$C^{0,\gamma}$ -Regularity for Vector-Valued Minimizers of Quasilinear Functionals

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Abstract. We discuss the interior $C^{0,\gamma}$ -everywhere regularity for minimizers of quasilinear functionals of the type

$$\mathcal{A}(u; \Omega) = \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j dx$$

where VMO dependence on the variable x and continuous dependence on the variable u are supposed.

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1. Introduction

The aim of this paper is to study the interior everywhere regularity of functions minimizing variational integrals

$$\mathcal{A}(u; \Omega) = \int_{\Omega} A_{ij}^{\alpha\beta}(x, u) D_{\alpha} u^i D_{\beta} u^j dx \quad (1)$$

where $u : \Omega \rightarrow \mathbb{R}^N$, $N > 1$, $\Omega \subset \mathbb{R}^n$, $n \geq 3$ is a bounded open set, $x = (x_1, \dots, x_n) \in \Omega$, $u(x) = (u^1(x), \dots, u^N(x))$, $Du = \{D_{\alpha} u^i\}$, $D_{\alpha} = \partial/\partial x_{\alpha}$, $\alpha = 1, \dots, n$, $i = 1, \dots, N$.

Throughout the whole text we use the summation convention over repeated indices. We call a function $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ a minimizer of the functional $\mathcal{A}(u; \Omega)$ if and only if $\mathcal{A}(u; \Omega) \leq \mathcal{A}(v; \Omega)$ for every $v \in W^{1,2}(\Omega, \mathbb{R}^N)$ with $u-v \in W_0^{1,2}(\Omega, \mathbb{R}^N)$. For more information see [8, 11].

On the functional $\mathcal{A}$ we assume the following conditions:

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- (i) $A_{ij}^{\alpha\beta} = A_{ji}^{\beta\alpha}$, $A_{ij}^{\alpha\beta}$ are continuous functions in $u \in \mathbb{R}^N$ for every $x \in \Omega$ and there exists $M > 0$ such that $|A_{ij}^{\alpha\beta}(x, u)| \leq M$, $\forall x \in \Omega$, $\forall u \in \mathbb{R}^N$.
- (ii) (ellipticity) There exists $\nu > 0$ such that

$$A_{ij}^{\alpha\beta}(x, u) \xi_\alpha^i \xi_\beta^j \geq \nu |\xi|^2, \quad \forall x \in \Omega, \quad \forall u \in \mathbb{R}^N, \quad \forall \xi \in \mathbb{R}^{nN}. \quad (2)$$

- (iii) (oscillation of coefficients) There exists a real function ω continuous on $[0, \infty)$, which is bounded, nondecreasing, concave, $\omega(0) = 0$ and such that for all $x \in \Omega$ and $u, v \in \mathbb{R}^N$

$$|A_{ij}^{\alpha\beta}(x, u) - A_{ij}^{\alpha\beta}(x, v)| \leq \omega(|u - v|). \quad (3)$$

We set $\omega_\infty = \lim_{t \rightarrow \infty} \omega(t) \leq 2M$.

- (iv) For all $u \in \mathbb{R}^N$, $A_{ij}^{\alpha\beta}(\cdot, u) \in VMO(\Omega)$ (uniformly with respect to $u \in \mathbb{R}^N$).

Here it is worth to recall that since C^0 is a proper subset of VMO , the continuity of coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x, u)$ with respect to x is not supposed. In the scalar case ($N = 1$) the continuity of the coefficients in x is not so crucial for regularity of minimizers as can be seen in [6, 16] where the Hölder continuity of weak solutions to uniformly elliptic linear equations with L^∞ coefficients is proved. Even for some nonlinear scalar problems the situation is similar (see [9]). Thus the Hölder continuity of minimizers of the functional $\mathcal{A}$ is possible even without the VMO condition.

The situation is completely different in the vector-valued case ($N > 1$). In the linear case, when the coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x)$ belong to $C^{0,\gamma}(\bar{\Omega})$ the regularity of minimizers functionals as (1) is well understood (see [8, Th. 3.1, Th. 3.2, p. 87, 88]). These results were later generalized to the case where the above coefficients are in VMO , hence possibly discontinuous (see [1, 4, 13]).

It is well known that even in the continuous case the dependence of coefficients $A_{ij}^{\alpha\beta}$ on u leads to weaker regularity results for minimizers. There are examples of vectorial quadratic functionals ($N > 1$) with analytic coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(u)$ whose minimizers are discontinuous in dimension $n \geq 3$ (see [11, p. 317], [12]). For the analytic coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x, u)$ see counterexample in [17] where the weak solutions to quasilinear system are unbounded.

In the above mentioned examples the singularity of minimizers occur by the peculiar way the discontinuous coefficients (x, u) mix-up with the components of the gradient variable Du . In this place it is worth to mention recent work [20] (and references therein) where the authors have shown that minimizers of a simple analytic functional may be even unbounded. This example is dimensionally optimal due to Hölder continuity of minimizers of such type of nonlinear functionals up to dimension $n = 4$. These examples imply that, in general, we cannot expect (without some additional structural assumptions) everywhere regularity of vector-valued minimizers even if the coefficients are regular. For such reason partial regularity properties for minimizers of vectorial functionals have been studied by many authors (see [7, 8, 11, 14, 18] and references therein). Here “partial regularity”

means that the minimizer u of functional $\mathcal{A}$ is regular in some specified sense, in an open subset $\Omega_u \subset \Omega$ such that $|\Omega \setminus \Omega_u| = 0$. In some special cases the Hausdorff dimension of the so-called singular set $\Omega \setminus \Omega_u$ can be estimated. Finally we recall the recent papers [7] and [18] (this paper extends [4] and [7]) where the authors obtained partial regularity results for minimizers of vectorial quasilinear (in the second paper with added nonlinear part which has some sublinear growth) functionals such as in (1) where the coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x, u)$ satisfy well known that VMO dependence on the variable x and the continuous one on u .

More precisely it follows from the last mentioned papers that there exist an open set $\Omega_u \subset \Omega$, such that $Du \in L_{loc}^{2,\lambda}(\Omega_u, \mathbb{R}^{nN})$, $\lambda < n$ and the Hausdorff dimension $\dim_H(\Omega \setminus \Omega_u) < n - 2$ (this fact implies Hölder continuity of minimizer u on Ω_u). Here $\Omega \setminus \Omega_u = \{x \in \Omega : \liminf_{R \rightarrow 0^+} R^{2-n} \int_{B_R(x)} |Du(y)|^2 dy > 0\}$.

It is obvious from the last mentioned results that the continuous dependence $A_{ij}^{\alpha\beta}$ on u is necessary for partial regularity of minimizer u but it doesn't suffice for everywhere regularity result (i.e. when the subset $\Omega \setminus \Omega_u$ is empty). If we want the everywhere regularity result for minimizer we will have to state some additional structural assumptions on the above mentioned continuous dependence.

In our paper we give some conditions formulated in terms of the modulus of continuity (with respect to u) of coefficients $A_{ij}^{\alpha\beta} = A_{ij}^{\alpha\beta}(x, u)$ that guarantee local Hölder continuity of minimizers u on Ω . The following results are motivated by the paper [5] where the analogous questions are studied for gradient of weak solution of the nonlinear elliptic system $-D_\alpha(A_i^\alpha(Du)) = 0$, with continuously differentiable coefficients A_i^α . A method of proof which is used in this paper is different from that one in [5]. We directly work with the functional $\mathcal{A}$ instead with its Euler equation and so we will not suppose any differentiability. In this paper Hölder continuity of the minimizers u of the functionals (1) is obtained as a consequence of the fact that the gradients of u belong to some Morrey spaces. Morrey space estimates for vectorial problems different from this one in our paper could be found in [3] and [14] as well.

About assumption (iii) we point out that it is well known (see [8, p. 169]) that for uniformly continuous (in u) coefficients $A_{ij}^{\alpha\beta}$ there exists a real function ω satisfying the assumption (iii) and, viceversa, (iii) implies uniform continuity of coefficients and absolute continuity of ω on $[0, \infty)$. Now we can formulate the result and the main structural assumptions about A . In what follows we will understand by pointwise derivative ω' of ω the right derivative which is finite on $(0, \infty)$.

Theorem 1.1. *Let $\delta \in [n - 2, n)$, $P > 1$ and $\overline{\Omega}_0 \subset \Omega$ with $\text{dist}(\Omega_0, \partial\Omega) \geq 2d > 0$. Let u be a minimizer of (1) and $g \in W^{1,2}(\Omega, \mathbb{R}^N)$ such that $u - g \in W_0^{1,2}(\Omega, \mathbb{R}^N)$.*

- (a) *Let assumptions (i), (ii), (iii), (iv) be satisfied with the constants M, ν such that $M/\nu \leq P$ and suppose that*

$$S_{p'} = \sup_{t \in (0, \infty)} \frac{d}{dt}(\omega^{p'})(t) < \infty$$

for $p' > 1$ ($p' = p/(p-1)$, p is from Lemma 2.7), then there exists a constant $\nu_0 = \nu_0(n, \delta, P, S_{p'}, d, \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}) > 0$ such that if $\nu \geq \nu_0$ then $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

- (b) Let assumptions (i), (ii), (iii), (iv) be satisfied with the constants M, ν such that $M/\nu \leq P$ and suppose that

$$S_0 = \sup_{t > t_0} \frac{d}{dt} \omega(t) < \infty$$

where $t_0 = \inf \omega^{-1}(\frac{\omega_\infty}{2KP} \epsilon_0)$, $K = K(n, p, N, P)$ and $\epsilon_0 = \frac{1}{4(2^{n+3}L)^{\frac{\delta}{n-\delta}}}(L = L(n, P))$, then there exists a constant $\nu_0 = \nu_0(n, p, \delta, N, P, S_0, d, \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}) > 0$ such that if $\nu \geq \nu_0$ then $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

- (c) If, in addition to assumptions in (b), we suppose that $\omega_\infty > 1$, $e^{\omega/\sqrt{\omega_\infty}}$ is concave on $[0, \infty)$ and $t_0 = \inf \omega^{-1}(\sqrt{\omega_\infty})$, then there exists a constant $\nu_0 = \nu_0(n, \delta, N, P, \omega'(t_0), d, \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}) > 0$ such that if $\nu \geq \nu_0$ then $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

Corollary 1.2. Let assumptions of Theorem 1.1 be satisfied and $\delta \in [n-2, n)$. Then

$$u \in \begin{cases} C^{0,(\delta-n+2)/2}(\overline{\Omega}_0, \mathbb{R}^N) & \text{if } n-2 < \delta < n \\ BMO(\Omega_0, \mathbb{R}^N) & \text{if } \delta = n-2. \end{cases}$$

Proof. The assertion follows from Theorem 1.1 and Proposition 2.4 (c) (see Preliminaries). $\square$

Remark 1.1.

- (a) It is clear that for u and g from the above theorem the following estimate holds

$$\|Du\|_{L^2(\Omega, \mathbb{R}^{nN})} \leq c(n, N, M/\nu) \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}.$$

- (b) As a typical example of modulus of continuity which satisfies the assumptions of Theorem 1.1(a) we can consider (see [5])

$$\omega(t) = \begin{cases} ct^\alpha & \text{for } 0 \leq t < t_1, \\ \omega_\infty & \text{for } t \geq t_1 \end{cases}$$

where $c > 0$, $\alpha \in (0, 1)$, $t_1 = (\omega_\infty/c)^{1/\alpha}$. Such a function satisfies the assumptions of Theorem 1.1(a) for $p' = 1/\alpha$.

- (c) A typical modulus of continuity which satisfies the assumptions of Theorem 1.1(b) and does not satisfy the assumptions of Theorem 1.1(a) is the modulus of continuity

$$\omega(t) = \begin{cases} \frac{c_1}{(1+\ln(\frac{c_2}{t}))^\beta} & \text{for } 0 < t < \frac{c_2}{e^\beta}, \\ \tilde{\omega}(t) & \text{for } t \geq \frac{c_2}{e^\beta} \end{cases}$$

where $0 < c_1 \leq \omega_\infty$, $c_2, \beta > 0$, $\tilde{\omega}(c_2/e^\beta) = c_1/(1+\beta)^\beta$ and $\tilde{\omega}$ is such that ω satisfies the assumption (iii).

- (d) A typical modulus of continuity which satisfies the assumptions of Theorem 1.1(c) is the modulus of continuity

$$\omega(t) = \begin{cases} c \ln(1 + \vartheta t) & \text{for } 0 < t \leq \frac{e^{\omega_\infty/c} - 1}{\vartheta}, \\ \omega_\infty & \text{for } t > \frac{e^{\omega_\infty/c} - 1}{\vartheta} \end{cases},$$

where $0 < c \leq \sqrt{\omega_\infty}$, $\vartheta > 0$.

The following example illustrates Theorem 1.1(a) and Theorem 1.1(b) as well.

Example 1.1. In $\Omega = B_1(0) \subset \mathbb{R}^3$ we consider quasilinear variational integral

$$\mathcal{A}(u; \Omega) = \int_{\Omega} A_{ij}^{\alpha\beta}(u) D_\alpha u^i D_\beta u^j dx$$

where

$$A_{ij}^{\alpha\beta}(u) = a \delta_{ij} \delta_{\alpha\beta} + b (\delta_{i\alpha} \arctan(\mu |u^i|^s) + \delta_{j\beta} \arctan(\mu |u^j|^s))$$

for $i, j = 1, 2, 3$, $\alpha, \beta = 1, 2, 3$, $u - g \in W_0^{1,2}(B_1(0), \mathbb{R}^3)$, $g \in W^{1,2}(B_1(0), \mathbb{R}^3)$, $0 < s \leq 1$, $a, b > 0$, $a > 9\pi b$, $\mu > 0$. In this case we have

$$M = a + \pi b, \quad \nu = a - 9\pi b, \quad \omega_\infty = \pi b$$

and modulus of continuity is

$$\omega(t) = \begin{cases} 2b\mu t^s, & \text{for } 0 < t < \left(\frac{\pi}{2\mu}\right)^{1/s}, \\ \pi b & \text{for } t \geq \left(\frac{\pi}{2\mu}\right)^{1/s}. \end{cases}$$

Let $B_r(0)$ be a ball with $0 < r < 1$ then for suitable $\mu > 0$ the above functional satisfies the assumptions of Theorem 1.1(a),(b) and so the minimizer u is Hölder continuous (in the case $\delta > n - 2$) in $B_r(0)$.

2. Preliminaries

If $x \in \mathbb{R}^n$ and r is a positive real number, we set $B_r(x) = \{y \in \mathbb{R}^n : |y - x| < r\}$, $\Omega_r(x) = \Omega \cap B_r(x)$. Denote by

$$u_{x,r} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} u(y) dy = \oint_{\Omega_r(x)} u(y) dy$$

the mean value of the function $u \in L^1(\Omega, \mathbb{R}^N)$ over the set $\Omega_r(x)$, where $|\Omega_r(x)|$ is the n -dimensional Lebesgue measure of $\Omega_r(x)$. Moreover, we set $\phi(r) = \phi(x, r) = \int_{B_r(x)} |Du(y)|^2 dy$, $U(r) = U(x, r) = r^{2-n} \phi(x, r)$ for $B_r(x) \subset \Omega$. Beside the standard space $C_0^\infty(\Omega, \mathbb{R}^N)$, Hölder space $C^{0,\alpha}(\bar{\Omega}, \mathbb{R}^N)$ and Sobolev spaces $W^{k,p}(\Omega, \mathbb{R}^N)$, $W_0^{k,p}(\Omega, \mathbb{R}^N)$ we use Campanato spaces $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ and Morrey spaces $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ (see, e.g. [15]). We will denote by $X_{loc}(\Omega, \mathbb{R}^N)$ the space of all functions which belong to $X(\bar{\Omega}, \mathbb{R}^N)$ for any bounded subdomain $\bar{\Omega}$ with smooth boundary which is compactly embedded in Ω .

We set for $f \in L^1(\Omega)$, $0 < a < \infty$

$$\mathcal{M}_a(f, \Omega) := \sup_{x \in \Omega, r < a} \int_{\Omega_r(x)} |f(y) - f_{x,r}| \, dy.$$

Definition 2.1 (See [19]). A function $f \in L^1(\Omega)$ is said to belong to $BMO(\Omega)$ if

$$\mathcal{M}_{\text{diam } \Omega}(f, \Omega) < \infty$$

a function $f \in L^1(\Omega)$ is said to belong to $VMO(\Omega)$ if

$$\lim_{a \rightarrow 0} \mathcal{M}_a(f, \Omega) = 0.$$

We will use conjugate Young functions Φ, Ψ

$$\Phi(t) = t \ln_+(at) \quad \text{for } t \geq 0, \quad \Psi(t) = \begin{cases} \frac{t}{a} & \text{for } 0 \leq t < 1, \\ \frac{e^t - 1}{a} & \text{for } t \geq 1 \end{cases} \quad (4)$$

where $a > 0$ is a constant,

$$\ln_+(at) = \begin{cases} 0 & \text{for } 0 \leq t < \frac{1}{a}, \\ \ln(at) & \text{for } t \geq \frac{1}{a}. \end{cases}$$

Then Young inequality for Φ, Ψ reads as

$$xy \leq \Phi(x) + \Psi(y), \quad \forall x, y \in \mathbb{R}. \quad (5)$$

Lemma 2.1 ([21, p. 37]). Let $\phi : [0, \infty) \rightarrow [0, \infty)$ be non decreasing function which is absolutely continuous on every closed interval of finite length, $\phi(0) = 0$. If $w \geq 0$ is measurable and $E(t) = \{y \in \mathbb{R}^n : w(y) > t\}$ then

$$\int_{\mathbb{R}^n} \phi \circ w \, dy = \int_0^\infty \mu(E(t)) \phi'(t) \, dt.$$

Proposition 2.2 (See [5]). Let $v \in L^1_{loc}(\Omega, \mathbb{R}^m)$, $m \geq 1$, $b > 0$, $B(x, \sigma) \subset\subset \Omega$ and ω be a function having properties described in (iii) of the introduction.

(a) Let $1 \leq p < \infty$, $\Psi(t) = t^p$ for $t \in [0, \infty)$. Then

$$\int_{B(x, \sigma)} \Psi(b\omega(|v(y) - v_{x, \sigma}|)) \, dy \leq \sup_{r > 0} \left[\Psi(b\omega(r)) \right]' \int_{B(x, \sigma)} |v(y) - v_{x, \sigma}| \, dy$$

if the supremum on the right hand side is finite.

- (b) Let $b > 1/\omega_\infty$ be a positive constant and Ψ be a Young function defined by (4). Then

$$\begin{aligned} \int_{B(x,\sigma)} \Psi\left(b\omega(|v(y) - v_{x,\sigma}|)\right) dy \\ \leq \frac{1}{a} \left(\kappa_n \sigma^n + \frac{1}{e} \sup_{r>t_0} \left(e^{b\omega(r)}\right)' \int_{B(x,\sigma)} |v(y) - v_{x,\sigma}| dy \right) \\ \leq \frac{1}{a} \left(\kappa_n \sigma^n + \frac{b}{e} e^{b\omega_\infty} \sup_{r>t_0} (\omega'(r)) \int_{B(x,\sigma)} |v(y) - v_{x,\sigma}| dy \right) \end{aligned}$$

where $t_0 = \inf \omega^{-1}(1/b)$ and κ_n is the volume of the unit ball in $\mathbb{R}^n$.

- (c) If, in addition to the assumptions of (b), we suppose that $e^{b\omega}$ is concave on $[0, \infty)$ then

$$\int_{B(x,\sigma)} \Psi\left(b\omega(|v(y) - v_{x,\sigma}|)\right) dy \leq \frac{1}{a} \left(\kappa_n \sigma^n + b\omega'(t_0) \int_{B(x,\sigma)} |v(y) - v_{x,\sigma}| dy \right)$$

with $t_0 = \inf \omega^{-1}(1/b)$.

Proof.

- (a) Taking in Lemma 2.1 $w(y) = |v(y) - v_{x,\sigma}|$ on $B(x, \sigma)$ and $w = 0$ otherwise, we have $E_\sigma(t) = \{y \in B(x, \sigma) : |v(y) - v_{x,\sigma}| > t\}$ and we get the assertion as an immediate consequence of Lemma 2.1.
- (b) Using once more Lemma 2.1 we get

$$\begin{aligned} \int_{B(x,\sigma)} \Psi\left(b\omega(|v(y) - v_{x,\sigma}|)\right) dy &= \frac{b}{a} \int_0^{t_0} \omega'(t) \mu(E_\sigma(t)) dt \\ &\quad + \frac{1}{ae} \int_{t_0}^\infty (e^{b\omega(r)})' \mu(E_\sigma(t)) dt \\ &= \frac{b}{a} I_1 + \frac{1}{ae} I_2. \end{aligned}$$

It is easy to get the following estimates

$$\begin{aligned} I_1 &\leq \kappa_n \sigma^n \int_0^{t_0} \omega'(t) dt = \kappa_n \sigma^n \omega(t_0) = \frac{\kappa_n}{b} \sigma^n, \\ I_2 &\leq \sup_{r>t_0} (e^{b\omega(r)})' \int_{t_0}^{\infty} \mu(E_\sigma(t)) dt \\ &\leq \sup_{r>t_0} (e^{b\omega(r)})' \int_{B(x,\sigma)} |v(y) - v_{x,\sigma}| dy \end{aligned}$$

which give immediately also the last inequality.

- (c) The proof follows from (b) using monotonicity of $(e^{b\omega})'$ and definition of t_0 . $\square$

The next lemma we will use in the proof of Theorem 1.1(b).

Lemma 2.3. *Let $v \geq 0$, $q \in [1, \infty)$ and $\delta > 0$ be arbitrary. Then*

$$\ln_+^q v \leq \left(\frac{q}{\delta e}\right)^q v^\delta.$$

Proof. Hint – calculate $\sup\{(\ln_+ v) v^{-\frac{\delta}{q}}; v \in (0, \infty)\}$. $\square$

Proposition 2.4 (See [2, 8, 15]). *For a bounded domain $\Omega \subset \mathbb{R}^n$ with a Lipschitz boundary, for $q \in (1, \infty)$ and $0 < \lambda < \mu < \infty$ we have the following:*

- (a) $L^{q,\mu}(\Omega, \mathbb{R}^N) \subset L^{q,\lambda}(\Omega, \mathbb{R}^N) \quad \mathcal{L}^{q,\mu}(\Omega, \mathbb{R}^N) \subset \mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N).$
- (b) $L^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$, for $0 < \lambda < n$.
- (c) $\mathcal{L}^{q,\lambda}(\Omega, \mathbb{R}^N)$ is isomorphic to the $C^{0,(\lambda-n)/q}(\bar{\Omega}, \mathbb{R}^N)$, for $n < \lambda \leq n + q$.
- (d) $L^{q,n}(\Omega, \mathbb{R}^N)$ is isomorphic to the $L^\infty(\Omega, \mathbb{R}^N)$.

In what follows we will use modifications of an algebraic lemma from [5]. Now we can state it.

Lemma 2.5. *Let A, d be positive numbers, $\beta \geq 0$, $\delta \in [n-2, n)$. Then there exist ϵ_0, C positive so that for any nonnegative, nondecreasing function ϕ defined on $[0, 2d]$ and satisfying with $B_1, B_2 U^\beta(2d) \in [0, \epsilon_0]$ the inequality*

$$\phi(\sigma) \leq \left(A \left(\frac{\sigma}{R} \right)^n + B_1 + B_2 U^\beta(2R) \right) \phi(2R) \quad (6)$$

for all σ, R such that $0 < \sigma < R \leq d$, it holds

$$\phi(\sigma) \leq C \sigma^\delta \phi(2d), \quad \forall \sigma : 0 < \sigma \leq d. \quad (7)$$

Remark 2.1. Note that we can take

$$\epsilon_0 = \frac{1}{4(2^{n+1}A)^{\frac{\delta}{n-\delta}}}, \quad C = \frac{(2^{n+1}A)^{\frac{\delta}{n-\delta}}}{(2d)^\delta}.$$

Proof.

- I. Without loss of generality we can suppose that $A \geq 1$. Choose $\tau \in (0, 1)$ so that $2^{n+1}A\tau^{n-\delta} = 1$, i.e. $\tau = (\frac{1}{2^{n+1}A})^{\frac{1}{n-\delta}}$, $\epsilon_0 = \tau^\delta/4$.
 II. We will prove by induction that

$$\begin{cases} \phi(2\tau^k d) \leq \tau^{k\delta} \phi(2d) \\ U(2\tau^k d) \leq U(2d). \end{cases} \quad (8)$$

Let $k = 1$. Putting $\sigma = 2\tau d$, $R = d$ in (6) we obtain thanks to the assumptions on τ , B_1 , B_2 , ϵ_0

$$\begin{aligned} \phi(2\tau d) &\leq (2^n A \tau^n + B_1 + B_2 U^\beta(2d)) \phi(2d) \\ &\leq \tau^\delta (2^n A \tau^{n-\delta} + B_1 \tau^{-\delta} + B_2 \tau^{-\delta} U^\beta(2d)) \phi(2d) \\ &\leq \tau^\delta (2^n A \tau^{n-\delta} + 2\epsilon_0 \tau^{-\delta}) \phi(2d) = \tau^\delta \phi(2d). \end{aligned}$$

Thus also

$$U(2\tau d) = (2\tau d)^{2-n} \phi(2\tau d) \leq (2\tau d)^{2-n} \tau^\delta \phi(2d) = \tau^{\delta-(n-2)} U(2d).$$

Suppose (8) is valid for $j = 1, \dots, k$ and put $\sigma = 2\tau^{k+1}d$, $R = \tau^k d$ into (6). We get

$$\phi(2\tau^{k+1}d) \leq (2^n A \tau^n + B_1 + B_2 U^\beta(2\tau^k d)) \phi(2\tau^k d).$$

Using now (8) for $j = k$, choice of τ , assumptions on B_1 , B_2 , ϵ_0 and estimates of $\phi(2\tau^k d)$ we have

$$\begin{aligned} \phi(2\tau^{k+1}d) &\leq (2^n A \tau^{n-\delta} + 2\epsilon_0 \tau^{-\delta}) \tau^\delta \phi(2\tau^k d) \\ &\leq \tau^\delta \phi(2\tau^k d) = \tau^{\delta(k+1)} \phi(2d). \end{aligned}$$

As $\delta \geq n - 2$ it immediately implies the estimate

$$U(2\tau^{k+1}d) \leq U(2d)$$

and we have (8).

- III. Let now σ be an arbitrary positive number less than d . Then there is an integer k such that $2\tau^{k+1}d \leq \sigma < 2\tau^k d$. Using monotonicity of ϕ , this inequality and (8) we get

$$\phi(\sigma) \leq \phi(2\tau^k d) \leq \tau^{k\delta} \phi(2d) \leq \sigma^\delta \frac{1}{(2\tau d)^\delta} \phi(2d).$$

Setting in this estimate $C = 1/(2\tau d)^\delta$ completes the proof. $\square$

Lemma 2.6 (See e.g. [2, 8]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution of the system*

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = 0, \quad i = 1, \dots, N$$

where $A_{ij}^{\alpha\beta}$ are constants satisfying (i) and (ii). Then there exists a constant $L = L(n, M/\nu) \geq 1$ such that for every weak solution $v \in W^{1,2}(\Omega, \mathbb{R}^N)$, for every $x \in \Omega$

and $0 < \sigma \leq R \leq \text{dist}(x, \partial\Omega)$ the following estimate

$$\int_{B_\sigma(x)} |Du(y)|^2 dy \leq L \left(\frac{\sigma}{R} \right)^n \int_{B_R(x)} |Du(y)|^2 dy$$

holds.

Lemma 2.7 (See [8, 11]). *Let $u \in W^{1,2}(\Omega, \mathbb{R}^N)$ be a minimum of the functional (1) under the assumptions (i) and (ii). Then $Du \in L_{loc}^{2p}(\Omega, \mathbb{R}^{nN})$ for some $p > 1$ and there exist a constant $c = c(n, p, M/\nu)$ such that for all balls $B_{2R}(x) \subset \Omega$*

$$\left(\int_{B_R(x)} |Du|^{2p} dy \right)^{1/2p} \leq c \left(\int_{B_{2R}(x)} |Du|^2 dy \right)^{1/2}$$

holds.

Let x_0 be any fixed point of Ω , $0 < R \leq \text{dist}(x_0, \partial\Omega)$. We set

$$A_{ij}^{\alpha\beta}(u_{x_0,R})_{x_0,R} = \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(y, u_{x_0,R}) dy.$$

If v is a solution to the following system

$$\begin{cases} D_\alpha(A_{ij}^{\alpha\beta}(u_{x_0,R})_{x_0,R} D_\beta v^j) = 0 & \text{in } B_R(x_0), \\ v - u \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N) \end{cases} \quad (9)$$

then the next lemma is true.

Lemma 2.8 (See [8, 11]). *Let $v \in W^{1,2}(B_R(x_0), \mathbb{R}^N)$ be the solution to the problem (9) with $u \in W^{1,2p}(B_R(x_0), \mathbb{R}^N)$, $p \geq 1$ then we have*

$$\int_{B_R(x)} |Dv|^{2p} dy \leq c(M/\nu) \int_{B_R(x)} |Du|^{2p} dy$$

holds.

Remark 2.2. Revising proofs of Lemmas 2.7 and 2.8 one can see that the constants from the above estimates depend increasingly on M/ν .

3. Proof of theorem

At first we recall that we set $\phi(r) = \phi(x_0, r) = \int_{B_r(x_0)} |Du(y)|^2 dy$, $U(r) = U(x_0, r) = r^{2-n}\phi(x_0, r)$ for $B_r(x_0) \subset \Omega$. Now let x_0 be any fixed point of $\bar{\Omega}_0 \subset \Omega$, $\text{dist}(\Omega_0, \partial\Omega) \geq 2d > 0$, $B_{2R}(x_0) \subset \Omega$ and v be a minimizer of the frozen functional

$$\mathcal{A}^0(v; B_R(x_0)) = \int_{B_R(x_0)} A_{ij}^{\alpha\beta}(u_R)_R D_\alpha v^i D_\beta v^j dx$$

among all the functions in $W^{1,2}(B_R(x_0), \mathbb{R}^N)$ taking the values u on $\partial B_R(x_0)$ where

$$A_{ij}^{\alpha\beta}(z)_R = \oint_{B_R(x_0)} A_{ij}^{\alpha\beta}(y, z) dy.$$

From Euler equation for v and from Lemma (2.6) we have

$$\int_{B_\sigma(x_0)} |Dv|^2 dx \leq L \left(\frac{\sigma}{R} \right)^n \int_{B_R(x_0)} |Dv|^2 dx, \quad \forall 0 < \sigma \leq R. \quad (10)$$

Put $w = u - v$. It is clear that $w \in W_0^{1,2}(B_R(x_0), \mathbb{R}^N)$. Using (10) by standard arguments we obtain

$$\begin{aligned} \int_{B_\sigma(x_0)} |Du|^2 dx &\leq 2 \int_{B_\sigma(x_0)} |Dw|^2 dx + 2 \int_{B_\sigma(x_0)} |Dv|^2 dx \\ &\leq 2 \left(1 + 2L \left(\frac{\sigma}{R} \right)^n \right) \int_{B_R(x_0)} |Dw|^2 dx \\ &\quad + 4L \left(\frac{\sigma}{R} \right)^n \int_{B_R(x_0)} |Du|^2 dx. \end{aligned} \quad (11)$$

In the following we will estimate the first integral on the right hand side of (11). From [10] (see Lemma 2.1) we have

$$\begin{aligned} \int_{B_R(x_0)} |Dw|^2 dx &\leq \frac{2}{\nu} \left(\mathcal{A}^0(u; B_R(x_0)) - \mathcal{A}^0(v; B_R(x_0)) \right) \\ &\leq \frac{2}{\nu} \left\{ \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(u_R)_R - A_{ij}^{\alpha\beta}(x, u_R)) D_\alpha u^i D_\beta u^j dx \right. \\ &\quad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x, u_R) - A_{ij}^{\alpha\beta}(x, u)) D_\alpha u^i D_\beta u^j dx \\ &\quad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x, u_R) - A_{ij}^{\alpha\beta}(u_R)_R) D_\alpha v^i D_\beta v^j dx \\ &\quad + \int_{B_R(x_0)} (A_{ij}^{\alpha\beta}(x, v) - A_{ij}^{\alpha\beta}(x, u_R)) D_\alpha v^i D_\beta v^j dx \\ &\quad \left. + \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\nu} \left\{ I + II + III + IV + \mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \right\} \\
&\leq \frac{2}{\nu} (I + II + III + IV). \tag{12}
\end{aligned}$$

Notice that $\mathcal{A}(u; B_R(x_0)) - \mathcal{A}(v; B_R(x_0)) \leq 0$, since u is a minimizer.

Now we will estimate the terms I, II, III and IV from (12). In the following we will denote $(A_{ij}^{\alpha\beta}) =: A$. Using Hölder inequality, higher integrability of gradient of minima ($p > 1$, $p' = p/(p-1)$)

$$\begin{aligned}
|I| &\leq \int_{B_R(x_0)} |A(u_R)_R - A(x, u_R)| |Du|^2 dx \\
&\leq cR^{n/p} \left(\int_{B_R(x_0)} |A(u_R)_R - A(x, u_R)|^{p'} dx \right)^{1/p'} \left(\int_{B_R(x_0)} |Du|^{2p} dx \right)^{1/p} \\
&\leq c(n, N, p, M/\nu) R^{n/p} \left(\int_{B_R(x_0)} |A(u_R)_R - A(x, u_R)|^{p'} dx \right)^{1/p'} \int_{B_{2R}(x_0)} |Du|^2 dx.
\end{aligned}$$

Taking into account the assumptions (i), (iv) and Definition 2.1 we obtain

$$\left(\int_{B_R(x_0)} |A(u_R)_R - A(x, u_R)|^{p'} dx \right)^{1/p'} \leq (2M)^{(p'-1)/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'}$$

and then using above two estimates we have

$$|I| \leq c(n, N, p, M/\nu) (2M)^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} \phi(2R). \tag{13}$$

A similarity of the terms I and III enables us to write (by means of Lemma 2.8) the following inequality

$$|III| \leq c(n, N, p, M/\nu) (2M)^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} \phi(2R). \tag{14}$$

The last two estimates are the same for all variants of Theorem 1.1. The estimates of the integrals II, IV are different and so we will divide the proof into three parts.

Case (a) Using Hölder inequality by the same way as in estimating of I and property (iii) we get

$$\begin{aligned} |II| &\leq \int_{B_R(x_0)} \omega(|u - u_R|) |Du|^2 dx \\ &\leq \left(\int_{B_R(x_0)} \omega^{p'}(|u - u_R|) dx \right)^{1/p'} \left(\int_{B_R(x_0)} |Du|^{2p} dx \right)^{1/p}. \end{aligned}$$

By means of Proposition 2.2(a), Hölder and Poincaré inequalities we get

$$\begin{aligned} \left(\int_{B_R(x_0)} \omega^{p'}(|u - u_R|) dx \right)^{1/p'} &\leq \left(\sup_{t>0} \frac{d}{dt} (\omega^{p'}(t)) \right)^{1/p'} \left(\int_{B_R(x_0)} |u(x) - u_R| dx \right)^{1/p'} \\ &\leq c(n) S_{p'}^{1/p'} R^{(n+2)/2p'} \left(\int_{B_R(x_0)} |Du|^2 dx \right)^{1/2p'}. \end{aligned}$$

Now Lemma 2.7 implies

$$\left(\int_{B_R(x_0)} |Du|^{2p} dx \right)^{1/p} \leq c(n, p, M/\nu) R^{-n/p'} \int_{B_{2R}(x_0)} |Du|^2 dx.$$

Last two estimates lead to

$$|II| \leq c(n, p, M/\nu) S_{p'}^{1/p'} U^{1/2p'} (2R) \phi(2R). \quad (15)$$

By means similar to those which are used for estimating of II we obtain

$$\begin{aligned} |IV| &\leq \int_{B_R(x_0)} \omega(|v - u_R|) |Dv|^2 dx \\ &\leq \left(\int_{B_R(x_0)} \omega^{p'}(|v - u_R|) dx \right)^{1/p'} \left(\int_{B_R(x_0)} |Dv|^{2p} dx \right)^{1/p}. \end{aligned} \quad (16)$$

The first term from the above inequality we can estimate using Poincaré inequality and Lemma 2.8 as follows

$$\begin{aligned}
& \left(\int_{B_R(x_0)} \omega^{p'} (|v - u_R|) dx \right)^{1/p'} \\
& \leq c(n) S_{p'}^{1/p'} R^{n/2p'} \left(\int_{B_R(x_0)} |v - u_R|^2 dx \right)^{1/2p'} \\
& \leq c(n) S_{p'}^{1/p'} R^{n/2p'} \left(\int_{B_R(x_0)} |w|^2 dx + \int_{B_R(x_0)} |u - u_R|^2 dx \right)^{1/2p'} \\
& \leq c(n) S_{p'}^{1/p'} R^{(n+2)/2p'} \left(\int_{B_R(x_0)} |Dw|^2 dx + \int_{B_R(x_0)} |Du|^2 dx \right)^{1/2p'} \\
& \leq c(n) S_{p'}^{1/p'} R^{(n+2)/2p'} \left(\int_{B_R(x_0)} |Du|^2 dx \right)^{1/2p'}. \tag{17}
\end{aligned}$$

From Lemma 2.7 and Lemma 2.8 we have

$$\begin{aligned}
\left(\int_{B_R(x_0)} |Dv|^{2p} dx \right)^{1/p} & \leq c \left(\int_{B_R(x_0)} |Du|^{2p} dx \right)^{1/p} \\
& \leq c(n, p, M/\nu) R^{-n(1-1/p)} \int_{B_{2R}(x_0)} |Du|^2 dx \tag{18}
\end{aligned}$$

and then from (16) by means of (17) and (18) we get

$$\begin{aligned}
|IV| & \leq c(n, p, M/\nu) S_{p'}^{1/p'} R^{(n+2)/2p' - n(1-1/p)} \phi^{1+1/2p'}(2R) \\
& \leq c(n, p, M/\nu) S_{p'}^{1/p'} U^{1/2p'}(2R) \phi(2R). \tag{19}
\end{aligned}$$

Now (12) together with (13), (15), (14) and (19) give us

$$\begin{aligned}
\int_{B_R(x_0)} |Dw|^2 dx & \leq \frac{2}{\nu} (I + II + III + IV) \tag{20} \\
& \leq \frac{c}{\nu} \left[M^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} + S_{p'}^{1/p'} U^{1/2p'}(2R) \right] \phi(2R)
\end{aligned}$$

where $c = c(n, N, p, M/\nu)$. From (11) and (20) we have

$$\begin{aligned} \phi(\sigma) &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(R) \\ &\quad + \frac{c(1+L)}{\nu} \left(M^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} + S_{p'}^{1/p'} U^{1/2p'}(2R) \right) \phi(2R) \\ &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) \\ &\quad + cL \left(\left(\frac{M}{\nu} \right)^{1-1/p'} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} + \frac{S_{p'}^{1/p'}}{\nu} U^{1/2p'}(2R) \right) \phi(2R). \end{aligned} \quad (21)$$

We remember that the constant L depends only on $n, M/\nu$ and from the assumptions of Theorem 1.1 we have

$$\begin{aligned} \phi(\sigma) &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) \\ &\quad + K \left(P^{1/p} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} + \frac{S_{p'}^{1/p'}}{\nu} U^{1/2p'}(2R) \right) \phi(2R) \end{aligned}$$

for all $\sigma < R \leq d$ and $K = K(n, N, p, P)$. Now we can use Lemma 2.5 in the following manner:

If ν_0 is such that

$$K P^{1/p} \left(\frac{\mathcal{M}_d(A(\cdot, u_d))}{\nu_0} \right)^{1/p'} \leq \epsilon_0 \quad \frac{K S_{p'}^{1/p'}}{\nu_0} \left(\frac{\|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}^2}{d^{n-2}} \right)^{1/2p'} \leq \epsilon_0$$

where the estimate from Remark 1.1(a) was used in the second inequality then for $\nu \geq \nu_0$ the assumptions of Lemma 2.5 are satisfied for $\delta \in [n-2, n)$, $\beta = 1/2p'$, $B_1 = K P^{1/p} (\mathcal{M}_d(A(\cdot, u_d)) / \nu_0)^{1/p'}$, $B_2 = K S_{p'}^{1/p'} / \nu_0$ and we get that $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

Case (b) Using property (iii) and Young inequality (5) we have

$$\begin{aligned} |II| &\leq \int_{B_R(x_0)} \omega(|u - u_R|) |Du|^2 dx \\ &\leq \int_{B_R(x_0)} \Phi(\varepsilon |Du|^2) dx + \int_{B_R(x_0)} \Psi\left(\frac{1}{\varepsilon} \omega(|u - u_R|)\right) dx = I_1 + I_2. \end{aligned}$$

By means of Hölder inequality, Lemma 2.7 and Lemma 2.3 with $q = p/(p-1)$, $\delta = 1$ we get

$$\begin{aligned}
I_1 &= \varepsilon \int_{B_R(x_0)} |Du|^2 \ln_+ (a\varepsilon |Du|^2) \, dx \\
&\leq \varepsilon \left(\int_{B_R(x_0)} |Du|^{2p} \, dx \right)^{1/p} \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)} (a\varepsilon |Du|^2) \, dx \right)^{1-1/p} \\
&\leq c(p, M/\nu) \varepsilon \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)} (a\varepsilon |Du|^2) \, dx \right)^{1-1/p} \phi(2R) \\
&\leq c(p, M/\nu) \varepsilon \frac{p}{e(p-1)} \left(a\varepsilon \int_{B_R(x_0)} |Du|^2 \, dx \right)^{1-1/p} \phi(2R).
\end{aligned}$$

From Proposition 2.2(b), Hölder and Poincaré inequalities we have

$$\begin{aligned}
I_2 &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{e^{\omega_\infty/\varepsilon}}{e\varepsilon} S_0 \int_{B_R(x_0)} |u - u_R| \, dx \right) \\
&\leq \frac{1}{a} \left(\kappa_n R^n + c(n) \frac{e^{\omega_\infty/\varepsilon}}{e\varepsilon} S_0 R^{1+n/2} \phi^{1/2}(2R) \right)
\end{aligned}$$

where $S_0 = \sup_{t>t_0} \omega'(t)$ ($\omega(t_0) = \varepsilon$). Last two estimates lead to

$$\begin{aligned}
|II| &\leq c(p, M/\nu) \varepsilon \left(a\varepsilon \int_{B_R(x_0)} |Du|^2 \, dx \right)^{1-1/p} \phi(2R) \\
&\quad + \frac{1}{a} \left(\kappa_n R^n + \frac{c(n)}{\varepsilon} e^{\omega_\infty/\varepsilon} S_0 R^{1+n/2} \phi^{1/2}(2R) \right).
\end{aligned}$$

Supposing $\int_{B_R(x_0)} |Du|^2 \, dx$ is positive we can choose $a = 1/(\varepsilon \int_{B_R(x_0)} |Du|^2 \, dx)$ in the last inequality. Then we obtain

$$|II| \leq \left[(c(p, M/\nu) + 1) \varepsilon + c(n) e^{\omega_\infty/\varepsilon} S_0 U^{1/2}(2R) \right] \phi(2R). \quad (22)$$

Now we can estimate the last term on the right-hand side (12) analogous to II

$$\begin{aligned} |IV| &\leq \int_{B_R(x_0)} \omega(|v - u_R|) |Dv|^2 dx \\ &\leq \int_{B_R(x_0)} \Phi(\varepsilon |Dv|^2) dx + \int_{B_R(x_0)} \Psi\left(\frac{1}{\varepsilon} \omega(|v - u_R|)\right) dx = J_1 + J_2. \end{aligned}$$

By means of Hölder inequality, Lemma 2.7, Lemma 2.3 with $q = p/(p-1)$, $\delta = 1$ and Lemma 2.8 we get

$$\begin{aligned} J_1 &= \varepsilon \int_{B_R(x_0)} |Dv|^2 \ln_+(a\varepsilon |Dv|^2) dx \\ &\leq \varepsilon \left(\int_{B_R(x_0)} |Dv|^{2p} dx \right)^{1/p} \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)}(a\varepsilon |Dv|^2) dx \right)^{1-1/p} \\ &\leq \varepsilon \left(\int_{B_R(x_0)} |Du|^{2p} dx \right)^{1/p} \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)}(a\varepsilon |Dv|^2) dx \right)^{1-1/p} \\ &\leq c(p, M/\nu) \varepsilon \left(\int_{B_R(x_0)} \ln_+^{p/(p-1)}(a\varepsilon |Dv|^2) dx \right)^{1-1/p} \phi(2R) \\ &\leq c(p, M/\nu) \varepsilon \frac{p}{e(p-1)} \left(a\varepsilon \int_{B_R(x_0)} |Dv|^2 dx \right)^{1-1/p} \phi(2R) \\ &\leq c(p, M/\nu) \varepsilon \left(a\varepsilon \int_{B_R(x_0)} |Du|^2 dx \right)^{1-1/p} \phi(2R). \end{aligned}$$

From Proposition 2.2(b), Hölder and Poincaré inequalities and Lemma 2.8 we have

$$\begin{aligned} J_2 &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{e^{\omega_\infty/\varepsilon}}{e\varepsilon} S_0 \int_{B_R(x_0)} |v - u_R| dx \right) \\ &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{e^{\omega_\infty/\varepsilon}}{e\varepsilon} S_0 \left(\int_{B_R(x_0)} |u - u_R| dx \right) \right) \end{aligned}$$

$$\begin{aligned}
& + \int_{B_R(x_0)} |v - u| dx \Bigg) \Bigg) \\
& \leq \frac{1}{a} \left(\kappa_n R^n + \frac{c(n)}{\varepsilon} e^{\omega_\infty/\varepsilon} S_0 R^{1+n/2} \phi^{1/2}(2R) \right).
\end{aligned}$$

Last two estimates lead to the same estimate as for II and we get

$$\begin{aligned}
|IV| & \leq c(p, M/\nu) \left(a\varepsilon \int_{B_R(x_0)} |Du|^2 dx \right)^{1-1/p} \phi(2R) \\
& + \frac{1}{a} \left(\kappa_n R^n + \frac{c(n)}{\varepsilon} e^{\omega_\infty/\varepsilon} S_0 R^{1+n/2} \phi^{1/2}(2R) \right).
\end{aligned}$$

Now we can choose $a = 1/(\varepsilon \int_{B_R(x_0)} |Du|^2 dx)$ in the last inequality again and we have

$$|IV| \leq \left[(c(p, M/\nu) + 1)\varepsilon + c(n)e^{\omega_\infty/\varepsilon} S_0 U^{1/2}(2R) \right] \phi(2R). \quad (23)$$

A substitution (13), (14), (22) and (23) into (12) gives finally

$$\begin{aligned}
\int_{B_R(x_0)} |Dw|^2 dx & \leq \frac{2}{\nu} (I + II + III + IV) \\
& \leq \frac{c}{\nu} \left[M^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} + \varepsilon + e^{\omega_\infty/\varepsilon} S_0 U^{1/2}(2R) \right] \phi(2R) \\
& \leq c \left[\left(\frac{M}{\nu} \right)^{1-1/p'} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\
& \quad \left. + \frac{\varepsilon}{\nu} + \frac{e^{\omega_\infty/\varepsilon}}{\nu} S_0 U^{1/2}(2R) \right] \phi(2R)
\end{aligned} \quad (24)$$

where $c = c(n, p, N, M/\nu)$.

From (11), (24) and from the assumptions of Theorem 1.1(b) (we remember that the constant L depends only on $n, M/\nu$) we have

$$\begin{aligned}
\phi(\sigma) & \leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) \\
& + c(1 + L) \left[\left(\frac{M}{\nu} \right)^{1-1/p'} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\
& \quad \left. + \frac{\varepsilon}{\nu} + \frac{e^{\omega_\infty/\varepsilon}}{\nu} S_0 U^{1/2}(2R) \right] \phi(2R) \\
& \leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R)
\end{aligned}$$

$$\begin{aligned}
& + K \left[P^{1/p} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\
& \left. + \frac{\varepsilon}{\nu} + \frac{e^{\omega_\infty/\varepsilon}}{\nu} S_0 U^{1/2}(2R) \right] \phi(2R)
\end{aligned} \tag{25}$$

for all $\sigma < R \leq d$ and $K = K(n, p, N, P)$. If we put in (25)

$$\varepsilon = \frac{\omega_\infty}{2KP} \epsilon_0$$

where $\epsilon_0 = \frac{1}{4(2^{n+3}L)^{\frac{\delta}{n-\delta}}}$, we get (we recall that $K \geq 2$) the final estimate

$$\begin{aligned}
\phi(\sigma) & \leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) + \left[KP^{1/p} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\
& \left. + \frac{1}{2}\epsilon_0 + K \frac{e^{2KP/\epsilon_0}}{\nu} S_0 U^{1/2}(2R) \right] \phi(2R)
\end{aligned} \tag{26}$$

for all $\sigma < R \leq d$.

Now we can use Lemma 2.5 in the following manner: If ν_0 is such that

$$\begin{aligned}
KP^{1/p} \left(\frac{\mathcal{M}_d(A(\cdot, u_d))}{\nu_0} \right)^{1/p'} & \leq \frac{1}{2}\epsilon_0 \\
KS_0 \frac{e^{2KP/\epsilon_0} \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}}{\nu_0 d^{-1+n/2}} & \leq \epsilon_0
\end{aligned}$$

where the estimate from Remark 1.1(a) was used in the second inequality then for $\nu \geq \nu_0$ the assumptions of Lemma 2.5 are satisfied for $\delta \in [n-2, n)$, $\beta = 1/2$, $B_1 = KP^{1/p}(\mathcal{M}_d(A(\cdot, u_d))/\nu_0)^{1/p'} + \epsilon_0/2$, $B_2 = KS_0 e^{2KP/\epsilon_0}/\nu_0$ and we get that $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

Case (c) Using property (iii) and Young inequality (5) we have

$$\begin{aligned}
|II| & \leq \int_{B_R(x_0)} \omega(|u - u_R|) |Du|^2 dx \\
& \leq \int_{B_R(x_0)} \Phi(\varepsilon |Du|^2) dx + \int_{B_R(x_0)} \Psi\left(\frac{1}{\varepsilon} \omega(|u - u_R|)\right) dx = I_1 + I_2.
\end{aligned}$$

In the same manner like in the previous case (b) we get

$$I_1 \leq c(p, M/\nu) \varepsilon \frac{p}{e(p-1)} \left(a \varepsilon \int_{B_R(x_0)} |Du|^2 dx \right)^{1-1/p} \phi(2R).$$

From Proposition 2.2(c), Hölder and Poincaré inequalities we have

$$\begin{aligned} I_2 &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{\omega'(t_0)}{\varepsilon} \int_{B_R(x_0)} |u - u_R| dx \right) \\ &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{\omega'(t_0)}{\varepsilon} R^{1+n/2} \phi^{1/2}(2R) \right) \end{aligned}$$

where $t_0 = \omega^{-1}(\varepsilon)$. Last two estimates lead to

$$\begin{aligned} |II| &\leq c(p, M/\nu) \varepsilon \left(a \varepsilon \int_{B_R(x_0)} |Du|^2 dx \right)^{1-1/p} \phi(2R) \\ &\quad + \frac{1}{a} \left(\kappa_n R^n + \frac{\omega'(t_0)}{\varepsilon} R^{1+n/2} \phi^{1/2}(2R) \right). \end{aligned}$$

The standard choice $a = 1/(\varepsilon \int_{B_R(x_0)} |Du|^2 dx)$ then gives

$$|II| \leq \left[(c(p, M/\nu) + 1) \varepsilon + c(n) \omega'(t_0) U^{1/2}(2R) \right] \phi(2R). \quad (27)$$

Combining the methods of estimating applied in the case (b) and in previous estimate we have

$$\begin{aligned} |IV| &\leq \int_{B_R(x_0)} \omega(|v - u_R|) |Dv|^2 dx \\ &\leq \int_{B_R(x_0)} \Phi(\varepsilon |Dv|^2) dx + \int_{B_R(x_0)} \Psi\left(\frac{1}{\varepsilon} \omega(|v - u_R|)\right) dx = J_1 + J_2. \end{aligned}$$

where

$$J_1 \leq c(p, M/\nu) \varepsilon \left(a \varepsilon \int_{B_R(x_0)} |Du|^2 dx \right)^{1-1/p} \phi(2R)$$

and

$$\begin{aligned} J_2 &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{\omega'(t_0)}{\varepsilon} \int_{B_R(x_0)} |v - u_R| dx \right) \\ &\leq \frac{1}{a} \left(\kappa_n R^n + \frac{\omega'(t_0)}{\varepsilon} \left(\int_{B_R(x_0)} |u - u_R| dx + \int_{B_R(x_0)} |v - u| dx \right) \right) \\ &\leq \frac{1}{a} \left(\kappa_n R^n + c(n) \frac{\omega'(t_0)}{\varepsilon} R^{1+n/2} \phi^{1/2}(2R) \right) \end{aligned}$$

and so

$$|IV| \leq \left[(c(p, M/\nu) + 1)\varepsilon + c(n)\omega'(t_0)U^{1/2}(2R) \right] \phi(2R). \quad (28)$$

Now (12) together with (13), (14), (27) and (28) give us

$$\begin{aligned} \int_{B_R(x_0)} |Dw|^2 dx &\leq \frac{2}{\nu} (I + II + III + IV) \\ &\leq \frac{c}{\nu} \left[M^{1-1/p'} \left(\mathcal{M}_R(A(\cdot, u_R)) \right)^{1/p'} \right. \\ &\quad \left. + \varepsilon + \omega'(t_0)U^{1/2}(2R) \right] \phi(2R) \end{aligned} \quad (29)$$

where $c = c(n, p, N, M/\nu)$. From (11), (29) and the assumptions of Theorem 1.1(c) (we remember that the constant L depends only on $n, M/\nu$) we have

$$\begin{aligned} \phi(\sigma) &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) \\ &\quad + c(1+L) \left[\left(\frac{M}{\nu} \right)^{1-1/p'} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\ &\quad \left. + \frac{\varepsilon}{\nu} + \frac{\omega'(t_0)}{\nu} U^{1/2}(2R) \right] \phi(2R) \\ &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) + K \left[P^{1/p} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\ &\quad \left. + \frac{\varepsilon}{\nu} + \frac{\omega'(t_0)}{\nu} U^{1/2}(2R) \right] \phi(2R) \end{aligned} \quad (30)$$

for all $\sigma < R \leq d$ and $K = K(n, p, N, P)$. If we put in (30) $\varepsilon = \sqrt{\omega_\infty}$ we have the final estimate

$$\begin{aligned} \phi(\sigma) &\leq 4L \left(\frac{\sigma}{R} \right)^n \phi(2R) + K \left[P^{1/p} \left(\frac{\mathcal{M}_R(A(\cdot, u_R))}{\nu} \right)^{1/p'} \right. \\ &\quad \left. + \frac{\sqrt{\omega_\infty}}{\nu} + \frac{\omega'(t_0)}{\nu} U^{1/2}(2R) \right] \phi(2R). \end{aligned}$$

Now we can use Lemma 2.5 in the following manner:

If ν_0 is such that

$$\begin{aligned} K \left[P^{1/p} \left(\frac{\mathcal{M}_d(A(\cdot, u_d))}{\nu_0} \right)^{1/p'} + \frac{\sqrt{2P}}{\sqrt{\nu_0}} \right] &\leq \epsilon_0 \\ K \frac{\omega'(t_0) \|Dg\|_{L^2(\Omega, \mathbb{R}^{nN})}}{\nu_0 d^{-1+n/2}} &\leq \epsilon_0 \end{aligned}$$

where the estimate from Remark 1.1(a) was used in the second inequality then for $\nu \geq \nu_0$ the assumptions of Lemma 2.5 are satisfied for $\delta \in [n-2, n)$, $\beta = 1/2$, $B_1 = K[P^{1/p}(\mathcal{M}_d(A(\cdot, u_d))/\nu_0)^{1/p'} + \sqrt{2P}/\sqrt{\nu_0}]$, $B_2 = K\omega'(t_0)/\nu_0$ and we get that $Du \in L^{2,\delta}(\Omega_0, \mathbb{R}^{nN})$.

Remark 3.1. The details of the proof of Theorem 1.1 indicate that one can assume $A_{ij}^{\alpha\beta}(\cdot, u) \in BMO(\Omega)$ with sufficiently small BMO -seminorms (uniformly with respect to $u \in \mathbb{R}^N$) instead of (iv) and all of the results remain in force.

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