Analogously, $u \in C([0, T]; H_0^2(\Omega^R))$. By analogy with [20, p.I, s.1,4] it is easy to prove that $u^R|_{t=0} = u_0^R(x)$, $u_t^R|_{t=0} = u_1^R(x)$. Hence the function u^R is the solution of the problem (2.1)–(2.3) in the sense of Definition 2.2. $\square$

Theorem 2.4. *If the conditions $(\mathbf{A}_1) - (\mathbf{A}_3), (\mathbf{B}), (\mathbf{C})$ hold, and $f, f_t \in L_{loc}^2(Q_T)$, $\frac{2n}{n-2} > q > p > 2$, $u_0 \in H_{0,loc}^2(\bar{\Omega})$, $u_1 \in V_{loc}(\bar{\Omega})$, then the problem (1.1)–(1.3) has a weak solution.*

Proof. Let $f^s(x, t) = \begin{cases} f(x, t), & \text{for } (x, t) \in Q_T^s, \\ 0, & \text{for } (x, t) \in Q_T \setminus Q_T^s, \end{cases} \quad s = 2, 3, 4 \dots$ and $u_{0,s}(x) = u_0(x)\theta(x)$, $u_{1,s}(x) = u_1(x)\theta(x)$, where $\theta \in C^4(\mathbb{R}^n)$, $0 \leq \theta(x) \leq 1$, and $\theta(x) = \begin{cases} 1, & \text{for } |x| \leq s-1 \\ 0, & \text{for } |x| \geq s. \end{cases}$ We shall consider in the domain Q_T^s the equation

$$A(u) = f^s(x, t) \quad (2.13)$$

with the following boundary and initial conditions

$$u|_{S_T^s} = 0, \quad \frac{\partial u}{\partial \nu}|_{S_T^s} = 0, \quad (2.14)$$

$$u|_{t=0} = u_{0,s}(x), \quad u_t|_{t=0} = u_{1,s}(x), \quad (2.15)$$

where $S_T^s = \partial\Omega^s \times (0, T)$. Due to Theorem 2.3 there is the solution u^s of the problem (2.13)–(2.15). If we extend this solution by 0 at the set $Q_T \setminus Q_T^s$ then for $\tau \in (0, T]$ we obtain

$$\begin{aligned} \int_{Q_\tau} & \left[u_{tt}^s v + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^s v_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^s v_{x_j} \right. \\ & + \sum_{i=1}^n b_i(x) |u_{tx_i}^s|^{p-2} u_{tx_i}^s v_{x_i} + a_0(x) u^s v \\ & \left. + c(x) |u_t^s|^{q-2} u_t^s v - f^s(x, t) v \right] dx dt = 0 \end{aligned} \quad (2.16)$$

for all v such that $v \in L^2((0, T); H_{0,\text{loc}}^2(\overline{\Omega}))$, $v_t \in L^p((0, T); W_{0,\text{loc}}^{1,p}(\overline{\Omega})) \cap L^q((0, T); L_{\text{loc}}^q(\overline{\Omega}))$. The function u^s satisfies the initial conditions

$$u^s(x, 0) = u_{0,s}(x), \quad u_t^s(x, 0) = u_{1,s}(x), \quad x \in \Omega.$$

For the fixed $R > 1$ we choose $v(x, t) = u_t^s(x, t)\Psi_R(x)e^{-\eta t}$, where

$$\Psi_R(x) = [h_R(x)]^\beta, \quad \beta > 1, \quad (2.17)$$

and

$$h_R(x) = \begin{cases} \frac{1}{R}(R^2 - |x|^2), & \text{for } 0 \leq |x| \leq R \\ 0, & \text{for } |x| > R. \end{cases}$$

From (2.17) we have

$$\Psi_{Rx_i} = \beta(h_R)^{\beta-1} h_{Rx_i}, \quad h_{Rx_i} = -\frac{2x_i}{R}, \quad |h_{Rx_i}| \leq 2, \quad \text{for } i = 1, 2, \dots, n.$$

If we consider the respective components of the equality (2.16) we will have

$$I_8 := \int_{Q_\tau} u_{tt}^s u_t^s \Psi_R(x) e^{-\eta t} dx dt = \frac{1}{2} \int_{\Omega_\tau} |u_t^s|^2 \Psi_R(x) e^{-\eta \tau} dx \\ - \frac{1}{2} \int_{\Omega_0} |u_{1,s}^s|^2 \Psi_R(x) dx + \frac{\eta}{2} \int_{Q_\tau} |u_t^s|^2 \Psi_R(x) e^{-\eta t} dx dt.$$

From $(\mathbf{A}_1)$ we have

$$I_9 := \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^s v_{x_k x_l} dx dt = I_9^1 + I_9^2 + I_9^3, \text{ where} \\ I_9^1 := \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^s u_{t x_k x_l}^s \Psi_R(x) e^{-\eta t} dx dt \\ \geq \frac{a_2}{2} \int_{\Omega_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta \tau} dx \\ - \frac{1}{2} \int_{\Omega_0} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0,s x_i x_j}^s u_{0,s x_k x_l}^s \Psi_R(x) dx \\ + \frac{\eta a_2}{2} \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt \quad \text{and} \\ I_9^2 := \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^s u_{t x_k}^s [\Psi_{R x_l}(x) + \Psi_{R x_k}(x)] e^{-\eta t} dx dt, \\ I_9^3 := \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^s u_t^s \Psi_{R x_l, x_k}(x) e^{-\eta t} dx dt.$$

Using Hölder's and Young's inequalities we from (2.15) obtain

$$I_9^2 \leq \left(\int_{Q_\tau} \sum_{i,j,k,l=1}^n |a_{ij}^{kl}(x)|^2 |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{2}} \\ \times \left(\int_{Q_\tau} \sum_{k=1}^n |u_{t x_k}^s|^p \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{p}} \cdot \left(\int_{Q_\tau} \sum_{l=1}^n \left(\frac{\Psi_{R x_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r e^{-\eta t} dx dt \right)^{\frac{1}{r}} \\ \leq \frac{\delta_2}{2} \int_{Q_\tau} \sum_{i,j,k,l=1}^n |a_{ij}^{kl}(x)|^2 |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt \\ + \frac{\delta_2}{2p} \int_{Q_\tau} \sum_{k=1}^n |u_{t x_k}^s|^p \Psi_R(x) e^{-\eta t} dx dt + \mu(\delta_2) \int_{Q_\tau} \sum_{l=1}^n \left(\frac{\Psi_{R x_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r e^{-\eta t} dx dt$$

$$\begin{aligned} &\leq \frac{\delta_2}{2} a_2^0 \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt + \frac{\delta_2}{2p} \int_{Q_\tau} \sum_{k=1}^n |u_{tx_k}^s|^p \Psi_R(x) e^{-\eta t} dx dt \\ &\quad + \mu(\delta_2) \int_{Q_\tau} \sum_{l=1}^n \left(\frac{\Psi_{Rx_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r e^{-\eta t} dx dt, \end{aligned}$$

where $r = \frac{2p}{p-2}$, $\delta_2 > 0$. Similarly,

$$\begin{aligned} I_9^3 &\leq \left(\int_{Q_\tau} \sum_{i,j,k,l=1}^n |a_{ij}^{kl}(x)|^2 |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{q}} \cdot \left(\int_{Q_\tau} \sum_{k,l=1}^n \left(\frac{\Psi_{Rx_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\eta t} dx dt \right)^{\frac{1}{r_1}} \\ &\leq \frac{a_2^0 \delta_2}{2} \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 \Psi_R(x) e^{-\eta t} dx dt + \frac{\delta_2}{2q} \int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \\ &\quad + \mu_1(\delta_2) \int_{Q_\tau} \sum_{k,l=1}^n \left(\frac{\Psi_{Rx_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\eta t} dx dt \end{aligned}$$

where $r_1 = \frac{2q}{q-2}$, $a_2^0 = \max_{i,j,k,l} \operatorname{ess\,sup}_\Omega (a_{ij}^{kl}(x))^2$.

From **(A₂)** we obtain

$$I_{10} := \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^s v_{x_j} dx dt := I_{10}^1 + I_{10}^2,$$

where

$$\begin{aligned} I_{10}^1 &:= \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^s u_{tx_j}^s \Psi_R(x) e^{-\eta t} dx dt \geq \frac{a_1}{2} \int_{\Omega_\tau^R} \sum_{i=1}^n |u_{x_i}^s|^2 \Psi_R(x) e^{-\eta \tau} dx \\ &\quad + \frac{a_1 \eta}{2} \int_{Q_\tau^R} \sum_{i=1}^n |u_{x_i}^s|^2 \Psi_R(x) e^{-\eta t} dx dt - \frac{1}{2} \int_{\Omega_0} \sum_{i,j=1}^n a_{ij}(x) u_{0,sx_i}^s u_{0,sx_j}^s \Psi_R(x) dx. \end{aligned}$$

By analogy to I_9^2 , we have

$$\begin{aligned} I_{10}^2 &:= \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^s u_t^s \Psi_{Rx_j}(x) e^{-\eta t} dx dt \\ &\leq \left(\int_{Q_\tau} \sum_{i,j=1}^n \left(\operatorname{ess\,sup}_\Omega (a_{ij}(x)) \right)^2 |u_{x_i}^s|^2 \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{2}} \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{q}} \cdot \left(\int_{Q_\tau} \sum_{j=1}^n \left(\frac{\Psi_{Rx_j}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\eta t} dx dt \right)^{\frac{1}{r_1}} \\
& \leq \frac{\delta_2}{2} \int_{Q_\tau} \sum_{i=1}^n |u_{x_i}^s|^2 \Psi_R(x) e^{-\eta t} dx dt + \frac{\delta_2}{2q} \int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \\
& \quad + \mu_2(\delta_2) \int_{Q_\tau} \sum_{j=1}^n \left(\frac{\Psi_{Rx_j}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\eta t} dx dt.
\end{aligned}$$

Next, from the assumption **(B)** we have

$$I_{11} := \int_{Q_\tau} \sum_{i=1}^n b_i(x) |u_{tx_i}^s|^{p-2} u_{tx_i}^s v_{x_i} dx dt := I_{11}^1 + I_{11}^2,$$

where

$$\begin{aligned}
I_{11}^1 &:= \int_{Q_\tau} \sum_{i=1}^n b_i(x) |u_{tx_i}^s|^{p-2} u_{tx_i}^s u_{tx_i}^s \Psi_R(x) e^{-\eta t} dx dt \\
&\geq b_0 \int_{Q_\tau} \sum_{i=1}^n |u_{tx_i}^s|^p \Psi_R(x) e^{-\eta t} dx dt.
\end{aligned}$$

Using Hölder's and Young's inequalities by analogy to I_9^2 we obtain

$$\begin{aligned}
I_{11}^2 &:= \int_{Q_\tau} \sum_{i=1}^n b_i(x) |u_{tx_i}^s|^{p-2} u_{tx_i}^s u_t^s \Psi_{Rx_i}(x) e^{-\eta t} dx dt \\
&\leq \left(\int_{Q_\tau} \sum_{i=1}^n |b_i(x)|^{p'} |u_{tx_i}^s|^{(p-1)p'} \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{p'}} \\
&\quad \cdot \left(\int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \right)^{\frac{1}{q}} \cdot \left(\int_{Q_\tau} \sum_{i=1}^n \left(\frac{\Psi_{Rx_i}}{\Psi_R^{\frac{1}{p'} + \frac{1}{q}}} \right)^{r_2} e^{-\eta t} dx dt \right)^{\frac{1}{r_2}} \\
&\leq \frac{b_0^1 \delta_2}{2p'} \int_{Q_\tau} \sum_{i=1}^n |u_{tx_i}^s|^{(p-1)p'} \Psi_R(x) e^{-\eta t} dx dt + \frac{\delta_2}{2q} \int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt \\
&\quad + \mu_3(\delta_2) \int_{Q_\tau} \sum_{i=1}^n \left(\frac{\Psi_{Rx_i}}{\Psi_R^{\frac{1}{p'} + \frac{1}{q}}} \right)^{r_2} e^{-\eta t} dx dt,
\end{aligned}$$

where $r_2 = \frac{p'q}{qp' - p' - q}$, $b_0^1 = \max_{i=1, \dots, n} \operatorname{ess\,sup}_{\Omega} |b_i(x)|^{p'}$. From **(A₃)** we obtain

$$\begin{aligned} I_{12} &:= \int_{Q_\tau} a_0(x) u^s u_t^s \Psi_R(x) e^{-\eta t} dx dt = \frac{1}{2} \int_{\Omega_\tau} a_0(x) |u^s|^2 \Psi_R(x) e^{-\eta \tau} dx \\ &\quad - \frac{1}{2} \int_{\Omega_0} a_0(x) |u_{0,s}^s|^2 \Psi_R(x) dx + \frac{\eta}{2} \int_{Q_\tau} a_0(x) |u^s|^2 \Psi_R(x) e^{-\eta t} dx dt \\ &\geq -\frac{1}{2} \int_{\Omega_0} a_0(x) |u_{0,s}^s|^2 dx + \frac{a_0}{2} \int_{\Omega_\tau} |u^s|^2 \Psi_R(x) e^{-\eta \tau} dx \\ &\quad + \frac{a_0 \eta}{2} \int_{Q_\tau} |u^s|^2 \Psi_R(x) e^{-\eta t} dx dt. \end{aligned}$$

From **(C)** we have

$$\begin{aligned} I_{13} &:= \int_{Q_\tau} c(x) |u_t^s|^{q-2} u_t^s u_t^s \Psi_R(x) e^{-\eta t} dx dt \geq c_0 \int_{Q_\tau} |u_t^s|^q \Psi_R(x) e^{-\eta t} dx dt. \\ I_{14} &:= \int_{Q_\tau} f^s(x, t) u_t^s \Psi_R(x) e^{-\eta t} dx dt \\ &\leq \frac{1}{2\delta_2} \int_{Q_\tau} |f^s(x, t)|^2 \Psi_R(x) e^{-\eta t} dx dt + \frac{\delta_2}{2} \int_{Q_\tau} |u_t^s|^2 \Psi_R(x) e^{-\eta t} dx dt \end{aligned}$$

where $\delta_2 > 0$. Let R_0 , $1 < R_0 < R$ be fixed. Thus from (2.16), taking into account the estimates of the integrals I_8 – I_{14} and assumptions of Theorem 2.4, we get the following inequality

$$\begin{aligned} &\int_{\Omega_\tau} \left[|u_t^s|^2 + a_2 \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 + a_1 \sum_{i=1}^n |u_{x_i}^s|^2 + a_0 |u^s|^2 \right] \Psi_R(x) e^{-\eta \tau} dx \\ &+ \int_{Q_\tau} \left[(\eta - \delta_2) |u_t^s|^2 + (\eta a_2 - 2a_2^0 \delta_2) \sum_{i,j=1}^n |u_{x_i x_j}^s|^2 + (\eta a_1 - \delta_2) \sum_{i=1}^n |u_{x_i}^s|^2 \right. \\ &+ \left(2b_0 - \frac{\delta_2}{p} - \frac{b_0^1 \delta_2}{p'} \right) \sum_{i=1}^n |u_{t x_i}^s|^p + \left(2c_0 - \frac{3\delta_2}{q} \right) |u_t^s|^q \\ &\left. + a_0 \eta |u^s|^2 \right] \Psi_R(x) e^{-\eta t} dx dt \leq \frac{1}{\delta_2} \int_{Q_\tau} |f^s(x, t)|^2 \Psi_R(x) e^{-\eta t} dx dt \\ &+ \int_{\Omega_0} \left[|u_{1,s}^s|^2 + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0,s x_i x_j}^s u_{0,s x_k x_l}^s \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{i,j=1}^n a_{ij}(x) u_{0,sx_i}^s u_{0,sx_j}^s + a_0(x) |u_{0,s}^s|^2 \Big] \Psi_R(x) dx \\
& + \mu_4(\delta_2) \int_{Q_\tau} \sum_{i=1}^n \left(\frac{\Psi_{Rx_i}}{\Psi_R^{\frac{1}{p'} + \frac{1}{q}}} \right)^{r_2} e^{-\eta t} dx dt,
\end{aligned} \tag{2.18}$$

where $\left(\frac{\Psi_{Rx_i}}{\Psi_R^{\frac{1}{p'} + \frac{1}{q}}} \right)^{r_2} \leq C \cdot ([h_R(x)]^{-r_2 + (1 - \frac{1}{p'} - \frac{1}{q})r_2\beta}) \leq C(2R)^{-r_2 + \beta}$. From (2.17) and the form of the function h_R we have

$$\Psi_R(x) = \left[\frac{1}{R} (R - |x|)(R + |x|) \right]^\beta \geq (R - R_0)^\beta \quad \text{for } |x| \leq R_0.$$

We choose δ_2, η from the inequalities $(\eta - \delta_2) \geq 1$, $\eta a_2^0 - 2\delta_2 a_0^2 \geq 1$, $2b_0 - \frac{\delta_2}{p} - \frac{b_0^1 \delta_2}{p'} \geq b_0$, $2c_0 - \frac{3\delta_2}{q} \geq c_0$, $\eta a_1 - \delta_2 \geq 1$.

Then from (2.18) we get that the sequence $\{u^s\}$ is bounded in $L^\infty((0, T); H_{0,1}^2(\Omega^{R_0}))$, $\{u_t^s\}$ is bounded in $L^p((0, T); W_{0,1}^p(\Omega^{R_0})) \cap L^q(Q_T^{R_0})$ and from (2.13) it follows that $\{u_{tt}^s\}$ is bounded in $L^{p'}((0, T); W_0^{-1,p}(\Omega^{R_0})) + L^{q'}(Q_T^{R_0})$. As for $L^{p'}((0, T); W_0^{-1,p}(\Omega^{R_0})) + L^{q'}(Q_T^{R_0}) \subset L^{q'}((0, T); W_0^{-1,p}(\Omega^{R_0}))$ and the imbedding $W_{0,1}^p(\Omega^{R_0}) \subset L^p(\Omega^R)$ is compact then by [20, p.I, s.5.2] the sequence $\{u_t^s\}$ is compact in $L^p(Q_T^{R_0})$.

Now we can choose the sequence of domain $Q_T^{R_0}$ such that $Q_T^{R_0} \subset Q_T^{R_0+1}$, $R_0 = 2, 3, \dots$. From the last inequality we have that the sequence $\{u^s\}$ is bounded in every bounded domain Q^{R_0} in the space $L^2((0, T); H_{0,1}^2(\Omega^{R_0}))$, $\{u_t^s\}$ is bounded in $L^p((0, T); W_{0,1}^{1,p}(\Omega^{R_0})) \cap L^q(Q_T^{R_0})$, $\{u_{tt}^s\}$ is bounded in $L^{q'}((0, T); W_0^{-1,p}(\Omega^{R_0}))$. Hence we can construct the diagonal subsequence $\{u^{ss}\}$ such that

$$\begin{aligned}
u^{s,s} & \rightarrow u \text{ weakly in } L^2((0, T); H_{0,loc}^2(\overline{\Omega})), \\
u_t^{s,s} & \rightarrow u_t \text{ weakly in } L^p((0, T); W_{0,loc}^{1,p}(\overline{\Omega})) \cap L^q((0, T); L_{loc}^q(\overline{\Omega})), \\
u_{tt}^{s,s} & \rightarrow u_{tt} \text{ in } L^p((0, T); L_{loc}^p(\overline{\Omega}))
\end{aligned}$$

as $s \rightarrow \infty$. Now, we can prove that $\{u^{s,s}\}$ is a Cauchy sequence.

Let $s, k \in N$ and $s > k > R$, $R > R_1 > 1$. Let us denote $u^{sk} = u^{s,s} - u^{k,k}$. It is easy to show that for every $\tau \in (0, T]$ the following equality is satisfied

$$\begin{aligned}
& \int_{Q_\tau} \left[u_{tt}^{sk} v + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^{sk} v_{x_k x_l} + \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{sk} v_{x_j} \right. \\
& + \sum_{i=1}^n \left[b_i(x) |u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} v_{x_i} - b_i(x) |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} v_{x_i} + a_0(x) u^{sk} v \right] \\
& \left. + c(x) \left(|u_t^{s,s}|^{q-2} u_t^{s,s} - |u_t^{k,k}|^{q-2} u_t^{k,k} \right) v - (f^s(x, t) - f^k(x, t)) v \right] dx dt = 0,
\end{aligned} \tag{2.19}$$

where $v = u_t^{sk} \Psi_R(x) e^{-\alpha t}$.

If we consider the respective components of the equality (2.19), by analogy to $I_8 - I_{10}$, I_{12} we will obtain $I_{15} := \int_{Q_\tau} u_{tt}^{sk} u_t^{sk} \Psi_R(x) e^{-\alpha t} dx dt = \frac{1}{2} \int_{\Omega_\tau} |u_t^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx + \frac{\alpha}{2} \int_{Q_\tau} |u_t^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt$. Next, let $I_{16} := \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^{sk} v_{x_k x_l} dx dt = I_{16}^1 + I_{16}^2 + I_{16}^3$, where

$$\begin{aligned}
I_{16}^1 &:= \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^{sk} u_{tx_k x_l}^{sk} \Psi_R(x) e^{-\alpha t} dx dt \\
&\geq \frac{\alpha_2}{2} \int_{\Omega_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx \\
&\quad - \frac{1}{2} \int_{\Omega_0} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{0, sx_i x_j}^{sk} u_{0, sx_k x_l}^{sk} \Psi_R(x) dx \\
&\quad + \frac{\alpha \alpha_2}{2} \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \\
&\geq \frac{\alpha_2}{2} \int_{\Omega_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx \\
&\quad + \frac{\alpha \alpha_2}{2} \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt, \\
I_{16}^2 &\leq \frac{\delta}{2} \alpha_2^0 \int_{\Omega_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \frac{\delta}{2p} \int_{Q_\tau} \sum_{k=1}^n |u_{tx_k}^{sk}|^p \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \mu(\delta) \int_{Q_\tau} \sum_{l=1}^n \left(\frac{\Psi_{Rx_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}}} \right)^r e^{-\alpha t} dx dt,
\end{aligned}$$

$$\begin{aligned}
I_{16}^3 &:= \int_{Q_\tau} \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j}^{sk} u_t^{sk} \Psi_{Rx_l, x_k}(x) e^{-\alpha t} dx dt \\
&\leq \frac{a_2^0 \delta}{2} \int_{Q_\tau} \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \frac{\delta}{2q} \int_{Q_\tau} |u_t^{sk}|^q \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \mu_1(\delta) \int_{Q_\tau} \sum_{k,l=1}^n \left(\frac{\Psi_{Rx_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}}} \right)^{r_1} e^{-\alpha t} dx dt, \quad \delta > 0.
\end{aligned}$$

From I_{10} we obtain $I_{17} := \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{sk} v_{x_j} dx dt = I_{17}^1 + I_{17}^2$, where

$$\begin{aligned}
I_{17}^1 &:= \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{sk} u_{tx_j}^{sk} \Psi_R(x) e^{-\alpha t} dx dt \\
&\geq \frac{a_1}{2} \int_{\Omega_\tau^R} \sum_{i=1}^n |u_{x_i}^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx \\
&\quad + \frac{a_1 \alpha}{2} \int_{Q_\tau^R} \sum_{i=1}^n |u_{x_i}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \quad \text{and} \\
I_{17}^2 &:= \int_{Q_\tau} \sum_{i,j=1}^n a_{ij}(x) u_{x_i}^{sk} u_t^{sk} \Psi_{Rx_j}(x) e^{-\alpha t} dx dt \\
&\leq \frac{\delta}{2} \int_{Q_\tau} \sum_{i=1}^n |u_{x_i}^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \frac{\delta}{2q} \int_{Q_\tau} |u_t^{sk}|^q \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \mu_2(\delta) \int_{Q_\tau} \sum_{j=1}^n \left(\frac{\Psi_{Rx_j}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}}} \right)^{r_1} e^{-\alpha t} dx dt.
\end{aligned}$$

Now, we consider the following component

$$\begin{aligned}
I_{18} &:= \int_{Q_\tau} \sum_{i=1}^n b_i(x) \left(|u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right) v_{x_i} dx dt \\
&= \int_{Q_\tau} \sum_{i=1}^n b_i(x) \left(|u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right) u_{tx_i}^{sk} \Psi_R(x) e^{-\alpha t} dx dt \\
&\quad + \int_{Q_\tau} \sum_{i=1}^n b_i(x) \left(|u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right) u_t^{sk} \Psi_{Rx_i}(x) e^{-\alpha t} dx dt \\
&= I_{18}^1 + I_{18}^2,
\end{aligned}$$

where

$$\begin{aligned}
 I_{18}^1 &:= \int_{Q_\tau} \sum_{i=1}^n b_i(x) \left(|u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right) \cdot u_{tx_i}^{sk} \Psi(x) e^{-\alpha t} dx dt \\
 &\geq 2^{2-p} b_0 \int_{Q_\tau} \sum_{i=1}^n |u_{tx_i}^{sk}|^p \Psi_R(x) e^{-\alpha t} dx dt \text{ and} \\
 I_{18}^2 &:= \int_{Q_\tau} \sum_{i=1}^n b_i(x) \left(|u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right) u_t^{sk} \Psi_{R x_i}(x) e^{-\alpha t} dx dt \\
 &\leq \left(\int_{Q_\tau} \sum_{i=1}^n |b_i(x)|^{p'} \left| |u_{tx_i}^{s,s}|^{p-2} u_{tx_i}^{s,s} - |u_{tx_i}^{k,k}|^{p-2} u_{tx_i}^{k,k} \right|^{p'} |\Psi_{R x_i}(x)| e^{-\alpha t} dx dt \right)^{\frac{1}{p'}} \\
 &\quad \times \left(\int_{Q_\tau} \sum_{i=1}^n |u_t^{sk}|^p |\Psi_{R x_i}(x)| e^{-\alpha t} dx dt \right)^{\frac{1}{p}} \equiv p_{sk}.
 \end{aligned}$$

We can choose $k_0 > R + 1$ such that for all $s, k \geq k_0$, p_{sk} will be small enough. From I_{12} we obtain

$$\begin{aligned}
 I_{19} &:= \int_{Q_\tau} a_0(x) u^{sk} u_t^{sk} \Psi_R(x) e^{-\alpha t} dx dt \geq -\frac{1}{2} \int_{\Omega_0} a_0(x) |u_{0,s}^{sk}|^2 \Psi_R(x) dx \\
 &\quad + \frac{a_0}{2} \int_{\Omega_\tau} |u^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx + \frac{a_0 \alpha}{2} \int_{Q_\tau} |u^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt \\
 &\geq \frac{a_0}{2} \int_{\Omega_\tau} |u^{sk}|^2 \Psi_R(x) e^{-\alpha \tau} dx + \frac{a_0 \alpha}{2} \int_{Q_\tau} |u^{sk}|^2 \Psi_R(x) e^{-\alpha t} dx dt.
 \end{aligned}$$

Similarly to I_{18}^1 from **(C)** we have

$$\begin{aligned}
 I_{20} &:= \int_{Q_\tau} c(x) \left(|u_t^{s,s}|^{q-2} u_t^{s,s} - |u_t^{k,k}|^{q-2} u_t^{k,k} \right) \cdot u_t^{sk} \Psi_R(x) e^{-\alpha t} dx dt \\
 &\geq c_0 2^{2-q} \int_{Q_\tau} |u_t^{sk}|^q \Psi_R(x) e^{-\alpha t} dx dt.
 \end{aligned}$$

From estimates of the integrals I_{15} – I_{20} and from the initial condition the following inequality is satisfied

$$\begin{aligned}
& \int_{\Omega_\tau^{R_1}} \left[|u_t^{sk}|^2 + a_2 \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 + a_1 \sum_{i=1}^n |u_{x_i}^{sk}|^2 + a_0 |u^{sk}|^2 \right] e^{-\alpha\tau} dx \\
& + \int_{Q_\tau^{R_1}} \left[\alpha |u_t^{sk}|^2 + (\alpha a_2 - 2a_2^0 \delta) \sum_{i,j=1}^n |u_{x_i x_j}^{sk}|^2 + (\alpha a_1 - \delta) \sum_{i=1}^n |u_{x_i}^{sk}|^2 \right. \\
& + \left(2^{3-p} b_0 - \frac{\delta}{p} \right) \sum_{i=1}^n |u_{t x_i}^{sk}|^p + \left(2^{3-q} c_0 - \frac{\delta}{q} \right) |u_t^{sk}|^q \\
& \left. + a_0 \alpha |u^{sk}|^2 \right] e^{-\alpha t} dx dt \leq \frac{2}{(R - R_1)^\beta} \left\{ \int_{Q_\tau} \left[\mu(\delta) \sum_{l=1}^n \left(\frac{\Psi_{R x_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r \right. \right. \\
& \left. \left. + \mu_1(\delta) \sum_{k,l=1}^n \left(\frac{\Psi_{R x_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} + \mu_2(\delta) \sum_{j=1}^n \left(\frac{\Psi_{R x_j}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} \right] e^{-\alpha t} dx dt + p_{sk} \right\}. \tag{2.20}
\end{aligned}$$

We can choose α and δ such that the inequalities $\alpha a_2 - 2a_2^0 \delta \geq 1$, $2^{3-p} b_0 - \frac{\delta}{p} \geq 2^{2-p} b_0$, $2^{3-q} c_0 - \frac{\delta}{q} \geq 2^{2-q} c_0$, $\alpha a_1 - \delta \geq 1$ hold. Moreover from (2.20) we have

$$\left(\frac{\Psi_{R x_i}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r \leq C(2R)^{-r+\beta}, \quad \left(\frac{\Psi_{R x_i}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} \leq C(2R)^{-r_1+\beta}.$$

Next, since $|\Psi_{R x_l x_k}| \leq C \cdot \Psi_R [h_R]^{-2}$, we get $\left(\frac{\Psi_{R x_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} \leq C(2R)^{-2r_1+\beta}$. Hence

$$\begin{aligned}
& \mu(\delta) \int_{Q_\tau} \sum_{l=1}^n \left(\frac{\Psi_{R x_l}}{\Psi_R^{\frac{1}{2} + \frac{1}{p}}} \right)^r e^{-\alpha t} dx dt + \mu_1(\delta) \int_{Q_\tau} \sum_{k,l=1}^n \left(\frac{\Psi_{R x_l x_k}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\alpha t} dx dt \\
& + \mu_2(\delta) \int_{Q_\tau} \sum_{j=1}^n \left(\frac{\Psi_{R x_j}}{\Psi_R^{\frac{1}{2} + \frac{1}{q}}} \right)^{r_1} e^{-\alpha t} dx dt \\
& \leq \mu_1(\delta) C_1 (2R)^{-r+\beta+n} + \mu_1(\delta) C_1 (2R)^{-2r_1+\beta+n} + \mu_2(\delta) C_1 (2R)^{-r_1+\beta+n}.
\end{aligned}$$

Let $\epsilon > 0$ be an arbitrary small number. From (2.20) we get that there exist $R > 0$ and $\kappa_0(R)$ such that the right hand of (2.20) is less than ϵ . Hence from (2.20) we have

$$\|u_t^{s,s} - u_t^{k,k}\|_{C([0,T];L^2(\Omega^R))} < \epsilon, \quad \|u^{s,s} - u^{k,k}\|_{L^2((0,T);H_{0,1}^2(\Omega^R))} < \epsilon$$

and

$$\|u_t^{s,s} - u_t^{k,k}\|_{L^q(Q_T^R)} < C_2 \epsilon, \quad \|u_t^{s,s} - u_t^{k,k}\|_{L^p((0,T);W_{0,1}^{1,p}(\Omega^R))} < \epsilon.$$

Thus the diagonal sequence $\{u^{s,s}\}$ satisfies the Cauchy condition. Taking into account that R is arbitrary, it follows that

$$\begin{aligned} u^{s,s} &\rightarrow u \text{ in } L^2((0, T); H_{0,\text{loc}}^2(\overline{\Omega})), \quad u_t^{s,s} \rightarrow u_t \text{ in } C([0, T]; L_{\text{loc}}^2(\overline{\Omega})), \\ u^{s,s} &\rightarrow u \text{ in } L^p((0, T); W_{0,\text{loc}}^{1,p}(\overline{\Omega})) \cap L^q((0, T); L_{\text{loc}}^q(\overline{\Omega})). \end{aligned}$$

If we put with s to infinity then from (2.16) we obtain

$$\begin{aligned} &\int_{\Omega_\tau} u_t(x, \tau) v(x, \tau) dx + \int_{Q_\tau} \left[-u_t v_t + \sum_{i,j,k,l=1}^n a_{ij}^{kl}(x) u_{x_i x_j} v_{x_k x_l} \right. \\ &\quad + \sum_{i,j=1}^n a_{ij}(x) u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x) |u_{tx_i}|^{p-2} u_{tx_i} v_{x_i} \\ &\quad \left. + a_0(x) uv + c(x) |u_t|^{q-2} u_t v \right] dx dt \\ &= \int_{Q_\tau} f(x, t) v dx dt + \int_{\Omega_0} u_1(x) v(x, 0) dx \end{aligned} \quad (2.21)$$

$\forall v \in L^2((0, T); H_{0,\text{loc}}^2(\overline{\Omega})) \cap L^q((0, T); L_{\text{loc}}^q(\overline{\Omega})) \cap L^p((0, T); W_{0,\text{loc}}^{1,p}(\overline{\Omega}))$, $v_t \in L^2((0, T); L_{\text{loc}}^2(\overline{\Omega}))$, where $\text{supp } v$ is bounded.

Besides that $u^{s,s}(x, 0) = u_{0s}(x)$, $u_{0s} \rightarrow u_0$ in $L_{\text{loc}}^2(\overline{\Omega})$, hence $u(x, 0) = u_0(x)$. In virtue of Definition 1, the function u is a solution of the problem (1.1)–(1.3). $\square$

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Received: 06 June 2008.

Accepted: 25 February 2009.