Uniqueness of Positive Solutions for a Nonlinear Elliptic System

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Abstract. In this paper we study the uniqueness of nontrivial positive solutions for the following second order nonlinear elliptic system:

$$\begin{cases} -\Delta u_1 + V_1(|x|)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2(|x|)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbb{R}^N. \end{cases}$$

We prove that for sufficiently small $\beta > 0$, the above system has a unique nontrivial positive solution.

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1. Introduction

In this paper, we consider the uniqueness of nontrivial positive solution of

$$-\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbb{R}^N, \quad (1)$$

$$-\Delta u_2 + V_2(x)u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 \quad \text{in } \mathbb{R}^N, \quad (2)$$

$$u_1(x), u_2(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty, \quad (3)$$

$$u_1(x), u_2(x) > 0 \quad \text{in } \mathbb{R}^N, \quad (4)$$

where $\mu_1, \mu_2, \beta > 0$ and $N = 1, 2, 3$. Here, we say $u = (u_1, u_2)$ is a *nontrivial positive solution* of (1)–(4) if u satisfies (1)–(4). This system appears in the study of the existence of standing waves for the following coupled nonlinear Schrödinger equations:

$$\begin{cases} i \frac{\partial \psi_1}{\partial t} + \Delta_x \psi_1 + \lambda_1(x) \psi_1 + (\mu_1 |\psi_1|^2 + \beta |\psi_2|^2) \psi_1 = 0 & \text{in } (0, \infty) \times \mathbb{R}^N, \\ i \frac{\partial \psi_2}{\partial t} + \Delta_x \psi_2 + \lambda_2(x) \psi_2 + (\beta |\psi_1|^2 + \mu_2 |\psi_2|^2) \psi_2 = 0 & \text{in } (0, \infty) \times \mathbb{R}^N, \end{cases} \quad (5)$$

Indeed, $\psi_j(t, x) = \exp(i\bar{\lambda}_j t) u_j(x)$ is a solution of (5) if and only if $(u_1(x), u_2(x))$ solves (1) and (2) with $V_j(x) = \bar{\lambda}_j - \lambda_j(x)$. The system (5) arises in many physical problems, especially in the Hartree–Fock theory and nonlinear optics. We refer to [1, 3, 6, 8–10, 13, 14] and references therein.

Recently the existence of nontrivial positive solutions has been studied extensively in [1, 3, 6, 8–10, 13, 14]. In particular, the case where $V_1(x)$, $V_2(x)$ are positive and independent of x is well studied and it is shown in [1, 3, 8–10, 13] that there exist positive constants $\beta_2 \geq \beta_1 > 0$ such that for $\beta \in [0, \beta_1) \cup (\beta_2, \infty)$, (1)–(4) has a nontrivial positive solution. And it has been extended to x -dependent situations in [6, 14]. However the uniqueness of nontrivial positive solution is not studied and the main purpose of this paper is to establish the uniqueness for small $\beta > 0$.

Firstly we consider the uniqueness of nontrivial positive solutions in the constant coefficient case:

$$\begin{cases} -\Delta u_1 + V_1 u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 & \text{in } \mathbb{R}^N, \\ -\Delta u_2 + V_2 u_2 = \beta u_1^2 u_2 + \mu_2 u_2^3 & \text{in } \mathbb{R}^N, \\ u_1(x), u_2(x) > 0 & \text{in } \mathbb{R}^N, \\ u_1(x), u_2(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty. \end{cases} \quad (6)$$

Here V_1, V_2 are positive constants and $N = 1, 2, 3$.

Now we state our result in the constant coefficient case.

Theorem 1.1. *Suppose that $N = 1, 2, 3$. Then there exists a $\beta_0 > 0$ such that if $\beta \in [0, \beta_0)$, then nontrivial positive solutions of (6) are unique up to translation.*

Next we deal with the variable coefficient case. Here we assume that $N = 2, 3$ and $V_j(x)$ satisfies the following condition:

- (V1) $V_j \in C^2(\mathbb{R}^N)$ and $V_j(x)$ is radially symmetric, i.e., $V_j(x) = V_j(|x|)$ for $j = 1, 2$.
- (V2) $0 < \inf V_j(r) \leq \sup V_j(r) < \infty$ for $j = 1, 2$ where $r = |x|$.
- (V3) $V_j'(r) \geq 0$ for $r \geq 0$ and $j = 1, 2$.
- (V4) When $N = 3$ we assume

$$\inf_{r>0} \left\{ V_j''(r)r^3 + \frac{13}{3}V_j'(r)r + \frac{8}{3}V_j(r) \right\} > 0 \quad \text{for } j = 1, 2.$$

Our main result is the following.

Theorem 1.2. *Suppose $N = 2, 3$ and $V_j(x)$ satisfies (V1)–(V4). Then there exists a $\beta_0 > 0$ such that for $\beta \in [0, \beta_0)$ (1)–(4) has a unique nontrivial positive solution.*

Remark 1.1. (i) When $N = 2, 3$, by the result of [2], it is known that any nontrivial positive solution of (1)–(4) is radially symmetric and monotone decreasing under (V1) and $V_j(r) > 0$, $V_j'(r) \geq 0$, $V_j'(r) \not\equiv 0$, and radially symmetric with respect to some point $x_0 \in \mathbb{R}^N$ if $V_j(x) \equiv V_j$ is independent of x and positive.

(ii) We remark that the argument in [2] also works for $N = 1$ after suitable modification. However, it is not clearly stated in [2], we give the symmetry and monotonicity result for $N = 1$ in Appendix for the sake of readers.

- (iii) Uniqueness of nontrivial positive solution of (1)–(4) does not hold for all $\beta > 0$. Indeed, the following example is given in [13]; let $V_1(x) \equiv V_2(x) \equiv 1$, $\mu_1 = \mu_2 = \beta = 1$ and ω_1 be a positive solution of

$$-\Delta u_1 + u_1 = u_1^3 \quad \text{in } \mathbb{R}^N.$$

Then $u(x) = (u_1(x), u_2(x)) = (\omega_1(x) \cos \theta, \omega_1(x) \sin \theta)$ is a nontrivial positive solution of (1)–(4) for any $\theta \in (0, \pi/2)$.

We say that a radial nontrivial positive solution $u = (u_1, u_2)$ of (1)–(4) is *nondegenerate* in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ if and only if the linearized system

$$\begin{cases} -\Delta w_1 + V_1(x)w_1 = 3\mu_1 u_1^2 w_1 + \beta(u_2^2 w_1 + 2u_1 u_2 w_2) & \text{in } \mathbb{R}^N, \\ -\Delta w_2 + V_2(x)w_2 = \beta(2u_1 u_2 w_1 + u_1^2 w_2) + 3\mu_2 u_2^2 w_2 & \text{in } \mathbb{R}^N, \\ w_1, w_2 \in H_r^1(\mathbb{R}^N) \end{cases}$$

has only trivial solution $w_1 \equiv w_2 \equiv 0$. Here

$$H_r^1(\mathbb{R}^N) = \{u_1 \in H^1(\mathbb{R}^N) \mid u_1(x) = u_1(|x|) \text{ is radially symmetric}\}.$$

As to the nondegeneracy of our unique positive radial solution, we have

Theorem 1.3. *The unique nontrivial positive radial solution of (1)–(4) is nondegenerate in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ for $\beta \in [0, \beta_0)$.*

To prove our Theorems 1.1–1.3, a priori estimates and uniform exponential decays of nontrivial positive solutions are important. In Sect. 2, we will obtain a priori L^∞ estimates and uniform exponential decays of

$$\mathcal{S}_{\bar{\beta}} := \{u = (u_1, u_2) \mid u \text{ is a nontrivial positive solution of (1)–(4) with } \beta \in [0, \bar{\beta}], \\ u \text{ is a radially symmetric function}\}.$$

Here Liouville type result (Lemma 2.1) and monotonicity of solutions play roles. In Sect. 3, we prove Theorems 1.1–1.3. The behavior of nontrivial positive solution as $\beta \rightarrow 0$ is a key of our argument.

2. A priori bounds for nontrivial positive solutions

In this section, we assume that $1 \leq N \leq 3$ and $V_j(x)$ satisfies (V1)–(V4).

2.1. L^∞ bound of nontrivial positive solutions

For $\bar{\beta} > 0$ we define $\mathcal{S}_{\bar{\beta}}$ as follows:

$$\mathcal{S}_{\bar{\beta}} := \{u = (u_1, u_2) \mid u \text{ is a nontrivial positive solution of (1)–(4) with } \beta \in [0, \bar{\beta}], \\ u \text{ is a radially symmetric function}\}.$$

The following lemma is essential to prove uniform L^∞ estimates and uniform exponential decays of $u \in \mathcal{S}_{\bar{\beta}}$.

Lemma 2.1. (c.f. [4, 5]) *Let $1 < \alpha \leq 3$ if $N = 3$ and $1 < \alpha < \infty$ if $N = 1, 2$. Then there is no positive function such that*

$$-\Delta u \geq |u|^{\alpha-1}u \text{ in } \mathbb{R}^N. \quad (7)$$

Proof. When $N = 3$, we refer to Theorem 8.4 in [12]. When $N = 2$, the conclusion of Lemma 2.1 follows from Liouville's theorem. For example, see [11]. When $N = 1$, Lemma 2.1 can be shown easily. $\square$

The following proposition gives us an a priori L^∞ -bound of $\mathcal{S}_{\bar{\beta}}$.

Proposition 2.2. *For any $\bar{\beta} > 0$ there exists an $M_{\bar{\beta}} > 0$ such that*

$$\|u_1\|_{L^\infty} + \|u_2\|_{L^\infty} \leq M_{\bar{\beta}}$$

for all $u \in \mathcal{S}_{\bar{\beta}}$.

Proof. We prove indirectly and assume that there exist $(u_k) \subset \mathcal{S}_{\bar{\beta}}$ and (β_k) such that $\beta_k \rightarrow \beta_0$ and $\|u_{k,2}\|_{L^\infty} \leq \|u_{k,1}\|_{L^\infty} \rightarrow \infty$. For $j = 1, 2$, we set

$$\eta_k = \frac{1}{\|u_{k,1}\|_{L^\infty}}, \quad v_{k,j}(x) = \eta_k u_{k,j}(\eta_k x).$$

By Remark 1.1, (u_k) is radially symmetric and monotone decreasing, we have $v_{k,1}(0) = 1$ and $\|v_{k,j}\|_{L^\infty} \leq 1$ ($j = 1, 2$). Moreover, since u_k is a solution of (1) and (2), v_k satisfies

$$\begin{cases} -\Delta v_{k,1} + \eta_k^2 V_1(\eta_k x) v_{k,1} = \mu_1 v_{k,1}^3 + \beta_k v_{k,1} v_{k,2}^2, \\ -\Delta v_{k,2} + \eta_k^2 V_2(\eta_k x) v_{k,2} = \beta_k v_{k,1}^2 v_{k,2} + \mu_2 v_{k,2}^3. \end{cases}$$

By the standard elliptic argument, if necessary take a subsequence, $v_{k,j} \rightarrow v_{0,j}$ in $C_{\text{loc}}^2(\mathbb{R}^N)$ where $v_{0,j}$ is a nonnegative solution of

$$\begin{cases} -\Delta v_{0,1} = \mu_1 v_{0,1}^3 + \beta_0 v_{0,1} v_{0,2}^2, \\ -\Delta v_{0,2} = \beta_0 v_{0,1}^2 v_{0,2} + \mu_2 v_{0,2}^3. \end{cases}$$

Since $v_{0,1}(0) = 1$ and the maximum principle, $v_{0,1} > 0$ in $\mathbb{R}^N$. So $v_{0,1}(x)$ satisfies the following differential inequality:

$$-\Delta v_{0,1} \geq \mu_1 v_{0,1}^3 \text{ in } \mathbb{R}^N.$$

This contradicts to Lemma 2.1, so the conclusion of Proposition 2.2 holds. $\square$

2.2. Uniform exponential decay estimates in $\mathcal{S}_{\bar{\beta}}$

As pointed in introduction, any nontrivial positive solution of (1)–(4) is a radially symmetric function. So we rewrite (1) and (2) as follows:

$$\begin{cases} -u_1''(r) - \frac{N-1}{r} u_1'(r) + V_1(r) u_1(r) = \mu_1 u_1^3(r) + \beta u_1(r) u_2^2(r), \\ -u_2''(r) - \frac{N-1}{r} u_2'(r) + V_2(r) u_2(r) = \beta u_1^2(r) u_2(r) + \mu_2 u_2^3(r). \end{cases} \quad (8)$$

The following proposition gives us a uniform exponential decay of $\mathcal{S}_{\bar{\beta}}$.

Proposition 2.3. *For any $\bar{\beta} > 0$ there exist $C_1 = C_1(\bar{\beta}), C_2 = C_2(\bar{\beta}) > 0$ such that*

$$|u_1(r)| + |u_2(r)| \leq C_1 \exp(-C_2 r), \quad |u'_1(r)| + |u'_2(r)| \leq C_1 \exp(-C_2 r)$$

for all $u \in \mathcal{S}_{\bar{\beta}}$.

Proof. The proof is divided into 2 steps.

Step 1 Let $\beta \geq 0$ and $u(r)$ be a nontrivial positive solution of (1)–(4). There exist $C_3 = C_3(u), C_4 = C_4(u) > 0$ such that

$$|u_1(r)| + |u_2(r)| \leq C_3 \exp(-C_4 r), \quad |u'_1(r)| + |u'_2(r)| \leq C_3 \exp(-C_4 r). \quad (9)$$

Let $\beta \geq 0$ and u be a nontrivial positive solution of (8). We prove that $u_1(r)$ and $u'_1(r)$ decays exponentially. Since $u_j(r) \rightarrow 0$ as $r \rightarrow \infty$ and (V2)–(V3) holds, there exists an $r_0 \geq 1$ such that for $r \geq r_0$

$$u_1(r), u_2(r) \leq 1, \quad 0 < \frac{V_1(0)}{2} \leq V_1(r) - \mu_1 u_1^2(r) - \beta u_2^2(r). \quad (10)$$

Thus it holds

$$\begin{aligned} 0 &= -u''_1(r) - \frac{N-1}{r} u'_1(r) + (V_1(r) - \mu_1^2 u_1^2(r) - \beta u_2^2(r)) u_1(r) \\ &\geq -u''_1(r) - \frac{N-1}{r} u'_1(r) + \frac{V_1(0)}{2} u_1(r) \quad \text{for } r \geq r_0. \end{aligned} \quad (11)$$

Let $\delta > 0$ satisfy $\max\{\delta^2, (N-1)\delta\} < V_1(0)/4$ and set

$$\varphi_{R,\delta}(r) := \exp(-\delta(r-r_0)) + \exp(-\delta(R-r))$$

for any $R > r_0$. Then we have

$$-\varphi''_{R,\delta}(r) - \frac{N-1}{r} \varphi'_{R,\delta}(r) + \frac{V_1(0)}{2} \varphi_{R,\delta}(r) \geq 0 \quad \text{for all } r \in [r_0, R]. \quad (12)$$

Since $\varphi_{R,\delta}(r_0) \geq 1$, $\varphi_{R,\delta}(R) \geq 1$ and (10)–(12) holds, by the comparison theorem, it follows

$$u_1(r) \leq \varphi_{R,\delta}(r) \quad \text{for all } r \in [r_0, R]. \quad (13)$$

Since $R > r_0$ is arbitrary, let $R \rightarrow \infty$, then we have

$$u_1(r) \leq \exp(-\delta(r-r_0)) \quad \text{for all } r \geq r_0.$$

Thus $u_1(r)$ has an exponential decay. In a similar way, we can show that $u_2(r)$ has an exponential decay. By (8), it holds

$$r^{N-1} u'_1(r) = \int_0^r (s^{N-1} u'_1(s))' ds = \int_0^r s^{N-1} u_1(s) (V_1(s) - \mu_1 u_1^2(s) - \beta u_2^2(s)) ds. \quad (14)$$

Since the integrand of right-hand side of (14) has an exponential decay, we conclude that $u'_1(r)$ also has an exponential decay. In a similar way, it follows that $u'_2(r)$ has an exponential decay.

Next we prove that for any $\bar{\beta} > 0$ we can choose C_3, C_4 in (9) uniformly with respect to $u \in \mathcal{S}_{\bar{\beta}}$, that is, we complete the proof of Proposition 2.3.

Step 2 Conclusion.

We prove that there exist $C_3 = C_3(\bar{\beta}), C_4 = C_4(\bar{\beta}) > 0$ such that

$$|u_1(r)| + |u_2(r)| \leq C_3 \exp(-C_4 r) \quad (15)$$

for all $u \in \mathcal{S}_{\bar{\beta}}$. From (14) and (15) it follows that

$$|u'_1(r)| + |u'_2(r)| \leq C_3 \exp(-C_4 r)$$

for all $u \in \mathcal{S}_{\bar{\beta}}$.

We prove (15) indirectly and assume that there exist $(u_k) \subset \mathcal{S}_{\bar{\beta}}$ and (β_k) such that $\beta_k \rightarrow \beta_0$ and $u_{k,1}(r_k) + u_{k,2}(r_k) > k \exp(-r_k/k)$ holds for some $r_k \geq 0$. Since (u_k) satisfies (8) with $\beta = \beta_k$ and $\|u_k\|_{L^\infty}$ is bounded by Proposition 2.2, we can also assume that $u_{k,j} \rightarrow u_{0,j}$ in $C_{\text{loc}}^2(\mathbb{R}^N)$, where u_0 is a nonnegative solution of

$$\begin{cases} -u''_{0,1}(r) - \frac{N-1}{r} u'_{0,1}(r) + V_1(r) u_{0,1}(r) = \mu_1 u_{0,1}^3(r) + \beta_0 u_{0,1}(r) u_{0,2}^2(r), \\ -u''_{0,2}(r) - \frac{N-1}{r} u'_{0,2}(r) + V_2(r) u_{0,2}(r) = \beta_0 u_{0,1}^2(r) u_{0,2}(r) + \mu_2 u_{0,2}^3(r). \end{cases} \quad (16)$$

We set

$$\begin{aligned} E_k(r) &= \sum_{j=1}^2 \left(\frac{1}{2} (u'_{k,j}(r))^2 + \frac{\mu_j}{4} u_{k,j}^4 - \frac{1}{2} V_j(r) u_{k,j}^2(r) \right) + \frac{\beta_k}{2} u_{k,1}^2(r) u_{k,2}^2(r), \\ E_0(r) &= \sum_{j=1}^2 \left(\frac{1}{2} (u'_{0,j}(r))^2 + \frac{\mu_j}{4} u_{0,j}^4 - \frac{1}{2} V_j(r) u_{0,j}^2(r) \right) + \frac{\beta_0}{2} u_{0,1}^2(r) u_{0,2}^2(r). \end{aligned}$$

Since $u_k \rightarrow u_0$ in $C_{\text{loc}}^2(\mathbb{R}^N)$, $E_k \rightarrow E_0$ in $C_{\text{loc}}^1(\mathbb{R}^N)$. By (V3) and (8) with $\beta = \beta_k$, we have

$$\begin{aligned} E'_k(r) &= \sum_{j=1}^2 (u''_{k,j}(r) + \mu_j u_{k,j}^3(r) - V_j(r) u_{k,j}(r)) u'_{k,j}(r) - \sum_{j=1}^2 \frac{1}{2} V'_j(r) u_{k,j}^2(r) \\ &\quad + \beta_k u_{k,1} u_{k,2}^2(r) u'_{k,1}(r) + \beta_k u_{k,1}^2(r) u_{k,2}(r) u'_{k,2}(r) \\ &= -\frac{N-1}{r} \sum_{j=1}^2 (u'_{k,1}(r))^2 - \sum_{j=1}^2 \frac{1}{2} V'_j(r) u_{k,j}^2(r) \leq 0. \end{aligned}$$

Since u_k and u'_k have an exponential decay for each $k \geq 1$ by Step 1, we have $\lim_{r \rightarrow \infty} E_k(r) = 0$. By the monotonicity of E_k , it follows that $E_k(r) \geq 0$ for $r \geq 0$, which implies that $E_0(r) \geq 0$ for $r \geq 0$.

By [2], $u_k(r)$ is a decreasing function when $N = 2, 3$. We remark that this result holds even when $N = 1$. See Appendix. Thus $u_0(r)$ is also a nonincreasing function. So we set

$$\lim_{|x| \rightarrow \infty} u_{0,j}(x) = u_{0,\infty,j} \geq 0 \quad (j = 1, 2) \quad (17)$$

and next we will show $u_{0,\infty,j} = 0$ ($j = 1, 2$).

Since $E_0(r)$ and $u_{0,j}(r)$ are bounded in $[0, \infty)$, $u'_0(r)$ is also bounded in $[0, \infty)$. By (16), $u_{0,\infty,1}$ and $u_{0,\infty,2}$ satisfy

$$\begin{aligned} u_{0,\infty,1}(\mu_1 u_{0,\infty,1}^2 + \beta_0 u_{0,\infty,2}^2 - V_{1,\infty}) &= 0, \\ u_{0,\infty,2}(\beta_0 u_{0,\infty,1}^2 + \mu_2 u_{0,\infty,2}^2 - V_{2,\infty}) &= 0. \end{aligned} \quad (18)$$

Then we have

$$(u_{0,\infty,1}, u_{0,\infty,2}) \in \left\{ (0, 0), \left(\sqrt{\frac{V_{1,\infty}}{\mu_1}}, 0 \right), \left(0, \sqrt{\frac{V_{2,\infty}}{\mu_2}} \right), (u_1^*, u_2^*) \right\}$$

where (u_1^*, u_2^*) satisfies

$$\begin{aligned} V_{1,\infty} &= \mu_1 (u_1^*)^2 + \beta_0 (u_2^*)^2 \\ V_{2,\infty} &= \beta_0 (u_1^*)^2 + \mu_2 (u_2^*)^2. \end{aligned}$$

If $(u_{0,\infty,1}, u_{0,\infty,2}) \neq (0, 0)$, then it is easily seen that $\lim_{r \rightarrow \infty} E_0(r) < 0$, which contradicts to $E_0(r) \geq 0$. Thus $u_{0,\infty,j} = 0$ ($j = 1, 2$).

Since $\lim_{r \rightarrow \infty} u_{0,j}(r) = 0$, we can prove that there exist $\bar{C}_1, \bar{C}_2 > 0$ such that

$$|u_{0,1}(r)| + |u_{0,2}(r)| \leq \bar{C}_1 \exp(-\bar{C}_2 r).$$

Since $u_{k,j}(r)$ is a decreasing function and $u_{k,j} \rightarrow u_{0,j}$ in $C_{\text{loc}}^2(\mathbb{R}^N)$, there exist $r_1 > 0$, $k_0 \in \mathbb{N}$ such that if $r \geq r_1$ and $k \geq k_0$, then

$$u_{k,1}(r), u_{k,2}(r) \leq 1, \quad 0 < \frac{V_1(0)}{2} \leq V_1(r) - \mu_1 u_{k,1}^2(r) - \beta_k u_{k,2}^2(r).$$

As in Step 1, we can show that there exist $\bar{C}_3, \bar{C}_4 > 0$ such that

$$|u_{k,1}(r)| + |u_{k,2}(r)| \leq \bar{C}_3 \exp(-\bar{C}_4 r)$$

for all $k \geq 1$ and $r \geq 0$. However this is a contradiction and (15) holds. $\square$

As a corollary to Proposition 2.3, we have

Corollary 2.4. *For any $\bar{\beta} > 0$, $\mathcal{S}_{\bar{\beta}}$ is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Moreover it has the following compactness property: for any sequence $(u_k) \subset \mathcal{S}_{\bar{\beta}}$, there exists a strongly convergent subsequence (u_{k_m}) in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.*

Proof. By Proposition 2.3, it is clear that (u_k) is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and for any $\varepsilon > 0$ there exists an $R_\varepsilon > 0$ such that

$$\sum_{j=1}^2 \int_{|x| \geq R_\varepsilon} |\nabla u_{k,j}(x)|^2 + u_{k,j}^2(x) dx < \varepsilon \quad \text{for } k \geq 1.$$

Moreover, since (u_k) is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$, there exists a subsequence (u_{k_m}) such that $u_{k_m} \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. Since (u_k) satisfies (8) and is bounded in L^∞ , we can assume $u_{k_m} \rightarrow u_0$ in $C_{\text{loc}}^2(\mathbb{R}^N)$. Therefore we have $u_{k_m} \rightarrow u_0$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$. $\square$

3. Proof of Theorems 1.1–1.3

In this section, we assume that $V_j(x)$ is a positive constant function if $N = 1$ and $V_j(x)$ satisfies (V1)–(V4) if $N = 2, 3$.

3.1. Nondegeneracy of solutions when $\beta = 0$

In this subsection, we consider the limit equations

$$\begin{cases} -\Delta v_j + V_j(x)v_j = \mu_j v_j^3 & \text{in } \mathbb{R}^N, \\ v_j \in H^1(\mathbb{R}^N), \quad v_j > 0, \end{cases} \quad (19)$$

which appears as $\beta \rightarrow 0$ in (1)–(4).

Definition 3.1. A solution ω_j of (19) is *nondegenerate* in $H_r^1(\mathbb{R}^N)$ if the following equation has only a trivial solution $\psi_j \equiv 0$:

$$\begin{cases} -\Delta \psi_j + V_j(x)\psi_j = 3\mu_j \omega_j^2 \psi_j & \text{in } \mathbb{R}^N, \\ \psi_j \in H_r^1(\mathbb{R}^N). \end{cases}$$

The following proposition is due to [7]

Proposition 3.1. *If $V_j(x)$ satisfies (V1)–(V4) and $N = 2, 3$, then (19) has a unique positive radial solution $\omega_j(x)$. Moreover, $\omega_j(x)$ is nondegenerate in $H_r^1(\mathbb{R}^N)$.*

The following proposition is well-known. See [15].

Proposition 3.2. *Suppose that $N \geq 1$, and $V_j(x)$ is positive and independent of x . Then (19) has a unique positive solution $\omega_j(x)$. Moreover, $\omega_j(x)$ is nondegenerate in $H_r^1(\mathbb{R}^N)$.*

By Propositions 3.1 and 3.2, we have

Corollary 3.3. *Under the assumptions in Theorem 1.1 or 1.2, (1)–(4) with $\beta = 0$ has a unique nontrivial positive solution $\omega(x) = (\omega_1(x), \omega_2(x))$ in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ and $\omega(x)$ is nondegenerate in $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$.*

Remark 3.1. We set

$$\begin{aligned} I_j(\psi_j) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla \psi_j|^2 + V_j(x)\psi_j^2 dx - \frac{\mu_j}{4} \int_{\mathbb{R}^N} \psi_j^4 dx, \\ I_0(\psi) &= I_0(\psi_1, \psi_2) = I_1(\psi_1) + I_2(\psi_2). \end{aligned}$$

Then solutions ω_j of (19) and $\omega(x)$ of (1)–(4) with $\beta = 0$ are nondegenerate in $H_r^1(\mathbb{R}^N)$ and $H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N)$ if and only if the mapping

$$\begin{aligned} \psi_j &\mapsto I_j''(\omega_j)[\psi_j, \cdot] : H_r^1(\mathbb{R}^N) \rightarrow (H_r^1(\mathbb{R}^N))^*, \\ \psi &\mapsto I_0''(\omega)[\psi, \cdot] : (H_r^1(\mathbb{R}^N))^2 \rightarrow ((H_r^1(\mathbb{R}^N))^2)^* \end{aligned}$$

are invertible.

3.2. Proof of Theorems 1.1–1.3

We prove Theorems 1.1–1.3.

Proof of Theorems 1.1–1.3. Let $(u_\beta) = (u_{\beta,1}, u_{\beta,2})$ be a nontrivial positive solution of (1)–(4). We consider the behavior of u_β as $\beta \rightarrow 0$. By Corollary 2.4, there exist (u_k) and (β_k) such that $u_k \rightarrow u_0$ in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $\beta_k \rightarrow 0$. Since (u_k) satisfies (1)–(4) with $\beta = \beta_k$ and 0 is a maximum point of u_k , we have

$$0 < V_1(0) \leq \mu_1 u_{k,1}^2(0) + \beta_k u_{k,2}^2(0), \quad 0 < V_2(0) \leq \beta_k u_{k,1}^2(0) + \mu_2 u_{k,2}^2(0).$$

Since $\beta_k \rightarrow 0$ and $u_{k,j} \rightarrow u_{0,j}$ in $C_{\text{loc}}^2(\mathbb{R}^N)$, we deduce that $u_{0,j}(0) \geq \sqrt{V_j(0)/\mu_j} > 0$. By the maximum principle, we have $u_{0,j}(x) > 0$ in $\mathbb{R}^N$. By Propositions 3.1 and 3.2, we have $u_{0,j} = \omega_j$. This implies that

$$u_\beta \rightarrow \omega = (\omega_1, \omega_2) \text{ strongly in } H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) \text{ as } \beta \rightarrow 0.$$

Thus, for any $\varepsilon > 0$ there exists a $\tilde{\beta} = \tilde{\beta}(\varepsilon) > 0$ such that for any $\beta \in [0, \tilde{\beta}]$ and any nontrivial positive solution u of (1)–(4) satisfies

$$\|u_j - \omega_j\|_{H^1} < \varepsilon. \quad (20)$$

Set

$$I_\beta(u) = I_1(u_1) + I_2(u_2) - \frac{\beta}{2} \int_{\mathbb{R}^N} u_1^2 u_2^2 dx,$$

$$\Phi(\beta, u) = I'_\beta(u) : \mathbb{R} \times H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N) \rightarrow (H_r^1(\mathbb{R}^N) \times H_r^1(\mathbb{R}^N))^*.$$

Then it is clear that $\Phi(0, \omega) = 0$, where $\omega = (\omega_1, \omega_2)$. Moreover, by Corollary 3.3 and Remark 3.1, $\Phi_u(0, \omega) = I''_0(\omega)$ is invertible. By the implicit function theorem, there exist $\beta_0 > 0$, $r_0 > 0$ and $\phi : (-\beta_0, \beta_0) \rightarrow B_{r_0}(\omega)$ such that

1. $\phi(0) = \omega$,
2. $\Phi(\beta, \phi(\beta)) = 0$ for all $\beta \in (-\beta_0, \beta_0)$,
3. For any $\beta \in (-\beta_0, \beta_0)$, $\Phi(\beta, u) = 0$ has a unique solution $u = \phi(\beta)$ in $B_{r_0}(\omega)$.
4. The mapping $\psi \mapsto I''_0(u)[\psi, \cdot] : (H_r^1(\mathbb{R}^N))^2 \rightarrow ((H_r^1(\mathbb{R}^N))^2)^*$ is invertible in $B_{r_0}(\omega)$.

By (20) and the above properties, we conclude that (1)–(4) has a unique radial nontrivial solution for any $\beta \in [0, \beta_0]$. The nondegeneracy of solution also follows from the above properties.

4. Appendix: Symmetry and monotonicity of nontrivial positive solutions in $\mathbb{R}$

In this appendix, we consider the symmetry and monotonicity of nontrivial positive solutions in $\mathbb{R}$. We consider the following equations:

$$\begin{cases} u_1'' + f_1(x, u_1(x), u_2(x)) = 0 & \text{in } \mathbb{R}, \\ u_2'' + f_2(x, u_1(x), u_2(x)) = 0 & \text{in } \mathbb{R}, \\ u_1(x), u_2(x) > 0 & \text{in } \mathbb{R}, \\ u_1(x), u_2(x) \rightarrow 0 & \text{as } |x| \rightarrow \infty. \end{cases} \quad (21)$$

Here we assume that f_i satisfies the following conditions:

- (f0) $f_i(x, s_1, s_2) \in C^1(\mathbb{R} \times (0, \infty)^2)$ and $f_i(-x, s_1, s_2) = f_i(x, s_1, s_2)$ for all $x \in \mathbb{R}$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.
- (f1) $\frac{\partial f_i}{\partial x}(x, s_1, s_2) \leq 0$ for all $x \geq 0$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.
- (f2) $\frac{\partial f_1}{\partial s_2}(x, s_1, s_2), \frac{\partial f_2}{\partial s_1}(x, s_1, s_2) \geq 0$ for all $x \in \mathbb{R}$, $(s_1, s_2) \in (0, \infty)^2$ and $i = 1, 2$.
- (f3) There exist $R_1 > 0$, $\varepsilon_1 > 0$ and $\delta_1 > 0$ such that if $|x| \geq R_1$ and $s_1^2 + s_2^2 < \varepsilon_1^2$, then

$$\frac{\partial f_i}{\partial s_i}(x, s_1, s_2) < -\delta_1 \text{ for } i = 1, 2, \quad 0 < \frac{\partial f_i}{\partial s_j}(x, s_1, s_2) \leq \delta_1 \text{ for } i \neq j.$$

In this appendix we will prove the following theorem:

Theorem 4.1. *Suppose that f_i satisfies (f0)–(f3). Then any nontrivial solution of (21) is radially symmetric and monotone decreasing.*

Remark 4.1. We can apply Theorem 4.1 to (1)–(4) with $N = 1$.

In [2], they considered the symmetry and monotonicity of nontrivial positive solutions for $N \geq 2$. In the following, we show the argument in [2] still works for $N = 1$ after modification and give an outline of the proof of Theorem 4.1.

We use the moving plane method to prove the symmetry and monotonicity of nontrivial positive solutions of (21). Let $u(x) = (u_1(x), u_2(x))$ be a solution of (21) and we set

$$\begin{aligned} x^\lambda &= 2\lambda - x, \\ U_i^\lambda(x) &= u_i(x^\lambda) - u_i(x) = u_i(2\lambda - x) - u_i(x) \end{aligned}$$

for $\lambda \geq 0$ and $x \geq \lambda$.

Outline of the proof of Theorem 4.1. First, we observe $U_i^\lambda(x)$ satisfies

$$(U_i^\lambda)'' + \sum_{k=1}^2 c_{ik}(x) U_k^\lambda \leq 0 \text{ in } (\lambda, \infty), \quad (22)$$

where

$$c_{ik}(x) = \int_0^1 \frac{\partial f_i}{\partial u_k}(|x| + t(|x^\lambda| - |x|), u(x) + tU^\lambda(x)) dt,$$

$$U^\lambda(x) = (U_1^\lambda(x), U_2^\lambda(x)).$$

These inequalities are corresponding to the inequality (13) in [2] and we can show (22) in a similar way to [2].

Next we define

$$A = \{\lambda \geq 0 \mid U_i^\mu \geq 0 \text{ in } (\mu, \infty) \text{ for all } \mu \geq \lambda, i = 1, 2\},$$

and a key step of the proof of Theorem 4.1 is to show $A \neq \emptyset$.

Here we need to modify the argument in [2]. In [2], they introduced the auxiliary function $\bar{U}^\lambda(x) = U^\lambda(x)/g(x)$, where

$$g(x) = \begin{cases} |x|^{-(N-2)/2} + 1 & \text{if } N \geq 3, \\ \log(\log(|x| + 27)) & \text{if } N = 2. \end{cases}$$

When $N = 1$, $g(x)$ is not given in [2].

Under our assumption (f3), which is stronger than those in [2], we can show there exist $R_2 \geq R_1$ and $\lambda_1 > R_2$ such that

$$\begin{aligned} u_1^2(x) + u_2^2(x) &< \varepsilon_1^2 \quad \text{for } |x| \geq R_2, \\ \max_{[2\lambda - R_2, 2\lambda + R_2]} u_i &< \min_{[-R_2, R_2]} u_i \quad \text{for } \lambda \geq \lambda_1, \\ c_{11}(x), c_{22}(x) &< -\delta_1, \quad 0 < c_{12}(x), c_{21}(x) \leq \delta_1 \\ &\text{for } \lambda \geq \lambda_1 \text{ and } x, x^\lambda \notin [-R_2, R_2]. \end{aligned} \quad (23)$$

From (23), we have

$$\det \begin{pmatrix} c_{11}(x) & c_{12}(x) \\ c_{21}(y) & c_{22}(y) \end{pmatrix} > 0 \quad \text{for } \lambda \geq \lambda_1 \text{ and } x, x^\lambda, y, y^\lambda \notin [-R_2, R_2]. \quad (24)$$

We will see that (24) enables us to show $A \neq \emptyset$ without introducing $\bar{U}^\lambda(x)$.

Indeed, we show $\lambda_1 \in A$ to prove $A \neq \emptyset$. We show indirectly and assume $\lambda_1 \notin A$. Then there exist a $\lambda_2 \geq \lambda_1$ and $x_1, x_2 \in [\lambda_2, \infty)$ and one of the following three cases takes place:

- (i) $U_1^{\lambda_2}(x_1) = \min_{[\lambda_2, \infty)} U_1^{\lambda_2} < 0$, $\min_{[\lambda_2, \infty)} U_2^{\lambda_2} = 0$,
- (ii) $\min_{[\lambda_2, \infty)} U_1^{\lambda_2} = 0$, $U_2^{\lambda_2}(x_2) = \min_{[\lambda_2, \infty)} U_2^{\lambda_2} < 0$,
- (iii) $U_1^{\lambda_2}(x_1) = \min_{[\lambda_2, \infty)} U_1^{\lambda_2} < 0$, $U_2^{\lambda_2}(x_2) = \min_{[\lambda_2, \infty)} U_2^{\lambda_2} < 0$.

We consider the case (i). Since x_1 is a minimum point of $U_1^{\lambda_2}$ in (λ_2, ∞) , $U_2^{\lambda_2}(\lambda_2) = 0$ and the inequality (22) holds, we have

$$c_{11}(x_1)U_1^{\lambda_2}(x_1) \leq 0. \quad (25)$$

On the other hand, by (23), $U_1^{\lambda_2}(x_1) < 0$ implies that $x_1, x_1^{\lambda_2} \notin [-R_2, R_2]$ and $c_{11}(x_1) < -\delta_1$, which contradicts to (25). Thus the case (i) never occurs. Similarly, the case (ii) cannot occur.

In the case (iii), Since x_i is a minimum point of $U_i^{\lambda_2}$, it follows

$$\begin{pmatrix} c_{11}(x_1) & c_{12}(x_1) \\ c_{21}(x_2) & c_{22}(x_2) \end{pmatrix} \begin{pmatrix} U_1^{\lambda_2}(x_1) \\ U_2^{\lambda_2}(x_2) \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \text{where } \xi_1, \xi_2 \leq 0.$$

By (23), we have

$$x_1, x_2, x_1^{\lambda_2}, x_2^{\lambda_2} \notin [-R_2, R_2], \quad c_{ii}(x_i) < -\delta_1, \quad 0 \leq c_{ij}(x_i) < \delta_1 \quad (j \neq i). \quad (26)$$

By (24) and (26), we have

$$U_1^{\lambda_2}(x_1) = \frac{c_{22}(x_2)\xi_1 - c_{12}(x_1)\xi_2}{c_{11}(x_1)c_{22}(x_2) - c_{12}(x_1)c_{21}(x_2)} \geq 0.$$

However, this is a contradiction. Thus $\lambda_1 \in A$.

Thus $A \neq \emptyset$ and $\lambda_* = \inf A$ is well-defined. Using $U^{\lambda_*}(x)$, we can show the symmetry and monotonicity of $u(x)$ along the argument in [2] after slight modification. $\square$

Note added in proof

After we submitted this paper, we learned the following paper from Professor J. Wei, in which Wei and Yao established related uniqueness result for systems with constant potentials. Wei and Yao, note on uniqueness of positive solutions for some coupled nonlinear Schrödinger equations, *Methods Anal. Appl.* (to appear).

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