Touchdown and related problems in electrostatic MEMS device equation

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Abstract. We study the electrostatic MEMS-device equation, $u_t - \Delta u = \frac{\lambda|x|^\beta}{(1-u)^p}$, with Dirichlet boundary condition. First, we describe the touchdown of non-stationary solution in accordance with the total set of stationary solutions. Then, we classify radially symmetric stationary solutions and their radial Morse indices. Finally, we show the Morse-Smale property for radially symmetric non-stationary solutions.

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1. Introduction

The purpose of the present paper is to study the global-in-time behaviour of the solution to the parabolic problem

$$\begin{aligned} u_t - \Delta u &= \frac{\lambda f(x)}{(1-u)^p}, \quad 0 \leq u < 1 \quad \text{in } \Omega \times (0, T) \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T) \\ u|_{t=0} &= u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a constant, $p > 1$, $f(x) \geq 0$, $f(x) \not\equiv 0$ is a continuous function, $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary $\partial\Omega$, and $u_0 = u_0(x) \in [0, 1]$ is a continuous function. This equation models the dynamic deflection of an elastic mebrane inside a micro-electro mechanical system (MEMS). This kind of systems combine electronics with various types of micro-size mechanical devices and could be found in accelerometers for airbag deployment in automobiles, in ink jet printer heads, in optical switches, in chemical sensors and so on, for more details see [21] and the references therein.

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The structure of the set of stationary solutions has been studied for $p = 2$ ([8, 11]). We obtain, similarly, an upper bound of λ for the existence of the solution denoted by $\bar{\lambda} > 0$ for general $p > 1$. Thus, if $0 < \lambda < \bar{\lambda}$ and $\lambda > \bar{\lambda}$, then $\mathcal{C}^\lambda \neq \emptyset$ and $\mathcal{C}^\lambda = \emptyset$, respectively, where

$$\mathcal{C}^\lambda = \{u \in C^2(\Omega) \cap C_0(\bar{\Omega}) \mid u \text{ solves (1.1)}\}.$$

Also, if $\mathcal{C}^\lambda \neq \emptyset$ there is a unique minimal element, denoted by u_λ .

The global-in-time behaviour of the non-stationary solution has been studied also for $p = 2$ ([9]). More precisely, for $\lambda < \bar{\lambda}$ and $\lambda > \bar{\lambda}$, there is a solution to (1.1) converging to u_λ uniformly and any solution must touchdown in finite time, i.e.,

$$T < +\infty, \quad \lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = 1, \quad (1.2)$$

respectively. Here and henceforth, $T > 0$ denotes the existence time of the solution. We can refine these results on the global-in-time behaviour of the solution even for general $p > 1$.

Theorem 1.1. *We have the following.*

1. If $\lambda > \bar{\lambda}$, then (1.2) occurs.
2. If there is $u^* \in \mathcal{C}^\lambda \setminus \{u_\lambda\}$ and $u_0 \leq u^*$, $u_0 \not\equiv u^*$ or $\mathcal{C}^\lambda = \{u_\lambda\}$ and $u_0 \leq u_\lambda$, then $T = +\infty$ and $u(\cdot, t) \rightarrow u_\lambda$ uniformly as $t \uparrow +\infty$.
3. If there is $u^* \in \mathcal{C}^\lambda \setminus \{u_\lambda\}$, $u_0 \geq u^*$, $u_0 \not\equiv u^*$, then (1.2) occurs.

Next, we take the case of

$$\Omega = B \equiv \{x \in \mathbb{R}^n \mid |x| < 1\}, \quad f(x) = |x|^\beta, \quad \beta \geq 0, \quad (1.3)$$

and prove existence of non-minimal radially symmetric stationary solutions. To state the results, we set

$$\begin{aligned} \mathcal{C}_r^\lambda &= \{u \in C^2(\Omega) \cap C_0(\bar{\Omega}) \mid u = u(|x|) \text{ solves (1.1)}\} \\ \mathcal{C}_r &= \bigcup_{\lambda > 0} \{\lambda\} \times \mathcal{C}_r^\lambda \end{aligned}$$

and

$$\begin{aligned} \bar{\beta} &= \frac{p\{2(p-1) - (p+1)n\} + (n-2)(p+1)\sqrt{p(p+1)}}{2p} \\ \bar{p} &= \frac{-(n^2 - 8n + 4) - 8\sqrt{n-1}}{(n-2)(n-10)}, \end{aligned}$$

where $\bar{\beta}$ and $\bar{p}$ are the maximum solutions of the equations $f(p, n, \beta) = 0$ and $f(p, n, 0) = 0$ respectively, for $f(p, n, \beta) = (p+1)^2 n^2 - 4(p+1)(\beta p + 3p + 1)n - 4\{\beta^2 p + 2p\beta(1-p) - (5p^2 + 2p + 1)\}$. Note that $\bar{\beta}, \bar{p}$ are defined for $n \geq 7$ and $n = 7, 8, 9$ respectively.

Theorem 1.2. *If $n \geq 2$, then $\mathcal{C}_r$ is homeomorphic to $\mathbb{R}$ and has end points $(0, 0)$ and $(\lambda^*, 1 - |x|^{\frac{\beta+2}{p+1}})$, where*

$$\lambda^* = \frac{(\beta + 2) \{ \beta + 2 + (n - 2)(p + 1) \}}{(p + 1)^2}.$$

In the cases of

$$\begin{aligned} n \in [2, 6], \quad p > 1, \quad \beta > 0, \\ n \in [7, 9], \quad p > \bar{p}, \quad \beta > 0, \quad \text{or} \quad p \in (1, \bar{p}), \quad \beta > \bar{\beta}, \\ n \geq 10, \quad p > 1, \quad \beta > \bar{\beta}, \end{aligned}$$

whenever $\bar{\beta} > 0$, $\mathcal{C}_r$ bends infinitely many times with respect to λ around λ^ , while at most a finite number of bendings occur to $\mathcal{C}_r$ when*

$$\begin{aligned} n \in [7, 9], \quad p \in (1, \bar{p}], \quad \beta \in (0, \bar{\beta}], \\ n \geq 10, \quad p > 1, \quad \beta \in (0, \bar{\beta}]. \end{aligned}$$

The case $p = 2$ has also been studied on $\mathcal{C} = \bigcup_{\lambda > 0} \{\lambda\} \times \mathcal{C}^\lambda$. First, the above profile of $\mathcal{C}_r$ is observed numerically [8]. Next, some estimates on $\bar{\lambda}$ are given for non-radially symmetric case [8, 9]. Finally, existence of the non-minimal non-radially symmetric stationary solution is proven by the variational method [4].

Given $u \in \mathcal{C}^\lambda$, the linearized eigenvalue problem is defined as follows:

$$-\Delta \phi - \frac{p\lambda f(x)}{(1-u)^{p+1}} \phi = \mu \phi \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega. \quad (1.4)$$

Then, the number of its negative eigenvalues, denoted by $i = i(\lambda, u)$, is called Morse index. In case (1.3) the number of negative eigenvalues corresponding to radially symmetric eigenfunctions, denoted by $i_R = i_R(\lambda, u)$, is called radial Morse index.

Theorem 1.3. *Under the assumptions of Theorem 1.2, $i_R = i_R(\lambda, u)$ increases one by one at each bending point.*

A general theorem guarantees $i(u, \lambda) = i_R(u, \lambda)$ for

$$-\Delta u = \lambda f(r, u) \geq 0 \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B,$$

if $f_r \leq 0$, where $r = |x|$, see [16]. This is not the case of (1.3) with $\beta > 0$. In fact, general theorems of radial symmetry of the solution to the stationary problem are not valid for $\beta > 0$ ([10, 20]). There may be non-radial bifurcation and $i(\lambda, u) = i_R(\lambda, u)$ would not be always valid.

This paper is composed of five sections. Theorems 1.1, 1.2, and 1.3 are proven in sections 2, 3, and 4, respectively. In the final section, §5, we confirm the role of radial Morse indices in the formation of connecting orbits.

2. Proof of Theorem 1.1

The original proof for $p = 2$ to the first case of this theorem is quite technical [9], it provides with some bounds of the blow-up time though. Here we give a simpler proof by applying Kaplan's method adopted by [2]. The proof of the second case is obtained by using classical techniques like maximum-principle or dynamical-systems arguments, [7, 9], and we omit it. The final case is obtained by the method of [15].

Before proceeding to these cases, we recall that Kaplan's method guarantees touchdown for any $\lambda > 0$ if the initial value is close to 1 and $f > 0$ on $\bar{\Omega}$ (see also [3]). In fact, let $\mu_1 > 0$ be the principal eigenvalue of $-\Delta$, and take the solution to

$$-\Delta \varphi_1 = \mu_1 \varphi_1, \quad \varphi_1 > 0 \quad \text{in } \Omega, \quad \varphi_1 = 0 \quad \text{on } \partial\Omega,$$

normalized by $\int_{\Omega} \varphi_1 = 1$. Then, using $m = \inf_{\Omega} f > 0$ and $A = \int_{\Omega} u \varphi_1 dx$, we obtain via Jensen's inequality

$$\frac{dA}{dt} \geq -\mu_1 + \lambda m \int_{\Omega} (1-u)^{-p} \varphi_1 dx \geq -\mu_1 A + \lambda m (1-A)^{-p}.$$

Let $H(s) = -\mu_1 s + \lambda m (1-s)^{-p}$, $s \in (0, 1)$. Then there exists $a \in (0, 1)$ such that $H(s) > 0$ in $(a, 1)$. Hence if we choose u_0 close enough to 1 so that $A(0) > a$, then $\frac{dA}{dt} > 0$ for all t when u is global. But, then we have

$$t = \int_0^t d\sigma \leq \int_{A(0)}^{A(t)} \frac{ds}{H(s)} \leq \int_{A(0)}^1 \frac{ds}{H(s)} < +\infty$$

which is a contradiction. Kaplan's method could be also used to prove touchdown for big enough λ , see [11].

Now, we show the first case of Theorem 1.1, i.e., $\lambda > \bar{\lambda}$ implies (1.2). It suffices to assume $u_0 = 0$ for this purpose. Then, it holds that

$$\begin{aligned} v_t - \Delta v &= \frac{p\lambda f(x)}{(1-u)^{p+1}} v && \text{in } \Omega \times (0, T) \\ v &= 0 && \text{on } \partial\Omega \times (0, T) \\ v|_{t=0} &= \lambda f(x) \geq 0 && \text{in } \Omega, \end{aligned}$$

for $v = u_t$ and hence $u_t \geq 0$. In particular, $\lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = 1$ follows from $T < +\infty$, because $\mathcal{C}^\lambda = \emptyset$.

We assume $T = +\infty$ and multiply the first equation of (1.1) by φ_1 :

$$\int_t^{t+1} \int_\Omega u_t \varphi_1 dx ds = \int_t^{t+1} \int_\Omega (\Delta u) \varphi_1 dx ds + \lambda \int_t^{t+1} \int_\Omega \frac{f(x) \varphi_1}{(1-u)^p} dx ds.$$

Then, it holds that

$$\begin{aligned} \lambda \int_t^{t+1} \int_\Omega \frac{f(x) \varphi_1}{(1-u)^p} dx ds \\ &= \int_\Omega u(\cdot, t+1) \varphi_1 dx - \int_\Omega u(\cdot, t) \varphi_1 dx + \mu_1 \int_t^{t+1} \int_\Omega u \varphi_1 dx ds \\ &\leq \|u(\cdot, t+1)\|_\infty + \mu_1 \|u(\cdot, t)\|_\infty \leq 1 + \mu_1 \end{aligned}$$

by $0 \leq u(x, t) < 1$ in $\Omega \times (0, +\infty)$, and therefore,

$$\lambda \int_\Omega \frac{f(x) \varphi_1}{(1-u(x, t))^p} dx \leq 1 + \mu_1$$

by $u_t \geq 0$. This implies

$$\sup_{t \geq 0} \int_\Omega \frac{f(x) \varphi_1}{(1-u(\cdot, t))^p} dx \leq \frac{1 + \mu_1}{\lambda} < +\infty.$$

From the monotone convergence theorem, and

$$\|u(\cdot, t)\|_1 = \int_\Omega |u(x, t)| dx \leq |\Omega| < +\infty,$$

it follows that $u(\cdot, t)$ has a limit w in $L^1(\Omega)$ as $t \rightarrow \infty$, while $K(x, t) = \frac{f(x)}{(1-u)^p}$ converges to $\tilde{K}(x) = \frac{f(x)}{(1-w)^p}$ in $L^1(\Omega, \delta(x) dx)$ as $t \uparrow +\infty$, where $\delta(x) = \text{dist}(x, \partial\Omega)$. Testing (1.1) by $\varphi \in C^2(\bar{\Omega})$ with $\varphi|_{\partial\Omega} = 0$, we obtain

$$\left[\int_\Omega u \varphi dx \right]_t^{t+1} = \int_t^{t+1} \int_\Omega u \Delta \varphi dx ds + \lambda \int_t^{t+1} \int_\Omega \frac{f(x) \varphi}{(1-u)^p} dx ds.$$

Letting $t \rightarrow \infty$ we derive

$$\int_\Omega w(-\Delta \varphi) dx = \int_\Omega \frac{\lambda f(x) \varphi}{(1-w)^p} dx. \quad (2.1)$$

This means that w is a weak solution to

$$-\Delta w = \frac{\lambda f(x)}{(1-w)^p} \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial\Omega, \quad (2.2)$$

and we obtain a contradiction by the following lemma, see also [2, 9].

Lemma 2.1. *Problem (2.2) has no weak solutions for $\lambda > \bar{\lambda}$.*

Proof. We define

$$g(w) = \frac{1}{(1-w)^p}, \quad h(w) = \int_0^w \frac{ds}{g(s)} = \int_0^w (1-s)^p ds$$

$$g_\varepsilon(w) = (1-\varepsilon)g(w), \quad h_\varepsilon(w) = \int_0^w \frac{ds}{g_\varepsilon(s)} = \frac{1}{1-\varepsilon}h(w)$$

and put $\Phi_\varepsilon(w) = h_\varepsilon^{-1}(h(w))$, where $\varepsilon > 0$. The following properties of this mapping $w \in [0, 1] \mapsto \Phi_\varepsilon(w)$ are obtained by $g'' > 0$:

$$\begin{aligned} \Phi_\varepsilon(0) &= 0, \quad 0 \leq \Phi_\varepsilon(w) < w, \\ \Phi'_\varepsilon(w) &= (1-\varepsilon)g(\Phi_\varepsilon(w))g(w)^{-1} > 0, \\ \Phi''_\varepsilon(w) &= (1-\varepsilon)g(\Phi_\varepsilon(w))((1-\varepsilon)g'(\Phi_\varepsilon(w)) - g'(w))g(w)^{-2} \\ &< (1-\varepsilon)g(\Phi_\varepsilon(w))((1-\varepsilon)g'(w) - g'(w))g(w)^{-2} \leq 0. \end{aligned}$$

Given a weak solution $w = w(x) \in [0, 1]$ to (2.2) satisfying $\frac{f(x)}{(1-w)^p} \in L^1(\Omega, \delta(x)dx)$, we define $w_\varepsilon = \Phi_\varepsilon(w)$. This means $h(w_\varepsilon) = (1-\varepsilon)h(w)$ and there is $\delta > 0$ such that $0 \leq w_\varepsilon \leq 1 - \delta$. It also holds by the concavity of Φ_ε that

$$-\Delta w_\varepsilon \geq \Phi'_\varepsilon(w) \frac{\lambda f(x)}{(1-w)^p} \quad \text{in } \Omega, \quad w_\varepsilon = 0 \quad \text{on } \partial\Omega$$

in weak sense, i.e.,

$$\begin{aligned} - \int_{\Omega} w_\varepsilon \Delta J dx &\geq \lambda \int_{\Omega} \Phi'_\varepsilon(w) \frac{f(x)}{(1-w)^p} J dx \\ &= \lambda(1-\varepsilon) \int_{\Omega} \frac{f(x)J}{(1-w_\varepsilon)^p} dx \end{aligned}$$

for any $J \in C^2(\bar{\Omega})$ satisfying $J \geq 0$ and $J|_{\partial\Omega} = 0$. Thus, w_ε is a weak supersolution to (2.2) and therefore, the iteration sequence $\{v_k\}_{k=0}^\infty$ defined by

$$-\Delta v_{k+1} = \frac{\lambda(1-\varepsilon)f(x)}{(1-v_k)^p} \quad \text{in } \Omega, \quad v_{k+1} = 0 \quad \text{on } \partial\Omega \quad (2.3)$$

with $v_0 = w_\varepsilon$ is monotone decreasing. It also satisfies $v_k \geq 0$ and converges uniformly to a solution to (2.2) for $\lambda(1 - \varepsilon)$ by Dini's theorem. This implies $C^{\lambda(1-\varepsilon)} \neq \emptyset$, a contradiction by the definition of $\bar{\lambda}$ since $\varepsilon > 0$ is arbitrary. $\square$

The third case of Theorem 1.1 is a consequence of the following theorem.

Theorem 2.2. *If there is a pair of super and sub-solutions to (2.2), $\bar{v}, \underline{v} \in C^2(\Omega) \cap C_0(\bar{\Omega})$ satisfying $0 \leq \bar{v} \leq \underline{v} < 1$ and $\bar{v} \not\equiv \underline{v}$, and if the initial value satisfies $\underline{v} \leq u_0 < 1$ and $\underline{v} \not\equiv u_0$ in (1.1), then it holds that $T < +\infty$ and $\lim_{t \uparrow T} \|u(\cdot, t)\|_\infty = 1$.*

Proof. Since $\bar{v}$ is a super-solution, the iteration sequence $\{v_k\}_{k=0}^\infty$ defined by (2.3) with $v_0 = \bar{v}(x) \in [0, 1]$ is monotone decreasing. It satisfies $v_k \geq 0$, and converges uniformly to a solution to (2.2) by Dini's theorem. Thus, we can assume that $\bar{v}$ is a classical solution to (2.2).

Let u_1 and u_2 be the local in time solutions to (1.1) with $u_0 = \underline{v}$ and $u_0 = \bar{v}$, respectively. Then, by the strong maximum principle and the Hopf's lemma we have $u(\cdot, t_0) \gg u_1(\cdot, t_0) \gg u_2(\cdot, t_0)$ for $0 < t_0 \ll 1$, i.e.,

$$\begin{aligned} u(\cdot, t_0) &> u_1(\cdot, t_0) > u_2(\cdot, t_0) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}(\cdot, t_0) &< \frac{\partial u_1}{\partial \nu}(\cdot, t_0) < \frac{\partial u_2}{\partial \nu}(\cdot, t_0) \quad \text{on } \partial\Omega, \end{aligned}$$

where ν denotes the outer unit normal vector. Since $\underline{v}$ is a sub-solution of (2.2) we obtain, by the maximum principle, that u_1 is increasing in time, i.e., $u_{1t} > 0$, and hence $u_1(\cdot, t_0)$ is a strict sub-solution of problem (2.2). On the other hand, we obtain $u_2(\cdot, t) = \bar{v}$ because $\bar{v}$ is supposed to be a stationary solution. Therefore, we may assume that $\underline{v}$ and $\bar{v}$ are a strict sub-solution and a solution of (2.2), respectively, and that $u_0, \underline{v}, \bar{v}$ are C^2 functions on $\bar{\Omega}$ satisfying

$$u_0 \gg \underline{v} \gg \bar{v}.$$

In this case, we can take a constant $\theta > 1$ such that

$$u_0 \gg \theta \underline{v} + (1 - \theta) \bar{v} > \bar{v},$$

and therefore, we can assume $u_0 = \theta \underline{v} + (1 - \theta) \bar{v}$ by the comparison theorem. This $u_0 = u_0(x) \in [0, 1]$ is a strict sub-solution because $g(u) = (1 - u)^{-p}$ is convex. In fact,

$$\begin{aligned} -\Delta u_0 &= -\theta \Delta \underline{v} - (1 - \theta) \Delta \bar{v} < \frac{\lambda \theta f(x)}{(1 - \underline{v})^p} + \frac{\lambda (1 - \theta) f(x)}{(1 - \bar{v})^p} \\ &= \lambda f(x) [\theta g(\underline{v}) + (1 - \theta) g(\bar{v})] < \lambda f(x) g(u_0). \end{aligned}$$

Then, $u(x, t)$ is strict increasing in t for each $x \in \Omega$ and so we have a measurable function $v(x)$ such that

$$\lim_{t \uparrow +\infty} u(x, t) = v(x) \in (\bar{v}(x), 1] \quad \text{for } x \in \Omega \quad (2.4)$$

under the assumption $T = +\infty$. Then, similarly to the first case, it holds that

$$\int_{\Omega} \frac{f(x)\varphi_1}{(1-v)^p} dx \leq \frac{1+\mu_1}{\lambda} < +\infty.$$

Using Duhamel's principle, we can write the solution of problem (1.1) in the form

$$u(x, t) = \int_{\Omega} U(x, y; t) u_0(y) dy + \int_0^t ds \int_{\Omega} U(x, y; s) \frac{\lambda f(y)}{(1-u(y, t-s))^p} dy,$$

where $U(x, y; t) > 0$ is the fundamental solution of the heat operator. The second term of the right-hand side is written as

$$\int_0^{\infty} \int_{\Omega} U(x, y; s) \frac{\lambda f(y)}{(1-u(y, t-s))^p} \chi_{[0, t]}(s) dy ds,$$

where the monotone convergence theorem is applicable by (2.4). Since

$$\lim_{t \uparrow +\infty} \int_{\Omega} U(x, y; t) u_0(y) dy = 0$$

it holds that

$$v(x) = \int_0^{\infty} ds \int_{\Omega} U(x, y; s) \frac{\lambda f(y)}{(1-v(y))^p} dy$$

for each $x \in \Omega$. Here, using Fubini's theorem, we obtain

$$v(x) = \lambda \int_{\Omega} G(x, y) \frac{f(y)}{(1-v(y))^p} dy \quad (2.5)$$

by $\int_0^{\infty} U(x, y; t) dt = G(x, y)$, where $G(x, y)$ denotes the Green's function of $-\Delta$. This implies that $v = v(x) \in (\bar{v}(x), 1]$ is a weak solution to (2.2).

We have

$$\bar{v} \ll u_0 \ll u(\cdot, t_0) \ll v$$

and therefore, there is $\beta \in (0, 1)$ such that

$$u_0 \leq \beta v + (1 - \beta) \bar{v} \equiv z.$$

Here, $\bar{v}$ is a classical solution to (2.2), while $v = v(x) \in [\bar{v}(x), 1]$ is a weak solution to (2.2) satisfying $\frac{f(x)}{(1-v)^p} \in L^1(\Omega, \delta(x)dx)$. Then, we can define the minimal solution, denoted by $\tilde{u}$ to (1.1) with the initial value z by the iteration scheme

$$\begin{aligned}\tilde{u}_{k+1}(x, t) &= \int_{\Omega} U(x, y; t) z(y) dy + \int_0^t \int_{\Omega} \frac{U(x, y; s) \lambda f(y)}{(1 - \tilde{u}_k(y, t - s))^p} dy ds \\ \tilde{u}_0(x, t) &= \int_{\Omega} U(x, y; t) z(y) dy.\end{aligned}$$

In fact, this $\{\tilde{u}_k\}_{k=0}^{\infty}$ is monotone increasing in k and satisfies

$$\tilde{u}_k(x, t) \leq z(x)$$

because $g(u) = (1 - u)^{-p}$ is convex, and then $\tilde{u}(x, t)$ is defined by

$$\lim_{k \rightarrow \infty} \tilde{u}_k(x, t) = \tilde{u}(x, t).$$

If $\{u_k\}_{k=0}^{\infty}$ is defined by

$$\begin{aligned}u_{k+1}(x, t) &= \int_{\Omega} U(x, y; t) u_0(y) dy + \int_0^t \int_{\Omega} \frac{U(x, y; s) \lambda f(y)}{(1 - u_k(y, t - s))^p} dy ds \\ u_0(x, t) &= \int_{\Omega} U(x, y; t) u_0(y) dy,\end{aligned}$$

then the monotonicity of $g(s)$ implies that $u_k \leq \tilde{u}_k$ for each k and hence $\lim_{k \rightarrow \infty} u_k(x, t) = u(x, t)$. This implies

$$v \leq \beta v + (1 - \beta) \bar{v} \quad \text{in } \Omega$$

by letting $k \rightarrow \infty$ and then $t \uparrow +\infty$. Then, it follows that $v \leq \bar{v}$, a contradiction to (2.4). $\square$

In the case that problem (2.2) has only a weak solution w^* , i.e. $\|w^*\|_{\infty} = 1$ and w^* satisfies (2.2) in the sense of (2.1), we can prove that an infinite-time touchdown occurs for the solution of problem (1.1). More precisely we have the following.

Theorem 2.3. *If problem (2.2) has only a weak solution w^* , then for $u_0(x) < w^*(x)$ there exists a unique global solution u^* of (1.1) such that $u^*(x, t) \leq w^*(x)$ in Ω for all $0 < t < \infty$, while $u^*(x, t)$ converges pointwise to w^* as $t \rightarrow \infty$. In particular, $\lim_{t \uparrow \infty} \|u^*(\cdot, t)\|_{\infty} = 1$.*

The proof of Theorem 2.3 follows the same steps with the proof of Theorem 2.5, for $p = 2$, in [9] and uses some arguments introduced in [2].

An natural question arises, from the above theorem, if the rate of convergence of u^* towards the singular steady state w^* could be determined. Since this is a very interesting question and the answer requires a delicate analysis it will be addressed in a forthcoming paper.

The problem of the characterization of the set of the touchdown points, for the case $p = 2$, has been first tackled in [9], while some more refined results were given in [12]. In particular, in [12], it is proven, for problem

$$\begin{aligned} v_t - \Delta v &= -\frac{\lambda f(x)}{v^2}, \quad 0 \leq v < 1 \quad \text{in } \Omega \times (0, T) \\ v &= 1 \quad \text{on } \partial\Omega \times (0, T) \\ v|_{t=0} &= 1 \quad \text{in } \Omega, \end{aligned} \quad (2.6)$$

that the finite-time touchdown set $Q := \{x \in \Omega : v(x, T) = 0, T = T_{max} < \infty\} \subset A := \{x \in \Omega : f(x) \neq 0\}$ under the hypothesis that Q is a compact set, which is the case when Ω is convex and f satisfies some extra condition. Under the same hypothesis some upper and lower estimates of the touchdown profile are given as well as some refined touchdown profiles are provided, in the radial symmetric case, coming out by classical arguments like similarity-variables and central-manifolds techniques. In the case that Ω is not convex the question of determining the touchdown rate is still open, and probably some scaling arguments, which have been introduced for general domains, could work in this case. Now, for the radial symmetric case, it is proven in [9], by using ideas from [5], that the origin $r = 0$ is the only touchdown point provided that $f'(r) \leq 0$. Most of the above results, existing in [9] for $p = 2$, could be easily extended to the case $p > 1$.

3. Proof of Theorem 1.2

We study the structure of the set of radially symmetric stationary solutions in the case of (1.3). This problem is reduced to

$$\begin{aligned} (r^{n-1}u_r)_r + \frac{\lambda r^{n+\beta-1}}{(1-u)^p} &= 0, \quad 0 \leq u < 1, \quad \text{for } r \in (0, 1) \\ u(1) &= 0, \quad u_r(0) = 0 \end{aligned} \quad (3.1)$$

where $u = u(r)$ for $r = |x|$.

We apply the phase-plane analysis [13, 17, 18, 19]. First, putting $r = y^{\frac{2}{\beta+2}}$, we obtain

$$\begin{aligned} (y^a u_y)_y + \frac{4\lambda}{(\beta+2)^2} \frac{y^a}{(1-u)^p} &= 0, \quad 0 \leq u < 1 \quad \text{for } y \in (0, 1) \\ u(1) &= 0, \quad \lim_{y \rightarrow 0} y^{\frac{\beta}{\beta+2}} u_y = 0 \end{aligned} \quad (3.2)$$

for $a = \frac{2n+\beta-2}{\beta+2}$.

We note that every positive solution of problem (3.2) can be obtained as a solution of the following initial value problem with certain positive constant A :

$$\begin{aligned} (y^a u_y)_y + \frac{4\lambda}{(\beta+2)^2} \frac{y^a}{(1-u)^p} &= 0, \quad 0 \leq u < 1, \quad \text{for } y \in (0, 1) \\ u(0) &= A \in (0, 1), \quad \lim_{y \rightarrow 0} y^{\frac{\beta}{\beta+2}} u_y = 0. \end{aligned} \quad (3.3)$$

Putting

$$k = \sqrt{\frac{\{\beta+2+(n-2)(p+1)\}(\beta+2)(1-A)^{p+1}}{\lambda(p+1)^2}},$$

we apply the Emden transformation

$$u(y) = 1 - (1-A)e^{\frac{2}{p+1}t} w(t), \quad y = ke^t.$$

Then, (3.3) is reduced to

$$\begin{aligned} \ddot{w} + \frac{2(np+n-2p+2\beta+2)}{(\beta+2)(p+1)} \dot{w} + \frac{4(np+n-2p+\beta)}{(\beta+2)(p+1)^2} \left(w - \frac{1}{w^p} \right) &= 0 \\ 0 < w < \frac{1}{1-A} e^{-\frac{2}{p+1}t} &\quad \text{for } t \in \mathbb{R} \\ \lim_{t \rightarrow -\infty} e^{\frac{2}{p+1}t} w &= 1, \quad \lim_{t \rightarrow -\infty} e^{\frac{2}{p+1}t} \dot{w} = -\frac{2}{p+1}. \end{aligned} \quad (3.4)$$

The profile of this orbit is studied in detail in [13]. First, we show the following lemma.

Lemma 3.1. *Problem (3.4) admits a unique solution for $t \ll -1$.*

Proof. Through the transformation,

$$v(z) = e^{\frac{2}{p+1}t} w(t), \quad z = e^t,$$

(3.4) is reduced to

$$\begin{aligned} (z^b v_z)_z - \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} z^b v^{-p} &= 0 \\ 0 < v < \frac{1}{1-A}, \quad z > 0, \\ v(0) &= 1, \quad \lim_{z \downarrow 0} z v_z = 0, \end{aligned}$$

where $b = \frac{p\beta+\beta+2np+2n-2p-2}{(p+1)(\beta+2)}$ and then we obtain the integral equation

$$v(z) = 1 + \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \cdot \int_0^z \sigma^{-b} d\sigma \int_0^\sigma \zeta^b v(\zeta)^{-p} d\zeta. \quad (3.5)$$

Now, we consider the Banach space

$$X = \{v = v(z) \mid v \in C([0, \varepsilon]) \text{ and } v(0) = 1\},$$

equipped with the supremum norm $\|\cdot\|_\infty$, where $\varepsilon > 0$ is a constant to be determined later. Let $\Psi(v) \in X$ be the right-hand side of (3.5) defined for $v \in X$. For, given

$$v_i \in Y \equiv \left\{v \in X \mid 1 \leq v \leq \frac{1}{1-A}\right\}, \quad (i = 1, 2),$$

we have

$$\begin{aligned} |\Psi(v_1) - \Psi(v_2)| &\leq \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \int_0^z \sigma^{-b} d\sigma \int_0^\sigma \zeta^b |v_1(\zeta)^{-p} - v_2(\zeta)^{-p}| d\zeta \\ &\leq \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \int_0^z \sigma^{-b} d\sigma \\ &\quad \cdot \int_0^\sigma \zeta^b p \left(\int_0^1 |\theta v_1(\zeta) + (1-\theta)v_2(\zeta)|^{p-1} d\theta \right) \frac{|v_1(\zeta) - v_2(\zeta)|}{v_1(\zeta)^p v_2(\zeta)^p} d\zeta \\ &\leq \frac{4p(np+n-2p+\beta)}{(p+1)^2(\beta+2)(1-A)^{p-1}} \int_0^z \sigma^{-b} d\sigma \int_0^\sigma \zeta^b \|v_1 - v_2\|_\infty d\zeta \\ &\leq \frac{4p(np+n-2p+\beta)}{(p+1)^2(\beta+2)(1-A)^{p-1}} \int_0^\varepsilon \sigma d\sigma \|v_1 - v_2\|_\infty \\ &\leq \frac{2p(np+n-2p+\beta)}{(p+1)^2(\beta+2)(1-A)^{p-1}} \varepsilon^2 \|v_1 - v_2\|_\infty. \end{aligned}$$

Therefore

$$\|\Psi(v_1) - \Psi(v_2)\|_\infty \leq \frac{2p(np+n-2p+\beta)}{(p+1)^2(\beta+2)(1-A)^{p-1}} \varepsilon^2 \|v_1 - v_2\|_\infty.$$

Similarly, we obtain

$$\begin{aligned} |\Psi(v)| &\leq 1 + \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \cdot \int_0^z \sigma^{-b} d\sigma \int_0^\sigma \zeta^b |v(\zeta)|^{-p} d\zeta \\ &\leq 1 + \frac{2(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \varepsilon^2, \end{aligned}$$

and hence

$$\|\Psi(v)\|_{\infty} \leq 1 + \frac{2(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \varepsilon^2.$$

For

$$\varepsilon \leq \min \left(\sqrt{\frac{(p+1)^2(\beta+2)(1-A)^{p-1}}{4p(np+n-2p+\beta)}}, \sqrt{\frac{(p+1)^2(\beta+2)A}{2(np+n-2p+\beta)(1-A)}} \right)$$

it holds that $\Psi(v_i) \in Y$ for $i = 1, 2$ and

$$\|\Psi(v_1) - \Psi(v_2)\|_{\infty} \leq \frac{1}{2} \|v_1 - v_2\|_{\infty}.$$

Hence Ψ has a unique fixed point $v \in X$ by the contraction mapping principle, and the proof is complete. $\square$

Now we write (3.4) as an autonomous system

$$\frac{d}{dt} \begin{pmatrix} w \\ \dot{w} \end{pmatrix} = \begin{pmatrix} -\frac{2(np+n-2p+2\beta+2)}{(p+1)(\beta+2)} \dot{w} - \frac{4(np+n-2p+\beta)}{(p+1)^2(\beta+2)} \left(w - \frac{1}{w^p}\right) \end{pmatrix}.$$

Note that $(w, \dot{w}) = (1, 0)$ is a singular-fixed point of this system. The linearized equation around this fixed point is given by

$$\frac{d}{dt} \begin{pmatrix} W \\ \dot{W} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{4(np+n-2p+\beta)}{(p+1)(\beta+2)} & -\frac{2(np+n-2p+2\beta+2)}{(p+1)(\beta+2)} \end{pmatrix} \begin{pmatrix} W \\ \dot{W} \end{pmatrix},$$

and the eigenvalues $\mu_{\pm}$ of the coefficient matrix

$$\begin{pmatrix} 0 & 1 \\ -\frac{4(np+n-2p+\beta)}{(p+1)(\beta+2)} & -\frac{2(np+n-2p+2\beta+2)}{(p+1)(\beta+2)} \end{pmatrix},$$

are given as

$$\mu_{\pm} = \frac{-(np+n-2p+2\beta+2) \pm \sqrt{f(p, n, \beta)}}{(p+1)(\beta+2)}$$

where

$$f(p, n, \beta) = (p+1)^2 n^2 - 4(p+1)(\beta p + 3p + 1)n - 4\{\beta^2 p + 2p\beta(1-p) - (5p^2 + 2p + 1)\}.$$

Since $n \geq 2$, we obtain $np + n - 2p + 2\beta + 2 > 0$. Thus, $(w, \dot{w}) = (1, 0)$ is an attractor.

The curve $\dot{w} = \varphi(w)$ defined for

$$\varphi(w) = -a \left(w - \frac{1}{w^p} \right)$$

$$a = \frac{(p+1)(\beta+2)}{np+n-2p+2\beta+2} \cdot \frac{2(np+n-2p+\beta)}{(p+1)^2(\beta+2)} > 0$$

passes through $(1, 0)$ and the right-half plane $\{(w, \dot{w}) \mid w > 0\}$ of the phase space is separated into four regions:

$$\begin{aligned} I &= \{(w, \dot{w}) \mid w > 0, \dot{w} > 0, \dot{w} > \varphi(w)\}, \\ II &= \{(w, \dot{w}) \mid w > 0, \dot{w} < 0, \dot{w} > \varphi(w)\}, \\ III &= \{(w, \dot{w}) \mid w > 0, \dot{w} < 0, \dot{w} < \varphi(w)\}, \\ IV &= \{(w, \dot{w}) \mid w > 0, \dot{w} > 0, \dot{w} < \varphi(w)\}. \end{aligned}$$

According to these cases, the directions of the orbits $\mathcal{O}$ are as follows:

$$\begin{aligned} (w, \dot{w}) \in \mathcal{O} \cap I &\Rightarrow \frac{dw}{dt} > 0, \frac{d\dot{w}}{dt} < 0, \\ (w, \dot{w}) \in \mathcal{O} \cap II &\Rightarrow \frac{dw}{dt} < 0, \frac{d\dot{w}}{dt} < 0, \\ (w, \dot{w}) \in \mathcal{O} \cap III &\Rightarrow \frac{dw}{dt} < 0, \frac{d\dot{w}}{dt} > 0, \\ (w, \dot{w}) \in \mathcal{O} \cap IV &\Rightarrow \frac{dw}{dt} > 0, \frac{d\dot{w}}{dt} > 0. \end{aligned}$$

If $(w, \dot{w}) \in \mathcal{O}$ approaches $w = 0$, then $\frac{d\dot{w}}{dt} \rightarrow +\infty$, and therefore it enters region *IV*. Thus, $\mathcal{O}$ in the right-half plane does not touch $w = 0$ in finite time. If $(w, \dot{w}) \in \mathcal{O} \cap I$ with $w \gg 1$, then it must eventually come into region *II* because $\frac{dw}{dt} = \dot{w} > 0$ and $\frac{d\dot{w}}{dt} = \gamma(-\dot{w} + \varphi(w)) \sim \gamma(-\dot{w} - aw) < 0$, where $\gamma = 2(np+n-2p+2\beta+2)/(p+1)(\beta+2)$. From those considerations, we see that the orbit $\mathcal{O}$ is compact in the right-half space, and eventually approaches the singular point $(w, \dot{w}) = (1, 0)$ as $t \rightarrow +\infty$.

The following elementary lemma controls the (local) behaviour of the orbit $\mathcal{O}$ around the singular-fixed point $(w, \dot{w}) = (1, 0)$.

Lemma 3.2. *We have the following.*

1. If $n \in [2, 6]$, then $f(p, n, \beta) < 0$ for any $p > 1$ and $\beta > 0$.
2. If $n \in [7, 9]$, $f(p, n, \beta) \geq 0$ for $p \in (1, \bar{p}]$ and $\beta \in (0, \bar{\beta}]$, and $f(p, n, \beta) < 0$ otherwise, i.e. for $p > \bar{p}$ and $\beta > 0$ or for $p \in (1, \bar{p})$ and $\beta > \bar{\beta}$, whenever $\bar{\beta} > 0$.
3. If $n \geq 10$, $f(p, n, \beta) \geq 0$ for $p > 1$ and $\beta \in (0, \bar{\beta}]$, and $f(p, n, \beta) < 0$ otherwise.

Proof. We have

$$f(p, n, \beta) = -4p \left(\beta - \frac{2(p-1) - (p+1)n}{2} \right)^2 + (p+1)^3 (n-2)^2.$$

Since

$$2(p-1) - (p+1)n = -(p+1)(n-2) - 4 < 0,$$

$f(p, n, \beta)$ achieves the maximum at $\beta = 0$ for fixed $p > 1$ and $n \geq 2$ with the value

$$\begin{aligned} f(p, n, 0) &= (p+1)^2 n^2 - 4(p+1)(3p+1)n + 4(5p^2 + 2p + 1) \\ &= (n-2)(n-10)p^2 + 2(n^2 - 8n + 4)p + (n-2)^2. \end{aligned}$$

By a simple computation, we have

$$f(p, n, 0) < 0 \quad \text{for } n = 2, \dots, 6 \text{ and } p > 1,$$

and

$$\begin{aligned} f(p, n, 0) &\geq 0 && \text{for } p \in (1, \bar{p}] \text{ and } n = 7, 8, 9 \\ f(p, n, 0) &< 0 && \text{for } p > \bar{p} \text{ and } n = 7, 8, 9. \end{aligned}$$

Then, the first and second cases of the lemma are obtained by the definition of $\bar{\beta}$. Since

$$f(1, n, 0) = 4(n^2 - 8n + 8),$$

and $f(p, n, 0)$ is increasing with respect to $p > 0$ for $n \geq 10$, we obtain $f(p, n, \beta) \geq 0$ for $p \geq 1$, $n \geq 10$, and $\beta \in (0, \bar{\beta}]$, while $f(p, n, \beta) < 0$ for $n \geq 10$, $p > 1$ and $\beta > \bar{\beta}$. This completes the proof of the lemma. $\square$

Lemma 3.2 guarantees that the singular point $(1, 0)$ is a spiral attractor if and only if

$$\begin{aligned} n &\in [2, 6], \quad p > 1, \quad \beta > 0, \\ n &\in [7, 9], \quad p > \bar{p}, \quad \beta > 0, \quad \text{or } p \in (1, \bar{p}), \quad \beta > \bar{\beta} \\ n &\geq 10, \quad p > 1, \quad \beta > \bar{\beta}, \end{aligned}$$

while in the secondary case, i.e. when

$$\begin{aligned} n &\in [7, 9], \quad p \in (1, \bar{p}], \quad \beta \in (0, \bar{\beta}], \\ n &\geq 10, \quad p > 1, \quad \beta \in (0, \bar{\beta}] \end{aligned}$$

the orbit $\mathcal{O}$ is absorbed into $(1, 0)$ exponentially.

Through the above transformation, the boundary condition $u(1) = 0$ in (3.1) corresponds to

$$w(\tau) = \left\{ \frac{[\beta + 2 + (n-2)(p+1)](\beta+2)}{\lambda(p+1)^2} \right\}^{\frac{1}{p+1}}$$

$$\tau = -\frac{1}{2} \log \frac{\{\beta + 2 + (n-2)(p+1)\}(1-A)^{p+1}}{\lambda(p+1)^2} = \log \frac{1}{k}.$$

In other words, for any $\tau \in \mathbb{R}$, (λ_τ, u_τ) defined by

$$u_\tau(r) = 1 - \frac{w(\tau + \frac{\beta+2}{2} \log r)}{w(\tau)} r^{\frac{\beta+2}{p+1}}$$

$$\lambda_\tau = \frac{\{\beta + 2 + (n-2)(p+1)\}(\beta+2)}{(p+1)^2 w(\tau)^{p+1}}$$

$$A_\tau = 1 - \frac{1}{e^{\frac{2}{p+1}\tau} w(\tau)} \quad (3.6)$$

satisfies (3.1), and conversely, every solution of (3.1) is written in the form of (3.6). Hence $\mathcal{C}_r$ is homeomorphic to $\mathcal{O}$. Since the singular point $(w, \dot{w}) = (1, 0)$ corresponds to $(\lambda, u) = (\lambda^*, 1 - |x|^{\frac{\beta+2}{p+1}})$, we obtain the conclusion of Theorem 1.2.

4. Proof of Theorem 1.3

Theorem 1.3 is proven by the method of [19]. Each $(\lambda, u) \in \mathcal{C}_r$ corresponds to $(w(\tau), \dot{w}(\tau)) \in \mathcal{O}$ and therefore, it is parametrized by $\tau \in \mathbb{R}$: $(\lambda, u) = (\lambda(\tau), u(\tau))$. We denote by μ_τ^ℓ , $\ell = 1, 2, \dots$, the ℓ -th eigenvalue of the linearized problem (1.4) corresponding to a radially symmetric eigenfunction, i.e. a function satisfying

$$-\phi_{rr} - \frac{n-1}{r} \phi_r - \frac{p\lambda r^\beta}{(1-u)^{p+1}} \phi = \mu \phi \quad (0 < r < 1)$$

$$\phi_r(0) = \phi(1) = 0. \quad (4.1)$$

Each μ_τ^ℓ is simple. If $(\lambda(\tau), u(\tau))$ is on the turning point of $\mathcal{C}_r$, then, by the implicit function theorem, there is $\ell \geq 1$ such that $\mu_\tau^\ell = 0$. If $\mu_\tau^\ell = 0$ holds for some $\ell \geq 1$ with $(\lambda(\tau), u(\tau)) \in \mathcal{C}_r$ not on the turning point, then, by the bifurcation theory on the critical point of odd multiplicity, [23, 24], it is actually the bifurcation point of $\mathcal{C}_r$. But, this is impossible by Theorem 1.2, and therefore, $(\lambda(\tau), u(\tau))$ is on the turning point of $\mathcal{C}_r$ if and only if (4.1) has the eigenvalue 0. Denoting the turning points of $\mathcal{C}_r$ by $T_k = (\lambda(\tau_k), u(\tau_k))$ ($\tau_1 < \tau_2 < \dots$), we only have to show $\dot{\mu}_{\tau=\tau_k}^{\ell(k)} < 0$ for $k \geq 1$ and $\ell = \ell(k)$ such that $\mu_{\tau_k}^{\ell(k)} = 0$, to prove Theorem 1.3.

Using (3.6), we have

$$\begin{aligned}\dot{\lambda}(\tau) &= -\frac{\{\beta + 2 + (n-2)(p+1)\}(\beta+2)}{p+1} \frac{\dot{w}(\tau)}{w(\tau)^{p+2}} \\ \ddot{\lambda}(\tau) &= -\frac{\{\beta + 2 + (n-2)(p+1)\}(\beta+2)}{p+1} \frac{\ddot{w}(\tau)w(\tau) - (p+2)\dot{w}(\tau)^2}{w(\tau)^{p+3}},\end{aligned}\quad (4.2)$$

where here and above $\dot{} = \frac{d}{d\tau}$, stands for the differentiation with respect to τ . On the other hand, from (1.1) it holds that

$$\begin{aligned}\dot{u}_{rr} + \frac{n-1}{r}\dot{u}_r + \frac{\dot{\lambda}r^\beta}{(1-u)^p} + \frac{p\lambda r^\beta}{(1-u)^{p+1}}\dot{u} &= 0, \quad (0 < r < 1) \\ \dot{u}_r(0) = \dot{u}(1) &= 0.\end{aligned}\quad (4.3)$$

Due to (3.6) we have

$$\dot{u}(r, \tau) = \frac{\dot{w}(\tau)w(\tau + \frac{\beta+2}{2}\log r) - \dot{w}(\tau + \frac{\beta+2}{2}\log r)w(\tau)}{w(\tau)^2} r^{\frac{\beta+2}{p+1}}$$

and since $\mathcal{O}$ does not contain a segment we derive that $\dot{u}(r, \tau) \not\equiv 0$. Moreover, since $\dot{\lambda}(\tau_k) = 0$ in (4.3), we can take $\phi(\cdot, \tau_k) = \dot{u}(\cdot, \tau_k)$. Thus, the perturbation theory, see [14], guarantees the existence of $\phi = \phi(\cdot, \tau)$ and $\mu = \mu(\tau)$ smooth in τ for $|\tau - \tau_k| \ll 1$ satisfying (4.1), $\phi(\cdot, \tau_k) = \dot{u}(\cdot, \tau_k)$, and $\mu(\tau_k) = \mu_{\tau_k}^{\ell(k)} = 0$ since $\mu_{\tau_k}^{\ell(k)}$ is simple.

Differentiating (4.3) and (4.1) with respect to τ , we have

$$\begin{aligned}\Delta_r \ddot{u} + \frac{\ddot{\lambda}r^\beta}{(1-u)^p} + \frac{p\dot{\lambda}r^\beta}{(1-u)^{p+1}}\dot{u} + \frac{p\dot{\lambda}r^\beta}{(1-u)^{p+1}}\dot{u} + \frac{p(p+1)\lambda r^\beta}{(1-u)^{p+2}}\dot{u}^2 \\ + \frac{p\lambda r^\beta}{(1-u)^{p+1}}\ddot{u} = 0 \quad \text{in } \Omega, \quad \ddot{u} = 0 \quad \text{on } \partial\Omega,\end{aligned}\quad (4.4)$$

and

$$\begin{aligned}-\Delta_r \dot{\phi} - \frac{p\dot{\lambda}r^\beta}{(1-u)^{p+1}}\dot{\phi} - \frac{p(p+1)\lambda r^\beta}{(1-u)^{p+2}}\dot{u}\dot{\phi} - \frac{p\lambda r^\beta}{(1-u)^{p+1}}\dot{\phi} \\ = \dot{\mu}\phi + \mu\dot{\phi} \quad \text{in } \Omega, \quad \dot{\phi} = 0 \quad \text{on } \partial\Omega,\end{aligned}\quad (4.5)$$

where $\Delta_r = \partial^2/\partial r^2 + \frac{n-1}{r}\partial/\partial r$.

Multiplying (4.4) and (4.5) by $\dot{u}$, integrating over Ω , and subtracting each other, we obtain

$$\ddot{\lambda}(\tau_k) \int_{\Omega} \frac{r^\beta}{(1-u_k)^p} \phi(\tau_k) dx = \dot{\mu}(\tau_k) \int_{\Omega} \phi(\tau_k)^2 dx$$

in the bending point $\tau = \tau_k$, where $u_k = u(\cdot, \tau_k)$.

Let

$$I_k = \int_{\Omega} \frac{r^\beta}{(1-u_k)^p} \phi(\tau_k) dx.$$

The following lemma implies

$$\begin{aligned} \dot{\mu}(\tau_k) \int_{\Omega} \phi(\tau_k)^2 dx &= \ddot{\lambda}(\tau_k) I_k \\ &= - \frac{\{\beta + 2 + (n-2)(p+1)\} (\beta+2)^2 \omega_n}{2(p+1)^2 \lambda_k} \frac{\ddot{w}(\tau_k)^2}{w(\tau_k)^{p+3}} \end{aligned}$$

by (4.2), where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$, and hence $\dot{\mu}(\tau_k) < 0$.

Lemma 4.1. *It holds that*

$$I_k = \frac{(\beta+2) \omega_n}{2(p+1) \lambda_k} \frac{\ddot{w}(\tau_k)}{w(\tau_k)},$$

where $\lambda_k = \lambda(\tau_k)$.

Proof. We have

$$\begin{aligned} \lambda(p-1) \int_{\Omega} \frac{r^\beta}{(1-u)^p} \dot{u} dx + \dot{\lambda} \int_{\Omega} \frac{r^\beta}{(1-u)^{p-1}} dx &= \frac{d}{d\tau} \left(\int_{\Omega} \frac{\lambda r^\beta}{(1-u)^{p-1}} dx \right) \\ &= \frac{d}{d\tau} \int_{\Omega} (\Delta_r u)(u-1) dx = - \int_{\Omega} \Delta_r \dot{u} dx + \frac{d}{d\tau} \int_{\Omega} u \Delta_r u dx \\ &= - \int_{\Omega} \Delta_r \dot{u} dx - \frac{d}{d\tau} \lambda \int_{\Omega} \frac{r^\beta}{(1-u)^p} u dx = - \int_{\Omega} \Delta_r \dot{u} dx - \dot{\lambda} \int_{\Omega} \frac{r^\beta}{(1-u)^p} u dx \\ &\quad - \lambda \int_{\Omega} \frac{r^\beta}{(1-u)^p} \dot{u} dx - p \lambda \int_{\Omega} \frac{r^\beta}{(1-u)^{p+1}} u \dot{u} dx \end{aligned}$$

and therefore,

$$(p-1) \lambda_k I_k = - \int_{\Omega} \Delta_r \dot{u}_k dx - \lambda_k I_k - p \lambda_k \int_{\Omega} \frac{r^\beta}{(1-u_k)^{p+1}} u_k \dot{u}_k dx,$$

by putting $\tau = \tau_k$, where $\dot{u}_k = \dot{u}(\cdot, \tau_k)$. This means

$$p \lambda_k I_k = - \int_{\Omega} \Delta_r \dot{u}_k dx - p \lambda_k \int_{\Omega} \frac{r^\beta}{(1-u_k)^{p+1}} u_k \dot{u}_k dx. \quad (4.6)$$

Since $\dot{u}_k = \dot{u}(r, \tau_k)$, we have

$$\int_{\Omega} \Delta_r \dot{u}_k dx = \omega_n (\dot{u}_k)_r (1),$$

while

$$u(r, \tau) = 1 - \frac{w(\tau + \frac{\beta+2}{2} \log r)}{w(\tau)} r^{\frac{\beta+2}{p+1}}$$

implies

$$\begin{aligned} \dot{u}_r(r, \tau) = & -\frac{\beta+2}{p+1} \frac{w(\tau) \dot{w}(\tau + \frac{\beta+2}{2} \log r) - \dot{w}(\tau) w(\tau + \frac{\beta+2}{2} \log r)}{w(\tau)^2} r^{\frac{\beta+2}{p+1}-1} \\ & - \frac{\beta+2}{2} \frac{w(\tau) \ddot{w}(\tau + \frac{\beta+2}{2} \log r) - \dot{w}(\tau) \dot{w}(\tau + \frac{\beta+2}{2} \log r)}{w(\tau)^2} r^{\frac{\beta+2}{p+1}-1}. \end{aligned}$$

Here, we have $\dot{w}(\tau_k) = 0$ by (4.2) and $\dot{\lambda}(\tau_k) = 0$ and hence

$$\int_{\Omega} \Delta_r \dot{u}_k dx = -\frac{(\beta+2) \omega_n}{2} \frac{\ddot{w}(\tau_k)}{w(\tau_k)}. \quad (4.7)$$

Multiplying (4.1) by u and integrating over Ω , we obtain

$$-\int_{\Omega} u \Delta_r \phi dx - \int_{\Omega} \frac{p \lambda r^{\beta}}{(1-u)^{p+1}} \phi u dx = \mu \int_{\Omega} \phi u dx$$

and hence

$$\begin{aligned} \int_{\Omega} \frac{p \lambda r^{\beta}}{(1-u)^{p+1}} \phi u dx &= -\mu \int_{\Omega} \phi u dx - \int_{\Omega} \Delta_r u \phi dx \\ &= -\mu \int_{\Omega} \phi u dx + \lambda \int_{\Omega} \frac{r^{\beta}}{(1-u)^p} \phi dx. \end{aligned}$$

Putting $\tau = \tau_k$, we have

$$\int_{\Omega} \frac{p \lambda_k r^{\beta}}{(1-u_k)^{p+1}} \dot{u}_k u_k dx = \lambda_k I_k. \quad (4.8)$$

Hence substituting (4.7) and (4.8) into (4.6), we finally derive

$$I_k = \frac{(\beta+2) \omega_n}{2(p+1) \lambda_k} \frac{\ddot{w}(\tau_k)}{w(\tau_k)}$$

and the proof is complete. $\square$

5. Connecting orbits

Problem (1.1) generates a local semi-flow in $X = W_0^{1,q}(\Omega)$ for $q > n$, and each $\varphi \in \mathcal{C}^\lambda$ is regarded as an equilibrium. It is said to be hyperbolic if (1.4) does not admit $\mu = 0$ as an eigenvalue. Given $\varphi, \psi \in \mathcal{C}^\lambda$, we define the stable manifold $W^s(\varphi)$ of φ and the unstable manifold $W^u(\psi)$ of ψ by

$$\begin{aligned} W^s(\varphi) &= \{\xi \in X \mid \text{there exists an orbit } u : [0, +\infty) \rightarrow X \text{ to (1.1)} \\ &\quad \text{such that } u(0) = \xi \text{ and } u(t) \rightarrow \varphi \text{ in } X \text{ as } t \rightarrow +\infty\}, \\ W^u(\psi) &= \{\xi \in X \mid \text{there exists an orbit } u : (-\infty, 0] \rightarrow X \text{ to (1.1)} \\ &\quad \text{such that } u(0) = \xi \text{ and } u(t) \rightarrow \psi \text{ in } X \text{ as } t \rightarrow -\infty\}. \end{aligned}$$

If there is a solution $u = u(\cdot, t)$ to (1.1) for $t \in \mathbb{R}$ satisfying $\lim_{t \rightarrow -\infty} u(t) = \psi$ and $\lim_{t \rightarrow +\infty} u(t) = \varphi$ in X , we call $\{u(\cdot, t) \mid t \in \mathbb{R}\}$ a connecting orbit between φ and ψ .

In the case of (1.3), we can define radially symmetric versions of the above notions. Here, we just note that $\varphi \in \mathcal{C}_r^\lambda$ is called radially hyperbolic if (4.1) does not admit $\mu = 0$ as an eigenvalue, and that radial stable and unstable manifolds of $\varphi \in \mathcal{C}_r^\lambda$ are denoted by $W_r^s(\varphi)$ and $W_r^u(\psi)$, respectively. Then, we obtain the Morse-Smale property of radially symmetric orbits [1, 6, 22].

Theorem 5.1. *Let $\varphi, \psi \in \mathcal{C}_r^\lambda$ be radially hyperbolic and assume the existence of a radial connecting orbit $\{u(\cdot, t) \mid t \in \mathbb{R}\}$ between φ and ψ . Then $W_r^s(\varphi)$ and $W_r^u(\psi)$ intersect transversally in $X_r = \{v \in X \mid v = v(|x|)\}$, and*

$$T_{u_0} W_r^s(\varphi) + T_{u_0} W_r^u(\psi) = X_r$$

for $u_0 = u(\cdot, 0)$, where $T_{u_0} W$ denotes the tangential space of W at u_0 . It also holds that $i_R(\lambda, \varphi) < i_R(\lambda, \psi)$.

To confirm this, we denote by μ_k the k -th eigenvalue of the linearized problem (4.1) around $\psi \in \mathcal{C}_r^\lambda$. For given $0 < k < i_R(\lambda, \psi)$ and $\mu_k < \mu < \min\{0, \mu_{k+1}\}$, we consider a finer invariant manifold than the unstable manifold as

$$\begin{aligned} W_{r,k}^u(\psi) &= \{u_0 \in W_r^u(\psi) \mid \lim_{t \rightarrow -\infty} e^{-\mu t} \|u(\cdot, t) - \psi\|_{X_r} = 0, \\ &\quad \text{where } u(\cdot, t) \text{ } (t \leq 0) \text{ is a solution to (1.1) satisfying } u(\cdot, 0) = u_0\}. \end{aligned}$$

Theorem 5.1 is obtained by the discrete Lyapunov function. To state this, for given $\Psi \in C[0, 1]$, we denote by $\ell(\Psi)$ the supremum of k such that there is $0 \leq r_1 < r_2 < \cdots < r_k \leq 1$ satisfying $\Psi(r_j)\Psi(r_{j+1}) < 0$ for $j = 1, 2, \dots, k$. Then, Lemma 2.9 of [6] is stated as follows.

Lemma 5.2. *Given radially hyperbolic $\varphi, \psi \in \mathcal{C}_r^\lambda$, we obtain the following.*

1. If $u_0, u_1 \in W_{r,k}^u(\psi)$, $w \in T_{u_0}W_{r,k}^u(\psi)$, $u_0 \neq u_1$, and $w \neq 0$, then it holds that $\ell(u_0 - u_1) \leq k$ and $\ell(w) \leq k$.
2. If $u_0, u_1 \in W_r^s(\varphi)$, $w \in T_{u_0}W_r^s(\varphi)$, $u_0 \neq u_1$, and $w \neq 0$, then it holds that $\ell(u_0 - u_1) \geq i_R(\lambda, \varphi) + 1$ and $\ell(w) \geq i_R(\lambda, \varphi) + 1$.

Proof of Theorem 5.1. The proof is similar to that of Lemma 3.1 in [6]. First, we note

$$u_0 \in W_r^s(\varphi) \cap W_r^u(\psi) \neq \emptyset$$

and hence $i_R(\lambda, \varphi) < i_R(\lambda, \psi)$ by Lemma 5.2. This implies

$$W_{r, i_R(\lambda, \varphi)}^u(\psi) \subset W_{r, i_R(\lambda, \psi)}^u(\psi),$$

where the dimension of $T_{u_0}W_r^u(\varphi)$ and the codimension of $T_{u_0}W_r^s(\varphi)$ are equal to $i_R(\lambda, \varphi)$. Therefore it is sufficient to show

$$T_{u_0}W_{r, i_R(\lambda, \varphi)}^u(\psi) \cap T_{u_0}W_r^s(\varphi) = \{0\},$$

but this is an immediate consequence of Lemma 5.2, hence

$$T_{u_0}W_r^s(\varphi) + T_{u_0}W_r^u(\psi) = X_r.$$

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