An Asymptotic Nondegeneracy Result for a Biharmonic Equation with the Nearly Critical Growth

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Abstract. We consider the problem $\Delta^2 u = c_0 u^{p_\varepsilon}$, $u > 0$ in Ω , $u = \Delta u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 5$), $c_0 = (N-4)(N-2)N(N+2)$, $p_\varepsilon = p - \varepsilon$, $p = (N+4)/(N-4)$ is the critical Sobolev exponent of the embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$.

We prove that along the solution sequence $\{u_\varepsilon\}$ minimizing the Sobolev quotient, each u_ε is nondegenerate for $\varepsilon > 0$ small, if the blow up point of $\{u_\varepsilon\}$ is a nondegenerate critical point of the Robin function. This is a Δ^2 counterpart of the former work by Grossi [9] for the $-\Delta$ case problem, and it needs some new argument.

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1. Introduction

In this paper, we consider the problem

$$(P_\varepsilon) \begin{cases} \Delta^2 u = c_0 u^{p_\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) is a smooth bounded domain, $c_0 = (N-4)(N-2)N(N+2)$, $\varepsilon > 0$ is a small positive parameter, $p_\varepsilon = p - \varepsilon$ where $p = (N+4)/(N-4)$ is the critical Sobolev exponent from the view point of the Sobolev embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. Here, $H^2 \cap H_0^1(\Omega)$ is a Hilbert space whose norm is given by

$$\|u\|_{H^2 \cap H_0^1(\Omega)} = \left(\int_{\Omega} |\Delta u|^2 dx \right)^{1/2}.$$

Since (P_ε) is a subcritical problem, the existence of solution for any $\varepsilon > 0$ small is rather easy to obtain. Let $\{u_\varepsilon\}$ be a solution sequence which satisfies

$$\frac{\|u_\varepsilon\|_{H^2 \cap H_0^1(\Omega)}^2}{\|u_\varepsilon\|_{L^{p_\varepsilon+1}(\Omega)}^2} \rightarrow S_N, \quad \text{as } \varepsilon \rightarrow 0, \quad (1.1)$$

where S_N is the best constant of the embedding $H^2 \cap H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$:

$$S_N = \inf_{u \in H^2 \cap H_0^1(\Omega)} \frac{\int_\Omega |\Delta u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{2/(p+1)}}.$$

When Ω is strictly convex, Chou and Geng [4] proved the precise asymptotic behavior of positive solutions of (P_ε) satisfying (1.1). Among other things, they proved that $\{u_\varepsilon\}$ blows up at exactly one point which is in the interior of Ω , and the blow up point has to be a critical point of the Robin function associated to the Green function of Δ^2 under the Navier boundary condition. Later, by using another method, Ben Ayed and El Mehdi [2], El Mehdi [6], Geng [8] succeeded to prove that the same result holds true without the convexity assumption of the domain.

In this article, we concern some qualitative property of the blowing-up solution u_ε . Recall that a solution u of (P_ε) is said to be *nondegenerate*, if the linearized problem around u :

$$(L_\varepsilon) \begin{cases} \Delta^2 v = c_0 p_\varepsilon u^{p_\varepsilon-1} v & \text{in } \Omega, \\ v = \Delta v = 0 & \text{on } \partial\Omega \end{cases}$$

admits only the trivial solution $v \equiv 0$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 5$) be a bounded smooth domain. Let $\{u_\varepsilon\}$ be a solution sequence to (P_ε) satisfying (1.1) and let x_0 be the blow up point of $\{u_\varepsilon\}$. If x_0 is a nondegenerate critical point of the Robin function, then u_ε is nondegenerate for $\varepsilon > 0$ sufficiently small.*

Grossi [9] considered the solutions $\{u_\varepsilon\}$ to

$$\begin{cases} -\Delta u = N(N-2)u^{(N+2)/(N-2)-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

satisfying

$$\frac{\|\nabla u_\varepsilon\|_{L^2(\Omega)}^2}{\|u_\varepsilon\|_{L^{2N/(N-2)-\varepsilon}(\Omega)}^2} \rightarrow S_N, \quad \text{as } \varepsilon \rightarrow 0.$$

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a smooth bounded domain and S_N is the best constant of the embedding $H_0^1(\Omega) \hookrightarrow L^{2N/(N-2)}(\Omega)$. Under the assumption that the blow up point of $\{u_\varepsilon\}$ is a nondegenerate critical point of the Robin function associated to the Green function of $-\Delta$ under the Dirichlet boundary condition, Grossi [9] proved the same nondegeneracy result for $\varepsilon > 0$ small.

Theorem 1.1 can be regarded as an extension of Grossi's theorem to the biharmonic case. However, some new arguments are needed to prove the result, which are not in [9]. Our argument also works well for the $-\Delta$ case problem, so it gives another proof of the result in [9].

2. Preliminaries

In this section, we recall some useful facts. Let $G = G(x, z)$ denote the Green function of Δ^2 under the Navier boundary condition:

$$\begin{cases} \Delta^2 G(\cdot, z) = \delta_z & \text{in } \Omega, \\ G(\cdot, z) = \Delta G(\cdot, z) = 0 & \text{on } \partial\Omega. \end{cases}$$

We decompose G as $G(x, z) = \Gamma(x, z) - H(x, z)$, where $\Gamma(x, z)$ is the fundamental solution of Δ^2 :

$$\Gamma(x, z) = \begin{cases} \frac{1}{(N-4)(N-2)\sigma_N} |x - z|^{4-N}, & N \geq 5, \\ \frac{1}{\sigma_4} \log |x - z|^{-1}, & N = 4, \end{cases} \quad (2.1)$$

where σ_N is the volume of the $(N-1)$ dimensional unit sphere in $\mathbb{R}^N$ and $H(x, z)$ is the regular part of the Green function. Finally, let $R(x) = H(x, x)$ denote the Robin function of the Green function of Δ^2 with the Navier boundary condition. By the maximum principle, we have $R > 0$ on Ω and $R(x) \rightarrow +\infty$ as x tends to the boundary of Ω . In the following, we set $\bar{G} = -\Delta G$. Then $\bar{G}$ is the Green function of $-\Delta$ under the Dirichlet boundary condition, and satisfy

$$\begin{cases} -\Delta G = \bar{G} & \text{in } \Omega, \\ -\Delta \bar{G} = \delta_z & \text{in } \Omega, \\ G > 0, \bar{G} > 0 & \text{in } \Omega, \\ G = \bar{G} = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 2.1 (Pohozaev identity for the Green function).

$$\int_{\partial\Omega} ((x - z) \cdot \nu_x) \left(\frac{\partial G}{\partial \nu_x} \right) \left(\frac{\partial \bar{G}}{\partial \nu_x} \right) (x, z) ds_x = (N - 4)R(z) \quad (2.2)$$

and

$$\int_{\partial\Omega} \frac{\partial G}{\partial \nu_x}(x, z) \frac{\partial \bar{G}}{\partial \nu_x}(x, z) \nu_i(x) ds_x = \frac{\partial R}{\partial z_i}(z) \quad (i = 1, \dots, N) \quad (2.3)$$

hold true for any $z \in \Omega$. Here ν_x is the outer unit normal at $x \in \partial\Omega$.

Proof. See [4, Lemma 3.1 and Lemma 3.3]. Note that our sign convention is different from that of [4]. Also note that there is a mistake in the claimed formula in [4]. $\square$

Lemma 2.2.

$$\begin{aligned} \frac{\partial^2 R}{\partial z_i \partial z_j}(z) &= \int_{\partial\Omega} \left(\frac{\partial \bar{G}}{\partial x_i} \right)(x, z) \frac{\partial}{\partial z_j} \left(\frac{\partial G}{\partial \nu_x} \right)(x, z) ds_x \\ &\quad + \int_{\partial\Omega} \left(\frac{\partial G}{\partial x_i} \right)(x, z) \frac{\partial}{\partial z_j} \left(\frac{\partial \bar{G}}{\partial \nu_x} \right)(x, z) ds_x \end{aligned} \quad (2.4)$$

holds true for any $z \in \Omega$ and for $1 \leq i, j \leq N$.

Proof. Differentiating (2.3) with respect to z_j , noting that $(\frac{\partial G}{\partial \nu_x}(x, z))\nu_i(x) = \frac{\partial G}{\partial x_i}(x, z)$, $(\frac{\partial \bar{G}}{\partial \nu_x}(x, z))\nu_i(x) = \frac{\partial \bar{G}}{\partial x_i}(x, z)$ on $\partial\Omega$, we obtain the result. $\square$

Next lemma is a classical elliptic regularity for a solution to the biharmonic equation.

Lemma 2.3. Let $u \in H_{loc}^2(\Omega)$ be a weak solution of

$$\Delta^2 u = a(x)u \quad \text{in } \Omega$$

where $a \in L^\alpha(\Omega)$ with $\alpha > N/4$. Then for any $q \in (0, +\infty)$, there exist $C = C(q) > 0$ and $R > 0$ such that for any $0 < r < R$ and $y \in \mathbb{R}^N$,

$$\sup_{B(y, r) \cap \Omega} |u| \leq C \left[\frac{1}{r^N} \int_{B(y, 2r) \cap \Omega} |u|^{q+1} dx \right]^{1/(q+1)}$$

holds true.

Proof. See [3]. $\square$

Lemma 2.4. Let u_ε be a solution to (P_ε) and v_ε be a solution to (L_ε) with $u = u_\varepsilon$. Denote $\bar{u}_\varepsilon = -\Delta u_\varepsilon$ and $\bar{v}_\varepsilon = -\Delta v_\varepsilon$. Then the following identities hold true:

$$\int_{\partial\Omega} ((x - z) \cdot \nu_x) \left\{ \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \bar{v}_\varepsilon}{\partial \nu_x} \right) + \left(\frac{\partial \bar{u}_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) \right\} ds_x = 0 \quad (2.5)$$

for any $z \in \mathbb{R}^N$ and

$$\int_{\partial\Omega} \left\{ \left(\frac{\partial \bar{u}_\varepsilon}{\partial x_i} \right) \left(\frac{\partial v_\varepsilon}{\partial \nu_x} \right) + \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \bar{v}_\varepsilon}{\partial \nu_x} \right) \right\} ds_x = 0, \quad i = 1, 2, \dots, N. \quad (2.6)$$

Proof. For smooth f, g , we have the formula

$$\begin{aligned} &\int_{\Omega} ((\Delta^2 f)g - (\Delta^2 g)f) dx \\ &= \int_{\partial\Omega} \left(\frac{\partial \Delta f}{\partial \nu_x} \right) g - \left(\frac{\partial \Delta g}{\partial \nu_x} \right) f ds_x + \int_{\partial\Omega} \left(\frac{\partial f}{\partial \nu_x} \right) \Delta g - \left(\frac{\partial g}{\partial \nu_x} \right) \Delta f ds_x. \end{aligned} \quad (2.7)$$

Set $w_\varepsilon(x) = (x - z) \cdot \nabla u_\varepsilon(x) + \alpha_\varepsilon u_\varepsilon(x)$ where $\alpha_\varepsilon = \frac{4}{p_\varepsilon - 1}$. Direct computation yields that

$$\Delta^2 w_\varepsilon = (\alpha_\varepsilon + 4)c_0 u_\varepsilon^{p_\varepsilon} + c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} (x - z) \cdot \nabla u_\varepsilon.$$

Since v_ε satisfies $\Delta^2 v_\varepsilon = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon - 1} v_\varepsilon$, we have

$$(\Delta^2 w_\varepsilon)v_\varepsilon - (\Delta^2 v_\varepsilon)w_\varepsilon = (\alpha_\varepsilon + 4 - p_\varepsilon \alpha_\varepsilon)c_0 u_\varepsilon^{p_\varepsilon} v_\varepsilon = 0.$$

Integrating this identity on Ω with the formula (2.7), and noting that

$$w_\varepsilon(x) = (x - z) \cdot \nu_x \left(\frac{\partial u_\varepsilon}{\partial \nu_x} \right), \quad \Delta w_\varepsilon(x) = (x - z) \cdot \nu_x \left(\frac{\partial \Delta u_\varepsilon}{\partial \nu_x} \right)$$

for $x \in \partial\Omega$, we have (2.5).

On the other hand, differentiating the equation in (P_ε) with respect to x_i , we have

$$\Delta^2 \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon-1} \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \quad \text{in } \Omega.$$

Multiplying this by v_ε , and the equation of v_ε by $\left(\frac{\partial u_\varepsilon}{\partial x_i} \right)$ and subtracting, we obtain

$$(\Delta^2 v_\varepsilon) \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) - \left(\Delta^2 \left(\frac{\partial u_\varepsilon}{\partial x_i} \right) \right) v_\varepsilon = 0.$$

Finally, integration by parts formula (2.7) yields (2.6). $\square$

Next is the main asymptotic result by Chou and Geng [4].

Theorem 2.5 ([4]). Assume $\Omega \subset \mathbb{R}^N$, $N \geq 5$ is strictly convex. Let u_ε be a solution to (P_ε) satisfying (1.1). Let $x_\varepsilon \in \Omega$ be a point such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty(\Omega)}$. Then after passing to a subsequence, we have

- (1) $\lim_{\varepsilon \rightarrow 0} x_\varepsilon = x_0$ for some interior point $x_0 \in \Omega$. Furthermore, x_0 is a critical point of the Robin function R , i.e., $\nabla R(x_0) = 0$.
- (2) There holds

$$\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon \rightarrow 2(N-4)(N-2)\sigma_N G(\cdot, x_0) \quad \text{in } C_{loc}^3(\overline{\Omega} \setminus \{x_0\}) \quad (2.8)$$

as $\varepsilon \rightarrow 0$.

- (3)

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon\|_{L^\infty(\Omega)}^\varepsilon = 1. \quad (2.9)$$

- (4) There exists a constant $C > 0$ independent of ε such that

$$u_\varepsilon(x) \leq C \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}}{\left(1 + \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{4}{N-4}} |x - x_\varepsilon|^2\right)^{\frac{N-4}{2}}} \quad (2.10)$$

holds for any $x \in \Omega$.

- (5)

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^2 = \frac{16\sigma_N^2 \Gamma(N) c_0}{\pi^{N/2} N(N-4)(N+2)^2 \Gamma(N/2)} R(x_0).$$

Remark. In [4], the assumption of the strict convexity of the domain is used in twofold: One is to exclude the possible boundary blow-ups and to obtain uniform L^∞ -estimate for solutions near the boundary by the method of moving planes, and the other is to obtain the pointwise estimate (2.10) by using the Kelvin transformation.

Now, thanks to the works of Ben Ayed and El Mehdi [2], El Mehdi [6] and Geng [8], we know that the blow up point x_0 for u_ε satisfying (1.1) must be

an interior point of Ω and is a critical point of the Robin function, without the convexity assumption of the domain. Therefore u_ε satisfying (1.1) is uniformly bounded near the boundary.

Also we can prove the pointwise estimate (2.10) via the theory of *isolated simple blow up points* performed by Geng [7]. For definitions of isolated, and isolated simple blow up points, we refer to [7]. We also refer to the original work by YanYan Li [10] for $-\Delta$ case problem. Though strict convexity of the domain is assumed in [7], it is simply because Geng had to treat solution sequences without energy bounds, which may blow up on the boundary. Thus if we a priori know that the blow up points of $\{u_\varepsilon\}$ are interior points of the domain, convexity assumption is not needed in applying the results in [7]. It is proved in [7] that the interior blow up points have to be isolated simple ones. Therefore we derive (2.10) from the estimates for isolated simple blow up points in [7].

In conclusion, we confirm that any convexity assumption is not needed for Theorem 2.5, and consequently, for Theorem 1.1 to hold.

In what follows, we use a symbol $\|\cdot\|$ to denote the L^∞ norm of functions. Now, let us consider the scaled function

$$\tilde{u}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} u_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon := \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}(\Omega - \varepsilon). \quad (2.11)$$

We see $0 < \tilde{u}_\varepsilon \leq 1$, $\tilde{u}_\varepsilon(0) = 1$, and $\tilde{u}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{u}_\varepsilon = c_0 \tilde{u}_\varepsilon^{p_\varepsilon} & \text{in } \Omega_\varepsilon, \\ \tilde{u}_\varepsilon = \Delta \tilde{u}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Since $\|u_\varepsilon\| \rightarrow \infty$ as $\varepsilon \rightarrow 0$ and x_ε does not approach to $\partial\Omega$, we see $\Omega_\varepsilon \rightarrow \mathbb{R}^N$. By standard elliptic estimates, we have a subsequence denoted also by $\tilde{u}_\varepsilon$ that

$$\tilde{u}_\varepsilon \rightarrow U \quad \text{compact uniformly in } \mathbb{R}^N \quad (2.12)$$

as $\varepsilon \rightarrow 0$ for some function U . Passing to the limit, we obtain that U is a solution of

$$\begin{cases} \Delta^2 U = c_0 U^{p_\varepsilon} & \text{in } \mathbb{R}^N, \\ 0 < U \leq 1, \quad U(0) = 1, \\ \lim_{|y| \rightarrow \infty} U(y) = 0. \end{cases}$$

Thus according to the uniqueness theorem by Chang Shou Lin [11], we obtain

$$U(y) = \left(\frac{1}{1 + |y|^2} \right)^{\frac{N-4}{2}}. \quad (2.13)$$

Also by (2.9), (2.10) can be written as

$$\tilde{u}_\varepsilon(y) \leq CU(y) \quad \text{for } y \in \Omega_\varepsilon. \quad (2.14)$$

Now, we recall the classification theorem by Bartsch, Weth and Willem [1].

Lemma 2.6. *Let v_0 be a solution to*

$$\begin{cases} \Delta^2 v_0 = c_0 p U^{p-1} v_0 & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |\Delta v_0|^2 dy \leq C. \end{cases}$$

Then there exist a_j ($j = 1, 2, \dots, N$), $b \in \mathbb{R}$ such that v_0 can be written as

$$v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1 + |y|^2)^{(N-2)/2}} + b \frac{1 - |y|^2}{(1 + |y|^2)^{(N-2)/2}}.$$

Next lemma is a unique solvability result for linear first order PDE's with the initial condition, and is well known.

Lemma 2.7. *Let $a = (a_1, a_2, \dots, a_N) \neq 0$ is a constant vector and $f, g \in C^1(\mathbb{R}^N)$. Let $\Gamma_a = \{x \in \mathbb{R}^N | a \cdot x = 0\}$ be the $(N - 1)$ -plane perpendicular to a through the origin. Then there exists a unique solution of the following initial value problem of the linear first order PDE*

$$\begin{aligned} a \cdot \nabla u &= f, \\ u|_{\Gamma_a} &= g. \end{aligned}$$

More precisely, this solution is obtained as

$$u(x) = \int_0^{\phi(x)} f(\tau a + \alpha(\psi(x))) d\tau + g(\alpha(\psi(x))), \quad x \in \mathbb{R}^N$$

where

$$\begin{aligned} \phi(x) &= \frac{a \cdot x}{|a|^2}, \quad \psi(x) = (\psi_1(x), \dots, \psi_{N-1}(x)), \\ \psi_j(x) &= \frac{|a|^2 x_j - (a \cdot x) a_j}{|a|^2}, \quad (j = 1, \dots, N - 1) \\ \alpha(s) &= \left(s, -\frac{1}{a_N} \sum_{j=1}^{N-1} a_j s_j \right) \in \mathbb{R}^N, \quad s = (s_1, \dots, s_{N-1}) \in \mathbb{R}^{N-1}, \end{aligned}$$

if we assume (w.l.o.g) $a_N \neq 0$. Furthermore, if $f(x) = O(|x|^\beta)$, $g(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$, then $u(x) = O(|x|^{\beta+1})$ as $|x| \rightarrow \infty$.

Proof. See, for example, [12]. □

3. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. As noticed earlier, we mainly follow the argument by Grossi [9], but some new argument is needed.

We argue by contradiction. We assume there exists a non-trivial solution v_ε to (L_ε) with $u = u_\varepsilon$ satisfying $\|v_\varepsilon\| = \|u_\varepsilon\|$ for any $\varepsilon > 0$.

Let us consider the scaled function

$$\tilde{v}_\varepsilon(y) := \frac{1}{\|u_\varepsilon\|} v_\varepsilon \left(\frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right), \quad y \in \Omega_\varepsilon = \|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}} (\Omega - x_\varepsilon). \quad (3.1)$$

We see $0 < \tilde{v}_\varepsilon \leq 1$ and $\tilde{v}_\varepsilon$ satisfies

$$\begin{cases} \Delta^2 \tilde{v}_\varepsilon = c_0 p_\varepsilon \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon & \text{in } \Omega_\varepsilon, \\ \tilde{v}_\varepsilon = \Delta \tilde{v}_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \\ \|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1. \end{cases} \quad (3.2)$$

By $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$ and the elliptic estimate, we see there exists v_0 such that

$$\tilde{v}_\varepsilon \rightarrow v_0 \quad \text{compact uniformly in } \mathbb{R}^N, \quad (3.3)$$

and v_0 satisfies

$$\Delta^2 v_0 = c_0 p U^{p-1} v_0 \quad \text{in } \mathbb{R}^N.$$

Now, we claim that

$$\int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy \leq C \quad (3.4)$$

for some $C > 0$ independent of $\varepsilon > 0$ small.

Set

$$c_\varepsilon(y) := c_0 p_\varepsilon \tilde{u}_\varepsilon^{p_\varepsilon-1}(y), \quad y \in \Omega_\varepsilon. \quad (3.5)$$

By the Sobolev inequality and (3.2), we have

$$S_N \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{2/(p+1)} \leq \int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy = \int_{\Omega_\varepsilon} c_\varepsilon(y) \tilde{v}_\varepsilon^2 dy \leq \int_{\Omega_\varepsilon} |c_\varepsilon(y)| |\tilde{v}_\varepsilon^{2-\delta}| dy$$

for some $0 < \delta < \min(2, 8/(N-4) - 2\varepsilon)$, because $\|\tilde{v}_\varepsilon\| \leq 1$. Now, by the Hölder inequality and (2.14), we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |c_\varepsilon(y)| |\tilde{v}_\varepsilon^{2-\delta}| dy &\leq \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{\frac{(2-\delta)}{(p+1)}} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1+\delta)} dy \right)^{\frac{(p-1+\delta)}{(p+1)}} \\ &\leq C \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{\frac{(2-\delta)}{(p+1)}} \\ &\quad \times \left(\int_{\Omega_\varepsilon} (U(y))^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy \right)^{\frac{(p-1+\delta)}{(p+1)}}, \end{aligned}$$

thus we obtain

$$\left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{\delta/(p+1)} \leq \left(\int_{\mathbb{R}^N} (U(y))^{(p_\varepsilon-1)(p+1)/(p-1+\delta)} dy \right)^{(p-1+\delta)/(p+1)}.$$

Note that $(p_\varepsilon - 1)(\frac{N-4}{2})\frac{p+1}{p-1+\delta} > N/2$ if $\delta < 8/(N-4) - 2\varepsilon$, so the last integral is bounded by a constant. Finally, again (3.2) and the Hölder inequality, we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy &= \int_{\Omega_\varepsilon} c_\varepsilon(y) \tilde{v}_\varepsilon^2 dy \\ &\leq \left(\int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \right)^{2/(p+1)} \left(\int_{\Omega_\varepsilon} |c_\varepsilon(y)|^{(p+1)/(p-1)} dy \right)^{(p-1)/(p+1)} \\ &\leq C \left(\int_{\mathbb{R}^N} (U(y))^{(p_\varepsilon-1)(p+1)/(p-1)} dy \right)^{(p-1)/(p+1)} \leq C \end{aligned}$$

if $\varepsilon > 0$ is small so that $(\frac{N-4}{2})(p_\varepsilon - 1)(p+1)/(p-1) > N/2$, i.e. $0 < \varepsilon < 4/(N-4)$. This proves (3.4).

By (3.4) and Fatou's lemma, we also have

$$\int_{\mathbb{R}^N} |\Delta v_0|^2 dy \leq C.$$

Thus by Lemma 2.6, we have

$$v_0 = \sum_{j=1}^N a_j \frac{y_j}{(1+|y|^2)^{(N-2)/2}} + b \frac{1-|y|^2}{(1+|y|^2)^{(N-2)/2}}. \quad (3.6)$$

In the following, we divide the proof into several steps.

Step 1. $b = 0$.

Step 2. $a_j = 0, j = 1, \dots, N$.

Step 3. $v_0 = 0$ leads to a contradiction.

We need the following pointwise estimate for the scaled function $\tilde{v}_\varepsilon$.

Lemma 3.1. *Let $\tilde{v}_\varepsilon$ be a solution of (3.2). Then we have the estimate*

$$|\tilde{v}_\varepsilon(y)| \leq CU(y), \quad \forall y \in \Omega_\varepsilon \quad (3.7)$$

for some $C > 0$.

Proof. Consider the Kelvin transformation of $\tilde{v}_\varepsilon$:

$$z_\varepsilon(z) = |z|^{4-N} \tilde{v}_\varepsilon \left(\frac{z}{|z|^2} \right), \quad z \in \Omega_\varepsilon^* := \left\{ \frac{y}{|y|^2} : y \in \Omega_\varepsilon \right\}. \quad (3.8)$$

Since $|\tilde{v}_\varepsilon(y)| \leq 1$ for any $y \in \Omega_\varepsilon$, it will be enough to show that $|z_\varepsilon|$ is bounded in $B(0, R) \cap \Omega_\varepsilon^*$ for some $R > 0$ to prove (3.7). Direct calculation shows that

$$\Delta_z^2 z_\varepsilon(z) = |z|^{-4-N} \Delta_y^2 \tilde{v}_\varepsilon(y), \quad z \in \Omega_\varepsilon^*,$$

$$\Delta_z z_\varepsilon(z) = 4|z|^{-2} (z, \nu_z) \frac{\partial z_\varepsilon}{\partial \nu_z}, \quad z \in \partial \Omega_\varepsilon^*,$$

$$\int_{\Omega_\varepsilon^*} |z_\varepsilon|^{p+1} dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy.$$

Thus, z_ε satisfies the equation

$$\begin{cases} \Delta^2 z_\varepsilon = |z|^{-8} c_\varepsilon \left(\frac{z}{|z|^2} \right) z_\varepsilon & \text{in } \Omega_\varepsilon^*, \\ z_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon^*, \\ \Delta z_\varepsilon = 4|z|^{-2} (z, \nu_z) \frac{\partial z_\varepsilon}{\partial \nu_z}, & \text{on } \partial\Omega_\varepsilon^* \end{cases} \quad (3.9)$$

where c_ε is in (3.5). Now, we claim that

$$a_\varepsilon(z) := |z|^{-8} c_\varepsilon \left(\frac{z}{|z|^2} \right) \in L^\infty(\Omega_\varepsilon^*). \quad (3.10)$$

Indeed, since $\Omega_\varepsilon \subset B(0, \gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/4})$ for some $\gamma > 0$, we see that the domain Ω_ε^* satisfies $\Omega_\varepsilon^* \subset \mathbb{R}^N \setminus B(0, \frac{1}{\gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/4}})$. By (2.14), we have

$$|c_\varepsilon(y)| \leq C U^{p_\varepsilon-1}(y) \quad \text{for } y \in \Omega_\varepsilon.$$

Therefore, we have

$$\begin{aligned} |z|^{-8} c_\varepsilon \left(\frac{z}{|z|^2} \right) &\leq C |z|^{-8} \left(\frac{|z|^2}{1+|z|^2} \right)^{(\frac{N-4}{2})(p_\varepsilon-1)} \\ &= C |z|^{-8+(N-4)(p_\varepsilon-1)} \frac{1}{(1+|z|^2)^{4-\varepsilon(\frac{N-4}{2})}} \\ &\leq C |z|^{-8+(N-4)(p_\varepsilon-1)} = C |z|^{-\varepsilon(N-4)}. \end{aligned}$$

Since $|z| \geq 1/(\gamma \|u_\varepsilon\|^{(p_\varepsilon-1)/4})$ for $z \in \Omega_\varepsilon^*$, we have

$$|z|^{-\varepsilon(N-4)} \leq \gamma^{\varepsilon(N-4)} \|u_\varepsilon\|^{\varepsilon(N-4)(p_\varepsilon-1)/4} \rightarrow 1$$

as $\varepsilon \rightarrow 0$. Here we have used (2.9). From these, we confirm the claim (3.10).

Now, for any $R > 0$, we have

$$\int_{\Omega_\varepsilon^* \cap B(0, 2R)} |z_\varepsilon|^{p+1} dz \leq \int_{\Omega_\varepsilon^*} |z_\varepsilon|^{p+1} dz = \int_{\Omega_\varepsilon} |\tilde{v}_\varepsilon|^{p+1} dy \leq \left(\frac{1}{S_N} \int_{\Omega_\varepsilon} |\Delta \tilde{v}_\varepsilon|^2 dy \right)^{\frac{p+1}{2}} \leq C$$

by (3.4). Let us take $q = p, y = 0, \Omega = \Omega_\varepsilon^*$ in Lemma 2.3. Thus for $R > 0$ in Lemma 2.3, we obtain

$$\sup_{B(0, R) \cap \Omega_\varepsilon^*} |z_\varepsilon| \leq C \left[\frac{1}{R^N} \int_{B(0, 2R) \cap \Omega} |z_\varepsilon|^{p+1} dz \right]^{1/(p+1)} \leq C. \quad \square$$

By Lemma 3.1 and Theorem 2.5 (2.10), we have the following convergence result.

Lemma 3.2. *Let $\omega \subset \Omega$ be any neighborhood of $\partial\Omega$ not containing x_0 . Then we have*

$$\|u_\varepsilon\| v_\varepsilon \rightarrow -2(N-2)(N-4)\sigma_N bG(\cdot, x_0) \quad \text{in } C^3(\omega). \quad (3.11)$$

Proof. We see

$$\Delta^2(\|u_\varepsilon\|v_\varepsilon) = \|u_\varepsilon\|c_0p_\varepsilon u_\varepsilon^{p_\varepsilon-1}v_\varepsilon =: f_\varepsilon(x) \quad (3.12)$$

for $x \in \Omega$ with the boundary condition

$$\|u_\varepsilon\|v_\varepsilon = \Delta(\|u_\varepsilon\|v_\varepsilon) = 0 \quad \text{on } \partial\Omega.$$

By using (2.14), (3.7), (2.9) and the dominated convergence theorem, we obtain

$$\begin{aligned} \int_{\Omega} f_\varepsilon(x) dx &= c_0p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon+1-(\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \tilde{u}_\varepsilon^{p_\varepsilon-1}(y) \tilde{v}_\varepsilon(y) dy \\ &\rightarrow c_0p \int_{\mathbb{R}^N} U^{p-1}v_0 dy = c_0pb \int_{\mathbb{R}^N} \frac{1-|y|^2}{(1+|y|^2)^{N/2+3}} dy \\ &= c_0pb\sigma_N \left(\int_0^\infty \frac{r^{N-1}}{(1+r^2)^{N/2+3}} dr - \int_0^\infty \frac{r^{N+1}}{(1+r^2)^{N/2+3}} dr \right) \\ &= -2(N-2)(N-4)b\sigma_N. \end{aligned}$$

Note that the integral involving the a_j terms of v_0 must vanish by the oddness of the integrand. Last integral can be computed by the formula

$$\int_0^\infty \frac{r^\alpha}{(1+r^2)^\beta} dr = \frac{\Gamma((\alpha+1)/2)\Gamma(\beta-(\alpha+1)/2)}{2\Gamma(\beta)}.$$

Furthermore, for any $x \neq x_0$, we have by (2.10) and (3.7),

$$\begin{aligned} f_\varepsilon(x) &\leq C\|u_\varepsilon\| \frac{\|u_\varepsilon\|^{-(p_\varepsilon-1)}}{|x-x_\varepsilon|^{(N-4)(p_\varepsilon-1)}} \frac{\|u_\varepsilon\|}{(\|u_\varepsilon\|^{(p_\varepsilon-1)/4}|x-x_\varepsilon|)^{N-4}} \\ &\leq C \frac{\|u_\varepsilon\|^{3-p_\varepsilon-(p_\varepsilon-1)(N-4)/4}}{|x-x_\varepsilon|^{(N-4)p_\varepsilon}} \rightarrow 0 \end{aligned}$$

since $3-p_\varepsilon-\frac{p_\varepsilon-1}{4}(N-4) = -8/(N-4) + \varepsilon(N/4) < 0$ for $\varepsilon > 0$ small. From these, we confirm that

$$f_\varepsilon \rightarrow -2(N-2)(N-4)\sigma_N b\delta_{x_0} \quad (3.13)$$

in the sense of distributions. On the other hand, from the equation (3.12) with the Navier boundary condition, L^p -theory of [5] can be applied to get

$$\|-\Delta(\|u_\varepsilon\|v_\varepsilon)\|_{C^{1,\alpha}(\omega)} \leq C(\omega) (\|f_\varepsilon\|_{L^1(\Omega)} + \|f_\varepsilon\|_{L^\infty(\omega')})$$

where $\omega \subset\subset \omega'$ is a neighborhood of $\partial\Omega$ not containing x_0 . Since we have seen that RHS of the above estimate is bounded by a constant independent of ε , Ascoli-Arzelá theorem implies that the function $-\Delta(\|u_\varepsilon\|v_\varepsilon)$ converges to some function in $C^{1,\alpha}$ -topology. Finally, (3.13) implies that this limit function is $-2(N-2)(N-4)\sigma_N b\overline{G}(x, x_0)$. $\square$

Proof of Step 1. Putting $z = x_0$ in (2.5) and multiplying $\|u_\varepsilon\|^2$, we have

$$\begin{aligned} \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| \bar{v}_\varepsilon}{\partial \nu_x} \right) ds_x \\ + \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial \|u_\varepsilon\| \bar{u}_\varepsilon}{\partial \nu_x} \right) \left(\frac{\partial \|u_\varepsilon\| v_\varepsilon}{\partial \nu_x} \right) ds_x = 0. \end{aligned} \quad (3.14)$$

By (2.8) and (3.11), the LHS of (3.14) tends to

$$-8(N-4)^2(N-2)^2\sigma_N^2b \int_{\partial\Omega} ((x - x_0) \cdot \nu_x) \left(\frac{\partial G(x, x_0)}{\partial \nu_x} \right) \left(\frac{\partial \bar{G}(x, x_0)}{\partial \nu_x} \right) ds_x$$

as $\varepsilon \rightarrow 0$. By (2.2), we obtain

$$-8(N-4)^3(N-2)^2\sigma_N^2bR(x_0) = 0.$$

Since $R(x_0) \neq 0$, we obtain $b = 0$. $\square$

Proof of Step 2. In this step, we prove $a_j = 0, j = 1, 2, \dots, N$ in (3.6) by using a lemma, which is not in [9].

Lemma 3.3. *Assume $b = 0$ and $a = (a_1, \dots, a_N) \neq 0$ in (3.6). Then we have*

$$\|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon \rightarrow 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial G}{\partial z_j}(x, z) \right) \Big|_{z=x_0}$$

in $C_{loc}^3(\bar{\Omega} \setminus \{x_0\})$.

Proof. Since $-\Delta \bar{v}_\varepsilon = c_0 p_\varepsilon u_\varepsilon^{p_\varepsilon-1} v_\varepsilon$ in Ω , $\bar{v}_\varepsilon = 0$ on $\partial\Omega$, the Green representation formula implies that

$$\bar{v}_\varepsilon(x) = c_0 p_\varepsilon \int_{\Omega} \bar{G}(x, z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz \quad (3.15)$$

for any $x \in \bar{\Omega} \setminus \{x_0\}$, here $\bar{G}(x, z) = -\Delta_x G(x, z)$ is the Green function of $-\Delta$ under the Dirichlet boundary condition. By a change of variables, we see

$$\begin{aligned} c_0 p_\varepsilon \int_{\Omega} \bar{G}(x, z) u_\varepsilon^{p_\varepsilon-1}(z) v_\varepsilon(z) dz \\ = c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \bar{G} \left(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) dy. \end{aligned}$$

By (2.12) and (3.3), we obtain

$$\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) \rightarrow \sum_{j=1}^N a_j \left(\frac{\partial}{\partial y_j} \frac{-1}{(N+4)} U^p(y) \right)$$

uniformly on compact subsets of $\mathbb{R}^N$.

Now, let us consider the following linear first order PDE

$$\sum_{j=1}^N a_j \frac{\partial w}{\partial y_j} = \tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y), \quad y \in \mathbb{R}^N$$

with the initial condition $w|_{\Gamma_a} = \frac{-1}{(N+4)} U^p(y)$, where $\Gamma_a = \{x \in \mathbb{R}^N | x \cdot a = 0\}$. Here, the right hand side is assumed to be 0 outside of Ω_ε . By Lemma 2.7, we have a solution w_ε of this problem with the estimate $w_\varepsilon(y) = O(|y|^{-(N+3)})$ as $|y| \rightarrow \infty$, since $\tilde{u}_\varepsilon^{p_\varepsilon-1} \tilde{v}_\varepsilon(y) = O(|y|^{-(N+4)})$ by (2.14) and (3.7). Also we have

$$w_\varepsilon \rightarrow \frac{-1}{(N+4)} U^p \quad \text{uniformly on compact subsets on } \mathbb{R}^N$$

and

$$\int_{\Omega_\varepsilon} w_\varepsilon(y) dy \rightarrow \frac{-1}{(N+4)} \int_{\mathbb{R}^N} U^p dy = \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right)$$

by the dominated convergence theorem. Using integration by parts, we have

$$\begin{aligned} \bar{v}_\varepsilon(x) &= c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \int_{\Omega_\varepsilon} \bar{G} \left(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \sum_{j=1}^N a_j \frac{\partial w_\varepsilon}{\partial y_j} dy \\ &= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N} \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \frac{\partial}{\partial y_j} \left\{ \bar{G} \left(x, \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon \right) \right\} w_\varepsilon(y) dy \\ &= -c_0 p_\varepsilon \|u_\varepsilon\|^{p_\varepsilon - (\frac{p_\varepsilon-1}{4})N - (\frac{p_\varepsilon-1}{4})} \\ &\quad \times \sum_{j=1}^N a_j \int_{\Omega_\varepsilon} \left(\frac{\partial \bar{G}}{\partial z_j} \right) (x, z) \Big|_{z = \frac{y}{\|u_\varepsilon\|^{\frac{p_\varepsilon-1}{4}}} + x_\varepsilon} w_\varepsilon(y) dy. \end{aligned}$$

Note that $p_\varepsilon - (\frac{p_\varepsilon-1}{4})N - (\frac{p_\varepsilon-1}{4}) = -(\frac{N-2}{N-4}) + \varepsilon(\frac{N-3}{4})$. Thus we obtain

$$\begin{aligned} \|u_\varepsilon\|^{\frac{N-2}{N-4}} \bar{v}_\varepsilon(x) &\rightarrow -c_0 p \left(\frac{-1}{N+4} \right) \left(\frac{2\sigma_N}{N(N+2)} \right) \sum_{j=1}^N a_j \left(\frac{\partial \bar{G}}{\partial z_j} (x, z) \right) \Big|_{z=x_0} \\ &= 2(N-2)\sigma_N \sum_{j=1}^N a_j \left(\frac{\partial \bar{G}}{\partial z_j} (x, z) \right) \Big|_{z=x_0} \end{aligned}$$

for $x \in \bar{\Omega} \setminus \{x_0\}$.

Standard elliptic estimate assures that this convergence also holds in $C_{loc}^1(\bar{\Omega} \setminus \{x_0\})$. Since $\bar{v}_\varepsilon(x) = -\Delta v_\varepsilon(x)$, again elliptic estimate proves Lemma 3.3. $\square$

Now, assume the contrary that $a = (a_1, \dots, a_N) \neq 0$. We multiply both sides of (2.6) in Lemma 2.4 by $\|u_\varepsilon\|^{(N-2)/(N-4)} \times \|u_\varepsilon\|$ to get

$$\int_{\partial\Omega} \left(\frac{\partial \|u_\varepsilon\| \bar{u}_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} v_\varepsilon}{\partial \nu_x} \right) ds_x + \left(\frac{\partial \|u_\varepsilon\| u_\varepsilon}{\partial x_i} \right) \left(\frac{\partial \|u_\varepsilon\|^{\frac{N-2}{N-4}} \bar{v}_\varepsilon}{\partial \nu_x} \right) ds_x = 0.$$

As $\varepsilon \rightarrow 0$, we see that the LHS of the above tends to

$$\begin{aligned} & 4(N-4)(N-2)^2\sigma_N^2 \sum_{j=1}^N a_j \times \int_{\partial\Omega} \left\{ \frac{\partial \bar{G}}{\partial x_i}(x, x_0) \frac{\partial}{\partial \nu_x} \left(\frac{\partial G}{\partial z_j} \right) (x, z) \Big|_{z=x_0} \right. \\ & \quad \left. + \frac{\partial G}{\partial x_i}(x, x_0) \frac{\partial}{\partial \nu_x} \left(\frac{\partial \bar{G}}{\partial z_j} \right) (x, z) \Big|_{z=x_0} \right\} ds_x \\ & = 4(N-4)(N-2)^2\sigma_N^2 \sum_{j=1}^N a_j \frac{\partial^2 R}{\partial z_i \partial z_j}(z) \Big|_{z=x_0}, \end{aligned}$$

here we have used Theorem 2.5 (2.8), Lemma 3.3 and Lemma 2.2 (2.4). Thus

$$\sum_{j=1}^N a_j \frac{\partial^2 R}{\partial z_i \partial z_j}(x_0) = 0.$$

Since we assumed that $a = (a_1, \dots, a_N) \neq 0$, this contradicts to our assumption that x_0 is a nondegenerate critical point of the Robin function. Thus we have proved Step 2. $\square$

Proof of Step 3. By Step 1 and Step 2, we have obtained that the limit function $\lim_{\varepsilon \rightarrow 0} \tilde{v}_\varepsilon = v_0 \equiv 0$. Since $\|\tilde{v}_\varepsilon\|_{L^\infty(\Omega_\varepsilon)} = 1$, there exists $y_\varepsilon \in \Omega_\varepsilon$ such that $\tilde{v}_\varepsilon(y_\varepsilon) = 1$ and $|y_\varepsilon| \rightarrow \infty$, because the above convergence $\tilde{v}_\varepsilon \rightarrow v_0 \equiv 0$ is uniform on compact sets of $\mathbb{R}^N$. But this is not possible because of Lemma 3.1. This proves Theorem 1.1. $\square$

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