On Existence and Multiplicity of Solutions for Elliptic Equations Involving the p -Laplacian

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Abstract. In the present paper, the following Dirichlet problem and Neumann problem involving the p -Laplacian

$$\begin{cases} -\Delta_p(u) = \lambda f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

and

$$\begin{cases} -\Delta_p(u) + a(x)|u|^{p-2}u = \lambda f(x, u), & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

are studied and some new multiplicity results of solutions for systems (1.1) and (2.1) are obtained. Moreover, by using the KKM principle we give also two new existence results of solutions for systems (1.1) and (2.1).

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1. Introduction and preliminaries

Consider the following Dirichlet problem and Neumann problem involving the p -Laplacian

$$\begin{cases} -\Delta_p(u) = \lambda f(x, u), & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (1.1)$$

and

$$\begin{cases} -\Delta_p(u) + a(x)|u|^{p-2}u = \lambda f(x, u), & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \quad (2.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $a \in L^\infty(\Omega)$ with $\text{ess inf}_\Omega a > 0$, $N \geq 1$, $p > 1$, $\lambda > 0$ and $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function, that is, $f(x, t)$ is measurable in x for every $t \in \mathbb{R}$ and continuous in t for a.e. $x \in \Omega$. Here, Δ_p is

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the p -Laplacian, i.e. $\Delta_p(u) = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Let $\lambda_1 = \min\{\int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), \int_{\Omega} V|u|^p dx = 1\}$ is the first eigenvalue of the problem

$$\begin{cases} -\Delta_p(u) = \lambda V(x)|u|^{p-2}u, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (3)$$

where $V \in L^s(\Omega)$ for some $s > \frac{N}{p}$ if $1 < p \leq N$ and $s = 1$ if $p > N$. Recall that λ_1 is simple, positive and isolated (see [4]). Moreover, there exists a unique positive eigenfunction w_1 whose norm in $W_0^{1,p}(\Omega)$ is 1 and $W_0^{1,p}(\Omega) = E_1 \oplus E_2$, where $E_1 = \operatorname{span}\{w_1\}$, $E_2 = \{u \in W_0^{1,p}(\Omega) : \int_{\Omega} V u w_1^{p-1} dx = 0\}$ and the following proposition holds:

Proposition 1. *There exists a number $\bar{\lambda} > \lambda_1$ such that*

$$\int_{\Omega} |\nabla u|^p dx \geq \bar{\lambda} \int_{\Omega} V|u|^p dx$$

for all $u \in E_2$.

Proof. Assume by contradiction that there exists a sequence $\{u_n\}$ in E_2 such that

$$\int_{\Omega} |\nabla u_n|^p dx < \left(\lambda_1 + \frac{1}{n}\right) \int_{\Omega} V|u_n|^p dx$$

for all $n \in \{1, 2, 3, \dots\}$. Let $v_n = \frac{u_n}{(\int_{\Omega} V|u_n|^p dx)^{\frac{1}{p}}}$. Then $\int_{\Omega} V|v_n|^p dx = 1$ and $\|v_n\|^p < \lambda_1 + \frac{1}{n}$. Hence $\{v_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Due to the reflexivity of $W_0^{1,p}(\Omega)$ and the compact embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, we can assume that there exists $v \in W_0^{1,p}(\Omega)$ such that

$$v_n \rightharpoonup v \quad (\text{i.e. weakly}) \quad \text{in } W_0^{1,p}(\Omega),$$

and

$$v_n \rightarrow v \quad (\text{i.e. strongly}) \quad \text{in } L^p(\Omega).$$

Hence $\int_{\Omega} V|v|^p dx = 1$ and so

$$\lambda_1 \leq \int_{\Omega} |\nabla v|^p dx \leq \liminf_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^p dx \leq \limsup_{n \rightarrow \infty} \int_{\Omega} |\nabla v_n|^p dx \leq \lambda_1.$$

Therefore, $\|v_n\| \rightarrow \|v\|$ and $\int_{\Omega} |\nabla v|^p dx = \lambda_1$. This implies that v is the first eigenfunction, i.e. $v \in E_1$. On the otherhand, since each $v_n \in E_2$,

$$\int_{\Omega} V v_n w_1^{p-1} dx = 0, \quad \forall n.$$

Note that $v_n \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Hence $\int_{\Omega} V v w_1^{p-1} dx = 0$, i.e. $v \in E_2$. This shows $v = 0$. It contradicts the fact that $\int_{\Omega} V|v|^p dx = 1$. This completes the proof. $\square$

The energy functionals associated to problems (1.λ) and (2.λ), namely $\varphi_{1\lambda}$ and $\varphi_{2\lambda}$, on $W_0^{1,p}(\Omega)$ and $W^{1,p}(\Omega)$ given by, respectively,

$$\varphi_{1\lambda}(u) = \frac{1}{p} \|u\|^p - \lambda \int_{\Omega} F(x, u) dx$$

and

$$\varphi_{2\lambda}(u) = \frac{1}{p}\|u\|_0^p - \lambda \int_{\Omega} F(x, u) dx$$

are differentiable and

$$\langle \varphi'_{1\lambda}(u), v \rangle = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx - \lambda \int_{\Omega} f(x, u) v dx$$

for $u, v \in W_0^{1,p}(\Omega)$,

$$\langle \varphi'_{2\lambda}(u), v \rangle = \int_{\Omega} [|\nabla u|^{p-2} \nabla u \nabla v + a(x)|u|^{p-2} uv] dx - \lambda \int_{\Omega} f(x, u) v dx$$

for $u, v \in W^{1,p}(\Omega)$, where $F(x, t) = \int_0^t f(x, s) ds$, $\|u\| = (\int_{\Omega} |\nabla u|^p dx)^{\frac{1}{p}}$ and $\|u\|_0 = (\int_{\Omega} [|\nabla u|^p + a(x)|u|^p] dx)^{\frac{1}{p}}$. It is well known that the weak solutions of the problems (1.λ) and (2.λ) correspond to the critical points of $\varphi_{1\lambda}$ and $\varphi_{2\lambda}$, respectively. As $\lambda = 1$ we denote $\varphi_{1\lambda} = \varphi_1$ and $\varphi_{2\lambda} = \varphi_2$.

In recent papers [2] and [3], problems (2.λ) and (1.λ) were researched and the following two multiplicity theorems were given, respectively.

Theorem A. Assume that $p > N$, f is such that, for each $r > 0$, $\sup_{|t| \leq r} |f(x, t)| \in L^1(\Omega)$, and assume that there exist three positive constants γ, δ and s , with $\gamma < \delta$ and $s < p$ and a function $\mu \in L^1(\Omega)$ such that

- (i) $\int_{\Omega} \sup_{t \in [-\gamma, \gamma]} F(x, t) dx < \frac{1}{k^p \|a\|_1} (\frac{\gamma}{\delta})^p \int_{\Omega} F(x, \delta) dx$, where $k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_0}$,
- (ii) $F(x, t) \leq \mu(x)(1 + |t|^s)$ for all $t \in R$ and a.e. $x \in \Omega$.

Then there exists an open interval $\Lambda \subset [0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, problem (2.λ) admits at least three solutions whose norms in $W^{1,p}(\Omega)$ are less than ρ .

Theorem B. Assume that $p > N$, f is such that, for each $s > 0$, $\sup_{|t| \leq s} |f(x, t)| \in L^1(\Omega)$, and assume that there exist two positive constants c, γ with $\gamma < p$, $a \in L^1(\Omega)$ and $u^* \in W_0^{1,p}(\Omega)$ such that

- (i) $\|u^*\| > \frac{c}{k}$, where $k = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|}$,
- (ii) $\frac{1}{c^p} \int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx < \frac{1}{k^p} \frac{\int_{\Omega} F(x, u^*(x)) dx}{\|u^*\|^p}$,
- (iii) $F(x, t) \leq a(x)(1 + |t|^\gamma)$ for all $t \in R$ and a.e. $x \in \Omega$.

Then there exists an open interval $\Lambda \subset [0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$ problem (1.λ) admits at least three solutions whose norms in $W_0^{1,p}(\Omega)$ are less than ρ .

In the present paper, our main purposes are to give some new multiplicity results of solutions for systems (1.λ) and (2.λ). Moreover, by using the KKM principle we give also some new existence results of solutions for systems (1.1) and (2.1).

2. Main results

Let us begin by considering the general case.

Theorem 1. Assume that $p > N$ and f is such that, for each $s > 0$, $\sup_{|t| \leq s} |f(x, t)| \in L^1(\Omega)$. If the following conditions hold:

- (i) there exist $g, h \in L^1(\Omega, R)$, $M > 0$ and $\alpha \in [0, p-1)$ such that

$$\frac{f(x, t)}{t} \leq \frac{g(x)|t|^{\alpha-1}t + h(x)}{t}$$

for all $|t| \geq M$ and a.e. $x \in \Omega$,

- (ii) there exist two numbers $c > 0$ and $\delta > 0$ with $c < \delta$ such that

$$\int_{\Omega} \sup_{t \in [-c, c]} F(x, t) dx < \frac{1}{k^p \|a\|_1} \left(\frac{c}{\delta}\right)^p \int_{\Omega} F(x, \delta) dx$$

where $k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_0}$, then there exists an open interval $\Lambda \subset [0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$, problem (2.1) admits at least three solutions whose norms in $W^{1,p}(\Omega)$ are less than ρ .

Proof. Let

$$\Phi(u) = \frac{\|u\|_0^p}{p}, \quad \Psi(u) = - \int_{\Omega} F(x, u) dx$$

for all $u \in W^{1,p}(\Omega)$. By the weakly lower semicontinuity of norm we know that Φ is weakly lower semicontinuous on $W^{1,p}(\Omega)$. Since $p > N$, the embedding operator $i : W^{1,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact. Hence $0 < k < +\infty$ and Ψ are weakly lower semicontinuous on $W^{1,p}(\Omega)$. Indeed, if the sequence $\{u_n\} \subset W^{1,p}(\Omega)$ with $u_n \rightharpoonup u$, then we may assume that $u_n \rightarrow u$ in $C^0(\Omega)$. We need to prove that $\Psi(u) \leq \sup_m \inf_{n \geq m} \Psi(u_n)$. If this is false, then there exist $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ such that $\Psi(u) - \Psi(u_{n_k}) > \varepsilon_0$. Note that $\sup_{|t| \leq s} |f(x, t)| \in L^1(\Omega)$ for each $s > 0$. By Lebesgue control convergence theorem we know that

$$0 = \lim_{k \rightarrow \infty} [\Psi(u) - \Psi(u_{n_k})] \geq \varepsilon_0,$$

a contradiction. This shows that Ψ is weakly lower semicontinuous on $W^{1,p}(\Omega)$.

Now, we prove that, for each $\lambda > 0$,

$$\lim_{\|u\| \rightarrow \infty} (\Phi(u) + \lambda \Psi(u)) = +\infty.$$

Indeed, let $\delta(x, t) = f(x, t) - g(x)|t|^{\alpha-1}t - h(x)$ and

$$\beta(x) = \max \left\{ \sup_{0 \leq t < M} \delta(x, t), - \inf_{-M < t \leq 0} \delta(x, t) \right\}.$$

Then

$$\delta(x, t)t \leq \begin{cases} 0, & \text{if } |t| \geq M, \\ \beta(x)|t|, & \text{if } |t| < M, \end{cases}$$

for a.e. $x \in \Omega$. Hence

$$\delta(x, st)t \leq \begin{cases} 0, & \text{if } s \geq \frac{M}{|t|}, \\ \beta(x)|t|, & \text{if } 0 < s < \frac{M}{|t|}. \end{cases}$$

Consequently,

$$\begin{aligned} F(x, t) &= \int_0^t f(x, s)ds \\ &= \int_0^1 f(x, ts)tds \\ &= \int_0^1 \delta(x, ts)tds + \int_0^1 g(x)|t|^{\alpha+1}s^\alpha ds + \int_0^1 h(x)tds \\ &\leq \beta(x)M + \frac{1}{\alpha+1}g(x)|t|^{\alpha+1} + h(x)t \end{aligned}$$

and so that

$$\begin{aligned} \int_{\Omega} F(x, u)dx &\leq \int_{\Omega} [\beta(x)M + \frac{1}{\alpha+1}g(x)u^{\alpha+1} + h(x)u]dx \\ &\leq M \int_{\Omega} |\beta(x)|dx + \frac{1}{\alpha+1} \int_{\Omega} |g(x)||u|^{\alpha+1}dx + \int_{\Omega} |h(x)||u|dx \\ &\leq M \int_{\Omega} |\beta(x)|dx + \frac{1}{\alpha+1} \|u\|_{\infty}^{\alpha+1} \int_{\Omega} |g(x)|dx + \|u\|_{\infty} \int_{\Omega} |h(x)|dx \\ &\leq c_1 + c_2 \|u\|_0^{\alpha+1} + c_3 \|u\|_0. \end{aligned}$$

Therefore, for each $u \in W^{1,p}(\Omega)$ and each $\lambda > 0$, we have

$$\begin{aligned} \Phi(u) + \lambda \Psi(u) &= \frac{1}{p} \|u\|_0^p - \lambda \int_{\Omega} F(x, u)dx \\ &\geq \frac{1}{p} \|u\|_0^p - \lambda [c_1 + c_2 \|u\|_0^{\alpha+1} + c_3 \|u\|_0]. \end{aligned}$$

Since $0 \leq \alpha < p-1$,

$$\lim_{\|u\|_0 \rightarrow +\infty} [\Phi(u) + \lambda \Psi(u)] = +\infty.$$

The rest parts of the proof are same with Theorem 2 in [2]. $\square$

Remark. There are functions satisfying Theorem 1 and not satisfying Theorem A.

Example. Consider the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u| \nabla u) + \frac{2+x}{\pi} |u|u = \lambda(x^2 + y^2)\beta(u), & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases}$$

where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and

$$\beta(t) = \begin{cases} 2e^{-t^2}t^{17}(9-t^2), & \text{if } |t| \leq 2, \\ 85 \times 2^{13} \times e^{-4\frac{4t^3}{1+t^4}}, & \text{if } |t| \geq 2. \end{cases}$$

In Theorem 1, we take $N = 2, p = 3, \alpha = 1, c = 1, \delta = 2$ and $f(x, y, t) = (x^2 + y^2)\beta(t), a(x, y) = \frac{2+x}{\pi}$ for all $(x, y) \in \Omega$ and $t \in R$. Then, all the assumptions of Theorem 1 are satisfied, but the condition (ii) of Theorem A is not satisfied.

Indeed, $\|a\|_1 = 2, \|a\|_\infty = \frac{3}{\pi}, \int_\Omega \sup_{|t| \leq 1} F(x, y, t) dx dy = \frac{\pi}{2e}, \int_\Omega F(x, y, \delta) dx dy = 2^{17} \times e^{-4}\pi$ and by Remark 1 in [2] we know that $k^3 \|a\|_1 \leq \frac{16 \times 27}{\pi}$. Hence the condition (ii) of Theorem 1 holds. Moreover, as M is enough large the condition (i) of Theorem 1 holds, too. But, the condition (ii) of Theorem A is not satisfied, obviously.

Theorem 2. Assume that $p > N$ and f is such that, for each $s > 0, \sup_{|t| \leq s} |f(x, t)| \in L^1(\Omega)$. If the following conditions hold:

- (i) there exist $g, h \in L^1(\Omega, R), M > 0$, and $\alpha \in [0, p-1)$ such that

$$\frac{f(x, t)}{t} \leq \frac{g(x)|t|^{\alpha-1}t + h(x)}{t}$$

for all $|t| \geq M$ and a.e. $x \in \Omega$,

- (ii) there exist a number $c > 0$ and an $u^* \in W_0^{1,p}(\Omega)$ such that

- (a) $\|u^*\| > \frac{c}{k}$, where

$$k = \sup_{u \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\max_{x \in \bar{\Omega}} |u(x)|}{\|u\|},$$

- (b) $\frac{1}{c^p} \int_\Omega \sup_{t \in [-c, c]} F(x, t) dx < \frac{1}{k^p} \frac{\int_\Omega F(x, u^*(x)) dx}{\|u^*\|^p}$,

then there exists an open interval $\Lambda \subset [0, +\infty)$ and a positive real number ρ such that, for each $\lambda \in \Lambda$ problem (1.1) admits at least three solutions whose norms in $W_0^{1,p}(\Omega)$ are less than ρ .

Proof. Let

$$\Phi(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = - \int_\Omega F(x, u) dx$$

for all $u \in W_0^{1,p}(\Omega)$. The rest parts of the proof are similar to Theorem 1. $\square$

Now, we consider the case that $\lambda = 1$. We have the following existence and multiplicity theorems.

As a new research method for the existence of solutions, we use KKM principle (see [5]) to prove the following Theorem 3. For this reason, we give the concept of KKM mapping. Set E be a linear space, X a nonempty subset of E and $G: X \rightarrow 2^E$ a multivalued mapping. G is said be a KKM mapping if for any finite set $\{x_1, x_2, \dots, x_n\} \subset X$, one has

$$co\{x_1, x_2, \dots, x_n\} \subset \bigcup_{i=1}^n G(x_i).$$

Theorem 3. If there exist $\alpha \in [0, p-1), g \in L^{\frac{p}{p-\alpha-1}}(\Omega, R^+)$ and $h \in L^q(\Omega, R^+)$ ($q = \frac{p}{p-1}$) such that

$$|f(x, t)| \leq g(x)|t|^\alpha + h(x)$$

for every $t \in R$ and a.e. $x \in \Omega$, then problem (2.1) has at least a solution in $W^{1,p}(\Omega)$.

Proof. For each $v \in W^{1,p}(\Omega)$, let

$$P(v) = \left\{ u \in W^{1,p}(\Omega) : \langle \varphi'_2(u), v - u \rangle \geq 0 \right\}.$$

Then $P : W^{1,p}(\Omega) \longrightarrow 2^{W^{1,p}(\Omega)}$ is a KKM mapping. Indeed, for each finite subset $\{v_1, v_2, \dots, v_n\}$ of $W^{1,p}(\Omega)$, if there exists a point $u_0 \in \text{co}\{v_1, v_2, \dots, v_n\}$ such that $u_0 \notin \bigcup_{i=1}^n P(v_i)$, then

$$\langle \varphi'_2(u_0), v_i - u_0 \rangle < 0, \quad i = 1, 2, \dots, n.$$

Let $u_0 = \sum_{i=1}^n \alpha_i v_i$, where each $\alpha_i \geq 0$ and $\sum_{i=1}^n \alpha_i = 1$. Then

$$0 = \langle \varphi'_2(u_0), u_0 - u_0 \rangle = \sum_{i=1}^n \alpha_i \langle \varphi'_2(u_0), v_i - u_0 \rangle < 0.$$

This is a contradiction. Hence $P : W^{1,p}(\Omega) \longrightarrow 2^{W^{1,p}(\Omega)}$ is a KKM mapping. $\square$

Moreover, for each $v \in W^{1,p}(\Omega)$, $P(v)$ is a weakly compact subset of $W^{1,p}(\Omega)$. In fact, note that $\alpha \in [0, p-1)$ and $\text{ess inf}_{\Omega} a > 0$, by Hölder inequality and the definition of $\|\cdot\|_0$ we have

$$\begin{aligned} \langle \varphi'_2(u), v - u \rangle &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (v - u) dx + \int_{\Omega} a(x) |u|^{p-2} u (v - u) dx \\ &\quad - \int_{\Omega} f(x, u) (v - u) dx \\ &= -\|u\|_0^p + \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx + \int_{\Omega} a(x) |u|^{p-2} u v dx \\ &\quad + \int_{\Omega} f(x, u) u dx - \int_{\Omega} f(x, u) v dx \\ &\leq -\|u\|_0^p + c_1 \|u\|_0^{p-1} + \int_{\Omega} [g(x) |u|^{\alpha+1} + h(x) |u|] dx \\ &\quad + \int_{\Omega} [g(x) |u|^{\alpha} |v| + h(x) |v|] dx \\ &\leq -\|u\|_0^p + c_1 \|u\|_0^{p-1} + c_2 \|u\|_0^{\alpha+1} + c_3 \|u\|_0 + c_4 \end{aligned}$$

for each $u \in W^{1,p}(\Omega)$, where c_1, c_2, c_3, c_4 are positive constants. This implies that $\langle \varphi'_2(u), v - u \rangle \rightarrow -\infty$ as $\|u\|_0 \rightarrow \infty$. Hence there exists a positive number $r > 0$ such that $\langle \varphi'_2(u), v - u \rangle < 0$ as $\|u\| > r$. This shows that $P(v)$ is a bounded subset of $W^{1,p}(\Omega)$. Note that φ_2 is continuously differentiable so that $P(v)$ is closed, obviously. Hence $P(v)$ is a weakly compact subset of $W^{1,p}(\Omega)$.

By virtue of FKKM theorem (see [5]) $\bigcap_{v \in W^{1,p}(\Omega)} P(v) \neq \emptyset$. Hence there exists a function $u \in W^{1,p}(\Omega)$ such that

$$\langle \varphi'_2(u), v - u \rangle \geq 0$$

for all $v \in W^{1,p}(\Omega)$. This implies that u is a critical point of φ_2 . Therefore, u is a solution of problem (2.1).

By using a similar method we can prove following Theorem 4.

Theorem 4. *If there exist $\alpha \in [0, p-1)$, $g \in L^{\frac{p}{p-\alpha-1}}(\Omega, R^+)$ and $h \in L^q(\Omega, R^+)$ ($q = \frac{p}{p-1}$) such that*

$$|f(x, t)| \leq g(x)|t|^\alpha + h(x)$$

for every $t \in R$ and a.e. $x \in \Omega$, then problem (1.1) has at least a solution in $W_0^{1,p}(\Omega)$.

Theorem 5. *Assume that $p > N$ and $f = f_1 + f_2$ is such that $f_i : \Omega \times R \rightarrow R$ is a Carathéodory function ($i = 1, 2$), and for each $s > 0$, $\sup_{|t| \leq s} |f_1(x, t)| \in L^1(\Omega)$. If the following conditions hold:*

(i) *there exist $g, h \in L^1(\Omega, R)$, $M > 0$ and $\alpha \in [0, p-1)$ such that*

$$\frac{f_1(x, t)}{t} \leq \frac{g(x)|t|^{\alpha-1}t + h(x)}{t}$$

for all $|t| \geq M$ and a.e. $x \in \Omega$,

(ii)

$$\sup_{\xi \in R} F_1(x, \xi) \leq 0$$

a.e. $x \in \Omega$ and there exist two numbers $r_0 > 0$ and $\xi_0 \in R$ such that

$$\frac{|\xi_0|^p \int_{\Omega} a(x) dx}{p} - \int_{\Omega} F_1(x, \xi_0) dx < r_0$$

and

$$F_2(x, \xi_0) = \sup_{|\xi| \leq k(pr_0)^{\frac{1}{p}}} F_2(x, \xi)$$

for a.e. $x \in \Omega$, where

$$k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_0},$$

then problem (2.1) admits at least one solution.

Proof. Let

$$\Psi(u) = \frac{\|u\|_0^p}{p} - \int_{\Omega} F_1(x, u) dx, \Phi(u) = - \int_{\Omega} F_2(x, u) dx$$

for all $u \in W^{1,p}(\Omega)$, and let

$$\varphi(r) = \inf_{u \in \Psi^{-1}((-\infty, r))} \frac{\Phi(u) - \inf_{cl_w(\Psi^{-1}((-\infty, r)))} \Phi(v)}{r - \Psi(u)},$$

where $cl_w(\Psi^{-1}((-\infty, r)))$ denotes the weakly closure of the set $\Psi^{-1}((-\infty, r))$. Since $p > N$, the embedding operator $i : W^{1,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact. Hence Φ and Ψ are weakly lower semicontinuous and Gateaux differentiable in $W^{1,p}(\Omega)$. From the proof of Theorem 1 we know that Ψ is coercive. For each

$v \in cl_w(\Psi^{-1}((-\infty, r_0)))$ there exists a sequence $\{v_m\} \subset \Psi^{-1}((-\infty, r_0))$ such that $v_m \rightharpoonup v$. By $\Psi(v_m) \leq r_0$, $\|v_m\|^p \leq pr_0 + \int_{\Omega} F_1(x, v_m) dx \leq pr_0$ and hence, $\|v\|^p \leq pr_0$. Hence

$$\varphi(r_0) \leq \inf_{u \in \Psi^{-1}((-\infty, r_0))} \frac{\Phi(u) - \inf_{\|v\|^p \leq pr_0} \Phi(v)}{r_0 - \Psi(u)}.$$

Let $w_0(x) \equiv \xi_0$ for all $x \in \Omega$. Then $\Psi(w_0) = \frac{\int_{\Omega} a(x) |\xi_0|^p dx}{p} - \int_{\Omega} F_1(x, \xi_0) dx < r_0$. Moreover, for each $u \in W^{1,p}(\Omega)$ with $\|u\|^p \leq pr_0$, by (ii), one has $\Phi(u) \geq \Phi(w_0)$ and hence $\varphi(r_0) = 0$. Note that $\inf_{u \in W^{1,p}(\Omega)} \Psi(u) = 0$. By virtue of Theorem 2.5(a) in [6] $\Phi + \Psi$ has a critical point, and hence problem (2.1) has a solution. $\square$

In Theorem 5, take $f_1 \equiv 0$ or $f_2 \equiv 0$, respectively. Then we have the following two interesting corollaries.

Corollary 6. Assume that $p > N$ and f is such that $f : \Omega \times R \rightarrow R$ is a Carathéodory function. If there exist two numbers $r_0 > 0$ and $\xi_0 \in R$ such that

$$\frac{|\xi_0|^p \int_{\Omega} a(x) dx}{p} < r_0$$

and

$$F(x, \xi_0) = \sup_{|\xi| \leq k(pr_0)^{\frac{1}{p}}} F(x, \xi)$$

for a.e. $x \in \Omega$, where

$$k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_0},$$

then problem (2.1) admits at least one solution.

Corollary 7. Assume that $p > N$ and $f : \Omega \times R \rightarrow R$ is a Carathéodory function, and for each $s > 0$, $\sup_{|t| \leq s} |f(x, t)| \in L^1(\Omega)$. If the following conditions hold:

- (i) there exist $g, h \in L^1(\Omega, R)$, $M > 0$ and $\alpha \in [0, p-1)$ such that

$$\frac{f(x, t)}{t} \leq \frac{g(x)|t|^{\alpha-1}t + h(x)}{t}$$

for all $|t| \geq M$ and a.e. $x \in \Omega$,

- (ii)

$$\sup_{\xi \in R} F(x, \xi) \leq 0$$

a.e. $x \in \Omega$ and there exist two numbers $r_0 > 0$ and $\xi_0 \in R$ such that

$$\frac{|\xi_0|^p \int_{\Omega} a(x) dx}{p} - \int_{\Omega} F(x, \xi_0) dx < r_0$$

for a.e. $x \in \Omega$, where

$$k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_{\infty}}{\|u\|_0},$$

then problem (2.1) admits at least one solution.

Theorem 8. *If there exist $g, h \in L^1(\Omega, R)$, $M > 0$, and $\alpha \in [0, p-1)$ such that*

$$\frac{f(x, t)}{t} \leq \frac{g(x)|t|^{\alpha-1}t + h(x)}{t}$$

for all $|t| \geq M$ and a.e. $x \in \Omega$, $p > N$ and there exists a number $\delta > 0$ such that, as $0 < |t| \leq \delta$,

$$\lambda_1 V(x)|t|^{p-2} \leq \frac{f(x, t)}{t} \leq \bar{\lambda} V(x)|t|^{p-2},$$

then problem (1.1) has at least two nontrivial solution in $W_0^{1,p}(\Omega)$.

Proof. Since $p > N$, the embedding operator $i : W_0^{1,p}(\Omega) \hookrightarrow C^0(\Omega)$ is compact. Hence there exists a number $c > 0$ such that

$$\|u\|_\infty \leq c\|u\|$$

for all $u \in W_0^{1,p}(\Omega)$, and φ_1 is weakly lower semicontinuous on $W_0^{1,p}(\Omega)$.

For each $u \in E_1$, there exists a real number k such that $u = k\varphi_1$, and as $\|u\| \leq \frac{\delta}{c}$,

$$\begin{aligned} \varphi_1(u) &= \frac{1}{p}\|u\|^p - \int_{\Omega} F(x, u)dx \\ &= \frac{1}{p}|k|^p - \int_{\Omega} \left[\int_0^1 f(x, kt\varphi_1)k\varphi_1 dt \right] dx \\ &\leq \frac{1}{p}|k|^p - \int_{\Omega} \left[\int_0^1 \lambda_1 V(x)t^{p-1}|k|^p\varphi_1^p dt \right] dx \\ &= \frac{1}{p}|k|^p - \frac{1}{p}|k|^p \int_{\Omega} \lambda_1 V(x)\varphi_1^p dx \\ &= 0. \end{aligned}$$

For each $u \in E_2$ with $\|u\| \leq \frac{\delta}{c}$,

$$\begin{aligned} \varphi_1(u) &= \frac{1}{p}\|u\|^p - \int_{\Omega} F(x, u)dx \\ &\geq \frac{1}{p}\|u\|^p - \int_{\Omega} \left[\int_0^1 \bar{\lambda} V(x)|u|^p t^{p-1} dt \right] dx \\ &= \frac{1}{p}\|u\|^p - \frac{1}{p} \int_{\Omega} \bar{\lambda} V(x)|u|^p dx \\ &\geq 0. \end{aligned}$$

Similar to Theorem 1, we can prove that φ_1 is coercive on $W_0^{1,p}(\Omega)$, and hence φ_1 satisfies (PS) and is bounded below.

If $\inf\{\varphi_1(u) : u \in W_0^{1,p}(\Omega)\} = 0$, then all $u \in E_1$ with $\|u\| \leq \frac{\delta}{c}$ are minima of φ_1 , which implies that φ_1 has infinite critical points. If $\inf\{\varphi_1(u) : u \in W_0^{1,p}(\Omega)\} < 0$, then by Theorem 4 in [1] φ_1 has at least two nonzero critical points. Therefore, problem (1.1) has at least two nontrivial solutions in $W_0^{1,p}(\Omega)$. $\square$

Remark. We need not assume $p > N$ in Theorem 3 and Theorem 4. But, there is the assumption $p > N$ in other theorems. The reason is that we need ensure $k = \sup_{u \in W^{1,p}(\Omega) \setminus \{0\}} \frac{\|u\|_\infty}{\|u\|_0} < +\infty$, which appear in our assumptions.

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