Multiple Existence of Nonradial Positive Solutions for a Coupled Nonlinear Schrödinger System

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Abstract. In this paper, we consider the multiple existence of nonradial positive solutions of coupled nonlinear Schrödinger system

$$\begin{cases} -\Delta u + \mu_1 u = u^3 + \beta v^2 u & \text{in } \mathbb{R}^3 \\ -\Delta v + \mu_2 v = v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 \end{cases} \quad (\text{P})$$

where $\mu_1, \mu_2 > 0$ with $\mu_1 \neq \mu_2$ and $\beta < 0$.

It is known that the solutions of (P) is not necessarily radial [12]. We show that problem (P) has multiple nonradial solutions in case that $|\beta|$ is sufficiently small.

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1. Introduction

In the present paper, we consider the multiple existence of nonradial positive solutions of coupled Schrödinger system

$$\begin{cases} -\Delta u + \mu_1 u = u^3 + \beta uv^2 & \text{in } \mathbb{R}^3 \\ -\Delta v + \mu_2 v = v^3 + \beta u^2 v & \text{in } \mathbb{R}^3 \end{cases} \quad (\text{P})$$

where $\mu_1, \mu_2 > 0$ and $\beta < 0$.

Coupled nonlinear Schrödinger system (P) models many physical problems. In nonlinear optics, the phenomenon in Kerr-like photorefractive media is described by system (P) (cf. [2]). In this case, the solution u and v denote the components of the beam in Kerr-like photorefractive media, and the coupling constant β is the interaction between two components u and v . In case $\beta > 0$, the interaction is attractive, while the interaction is repulsive if $\beta < 0$. The bimodal pulse in

optical fibers under birefringent effects is also governed by system (P) (cf. [15]). System (P) is also a model for a mixture of two Bose–Einstein condensates (cf. [6]).

Recently motivated by these physical interest, the existence of solutions of (P) has been investigated by several authors. In the case that $\beta > 0$, problem (P) was studied by Ambrosetti & Colorado [3], Maia, Montefusco & Pellacci [14] and Lin & Wei [10]. They proved the existence of least energy solutions (u, v) of (P) with $u, v > 0$. We note that in this case, all positive solutions (u, v) of (P) satisfying

$$u(x) \longrightarrow 0, \quad v(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty \quad (1.1)$$

are radially symmetric functions (cf. [19]). On the other hand, in case that $\beta < 0$, it is known that there is no least energy solution of (P) (cf. [10]). Moreover, positive solutions of (P) satisfying (1.1) is not always radial. In case that $\beta < 0$ and $|\beta|$ is small, there are positive solutions with one component concentrating on the origin and the other component concentrating around a regular polygon (cf. [12]). The existence of non radial positive solutions was also considered in [21] for the case that $\beta < 0$ and $\mu_1 = \mu_2$. In this case, there are infinitely many nonradial solutions if $\beta < -1$. The existence of sign changing solutions of (P) with $\beta > 0$ was considered in [8]. For the problem (P) with $\mathbb{R}^3$ replaced by a bounded domain, we refer to [11] and [16].

In the present paper, we are interested in the case that $\beta < 0$ and $|\beta|$ is small, i.e. the case that the interaction of two solutions are small and repulsive. We will show the multiple existence of nonradial solutions of (P) in this case with $\mu_1 \neq \mu_2$. Our results improve the results in [12]. (See Remark 1 and Remark 3).

To state our main results, we need some notations. We denote by $B_r(x)$ the open ball in $\mathbb{R}^3$ centered at $x \in \mathbb{R}^3$ with radius $r > 0$. The inner product in $\mathbb{R}^3$ is denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^3}$. We put $H = H^1(\mathbb{R}^3)$ and $\mathbb{H} = H \times H$. We set $\mu_0 = 1$. We denote by $\|\cdot\|_{\mu_i}$ the norm of H defined by $\|u\|_{\mu_i}^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu_i |u|^2) dx$ for $u \in H$ and $i \in \{0, 1, 2\}$. For simplicity of notations, we put $|u(x)|_{\mu_i}^2 = |\nabla u(x)|^2 + \mu_i |u(x)|^2$ for $u \in H$ and $x \in \mathbb{R}^3$. For each function $u \in H$, we set $u^+(x) = \max\{u(x), 0\}$, $u^-(x) = \max\{-u(x), 0\}$. For each $p \geq 1$, we denote by $|\cdot|_p$ the norm of the space $L^p(\mathbb{R}^3)$. The Hilbert space $\mathbb{H}$ is equipped with the norm defined by $\|\mathcal{U}\|^2 = \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2$ for $\mathcal{U} = (u, v) \in \mathbb{H}$. We recall that for each $i \in \{0, 1, 2\}$, problem

$$\begin{cases} -\Delta u + \mu_i u = u^3 & \text{in } \mathbb{R}^3 \\ u(x) > 0 & \text{in } \mathbb{R}^3 \\ u(x) \longrightarrow 0 & \text{as } |x| \longrightarrow \infty \end{cases} \quad (P_i)$$

has a radial solution, denoted by U_i (cf. [7, 9]). The function U_i is the unique smooth solution of (P_i) up to translation. Moreover we know that U_i satisfies $I_i(U_i) = c_i = \min\{I(v) : v \in \mathcal{S}_i\}$, where I_i is the functional associated with problem (P_i) defined by

$$I_i(v) = \frac{1}{2} \|v\|_{\mu_i}^2 - \frac{1}{4} |v^+|_4^4 \quad \text{for } v \in H \quad (1.2)$$

and $\mathcal{S}_i$ is the set defined by

$$\mathcal{S}_i = \left\{ v \in H : \|v\|_{\mu_i}^2 = |v^+|_4^4 \right\} \quad \text{for } i = 1, 2.$$

For each $x \in \mathbb{R}^3$ and $i \in \{0, 1, 2\}$, we put $U_{i,x}(\cdot) = U_i(\cdot - x)$. It is also known that

$$|U_i(x)|_{\mu_i} |x| \exp(\sqrt{\mu_i}|x|) \longrightarrow c > 0, \quad \text{as } |x| \longrightarrow \infty \quad \text{for } i \in \{0, 1, 2\}. \quad (1.3)$$

(cf. [9]). For each $u \in L^4(\mathbb{R}^3)$, we put $\widehat{u}(x) = \int_{B_1(x)} |u(x)|^4 dx$ for $x \in \mathbb{R}^3$. Then from (1.3), we can choose $R_0 > 0$ such that

$$\widehat{U}_i(z) < \frac{|\widehat{U}_i|_\infty}{3} \quad \text{for all } z \in \mathbb{R}^3 \setminus B_{R_0}(0) \quad \text{and } i \in \{1, 2\}. \quad (1.4)$$

Since we consider the case that $\mu_1 \neq \mu_2$, we may assume without any loss of generality that $\mu_2 < \mu_1$. We can now state our main results.

Theorem 1. *Suppose that $\sqrt{\mu_1/\mu_2}$ is irrational. Then for each $i \in \{2, 4, 6, 8, 12, 20\}$ there exists $\beta_i \in (-1, 0)$ such that for each $\beta \in (\beta_i, 0)$, there exists a positive solution $\mathcal{U}_i \in \mathbb{H}$ of (P) such that $ic_1 + c_2 < \Phi(\mathcal{U}_i) < ic_1 + 2c_2$ and $\mathcal{U}_i$ has the form*

$$\mathcal{U}_i = (U, V) = \left(\sum_{j=1}^i U_{1,x_j} + u, U_2 + v \right) \quad (1.5)$$

where $\{x_1, x_2, \dots, x_i\}$ forms a regular i -polyhedra in $\mathbb{R}^3$ in case $i \neq 2$ and $x_1 = -x_2$ in case that $i = 2$, and $u, v \in H$ such that $\|(u, v)\|$ is so small that

$$\begin{aligned} \widehat{U}(z) &< \frac{1}{2} |\widehat{U}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus \left(\bigcup_{j=1}^i B_{R_0}(x_j) \right) \quad \text{and} \\ \widehat{V}(z) &< \frac{1}{2} |\widehat{V}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0). \end{aligned} \quad (1.6)$$

Remark 1. The assertion of Theorem 1 implies that for $\beta < 0$ with $|\beta|$ sufficiently small, problem (P) possesses at least 6 nonradial positive solutions. In [12], the existence of positive solutions of (P) of the form (1.5) was established in the case that $\{x_1, x_2, \dots, x_i\}$ forms regular cube or tetrahedra under the assumption

$$\sqrt{\frac{\mu_1}{\mu_2}} < \begin{cases} \frac{\sqrt{3}}{3} & \text{for the cube} \\ \frac{\sqrt{3}}{2} & \text{for the tetrahedra.} \end{cases}$$

Our argument employed in this paper does not require the ratio of $\sqrt{\mu_1}$ and $\sqrt{\mu_2}$.

Remark 2. One can see, from the proof of Theorem 1, that under condition (1.6), the absolute value $|x_i|$ of x_i is uniquely determined.

Theorem 2. *Suppose that $\sqrt{\mu_1/\mu_2}$ is irrational. Then for each $k \in \mathbb{N}$, there exists $\tilde{\beta}_k \in (-1, 0)$ such that for each $\beta \in (\tilde{\beta}_k, 0)$, the problem (P) has a positive solution*

$\tilde{\mathcal{U}}_k$ such that $kc_1 + c_2 < \Phi(\tilde{\mathcal{U}}_k) < kc_1 + 2c_2$ and $\tilde{\mathcal{U}}_k$ has the form

$$\tilde{\mathcal{U}}_k = (U, V) = \left(\sum_{j=1}^k U_{1,x_j} + u, U_2 + v \right) \quad (1.7)$$

where $\{x_1, x_2, \dots, x_k\} \subset \mathbb{R}^3$ form a regular k -polygon in a two dimensional subspace of $\mathbb{R}^3$, and $u, v \in H$ such that $\|(u, v)\|$ is so small that

$$\widehat{U}(z) < \frac{1}{2}|\widehat{U}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus (\cup_{j=1}^k B_{R_0}(x_j))$$

and

$$\widehat{V}(z) < \frac{1}{2}|\widehat{V}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus B_{R_0}(0).$$

Remark 3. The existence of positive solutions of (P) of the form (1.7) was proved in [12] in the case that the spacial dimension is 2 and μ_1, μ_2 satisfy

$$\sqrt{\frac{\mu_1}{\mu_2}} < \sin \frac{\pi}{k}.$$

To avoid unnecessary complexity, we give a proof of Theorem 1 for the case $i = 2$ in Section 2 and Section 3. The proofs of Theorem 1 for $i \neq 2$ and the proof of Theorem 2 are slight modifications of that of the case $i = 2$ of Theorem 1. The sketch of proofs for these cases are given in Section 4.

2. Preliminaries

Throughout the rest of this paper, we assume that $\sqrt{\mu_1/\mu_2}$ is irrational. For each $u, v \in H = H^1(\mathbb{R}^3)$, we put $\langle u, v \rangle = \int_{\mathbb{R}^3} uv$. We denote by $\langle \cdot, \cdot \rangle_{\mu_i}$ the inner product of H defined by $\langle u, v \rangle_{\mu_i} = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + \mu_i uv)$ for $u, v \in H$ and $i \in \{0, 1, 2\}$. The inner product of $\mathbb{H}$ is defined by $\langle \mathcal{U}_1, \mathcal{U}_2 \rangle_{\mathbb{H}} = \langle U_1, U_2 \rangle_{\mu_1} + \langle V_1, V_2 \rangle_{\mu_2}$ for $\mathcal{U}_1 = (U_1, V_1), \mathcal{U}_2 = (U_2, V_2) \in \mathbb{H}$. For each $u \in L^4(\mathbb{R}^3)$, we put $\Omega(u) = \{x \in \mathbb{R}^3 : \widehat{u}(x) \geq \frac{|\widehat{u}|_\infty}{2}\}$ and

$$\mathcal{B}(u) = \frac{\int_{\Omega(u)} x(\widehat{u}(x) - \frac{|\widehat{u}|_\infty}{2}) dx}{\int_{\Omega(u)} (\widehat{u}(x) - \frac{|\widehat{u}|_\infty}{2}) dx}.$$

The mapping $\mathcal{B}$ is called generalized barycenter, which is introduced in [5] (cf. also [4]). By Sobolev's embedding theorem [1], for $p \in [2, 6]$ there exists $m_p > 0$ such that

$$|z|_{L^p(B_r(0))} \leq m_p \left(|\nabla z|_{L^2(B_r(0))}^2 + \mu_i |z|_{L^2(B_r(0))}^2 \right)^{1/2}$$

for $r > 1, z \in H^1(B_r(0))$, and $i \in \{0, 1, 2\}$. For each $a \in \mathbb{R}$, and a functional $F : \mathbb{H} \rightarrow \mathbb{R}$, we denote by F^a the level set defined by $F^a = \{v \in \mathbb{H} : F^a(v) \leq a\}$. The same notation is used for functionals defined on H .

It is easy to see that for each $u \in H \setminus \{0\}$ with $u^+ \neq 0$, there exists a unique positive number t such that $tu \in \mathcal{S}_i$ (cf. [22]). It follows from the definitions of U_i that $U_i(x) = \sqrt{\mu_i}U_0(\sqrt{\mu_i}x)$ on $\mathbb{R}^3$. Then one can see that

$$c_1 = \sqrt{\mu_1}c_0 > c_2 = \sqrt{\mu_2}c_0. \quad (2.1)$$

Let $i \in \{0, 1, 2\}$. It is known that $\{U_{i,x} : x \in \mathbb{R}^3\}$ is a nondegenerate critical set of I_i (cf. [20]). More precisely, we have there exists $\lambda > 0$ such that

$$\|u\|_{\mu_i}^2 - 3\langle U_i^2 u, u \rangle \geq \lambda \|u\|_{\mu_i}^2 \quad \text{for all } u \in \left\{U_i, \frac{\partial U_i}{\partial x_1}, \frac{\partial U_i}{\partial x_2}, \frac{\partial U_i}{\partial x_3}\right\}^\perp. \quad (2.2)$$

We define a functional $\Phi : \mathbb{H} \longrightarrow \mathbb{R}$ associated with problem (P) by

$$\begin{aligned} \Phi(\mathcal{U}) &= \frac{1}{2}(\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) - \frac{1}{4}(|U^+|_4^4 + |v^+|_4^4) - \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ &= \Phi_1(\mathcal{U}) + \Phi_2(\mathcal{U}) \quad \text{for } \mathcal{U} = (U, V) \in \mathbb{H}, \end{aligned}$$

where

$$\Phi_1(\mathcal{U}) = \frac{1}{2} \|U\|_{\mu_1}^2 - \frac{1}{4} |U^+|_4^4 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2$$

and

$$\Phi_2(\mathcal{U}) = \frac{1}{2} \|V\|_{\mu_2}^2 - \frac{1}{4} |v^+|_4^4 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2.$$

Then a direct computation shows

$$\begin{aligned} \langle \nabla \Phi(\mathcal{U}), \mathcal{V} \rangle_{\mathbb{H}} &= \left\langle \begin{pmatrix} \nabla_u \Phi(\mathcal{U}) \\ \nabla_v \Phi(\mathcal{U}) \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle_{\mathbb{H}} \\ &= \langle -\Delta U + \mu_1 U - (U^+)^3 - \beta U^+ (V^+)^2, W \rangle \\ &\quad + \langle -\Delta V + \mu_2 V - (V^+)^3 - \beta (U^+)^2 V^+, Z \rangle \end{aligned}$$

for $\mathcal{U} = (U, V), \mathcal{V} = (W, Z) \in \mathbb{H}$. We put

$$\begin{aligned} \mathcal{M}_+ &= \left\{ (U, V) \in \mathbb{H} \setminus \{0\} : \|U\|_{\mu_1}^2 = |U^+|_4^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2, \right. \\ &\quad \left. \|V\|_{\mu_2}^2 = |V^+|_4^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \right\}. \end{aligned}$$

Then one can see that $\mathcal{U} = (U, V) \in \mathcal{M}_+$ is a critical point of Φ if and only if $\mathcal{U}$ is a positive solution of problem (P). From the definition, we have

$$\Phi_1(\mathcal{U}) = \frac{1}{4} \|U\|_{\mu_1}^2, \quad \Phi_2(\mathcal{U}) = \frac{1}{4} \|V\|_{\mu_2}^2 \quad \text{and} \quad \Phi(\mathcal{U}) = \frac{1}{4} (\|U\|_{\mu_1}^2 + \|V\|_{\mu_2}^2) \quad (2.3)$$

for $\mathcal{U} = (U, V) \in \mathcal{M}_+$. We also have that for each $\mathcal{U} = (U, V) \in \mathcal{M}_+$, there exists $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that $(sU, tV) \in \mathcal{M}_+$. In fact, for each $\mathcal{U} = (U, V)$, $U \not\equiv 0, V \not\equiv 0$, $(sU, tV) \in \mathcal{M}_+$ if and only if

$$\begin{pmatrix} s^2 \\ t^2 \end{pmatrix} = A^{-1} \begin{pmatrix} \|U\|_{\mu_1}^2 \\ \|V\|_{\mu_2}^2 \end{pmatrix}$$

where

$$A = \begin{pmatrix} |U^+|_4^4 & \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 & |V^+|_4^4 \end{pmatrix}.$$

Since $\beta \in (-1, 0)$, we have by the Schwartz's inequality that A^{-1} exists and then there exists a unique solution $(s, t) \in \mathbb{R}^+ \times \mathbb{R}^+$. For given $\mathcal{U} = (U, V) \in \mathbb{H}$ with $U \not\equiv 0, V \not\equiv 0$, we put $\mathcal{N}\mathcal{U} = \mathcal{N}(U, V) = (\mathcal{N}_1 U, \mathcal{N}_2 V) = (sU, tV) \in \mathcal{M}_+$.

We start from the following lemma which is a simple consequence from the uniqueness of the positive solution of (P_i)

Lemma 1. *Let $i \in \{1, 2\}$ and $\{u_n\} \subset S_i$ such that $\lim_{n \rightarrow \infty} I_i(u_n) = c_i$. Then there exists $\{x_n\} \subset \mathbb{R}^3$, $\{\alpha_n\} \subset \mathbb{R}^+$ and $\{v_n\} \subset H$ such that $\lim_{n \rightarrow \infty} \alpha_n = 1$, $v_n \in \{U_{i,x_n}\}^\perp$, $\lim_{n \rightarrow \infty} \|v_n\|_{\mu_i} = 0$,*

$$x_n = \mathcal{B}(u_n) \quad \text{and} \quad u_n = \alpha_n U_{i,x_n} + v_n \quad \text{for each } n \geq 1.$$

Proof. The assertion follows from a standard concentrate compactness argument (cf. Proposition 2.1 of [18]). Let $i \in \{1, 2\}$ and let $\{u_n\}$ satisfy the assumption. Then since $\{u_n\} \subset S_i$ is a minimizing sequence for the functional I_i on S_i , we have, by the uniqueness of the positive solution of (P_i) up to translation, that there exists $\{\tilde{x}_n\} \subset \mathbb{R}^3$, $\{\alpha_n\} \subset \mathbb{R}^+$ and $\{\tilde{v}_n\} \subset H$ such that $\lim_{n \rightarrow \infty} \|\tilde{v}_n\| = 0$ and $u_n = \alpha_n U_{i,\tilde{x}_n} + \tilde{v}_n$ for $n \geq 1$. We may assume that $\alpha = 1$. Then by the definition of $\mathcal{B}$, we have $\lim_{n \rightarrow \infty} |\mathcal{B}(u_n) - \tilde{x}_n| = 0$. Then putting $x_n = \mathcal{B}(u_n)$, we find that $u_n = U_{i,x_n} + (U_{i,\tilde{x}_n} - U_{i,x_n} + \tilde{v}_n)$ for $n \geq 1$ and

$$\lim_{n \rightarrow \infty} \|U_{i,\tilde{x}_n} - U_{i,x_n} + \tilde{v}_n\|_{\mu_i} = 0.$$

Let $\{(\theta_n, v_n)\} \subset \mathbb{R} \times H$ such that $v_n \in \{U_{i,x_n}\}^\perp$ and

$$U_{i,\tilde{x}_n} - U_{i,x_n} + \tilde{v}_n = \theta_n U_{i,x_n} + v_n \quad \text{for } n \geq 1.$$

Then

$$u_n = (1 + \theta_n) U_{i,x_n} + v_n \quad \text{for } n \geq 1,$$

$\alpha_n = (1 + \theta_n) \rightarrow 1$ as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} \|v_n\|_{\mu_i} = 0$. This completes the proof. $\square$

Lemma 2. *Let $U \in S_1$ such that $U = \alpha U_{1,x} + u$ where $\alpha \in (\frac{3}{4}, 2)$, $x \in \mathbb{R}^3$ and $u \in \{U_{1,x}\}^\perp$ with $\|u\|_{\mu_1}$ sufficiently small. Then*

$$I_1(U) \geq c_1 + \frac{1}{6} \|u\|_{\mu_1}^2. \quad (2.4)$$

Proof. Let $U = \alpha U_{1,x} + u$ satisfy the assumption. If $\alpha \geq 1$, then noting that $u \in \{U_{1,x}\}^\perp$,

$$I_1(U) = \frac{1}{4} \|\alpha U_{1,x} + u\|_{\mu_1}^2 \geq \frac{1}{4} (\|U_{1,x}\|_{\mu_1}^2 + \|u\|_{\mu_1}^2) \geq c_1 + \frac{1}{4} \|u\|_{\mu_1}^2.$$

Then (2.4) holds. We next assume that $\alpha < 1$. Since $U \in S_1$, we have

$$\|\alpha U_{1,x} + u\|_{\mu_1}^2 = ((\alpha U_{1,x} + u)^+)^4 \leq (\alpha U_{1,x} + u)^4.$$

Then noting that $\|u\|_{\mu_1}^2 \geq 3 \int_{\mathbb{R}^3} U_{1,x}^2 u^2$ by (2.2) and $u \in \{U_{1,x}\}^\perp$, we have

$$\begin{aligned} (\alpha^2 - \alpha^4) \|U_{1,x}\|_{\mu_1}^2 + \|u\|_{\mu_1}^2 &= \int_{\mathbb{R}^3} (4\alpha^3 U_{1,x}^3 u + 6\alpha^2 U_{1,x}^2 u^2) + o(\|u\|_{\mu_1}^2) \\ &= 4\alpha^3 \langle (-\Delta + \mu_1) U_{1,x}, u \rangle + \int_{\mathbb{R}^3} 6\alpha^2 U_{1,x}^2 u^2 + o(\|u\|_{\mu_1}^2) \\ &\leq 2\alpha^2 \|u\|_{\mu_1}^2 + o(\|u\|_{\mu_1}^2). \end{aligned}$$

That is

$$\alpha^2(1 - \alpha^2) \|U_{1,x}\|_{\mu_1}^2 \leq (2\alpha^2 - 1) \|u\|_{\mu_1}^2 + o(\|u\|_{\mu_1}^2).$$

Then by an elemental computation, we find that

$$\alpha^2 \geq \frac{\|U_{1,x}\|_{\mu_1}^2 - \frac{2}{9} \|u\|_{\mu_1}^2}{\|U_{1,x}\|_{\mu_1}^2}$$

for $\|u\|_{\mu_1}^2$ sufficiently small. Therefore

$$\|\alpha U_{1,x} + u\|_{\mu_1}^2 \geq \|U_{1,x}\|_{\mu_1}^2 - \frac{2}{9} \|u\|_{\mu_1}^2 + \|u\|_{\mu_1}^2 \geq \|U_{1,x}\|_{\mu_1}^2 + \frac{2}{3} \|v\|_{\mu_1}^2$$

for $\|u\|_{\mu_1}^2$ sufficiently small. This completes the proof. $\square$

Lemma 3. (1) For each $\beta < 0$ and $\mathcal{U} = (U, V) \in \mathcal{M}_+$,

$$\Phi_1(U) = \frac{1}{4} \|U\|_{\mu_1}^2 > c_1 \quad \text{and} \quad \Phi_2(V) = \frac{1}{4} \|V\|_{\mu_2}^2 > c_2. \quad (2.5)$$

(2) Let $\{\mathcal{U}_n\} = \{(U_n, V_n)\} \subset \mathbb{H} \setminus \{0\}$ satisfy $\lim_{n \rightarrow \infty} \nabla \Phi(U_n, V_n) = 0$. Then

$$\liminf_{n \rightarrow \infty} \Phi_1(\mathcal{U}_n) \geq c_1 \quad \text{if} \quad \liminf_{n \rightarrow \infty} \|U_n\|_{\mu_1}^2 > 0$$

and

$$\liminf_{n \rightarrow \infty} \Phi_2(\mathcal{U}_n) \geq c_2 \quad \text{if} \quad \liminf_{n \rightarrow \infty} \|V_n\|_{\mu_2}^2 > 0.$$

Proof. (1) Let $\beta < 0$. Let $\mathcal{U} = (U, V) \in \mathcal{M}_+$. Then we have

$$\|U\|_{\mu_1}^2 = |U^+|_4^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \leq |U^+|_4^4. \quad (2.6)$$

Then there exists $t \in (0, 1]$ such that $\|U\|_{\mu_1}^2 = t^2 |U^+|_4^4$. Since $tU \in S_1$, we have that

$$c_1 \leq I_1(tU) = \frac{t^2}{4} \|U\|_{\mu_1}^2 \leq \frac{1}{4} \|U\|_{\mu_1}^2.$$

If $\|U\|_{\mu_1}^2 / 4 = c_1$, we have that $t = 1$ and $U = U_{1,x}$ for some $x \in \mathbb{R}^3$. Then since $\text{supp } U_{1,x} = \mathbb{R}^3$, $\int_{\mathbb{R}^3} U^2 V^2 \neq 0$. This implies $\|U\|_{\mu_1}^2 = |U^+|_4^4 + \int_{\mathbb{R}^3} \beta (U^+)^2 (V^+)^2 < |U^+|_4^4$ and then $t < 1$. This is a contradiction. Therefore strict inequality $\frac{1}{4} \|U\|_{\mu_1}^2 > c_1$ holds. Similarly, we have $\frac{1}{4} \|V\|_{\mu_2}^2 > c_2$. This completes the proof of (1).

(2) Let $\{\mathcal{U}_n\} = \{(U_n, V_n)\} \subset \mathbb{H} \setminus \{0\}$ satisfy the assumption. Then we have

$$\|U_n\|_{\mu_1}^2 - |U_n^+|_4^4 - \beta \int_{\mathbb{R}^3} (U_n^+)^+ (V_n^+)^2 \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.$$

Then by the same argument as in the proof of (1), we have that $\liminf_{n \rightarrow \infty} \Phi_1(\mathcal{U}_n) \geq c_1$ if $\liminf_{n \rightarrow \infty} \|U_n\|_{\mu_1}^2 > 0$. We also have by the same argument that $\liminf_{n \rightarrow \infty} \Phi_2(\mathcal{U}_n) \geq c_2$ if $\liminf_{n \rightarrow \infty} \|V_n\|_{\mu_2}^2 > 0$. $\square$

Now to prove Theorem 1 for the case that $i = 2$, we define $H_2 \subset H$, $\mathbb{H}_2 \subset \mathbb{H}$ and $\mathcal{M}_2 \subset \mathcal{M}_+$ by

$$H_2 = \{u \in H : u(x) = u(-x) \text{ for } x \in \mathbb{R}^3\}, \quad \mathbb{H}_2 = H_2 \times H_2,$$

and

$$\mathcal{M}_2 = \mathcal{M}_+ \cap \mathbb{H}_2.$$

Since $\sqrt{\mu_1/\mu_2}$ is irrational, we can choose $\delta_2 \in (0, c_2)$ so small that

$$c_1 + kc_2 \notin [2c_1 + c_2 - \delta_2, 2c_1 + c_2 + \delta_2] \quad \text{for all } k \in \mathbb{N}. \quad (2.7)$$

Lemma 4. (1) *There exists $\beta_1 \in (-1, 0)$ such that for each $\beta \in (\beta_1, 0)$ and each critical point $\mathcal{U} \in \mathcal{M}_2 \cap \Phi^{2c_1+2c_2}$ of Φ ,*

$$\Phi(\mathcal{U}) \in \cup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_2/2, ic_1 + jc_2 + \delta_2/2].$$

(2) *Let $\beta \in (\beta_1, 0)$ and $\{\mathcal{U}_n\} \subset \mathcal{M}_2$ such that $\lim_{n \rightarrow \infty} \nabla \Phi(\mathcal{U}_n) = 0$ and*

$$\lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) = 2c_1 + c_2 + \varepsilon \quad \text{with } \varepsilon \in (0, \delta_2/2).$$

Then there exists a convergent subsequence $\{\mathcal{U}_{n_i}\} \subset \{\mathcal{U}_n\}$.

Proof. (1) Let $\{\beta_n\} \subset (-1, 0)$ with $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\{\mathcal{U}_n\} = \{(U_n, V_n)\} \subset \mathcal{M}_2$ such that $\lim_{n \rightarrow \infty} \nabla \Phi_u(\mathcal{U}_n) = \lim_{n \rightarrow \infty} \nabla \Phi_v(\mathcal{U}_n) = 0$ and $\lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) \leq 2c_1 + 2c_2$. Then we have $\lim_{n \rightarrow \infty} \nabla I_1(U_n) = \lim_{n \rightarrow \infty} \nabla I_2(V_n) = 0$. Then by the uniqueness of the positive solution of (P_i) , we have by the concentrate compactness argument (cf. Proposition 2.1 of [18]) that there exists $k_1 \in \mathbb{N}$ and sequences $\{x_{n,i}\} \subset \mathbb{R}^3$, $1 \leq i \leq k_1$ such that $\lim_{n \rightarrow \infty} |x_{n,i} - x_{n,j}| = \infty$ for $i \neq j$, and

$$U_n - \sum_{i=1}^{k_1} U_{1,x_{n,i}} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty \quad \text{in } H.$$

That is we find $\lim_{n \rightarrow \infty} \Phi_1(\mathcal{U}_n) = \lim_{n \rightarrow \infty} I_1(U_n) = k_1 c_1$. Similarly, we have that $\lim_{n \rightarrow \infty} \Phi_2(\mathcal{U}_n) = k_2 c_2$ for some $k_2 \geq 1$. Then the assertion follows.

(2) Let $\{\mathcal{U}_n\} = \{(U_n, V_n)\} \subset \mathcal{M}_2$ satisfy the assumption. Then by subtracting subsequences, we may assume that (U_n, V_n) has the form

$$U_n = U + u_n, V_n = V + v_n,$$

where

$$(U_n, V_n) \longrightarrow (U, V) \in \mathbb{H}_2 \quad \text{weakly in } \mathbb{H} \quad \text{and strongly in } L^4(\mathbb{R}^3)$$

and $\{(u_n, v_n)\} \subset \mathbb{H}_2$ is such that

$$(u_n, v_n) \longrightarrow (0, 0) \quad \text{weakly in } \mathbb{H} \quad \text{and strongly in } L^4_{loc}(\mathbb{R}^3).$$

From the definition, we have that (U, V) is a solution of (P), and

$$\lim_{n \rightarrow \infty} \nabla \Phi(u_n, V) = \lim_{n \rightarrow \infty} \nabla \Phi(U, v_n) = \lim_{n \rightarrow \infty} \nabla \Phi(u_n, v_n) = 0. \quad (2.8)$$

Case 1. Assume that $U \not\equiv 0$ and $\liminf_{n \rightarrow \infty} \|u_n\| > 0$. In this case, we reaches to a contradiction by a standard concentrate compactness argument. For completeness, we give a proof. Since $(U, V) \in \mathcal{M}_+$, $\Phi_1(U, V) \geq c_1$. We first note that vanishing of $\{|u_n|_{\mu_1}^2\}$ does not happen. (See Lemma 1.1 of [13]). In fact, if $\{|u_n|_{\mu_1}^2\}$ is vanishing, i.e., $\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} |u_n|_{\mu_1}^2 = 0$ for each $R > 0$, then we have

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \frac{\int_{B_R(y)} |u_n^+|^4}{\int_{B_R(y)} |u_n|_{\mu_1}^2} = 0 \quad \text{for each } R > 0.$$

Then we find that

$$\lim_{n \rightarrow \infty} \frac{|u_n^+|^4 - \beta \int_{\mathbb{R}^3} (u_n^+)^2 (v_n^+)^2}{\|u_n\|_{\mu_1}^2} \leq 0.$$

On the other hand, we have by (2.8) that

$$\lim_{n \rightarrow \infty} \frac{|u_n^+|^4 - \beta \int_{\mathbb{R}^3} (u_n^+)^2 (v_n^+)^2}{\|u_n\|^2} = 1.$$

This is a contradiction. Then vanishing of $\{|u_n|_{\mu_1}\}$ does not happen. On the other hand, we can see, noting that $u_n(x) = u_n(-x)$ on $\mathbb{R}^3$, that compactness of $\{|u_n|_{\mu_1}\}$ does not happen. Therefore subtracting subsequence, we have again by Lemma 1.1 of [13] that there exist $\lambda_1, \lambda_2 > 0$ and sequences $\{u_n^1\}, \{u_n^2\} \subset H$ such that $\text{dist}(\text{Supp } u_n^1, \text{Supp } u_n^2) \rightarrow \infty$, as $n \rightarrow \infty$, and

$$\begin{aligned} \lim_{n \rightarrow \infty} \|u_n - (u_n^1 + u_n^2)\| &= 0, \\ \lim_{n \rightarrow \infty} \|u_n - u_n^1\| &= \lambda_1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \|u_n - u_n^2\| = \lambda_2. \end{aligned}$$

Then it follows that

$$\lim_{n \rightarrow \infty} \nabla \Phi(u_n^1, V_n) = \lim_{n \rightarrow \infty} \nabla \Phi(u_n^2, V_n) = 0.$$

Therefore we have by (2) of Lemma 3 that

$$\lim_{n \rightarrow \infty} \Phi_1(u_n, V_n) = \lim_{n \rightarrow \infty} \Phi_1(u_n^1, V_n) + \lim_{n \rightarrow \infty} \Phi_1(u_n^2, V_n) \geq 2c_1. \quad (2.9)$$

Thus we find

$$\lim_{n \rightarrow \infty} \Phi_1(\mathcal{U}_n) = \lim_{n \rightarrow \infty} \Phi_1(u_n, V_n) + \lim_{n \rightarrow \infty} \Phi_1(U, V) \geq 3c_1.$$

This is a contradiction. Then case 1 does not happen.

Case 2. Assume that $U \not\equiv 0$ and $\liminf_n \|u_n\|_{\mu_1} = 0$. Then subtracting subsequences, we have $\lim_{n \rightarrow \infty} \nabla_v \Phi(U_n, v_n) = \lim_{n \rightarrow \infty} \nabla I_2(v_n) = 0$. Assume that $\liminf_n \|v_n\|_{\mu_2} > 0$. Then we have $\lim_{n \rightarrow \infty} \Phi_2(u_n, v_n) = kc_2$ for some $k \in \mathbb{N}$. Then

$$\lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) = \Phi(U, V) + \lim_{n \rightarrow \infty} \Phi_2(u_n, v_n) = \Phi(U, V) + kc_2 \leq 2c_1 + c_2 + \varepsilon.$$

Then noting that $\Phi_1(U) \geq c_1$, we find by (1) that $\Phi(U, V) \in (c_1 + lc_2 - \delta_2/2, c_1 + lc_2 + \delta_2/2)$ for some $l \in \mathbb{N}$. Then

$$2c_1 + c_2 + \varepsilon = \lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) \in (c_1 + (l+k)c_2 - \delta_2/2, c_1 + (l+k)c_2 + \delta_2/2).$$

Since $\varepsilon \in (0, \delta_2/2)$, this contradicts to (2.7). Therefore there exists a subsequence $\{v_{n_i}\} \subset \{v_n\}$ such that $\lim_{i \rightarrow \infty} \|v_{n_i}\| = 0$. That is $\lim_{i \rightarrow \infty} \Phi(\mathcal{U}_{n_i}) = \Phi(U, V)$. This implies that $\{\mathcal{U}_{n_i}\}$ is a strongly convergent subsequence of $\{\mathcal{U}_n\}$.

Case 3. Assume that $U \equiv 0$. Then we have $\liminf_n \|u_n\| > 0$. Therefore (2.9) holds. That is $\liminf_n \Phi(\mathcal{U}_n) \geq 2c_1$. Then $\limsup_n \Phi_2(\mathcal{U}_n) \leq c_2 + \varepsilon$. This implies $\liminf_n \|v_n\| = 0$. In fact, if $\liminf_n \|v_n\| > 0$, we have by the same argument as above that $\limsup_n \Phi_2(\mathcal{U}_n) \geq 2c_2$. This is a contradiction. Therefore by the uniqueness of positive solution of (P_2) , we have

$$\begin{aligned} \liminf_n \Phi_2(\mathcal{U}_n) &= \liminf_n \Phi_2(0, V) = \liminf_n I_2(V) = c_2 \quad \text{and} \\ \liminf_n \Phi_1(\mathcal{U}_n) &= 2 \liminf_n I_1(u_n) = 2c_1. \end{aligned}$$

That is $\lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) = 2c_1 + c_2$. This is a contradiction. Therefore $U \equiv 0$ can not happen. This completes the proof. $\square$

Lemma 5. For given $\varepsilon > 0$, there exists $\beta_\varepsilon \in (\beta_1, 0)$ such that for each $\beta \in (\beta_\varepsilon, 0)$ and for each $x \in \mathbb{R}^3 \setminus \{0\}$,

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \varepsilon \quad (2.10)$$

and

$$\Phi(\mathcal{N}(U_1, U_2)) < \frac{7}{4}c_1 + c_2. \quad (2.11)$$

Proof. Let $\beta \in (\beta_1, 0)$. Let $x \in \mathbb{R}^3 \setminus \{0\}$ and $t_1(x), t_2(x) > 0$ such that $(t_1(x)(U_{1,x} + U_{1,-x}), t_2(x)U_2) \in \mathcal{M}_2$. That is

$$\begin{pmatrix} t_1(x)^2 \\ t_2(x)^2 \end{pmatrix} = A_x^{-1} \begin{pmatrix} \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 \\ \|U_2\|_{\mu_2}^2 \end{pmatrix}, \quad (2.12)$$

where

$$A_x = \begin{pmatrix} |U_{1,x} + U_{1,-x}|_4^4 & \beta \int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^2 \\ \beta \int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^2 & |U_2|_4^4 \end{pmatrix}.$$

Since $\beta \in (\beta_1, 0) \subset (-1, 0)$, we can see by Schwartz's inequality that $\inf_{x \in \mathbb{R}^3} |A_x| > 0$. Then one can see that there exists $M > 0$ such that

$$\max \{t_1(x), t_2(x)\} < M \quad \text{for all } x \in \mathbb{R}^3 \setminus \{0\}, \quad (2.13)$$

where M is independent of $\beta \in (\beta_1, 0)$. Then we have

$$L = \max_{x \in \mathbb{R}^3} \left\{ t_2(x)^2 \int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^2, t_1(x)^2 \int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^2 \right\} < \infty. \quad (2.14)$$

Then by the definition of U_1 , we have by

$$\begin{aligned} \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 &= \|U_{1,x}\|_{\mu_1}^2 + \|U_{1,-x}\|_{\mu_1}^2 \\ &\quad + 2 \langle U_{1,x}, U_{1,-x} \rangle_{\mu_1} \\ &= |U_{1,x}|_4^4 + |U_{1,-x}|_4^4 + 2 \int_{\mathbb{R}^3} U_{1,x}^3 U_{1,-x} \\ &= |U_{1,x}|_4^4 + |U_{1,-x}|_4^4 + 2 \int_{\mathbb{R}^3} U_{1,x} U_{1,-x}^3. \end{aligned}$$

Then noting that $\int_{\mathbb{R}^3} U_{1,x}^2 U_{1,-x}^2 \leq |U_{1,x}|_4^4$, we find

$$\frac{\|U_{1,x} + U_{1,-x}\|_{\mu_1}^4}{|U_{1,x} + U_{1,-x}|_4^4} \leq |U_{1,x}|_4^4 + |U_{1,-x}|_4^4 = 8c_1 \quad \text{for } x \in \mathbb{R}^3.$$

Therefore we have by (2.12), (2.14) and the inequality above that

$$\begin{aligned} t_1(x)^2 \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 &\leq \left(\frac{\|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 - \beta L}{|U_{1,x} + U_{1,-x}|_4^4} \right) \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 \\ &\leq 8c_1 \left(1 - \frac{\beta L}{|U_1|_4^4} \right). \end{aligned}$$

Similarly, we have that

$$t_2(x)^2 \|U_2\|_{\mu_1}^2 \leq 4c_2 \left(1 - \frac{\beta L}{|U_2|_4^4} \right).$$

Then combining the inequalities above, we have

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) \leq 2c_1 + c_2 - \left(\frac{\beta L c_1}{|U_1|_4^4} + \frac{\beta L c_2}{|U_2|_4^4} \right) = 2c_1 + c_2 - 8\beta L.$$

Then for given $\varepsilon > 0$, we can choose $\beta_\varepsilon < 0$ with $|\beta_2|$ sufficiently small that (2.10) holds. The inequality (2.11) can be proved by a parallel argument as above. Then we omit the proof. $\square$

3. Proof of Theorem 1 for $i = 2$.

Throughout this section we assume that $\beta \in (\beta_1, 0)$. We put

$$b_R(U) = \int_{\mathbb{R}^3 \setminus B_R(0)} |U|_{\mu_1}^2 \quad \text{for } U \in H \quad \text{and } R > 0$$

and

$$\Lambda_{2,\varepsilon}(R) = \left\{ \mathcal{U} = (U, V) \in \Phi^{2c_1+c_2+\varepsilon} \cap \mathcal{M}_2 : b_R(U) \geq 8c_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \right\}$$

for each $\varepsilon > 0$ and $R > 0$.

Proposition 1. *For $\varepsilon > 0$ sufficiently small, there exists $(R_\varepsilon, \delta_\varepsilon, \alpha_\varepsilon, \gamma_\varepsilon) \in (\mathbb{R}^+)^4$ such that $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon = 0$ and each $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ has the form*

$$\mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v) \quad (3.1)$$

where $\alpha \in (1 - \alpha_\varepsilon, 1 + \alpha_\varepsilon), \gamma \in (1 - \gamma_\varepsilon, 1 + \gamma_\varepsilon)$,

$$|x| \geq R_\varepsilon, \quad x = \mathcal{B}(U|_{B_{R_0}(x)}), \quad (3.2)$$

$$\widehat{U}(z) < \frac{1}{2}|\widehat{U}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus \bigcup_{i=\pm 1} B_{R_0}(ix), \quad (3.3)$$

and

$$(u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp \quad \text{with} \quad \|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\varepsilon. \quad (3.4)$$

Proof. Let $r > 1$ and $z \in \mathbb{R}^3$ with $|z| = 1$, we put

$$B_{r,z} = \left(B_{r+1}(0) \cup \{x \in \mathbb{R}^3 : |\langle x, z \rangle_{\mathbb{R}^3}| < 1\} \right) \setminus \overline{B_r(0)}.$$

Let $\varphi_{r,z} \in C^\infty(\mathbb{R}^3; [0, 1])$ such that $\varphi_{r,z}(x) = \varphi_{r,z}(-x)$ on $\mathbb{R}^3$, $|\nabla \varphi|_\infty \leq 4$, and

$$\varphi_{r,z} = \begin{cases} 1 & \text{on } \mathbb{R}^3 \setminus B_{r,z} \\ 0 & \text{on } \partial B_{r+\frac{1}{2}}(0) \cup \{x \in \mathbb{R}^3 : \langle x, z \rangle = 0, |x| \geq r + \frac{1}{2}\}. \end{cases}$$

Then we have that there exists $C_1 > 1$ such that

$$\max \left\{ \int_{B_{r,z}} |\varphi_{r,z} U|_{\mu_1}^2, \int_{B_{r,z}} |U^+|^4 \right\} \leq C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 \quad (3.5)$$

for $(r, z) \in \mathbb{R}^+ \times \partial B_1(0)$ and $U \in \Phi^{2c_1+2c_2} \cap \mathcal{M}_2$.

Now let $\varepsilon \in (0, \min\{1/8m_4^4, c_1\})$, R_ε be a positive number such that $R_\varepsilon > r + \frac{1}{2}$, and $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$. From the definition of $\varphi_{r,z}$, we have that $\varphi_{r,z}U$ has the form

$$\varphi_{r,z}U = W_0 + W_+ + W_-, \quad (3.6)$$

where $\text{supp } W_0, \text{supp } W_+, \text{supp } W_-$ are disjoint, $\text{supp } W_0 \subset B_{r+\frac{1}{2}}(0)$ and $W_+(x) = W_-(-x)$ for $x \in \mathbb{R}^3$. Since $\|V\|_{\mu_2}^2 \geq 4c_2$ by (1) of Lemma 3,

$$\|U\|_{\mu_1}^2 = 4\Phi_1(\mathcal{U}) \leq 8c_1 + 4\varepsilon. \quad (3.7)$$

We also have

$$\begin{aligned} \left(\int_{B_{r+\frac{1}{2}}(0)} |U^+|^4 \right)^{\frac{1}{2}} &\leq m_4^2 \int_{B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 \leq m_4^2 \left(8c_1 + 4\varepsilon - \int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 \right) \\ &\leq m_4^2 (8c_1 + 4\varepsilon - b_{R_\varepsilon}(U)) < \frac{1}{m_4^2}. \end{aligned}$$

Then we find

$$\begin{aligned} \int_{B_{r+\frac{1}{2}}(0)} |U^+|^4 &= \frac{1}{m_4^2} \left(m_4^2 \left(\int_{B_{r+\frac{1}{2}}(0)} |U^+|^4 \right)^{\frac{1}{2}} \right) \left(\int_{B_{r+\frac{1}{2}}(0)} |U^+|^4 \right)^{\frac{1}{2}} \\ &< \int_{B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2. \end{aligned}$$

Then

$$\int_{B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 > \int_{B_{r+\frac{1}{2}}(0)} |U^+|^4 + \beta \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2.$$

Therefore noting that $\mathcal{U} \in \mathcal{M}_2$, we have

$$\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} (|U|_{\mu_1}^2 - |U^+|^4) < 0. \quad (3.8)$$

On the other hand, we have from (3.5) and (3.6) that

$$\begin{aligned} \|W_+\|_{\mu_1}^2 &\leq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 + \int_{B_{r,z}} |\varphi_{r,z} U|_{\mu_1}^2 \right) \\ &\leq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 + C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 \right), \end{aligned} \quad (3.9)$$

and

$$\begin{aligned} |W_+|^4_4 &\geq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U^+|^4 - \int_{B_{r,z}} |U^+|^4 \right) \\ &\geq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U^+|^4 - C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 \right). \end{aligned} \quad (3.10)$$

Then noting $\beta < 0$, we have from (3.8), (3.9) and (3.10) that

$$\begin{aligned} \|W_+\|_{\mu_1}^2 &= \int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |W_+|_{\mu_1}^2 \leq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U|_{\mu_1}^2 + C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 \right) \\ &\leq \frac{1}{2} \left(\int_{\mathbb{R}^3 \setminus B_{r+\frac{1}{2}}(0)} |U^+|^4 + C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 \right) \leq |W_+|_4^4 + C_1 \int_{B_{r,z}} |U|_{\mu_1}^2. \end{aligned}$$

Now let $t > 0$ such that

$$\|W_+\|^2 = t^2 |W_+^+|_4^4.$$

Then we have

$$t^2 = \frac{\|W_+\|^2}{|W_+^+|_4^4} \leq \frac{\|W_+\|^2}{\|W_+\|^2 - C_1 \int_{B_{r,z}} |U|_{\mu_1}^2} \leq 1 + \frac{\alpha(r, z)}{\|W_+\|^2 - \alpha(r, z)}, \quad (3.11)$$

where $\alpha(r, z) = C_1 \int_{B_{r,z}} |U|_{\mu_1}^2$. Since (3.5), (3.7) and $tW_+ \in S_1$ hold, we find

$$\begin{aligned} 4c_1 \leq t^2 \|W_+\|_{\mu_1}^2 &\leq \left(1 + \frac{\alpha(r, z)}{\|W_+\|^2 - \alpha(r, z)} \right) \left(\frac{1}{2} \int_{\mathbb{R}^3 \setminus B_{r,z}} |U|_{\mu_1}^2 + \int_{B_{r,z}} |\varphi_{r,z}^+ U|_{\mu_1}^2 \right) \\ &\leq \left(1 + \frac{\alpha(r, z)}{\|W_+\|^2 - \alpha(r, z)} \right) (4c_1 + 2\varepsilon + \alpha(r, z)). \end{aligned}$$

We now choose $R_\varepsilon > 0$ sufficiently large that there exists $(r, z) \in (R_\varepsilon/2, R_\varepsilon) \times \partial B_1(0)$ such that

$$\alpha(r, z) = C_1 \int_{B_{r,z}} |U|_{\mu_1}^2 < \varepsilon. \quad (3.12)$$

Let (r, z) satisfy (3.12). Then

$$4c_1 \leq t^2 \|W_+\|_{\mu_1}^2 \leq \left(1 + \frac{\varepsilon}{\|W_+\|^2 - \varepsilon} \right) (4c_1 + 3\varepsilon). \quad (3.13)$$

Since

$$\begin{aligned} \|W_+\|_{\mu_1}^2 &\geq \frac{1}{2} \int_{\mathbb{R}^N \setminus B_{r,z}} |U|_{\mu_1}^2 \\ &= \frac{1}{2} \left(\|U\|_{\mu_1}^2 - \int_{B_{r,z}} |U|_{\mu_1}^2 \right) \\ &\geq \frac{1}{2} \left(b_{R_\varepsilon}(U) - \frac{\alpha(r, z)}{C_1} \right) \\ &\geq \frac{1}{2} \left(7c_1 - \frac{\varepsilon}{C_1} \right) \geq 3c_1, \end{aligned}$$

we have by (3.13) that

$$4c_1 \leq t^2 \|W_+\|_{\mu_1}^2 \leq \left(1 + \frac{\varepsilon}{3c_1 - \varepsilon}\right) (4c_1 + 3\varepsilon).$$

Then by Lemma 1 that for $\varepsilon > 0$ sufficiently small, tW_+ has the form

$$tW_+ = \tilde{\alpha}U_{1,\tilde{x}} + \tilde{u} \quad (3.14)$$

where $\tilde{\alpha} \rightarrow 1$ and $\|\tilde{u}\|_{\mu_1} \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $\tilde{x} \in \mathbb{R}^3$ satisfies $\mathcal{B}(W_+) = \tilde{x}$ and $|\tilde{x}| \rightarrow \infty$, as $\varepsilon \rightarrow 0$. On the other hand, we have by (3.7) and (3.9) that

$$\|W_+\|_{\mu_1}^2 \leq \frac{1}{2}(8c_1 + 4\varepsilon + \varepsilon) \leq 4c_1 + 4\varepsilon.$$

Then

$$\frac{c_1}{c_1 + \varepsilon} \leq \frac{4c_1}{\|W_+\|_{\mu_1}^2} \leq t^2.$$

That is we have by (3.11) and the inequalities above that $t \rightarrow 1$ as $\varepsilon \rightarrow 0$. Since $\varphi_{r,z} = 1$ on $\mathbb{R}^3 \setminus B_{r,z}$ and (3.12) holds, we find $\|U - \varphi_{r,z}U\|_{\mu_1}^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then noting that $\|W_+ + W_-\|^2 \rightarrow 8c_1$ as $\varepsilon \rightarrow 0$ by (3.14), and $\|U\|_{\mu_1}^2 \leq 8c_1 + 4\varepsilon$, we find that $\|W_0\|_{\mu_1}^2 \rightarrow 0$, as $\varepsilon \rightarrow 0$. Therefore U has the form

$$U = \tilde{\alpha}(U_{1,\tilde{x}} + U_{1,-\tilde{x}}) + \hat{u},$$

where $\hat{u} \in H$ with $\|\hat{u}\|_{\mu_1}^2 \rightarrow 0$ and $\tilde{\alpha} \rightarrow 1$, as $\varepsilon \rightarrow 0$. Then we have as in the proof of Lemma 1 that there exists $x \in \mathbb{R}^3$, $\alpha \in (0, 2)$ and $u \in \{U_{1,x}, U_{1,-x}\}^\perp$ such that $x = \mathcal{B}(U|_{B_{R_0}(x)})$ with $|x| \geq R_\varepsilon$,

$$U = \alpha(U_{1,x} + U_{1,-x}) + u,$$

and it follows that $\|u\|_{\mu_1}^2 \rightarrow 0$ and $\alpha \rightarrow 1$ as $\varepsilon \rightarrow 0$. From (1.4), One can see that $\widehat{U}(z) < \frac{1}{2}|\widehat{U}|_\infty$ on $\mathbb{R}^3 \setminus (B_{R_0}(x) \cup B_{R_0}(-x))$. Then (3.2) holds. By a parallel argument, one can see that the assertions for V also follows, and then (3.3) and (3.4) holds. $\square$

Remark 4. By (3.2) and the definition of $\mathcal{B}$, one can see that for each $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon)$, $(x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$ in (3.1) is uniquely determined, and the mapping $\mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon) \rightarrow (x, -x) \in \mathbb{R}^3 \times \mathbb{R}^3$ is continuous. We define a continuous mapping $\eta : \Lambda_{2,\varepsilon}(R_\varepsilon) \rightarrow \mathbb{R}^+$ by

$$\eta(\mathcal{U}) = |x| \quad \text{for } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon). \quad (3.15)$$

We next prove the following Proposition.

Proposition 2. *There exists $M_0 > 0$ satisfying that for $\varepsilon > 0$ sufficiently small,*

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 - \beta M_0 e^{-2\sqrt{\mu_2}|x|} \quad \text{for each } \mathcal{U} \in \Lambda_{2,\varepsilon}(R_\varepsilon),$$

where $x \in \mathbb{R}^3$ such that $\mathcal{U}$ has the form (3.1).

To prove Proposition 2, we need following two lemmas.

Lemma 6. *There exists $k_1 > 0$ such that for $\varepsilon > 0$ sufficiently small, each $\mathcal{U} = (U, V) \in \mathcal{M}_2 \cap \Phi^{2c_1+c_2+\varepsilon}$ satisfies*

$$\Phi_2(\mathcal{U}) \geq c_2 + k_1 \|v\|_{\mu_1}^2,$$

where $v \in \{U_2\}^\perp$ such that $V = \alpha U_2 + v$ for some $\alpha \in \mathbb{R}$.

Proof. Let $\varepsilon > 0$ sufficiently small and $\mathcal{U} = (U, V) \in \mathcal{M}_2 \cap \Phi^{2c_1+c_2+\varepsilon}$. We have from Proposition 1 that V has the form $V = (1 + \theta)U_2 + v$, where $v \in \{U_2\}^\perp$ and $|\theta|$ is small. If $\theta \geq 0$, then the assertion holds with $k_1 = 1$. We assume that $\theta < 0$. Since $v(x) = v(-x)$ on $\mathbb{R}^3$ and $\frac{\partial U_2}{\partial x_i}$ is odd for each $1 \leq i \leq 3$, we have that $\langle v, \frac{\partial U_2}{\partial x_i} \rangle = 0$ for $1 \leq i \leq 3$. Then noting that

$$\langle -\Delta U_2 + \mu_2 U_2, v \rangle = \langle U_2^3, v \rangle = 0,$$

we have

$$\begin{aligned} \|(1 + \theta)U_2 + v\|_{\mu_2}^2 &= \int_{\mathbb{R}^3} \left[\left(((1 + \theta)U_2 + v)^+ \right)^4 + \beta(U^+)^2(V^+)^2 \right] \\ &\leq \int_{\mathbb{R}^3} \left[(1 + \theta)^4 U_2^4 + 6(1 + \theta)^2 U_2^2 v^2 + 4(1 + \theta) U_2 v^3 + v^4 \right. \\ &\quad \left. + \beta(U^+)^2(V^+)^2 \right]. \end{aligned}$$

Then since $\|U_2\|_{\mu_2}^2 = \int_{\mathbb{R}^3} U_2^4$, we have

$$\theta \|U_2\|_{\mu_2}^2 + O(\theta^2) \geq \frac{1}{2} (\|v\|_{\mu_2}^2 - \int_{\mathbb{R}^3} [6U_2^2 v^2 + 4(1 + \theta) U_2 v^3 + v^4 + \beta(U^+)^2(V^+)^2]).$$

Then by (2.2),

$$\begin{aligned} \|(1 + \theta)U_2 + v\|_{\mu_2}^2 &= (1 + \theta)^2 \|U_2\|_{\mu_2}^2 + \|v\|_{\mu_2}^2 \\ &\geq 4c_2 + 2\|v\|_{\mu_2}^2 - \int_{\mathbb{R}^3} [6U_2^2 v^2 + 4(1 + \theta) U_2 v^3 + v^4 \\ &\quad + \beta(U^+)^2(V^+)^2] \\ &\geq 4c_2 + 2 \left(\|v\|_{\mu_2}^2 - 3 \int_{\mathbb{R}^3} U_2^2 v^2 \right) + o(\|v\|_{\mu_2}^2) \\ &\quad - \beta \int_{\mathbb{R}^3} (U^+)^2(V^+)^2 \geq 4c_2 + 2\lambda \|v\|_{\mu_2}^2 + o(\|v\|_{\mu_2}^2). \end{aligned}$$

Since $\|v\|_{\mu_2}^2 \rightarrow 0$, as $\varepsilon \rightarrow 0$, the assertion holds with $k_1 = \lambda/3$. This completes the proof. $\square$

Here we fix $\rho \in (0, \sqrt{\mu_1}/2)$ such that

$$\sqrt{\mu_2} < \sqrt{\mu_1} - 2\rho. \quad (3.16)$$

Lemma 7. *There exists $k_2 > 0$ such that for $\varepsilon > 0$ sufficiently small,*

$$\begin{aligned} \Phi_1(\mathcal{U}) &\geq 2c_1 - \frac{\beta}{8} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - k_2 e^{-2(\sqrt{\mu_1} - \rho)|x|} \\ \mathcal{U} &= (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon), \end{aligned} \quad (3.17)$$

where $x \in \mathbb{R}^3$ is the element satisfying (3.1).

Proof. For $r \geq 1$ and $x \in \partial B_1(0)$, we put

$$\Omega_0(r, x) = \{z \in \mathbb{R}^3 : |\langle x, z \rangle| \leq r\}, \quad \Omega_\pm(r, x) = \{z \in \mathbb{R}^3 : \langle \pm x, z \rangle \geq r\}$$

and

$$a_{n,x}(U) = \int_{\Omega_0(n,x)} |U|_{\mu_1}^2 dx \quad \text{for each } n \geq 1 \quad \text{and } U \in H.$$

We also fix functions $\psi_{n,z}^\pm \in C^\infty(\mathbb{R}^3, [0, 1])$ such that

$$\psi_{n,x}^\pm(z) = 1 \quad \text{on } \Omega_\pm(n+1, x) \quad \text{and} \quad \psi_{n,x}^\pm(z) = 0 \quad \text{on } \mathbb{R}^3 \setminus \Omega_\pm(n, x)$$

and $|\nabla \psi_{n,x}^\pm| \leq 2$ on $\mathbb{R}^3$.

Step 1: We first show that there exists $C_1 > 0$ such that for $\varepsilon > 0$ sufficiently small and $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$, the following inequality holds:

$$\begin{aligned} \Phi_1(U) &= \frac{1}{4} \|U_{1,x} + U_{1,-x} + u\|_{\mu_1}^2 \\ &\geq \frac{1}{4} \left(\int_{\Omega_+(n,x)} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 \right. \\ &\quad + \int_{\Omega_-(n,x)} |U_{1,-x} + \psi_{n,x}^-(U_{1,x} + u)|_{\mu_1}^2 \\ &\quad \left. + \int_{\Omega_0(n,x)} |U|_{\mu_1}^2 - 2C_1(a_{n+1,x}(U) - a_{n,x}(U)) - 2C_1 e^{-2\sqrt{\mu_1}(|x|-n)} \right), \end{aligned} \quad (3.18)$$

for each $1 \leq n \leq (\rho/\sqrt{\mu_1})|x|$, where $x \in \mathbb{R}^3$ satisfying (3.1) and $u \in H$ such that $U = U_{1,x} + U_{1,-x} + u$. Now let $\varepsilon > 0$, $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ and $1 \leq n \leq (\rho/\sqrt{\mu_1})|x|$. Noting that $|\nabla \psi_{n,x}^\pm| \leq 2$, we have

$$\begin{aligned} &\int_{\Omega_0(n+1,x) \setminus \Omega_0(n,x)} |\psi_{n,x}^+ U|_{\mu_1}^2 \\ &\leq 2 \int_{\Omega_0(n+1,x) \setminus \Omega_0(n,x)} (|U|_{\mu_1}^2 + 4|U|^2) \quad \text{for } U \in H. \end{aligned} \quad (3.19)$$

By choosing ε sufficiently small, we have by Proposition 1 that U has the form $U = U_{1,x} + U_{1,-x} + u$. We may assume that $|x|$ is sufficiently large and $\|u\|_{\mu_1} < \frac{1}{4m_4^4}$. Then by (1.3), we have that

$$|U_{1,\pm x}(z)| \leq e^{-\sqrt{\mu_1}(|x|-n)} \quad \text{for } z \in \Omega_0(n, x) \cup \Omega_\mp(n, x). \quad (3.20)$$

Since

$$\begin{aligned} |U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 &\leq 2|U_{1,\pm x} - \psi_{n,x}^\pm U_{1,\pm x}|_{\mu_1}^2 \\ &\quad + 2|\psi_{n,x}^\pm(U_{1,x} + U_{1,-x} + u)|_{\mu_1}^2 \\ &= 2|U_{1,\pm x} - \psi_{n,x}^\pm U_{1,\pm x}|_{\mu_1}^2 + 2|\psi_{n,x}^\pm U|_{\mu_1}^2, \end{aligned}$$

we have by (3.19) and (3.20) that there exists $C_1 > 0$ such that

$$\begin{aligned} \int_{\Omega_0(n+1,x) \setminus \Omega_0(n,x)} (|U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2) &\quad (3.21) \\ &\leq C_1 \left(\int_{\Omega_0(n+1,x) \setminus \Omega_0(n,x)} |U|_{\mu_1}^2 dx + e^{-2\sqrt{\mu_1}(|x|-n)} \right) \\ &= C_1 \left((a_{n+1,x}(U) - a_{n,x}(U)) \right. \\ &\quad \left. + e^{-2\sqrt{\mu_1}(|x|-n)} \right). \end{aligned}$$

Then

$$\begin{aligned} \int_{\Omega_\pm(n,x)} |U|_{\mu_1}^2 &\geq \int_{\Omega_\pm(n+1,x)} |U|_{\mu_1}^2 & (3.22) \\ &= \int_{\Omega_\pm(n+1,x)} |U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 \\ &\geq \int_{\Omega_\pm(n,x)} |U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 \\ &\quad - \int_{\Omega_0(n+1,x) \setminus \Omega_0(n,x)} |U_{1,x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 \\ &\geq \int_{\Omega_\pm(n,x)} |U_{1,x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 \\ &\quad - C_1 \left((a_{n+1,x}(U) - a_{n,x}(U)) + e^{-2\sqrt{\mu_1}(|x|-n)} \right). \end{aligned}$$

Then noting that

$$\Phi_1(\mathcal{U}) = \frac{1}{4} \left(\int_{\Omega_+(n,x)} |U|_{\mu_1}^2 + \int_{\Omega_-(n,x)} |U|_{\mu_1}^2 + \int_{\Omega_0(n,x)} |U|_{\mu_1}^2 \right),$$

we find from (3.22) that (3.18) holds. Then to prove the assertion of Lemma 7, it is sufficient to show that the right hand side of inequality (3.18) is greater than that of (3.17).

Step 2: We show that inequality (3.17) holds in case that

$$a_{n+1,x}(U) - a_{n,x}(U) < (8C_1)^{-1} a_{n,x}(U)$$

for some $1 \leq n \leq (\rho/\sqrt{\mu_1})|x|$. That is we assume that there exists $n \in \mathbb{N}$ such that $1 \leq n \leq (\rho/\sqrt{\mu_1})|x|$ and

$$\int_{\Omega_0(n,x)} |U|_{\mu_1}^2 \geq 8C_1 (a_{n+1,x}(U) - a_{n,x}(U)). \quad (3.23)$$

Recalling that $\int_{\Omega_0(n+1,x)} |u|_{\mu_1}^2 < \frac{1}{4m_4^4}$ and $|U_{1,\pm x}(z)| = O(e^{-\sqrt{\mu_1}(|x|-n)})$ on $\Omega_0(n+1, x)$, we have that for $|x|$ sufficiently large,

$$\int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2 < \frac{1}{2m_4^4}.$$

Then noting that

$$|U|_{L^4(\Omega_0(n+1,x))}^4 \leq m_4^4 \left(\int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2 \right)^2 \leq (1/2) \int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2,$$

we find

$$\int_{\Omega_0(n+1,x)} (|U|_{\mu_1}^2 - U^4) \geq \frac{1}{2} \int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2 dx.$$

Since $\mathcal{U} \in \mathcal{M}_2$, this yields

$$\begin{aligned} & \int_{\Omega_+(n+1,x)} |U|_{\mu_1}^2 \\ & \leq \int_{\Omega_+(n+1,x)} (U^+)^4 + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - \frac{1}{4} \int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2 dx. \end{aligned} \quad (3.24)$$

By (3.24),

$$\begin{aligned} & \int_{\Omega_+(n,x)} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 \\ & = \int_{\Omega_+(n+1,x)} |U_{1,x}|_{\mu_1}^2 + \int_{\Omega_+(n,x) \setminus \Omega_+(n+1,x)} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 \\ & \leq \int_{\Omega_+(n+1,x)} (U^+)^4 + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - \frac{1}{4} \int_{\Omega_0(n+1,x)} |U|_{\mu_1}^2 dx \\ & \quad + \int_{\Omega_+(n,x) \setminus \Omega_+(n+1,x)} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 \\ & \leq \int_{\Omega_+(n,x)} \left((U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u))^+ \right)^4 + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ & \quad - \frac{1}{4} \int_{\Omega_0(n,x)} |U|_{\mu_1}^2 dx + 2C_1 (a_{n+1,x}(U) - a_{n,x}(U)) + e^{-2\sqrt{\mu_1}(|x|-n)}. \end{aligned} \quad (3.25)$$

On the other hand, it follows from (3.21) that

$$\begin{aligned}
& \|U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)\|_{\mu_1}^2 \\
&= \int_{\Omega_\pm(n,x)} |U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 + \int_{\mathbb{R}^3 \setminus \Omega_\pm(n,x)} |U|_{\mu_1}^2 \\
&\leq \int_{\Omega_\pm(n,x)} |U_{1,\pm x} + \psi_{n,x}^\pm(U_{1,\mp x} + u)|_{\mu_1}^2 + O(e^{-2\sqrt{\mu_1}(|x|-n)}).
\end{aligned} \tag{3.26}$$

Then we find

$$\begin{aligned}
\|U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &\leq \int_{\Omega_+(n,x)} \left((U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u))^+ \right)^4 \\
&\quad + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\
&\quad - \frac{1}{4} \int_{\Omega_0(n,x)} |U|_{\mu_1}^2 dx + 2C_1 (a_{n+1,x}(U) - a_{n,x}(U)) \\
&\quad + O(e^{-2\sqrt{\mu_1}(|x|-n)}).
\end{aligned} \tag{3.27}$$

Now let $t > 0$ such that $W = t^{-1}(U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)) \in S_1$. That is

$$t^{-2} \int_{\mathbb{R}^3} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 = \int_{\mathbb{R}^3} t^{-4} \left((U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u))^+ \right)^4.$$

Since $|x|$ is large and $\|u\|_{\mu_1}^2$ is small, t is close to 1. Then we may assume that $t^2 \leq 2$. Since $W \in S_1$, we have

$$t^{-2} \int_{\mathbb{R}^3} |U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)|_{\mu_1}^2 = \|W\|_{\mu_1}^2 \geq 4c_1.$$

Then we find by (3.23), and (3.27) that there exists $C_2 > 0$ such that

$$\begin{aligned}
\|U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &= t^2 \|W\|_{\mu_1}^2 \leq t^4 \int_{\mathbb{R}^3} (W^+)^4 \\
&\quad + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 + C_2 e^{-2\sqrt{\mu_1}(|x|-n)}.
\end{aligned}$$

Thus it follows that

$$\begin{aligned}
t^2 &\geq \frac{\|W\|_{\mu_1}^2 - \frac{\beta}{2t^2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - C_2 e^{-2\sqrt{\mu_1}(|x|-n)}}{\int_{\mathbb{R}^3} (W^+)^4} \\
&\geq 1 - \frac{\frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 + C_2 e^{-2\sqrt{\mu_1}(|x|-n)}}{\|W\|_{\mu_1}^2}.
\end{aligned} \tag{3.28}$$

Then

$$\begin{aligned} \|U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &= t^2 \|W\|_{\mu_1}^2 \\ &\geq \|W\|_{\mu_1}^2 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - C_2 e^{-2\sqrt{\mu_1}(|x|-n)} \\ &\geq 4c_1 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - C_2 e^{-2\sqrt{\mu_1}(|x|-n)}. \end{aligned} \quad (3.29)$$

Similarly, we have

$$\|U_{1,-x} + \psi_{n,x}^-(U_{1,x} + u)\|_{\mu_1}^2 \geq 4c_1 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - C_2 e^{-2\sqrt{\mu_1}(|x|-n)}. \quad (3.30)$$

Then combining (3.23), (3.26), (3.29) and (3.30) with (3.18), we obtain that (3.17) holds.

Step 3: Lastly, we show that inequality (3.17) holds in case that

$$a_{n+1,x}(U) - a_{n,x}(U) \geq (8C_1)^{-1} a_{n,x}(U) \quad \text{for all } 1 \leq n \leq (\rho/\sqrt{\mu_1})|x|.$$

Assume that $a_{n+1,x}(U) - a_{n,x}(U) \geq (8C_1)^{-1} a_{n,x}(U)$ for all $1 \leq n \leq (\rho/\sqrt{\mu_1})|x|$. Then

$$a_{n,x}(U) \geq d^{n-2} M \quad \text{for } 1 \leq n \leq (\rho/\sqrt{\mu_1})|x|.$$

where $M = a_2(U)$ and $d = (1 + (8C_1)^{-1})$. Then instead of (3.27), we have that there exists $C_3 > 0$ such that

$$\begin{aligned} \|U_{1,x} + \psi_{1,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &\leq \int_{\Omega_+(1,x)} \left((U_{1,x} + \psi_{n,x}^+(U_{1,-x} + u))^+ \right)^4 \\ &\quad + \frac{\beta}{2} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 + 2C_1 M + C_3 e^{-2\sqrt{\mu_1}|x|}, \end{aligned}$$

Let W be as in the argument above. Then inequality (3.28) is rewritten as

$$\begin{aligned} t^2 &\geq \frac{\|W\|_{\mu_1}^2 - \frac{\beta}{4} \int_{\mathbb{R}^3} (V^+)^2 (W^+)^2 - 2C_1 M - C_3 e^{-2\sqrt{\mu_1}|x|}}{\int_{\mathbb{R}^3} W^4} \\ &= 1 - \frac{\frac{\beta}{4} \int_{\mathbb{R}^3} (V^+)^2 (W^+)^2 + 2C_1 M + C_3 e^{-2\sqrt{\mu_1}|x|}}{\|W\|_{\mu_1}^2}. \end{aligned} \quad (3.31)$$

Since $|x|$ is large and $\|u\|_{\mu_1}$ is small by the assumption, we have t is close to 1 and W has the form

$$W = \frac{1}{t} (U_{1,x} + \psi_{1,x}^+(U_{1,-x} + u)) = \alpha U_{1,x} + \tilde{u} \quad \text{with } \tilde{u} \in \{U_{1,x}\}^\perp.$$

Then noting that $|t-1|$ and $\|\psi_{1,x}^+(U_{1,-x} + u)\|_{\mu_1}^2$ are small, we have that $|\alpha-1|$ is small. Then we may assume that $\|\tilde{u}\|_{\mu_1}^2 \geq \frac{1}{4} \|\psi_{1,x}^+(U_{1,-x} + u)\|_{\mu_1}^2$. Let $n \in \mathbb{N}$ be the largest integer less than or equal to $(\rho/\sqrt{\mu_1})|x|$. Recalling that $|U_{1,-x}(z)| =$

$O(e^{-\sqrt{\mu_1}(|x|-n)})$ on $\Omega_0(n, x)$, we have that there exists $C_4 > 0$ such that

$$\begin{aligned} \|\psi_{1,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &\geq \frac{1}{2} \int_{\Omega_0(n,x) \setminus \Omega_0(2,x)} |U|_{\mu_1}^2 - \int_{\Omega_0(n,x) \setminus \Omega_0(2,x)} |U_{1,x}|^2 \\ &\geq \frac{1}{2} \int_{\Omega_0(n,x) \setminus \Omega_0(2,x)} |U|_{\mu_1}^2 - C_4 e^{-2\sqrt{\mu_1}(|x|-n)} \\ &\geq \frac{1}{2} (d^{n-2} - 1)M - C_4 e^{-2\sqrt{\mu_1}(|x|-n)}. \end{aligned}$$

Therefore by Lemma 2, we have that for $|x|$ sufficiently large,

$$\|W\|_{\mu_1}^2 \geq 4c_1 + \frac{2}{3} \|\tilde{u}\|_{\mu_1}^2 \geq 4c_1 + \frac{1}{6} (d^{n-2} - 1)M - C_4 e^{-2\sqrt{\mu_1}(|x|-n)}.$$

Then

$$\begin{aligned} \|U_{1,x} + \psi_{1,x}^+(U_{1,-x} + u)\|_{\mu_1}^2 &= t^2 \|W\|_{\mu_1}^2 \\ &\geq \left(1 - \frac{\frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 + 2C_1 M}{\|W\|_{\mu_1}^2}\right) \|W\|_{\mu_1}^2 - C_3 e^{-2\sqrt{\mu_1}|x|} \\ &\geq \left(1 - \frac{2C_1 M}{\|W\|_{\mu_1}^2}\right) \|W\|_{\mu_1}^2 - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - C_3 e^{-2\sqrt{\mu_1}|x|} \\ &\geq \left(4c_1 + \frac{1}{12} (d^{n-2} - 1)M\right) - 2C_1 M - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ &\quad - C_4 e^{-2\sqrt{\mu_1}(|x|-n)} - C_3 e^{-2\sqrt{\mu_1}|x|} \\ &\geq 4c_1 + \frac{1}{12} (d^{n-2} - 24C_1 - 1)M - \frac{\beta}{4} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 \\ &\quad - C_4 e^{-2\sqrt{\mu_1}(|x|-n)} - C_3 e^{-2\sqrt{\mu_1}|x|}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \|U_{1,-x} + \psi_{n,x}^-(U_{1,x} + u)\|_{\mu_1}^2 &\geq 4c_1 + \frac{1}{6} (d^{n-2} - 12C_1 - 1)M \\ &\quad - \frac{\beta}{4} \int_{\mathbb{R}^3} (V^+)^2 (W^+)^2 \\ &\quad - C_4 e^{-2\sqrt{\mu_1}(|x|-n)} - C_3 e^{-2\sqrt{\mu_1}|x|}. \end{aligned}$$

By choosing $\varepsilon > 0$ sufficiently small, we have that $|x|$ is so large that we can choose $n \geq 1$ such that $d^{n-2} - 12C_1 - 1 > 0$. Then again we have (3.29). We also have (3.30). Then again by the same argument in Step 2, we have that inequality (3.17) holds. This completes the proof. $\square$

Proof of Proposition 2. Let $\varepsilon > 0$ and $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$. By Proposition 1, we may assume, by choosing $\varepsilon > 0$ sufficiently small, that $\mathcal{U}$ has the form $\mathcal{U} =$

$(\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v)$, where $|x|$ is sufficiently large, α is close to 1 and $\|(u, v)\|^2 < \delta_\varepsilon$. Then by Lemma 6 and Lemma 7, we have

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 + k_1 \|v\|^2 - \frac{\beta}{8} \int_{\mathbb{R}^3} (U^+)^2 (V^+)^2 - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|}.$$

If $\|v\|_{\mu_2}^2 \geq (2k_2/k_1)e^{-2(\sqrt{\mu_1}-\rho)|x|}$, then we have that for $|x|$ sufficiently large,

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 + k_2 e^{-(2\sqrt{\mu_1}-\delta)|x|}. \quad (3.32)$$

Now assume that $\|v\|_{\mu_2}^2 < (2k_2/k_1)e^{-2(\sqrt{\mu_1}-\rho)|x|}$. Then we have

$$\begin{aligned} \Phi(\mathcal{U}) &\geq 2c_1 + c_2 + k_1 \|v\|_{\mu_2}^2 - \frac{\beta}{8} \int_{\mathbb{R}^3} U^2 (\gamma U_2 + v)^2 - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|} \\ &\geq 2c_1 + c_2 + k_1 \|v\|_{\mu_2}^2 - \frac{\beta}{8} \int_{\mathbb{R}^3} U^2 \left(\frac{\gamma^2}{2} U_2^2 - v^2 \right) - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|} \\ &\geq 2c_1 + c_2 - \frac{\beta\gamma^2}{8} \int_{\mathbb{R}^3} U^2 U_2^2 - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|}. \end{aligned}$$

Recalling that $\sqrt{\mu_2} < \sqrt{\mu_1} - 2\rho$, we have by (1.3) that there exists $C_1 > 0$ such that

$$\int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^2 \geq C_1 e^{-2\sqrt{\mu_2}|x|}.$$

Similarly, we have that there exists $C_2 > 0$ such that

$$\begin{aligned} \left| \int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x}) U_2^2 u \right| &\leq \left(\int_{\mathbb{R}^3} (U_{1,x} + U_{1,-x})^2 U_2^4 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} u^2 \right)^{\frac{1}{2}} \\ &\leq C_2 e^{-\min\{\sqrt{\mu_1}, 2\sqrt{\mu_2}\}|x|} \|u\|_{\mu_1}. \end{aligned}$$

Then noting that $U^2 U_2^2 = (U_{1,x} + U_{1,-x})^2 U_2^2 + 2(U_{1,x} + U_{1,-x}) U_2^2 u + u^2$, we find

$$\begin{aligned} \Phi(\mathcal{U}) &\geq 2c_1 + c_2 - \frac{\beta\gamma^2}{8} \left(C_1 e^{-2\sqrt{\mu_2}|x|} - C_2 e^{-\min\{\sqrt{\mu_1}, 2\sqrt{\mu_2}\}|x|} \|u\|_{\mu_1} + \|u\|_{\mu_1}^2 \right) \\ &\quad - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|}. \end{aligned}$$

If $\min\{\sqrt{\mu_1}, 2\sqrt{\mu_2}\} = 2\sqrt{\mu_2}$, then noting that $\|u\|_{\mu_1}^2 \rightarrow 0$ and $\gamma \rightarrow 1$ as $\varepsilon \rightarrow 0$, we have

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 - \frac{\beta C_2}{16} e^{-2\sqrt{\mu_2}|x|} \quad (3.33)$$

for $\varepsilon > 0$ sufficiently small. Suppose that $\min\{\sqrt{\mu_1}, 2\sqrt{\mu_2}\} = \sqrt{\mu_1}$. Then in case that $\|u\|_{\mu_1} \geq e^{-\sqrt{\mu_2}|x|}$, we have

$$\begin{aligned} \Phi(\mathcal{U}) &\geq 2c_1 + c_2 - \frac{\beta\gamma^2}{8} \left(C_1 e^{-2\sqrt{\mu_2}|x|} + e^{-\sqrt{\mu_2}|x|} (e^{-\sqrt{\mu_2}|x|} - C_2 e^{-\sqrt{\mu_1}|x|}) \right) \\ &\quad - k_2 e^{-2(\sqrt{\mu_1}-\rho)|x|} \end{aligned}$$

for $|x|$ sufficiently large. That is for $\varepsilon > 0$ sufficiently small, (3.33) holds. If $\|u\|_{\mu_1} \leq e^{-\sqrt{\mu_2}|x|}$, then

$$\Phi(\mathcal{U}) \geq 2c_1 + c_2 - \frac{\beta\gamma^2}{8}C_1e^{-2\sqrt{\mu_2}|x|} - C_2e^{-(\sqrt{\mu_1}+\sqrt{\mu_2})|x|} - k_2e^{-2(\sqrt{\mu_1}-\rho)|x|}.$$

Then recalling that $\sqrt{\mu_2} < \sqrt{\mu_1} - 2\rho$, we find (3.33) holds for $\varepsilon > 0$ sufficiently small. This completes the proof. $\square$

Now for $x \in \mathbb{R}^3 \setminus \{0\}$, we define a class $\Gamma_2(x) \subset C([0, 1], \mathcal{M}_2)$ by

$$\Gamma_2(x) = \left\{ p \in C([0, 1], \mathcal{M}_2) : p(0) = \mathcal{N}(U_1, U_2), p(1) = \mathcal{N}(U_{1,x} + U_{1,-x}, U_2) \right\}$$

and put

$$c_2(x) = \inf_{p \in \Gamma_2(x)} \sup_{t \in [0, 1]} \Phi(p(t)).$$

We also note that from the definitions of $\mathcal{N}$ and Φ , we have that $\mathcal{N}(U_{1,x} + U_{1,-x}, U_2) - (U_{1,x} + U_{1,-x}, U_2) \rightarrow 0$ in $\mathbb{H}$ as $|x| \rightarrow \infty$ and then

$$\lim_{|x| \rightarrow \infty} \Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) = 2I_1(U_1) + I_2(U_2) = 2c_1 + c_2. \quad (3.34)$$

We can now finish the proof of Theorem 1 for $i = 2$.

Proof of Theorem 1. Let $\varepsilon \in (0, \delta_2/2)$ sufficiently small. Let $\beta \in (\beta_\varepsilon, 0)$. To complete the proof, it is sufficient to show that there exists $\delta > 0$ and $R > 0$ such that

$$2c_1 + c_2 + \delta < c_2(x) < 2c_1 + c_2 + \delta_2/2 \quad \text{for } |x| > R. \quad (3.35)$$

In fact, if the inequalities above hold, we have by (3.34) that we can choose $x \in \mathbb{R}^3$ such that $|x| > R$ and

$$\Phi(\mathcal{N}(U_{1,x} + U_{1,-x}, U_2)) < 2c_1 + c_2 + \delta.$$

That is $\Phi(p(1)) < c_2(x)$ for all $p \in \Gamma_2(x)$. We also have $\Phi(p(0)) < \frac{7}{4}c_1 + c_2$. Then since the Palais-Smale condition holds by (2) of Lemma 4 on $\Phi^{(2c_1+c_2, 2c_1+c_2+\delta_2/2)}$, we have by a standard mountain pass argument that there exists a critical point $\mathcal{U}$ of Φ with $\Phi(\mathcal{U}) = c_2(x)$.

From the definition of ε and Lemma 5, one can see the pass $p \in \Gamma_2(x)$ defined by

$$p(s) = \mathcal{N}(U_{1,sx} + U_{1,-sx}, U_2), \quad s \in [0, 1]$$

satisfies $\max_{s \in [0, 1]} \Phi(p(s)) \leq 2c_1 + c_2 + \varepsilon$. Then the second inequality of (3.35) holds. We now show that the first inequality of (3.35) holds. We first see that there exists $\bar{R} > 2R_\varepsilon$ such that ,

$$\begin{aligned} b_{R_\varepsilon}(U) &\geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \\ \text{for } \mathcal{U} = (U, V) &\in \Lambda_{2,\varepsilon}(R_\varepsilon) \quad \text{with } \eta(U) \geq \bar{R}, \end{aligned} \quad (3.36)$$

where η is the function defined by (3.15). By Proposition 1, each $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ has the form

$$\mathcal{U} = (\alpha(U_{1,x} + U_{1,-x}) + u, \gamma U_2 + v) \quad (3.37)$$

with $\alpha \in (1 - \alpha_\varepsilon, 1 + \alpha_\varepsilon)$, $\gamma \in (1 - \gamma_\varepsilon, 1 + \gamma_\varepsilon)$ and $(u, v) \in \{U_{1,x}, U_{1,-x}\}^\perp \times \{U_2\}^\perp$ with $\|u\|_{\mu_1}^2 + \|v\|_{\mu_2}^2 \leq \delta_\varepsilon$. Since $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon = 0$, we may assume that $\varepsilon > 0$ is sufficiently small that

$$8\alpha_\varepsilon^2 c_1 - \delta_\varepsilon > 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}. \quad (3.38)$$

Since $\mathcal{U}$ has the form (3.37),

$$\begin{aligned} b_{R_\varepsilon}(U) &= \int_{\mathbb{R}^3 \setminus B_{R_\varepsilon}(0)} |U|_{\mu_1}^2 \\ &= \|\alpha(U_{1,x} + U_{1,-x}) + u\|_{\mu_1}^2 - \int_{B_{R_\varepsilon}(0)} |U|_{\mu_1}^2 \\ &\geq \alpha^2 \|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 - \|u\|_{\mu_1}^2 - 2 \int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|_{\mu_1}^2 \end{aligned}$$

holds. Then noting that

$$\|U_{1,x} + U_{1,-x}\|_{\mu_1}^2 \longrightarrow 8c_1 \quad \text{and} \quad \int_{B_{R_\varepsilon}(0)} |U_{1,x} + U_{1,-x}|_{\mu_1}^2 \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty,$$

we find by (3.38) that there exists $\bar{R}$ such that for each $\mathcal{U} = (U, V) \in \Lambda_{2,\varepsilon}(R_\varepsilon)$ with $\eta(\mathcal{U}) \geq \bar{R}$, (3.36) holds. Now we choose $x \in \mathbb{R}^3$ so large that $|x| > \bar{R}$. Then

$$b_{R_\varepsilon}(\mathcal{N}_1(U_{1,x} + U_{1,-x})) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}.$$

Let $p = (p_1, p_2) \in \Gamma_2(x)$ such that $\sup_{t \in [0,1]} \Phi(p(t)) \leq 2c_1 + c_2 + \varepsilon$. From the definition,

$$\eta(p_1(1)) = \eta(\mathcal{N}_1(U_{1,x} + U_{1,-x})) > \bar{R} \quad \text{and} \quad b_{R_\varepsilon}(p_1(1)) \geq 8c_1 - \frac{1}{2} \min \left\{ \frac{1}{2m_4^4}, c_1 \right\}.$$

On the other hand, recalling that $\Phi_2(\mathcal{U}) \geq c_2$, we have by (2.11) that $\Phi_1(\mathcal{U}) \leq \frac{7}{4}c_1$. Then by the definition of ε , $b_{R_\varepsilon}(p_1(0)) < 7c_1 \leq 8c_1 - \min\{\frac{1}{2m_4^4}, c_1\}$, there exists $t \in (0, 1)$ such that $b_{R_\varepsilon}(p_1(t)) = 8c_1 - \min\{\frac{1}{2m_4^4}, c_1\}$. Then by (3.36), $\eta(p_1(t)) < \bar{R}$. Therefore by the continuity of η , we have that there exists $t_0 \in (0, t)$ such that $\eta(p_1(t_0)) = \bar{R}$. By Proposition 2, we have

$$\Phi(p(t_0)) \geq 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\bar{R}}.$$

Therefore we obtain that $\sup_{t \in [0,1]} \Phi(p(t)) > 2c_1 + c_2 + \beta M_0 e^{-2\sqrt{\mu_2}\bar{R}}$. Thus by the mountain pass theorem (cf. [17]), we find that there exists a critical point $\mathcal{U}$ of Φ with $\Phi(\mathcal{U}) = c_2(x)$. $\square$

4. Multiple existence

In the previous section, we proved Theorem 1 in the case that $i = 2$. The proofs for the cases $i \in \{4, 6, 8, 12, 20\}$ are slight modifications of the proof for $i = 2$. We give a sketch of the proof for $i = 8$. Let $k \geq 1$ and $S^{(k)} = (1, \sigma_k, \sigma_k^2, \dots, \sigma_k^{k-1})$ be a finite group acting on $\mathbb{R}^3$ defined by

$$\sigma_k x = \left(\left(\cos \frac{2\pi}{k} \right) x_1 - \left(\sin \frac{2\pi}{k} \right) x_2, \left(\sin \frac{2\pi}{k} \right) x_1 + \left(\cos \frac{2\pi}{k} \right) x_2, x_3 \right)$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

We also define a mapping $\kappa : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ by

$$\kappa x = (x_1, x_2, -x_3) \quad \text{for } x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

We put

$$\tilde{H}_k = \{u \in H : u(\sigma x) = u(x) \text{ and } u(\kappa x) = u(x) \text{ for } x \in \mathbb{R}^3 \text{ and } \sigma \in S^{(k)}\},$$

and $\tilde{\mathbb{H}}_k = \tilde{H}_k \times \tilde{H}_k$. By the equivariance of Φ under the actions σ and κ , we have that each critical point u of Φ in $\mathbb{H}_k$ is a critical point of Φ in $\mathbb{H}$. We denote by G the finite group action on $\mathbb{R}^3$ which is generated by $S^{(4)}$ and τ . To prove the case that $i = 8$, we use, instead of b_r , the function $b_r^{(8)}$ which is defined by

$$b_r^{(8)}(U) = \int_{\mathbb{R}^3 \setminus A_r} |U|_{\mu_1}^2 \quad \text{for } r > 0 \text{ and } U \in H. \quad (4.1)$$

where $D_r = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 < r\}$ and

$$A_r = D_r \cup \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x_3| < r\}.$$

We put $\mathcal{M}_8 = \mathcal{M}_+ \cap \tilde{\mathbb{H}}_4$ and

$$\Lambda_{8,\varepsilon}(R) = \left\{ \mathcal{U} = (U, V) \in \Phi^{8c_1+c_2+\varepsilon} \cap \mathcal{M}_8 : b_R^{(8)}(U) > 32c_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \right\}$$

for $(\varepsilon, R) \in \mathbb{R}^+ \times \mathbb{R}^+$. We also choose $\delta_8 > 0$ so small that

$$c_1 + kc_2 \notin [8c_1 + c_2 - \delta_8, 8c_1 + c_2 + \delta_8] \quad \text{for all } k \in \mathbb{N}.$$

Then we have following lemmas and propositions instead of Lemma 4, Lemma 5, Proposition 1 and Proposition 2.

Lemma 8. (1) *There exists $\beta_1^{(8)} \in (-1, 0)$ such that for each critical point $\mathcal{U} \in \mathcal{M}_8 \cap \Phi^{8c_1+2c_2}$,*

$$\Phi(\mathcal{U}) \in \cup_{i \geq 1, j \geq 1} [ic_1 + jc_2 - \delta_8/2, ic_1 + jc_2 + \delta_8/2] :$$

(2) *Let $\beta \in (\beta_1^{(8)}, 0)$ and $\{\mathcal{U}_n\} \subset \mathcal{M}_8$ such that*

$$\lim_{n \rightarrow \infty} \nabla \Phi(\mathcal{U}_n) \longrightarrow 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \Phi(\mathcal{U}_n) = 8c_1 + c_2 + \varepsilon$$

with $\varepsilon \in (0, \delta_8/2)$. Then there exists a convergence subsequence $\{\mathcal{U}_{n_i}\} \subset \{\mathcal{U}_n\}$.

The proof of Lemma 8 is the same as that of Lemma 4. Now we fix $\beta \in (\beta_1^{(8)}, 0)$, Then we have the following two propositions.

Proposition 3. *For $\varepsilon > 0$ sufficiently small, there exists $(R_\varepsilon^{(8)}, \delta_\varepsilon^{(8)}, \alpha_\varepsilon^{(8)}, \gamma_\varepsilon^{(8)}) \in (\mathbb{R}^+)^4$ such that $\lim_{\varepsilon \rightarrow 0} \delta_\varepsilon^{(8)} = \lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon^{(8)} = \lim_{\varepsilon \rightarrow 0} \gamma_\varepsilon^{(8)} = 0$ and each $\mathcal{U} \in \Lambda_{8,\varepsilon}(R_\varepsilon^{(8)})$ has the form*

$$\mathcal{U} = \left(\alpha \sum_{i=1}^8 U_{1,x_i} + u, \gamma U_{2,0} + v \right) \quad (4.2)$$

where $\alpha \in (1 - \alpha_\varepsilon^{(8)}, 1 + \alpha_\varepsilon^{(8)})$, $\gamma \in (1 - \gamma_\varepsilon^{(8)}, 1 + \gamma_\varepsilon^{(8)})$, $\{x_i\}_{i=1}^8 \subset \mathbb{R}^3$ forms a octahedron,

$$|x_i| \geq R_\varepsilon^{(8)} \quad i = 1, \dots, 8, \quad x = \mathcal{B}(U|_{B_{R_0}(x)}),$$

$$\widehat{U}(z) < \frac{1}{2} |\widehat{U}|_\infty \quad \text{for } z \in \mathbb{R}^3 \setminus \left(\bigcup_{i=1}^8 B_{R_0}(x_i) \right)$$

and

$$(u, v) \in \{U_{1,x_i} : 1 \leq i \leq 8\}^\perp \times \{U_2\}^\perp \quad \text{with} \quad \|u\|^2 + \|v\|^2 \leq \delta_\varepsilon^{(8)}.$$

Sketch of proof. For $r > 0$, we put

$$B_{r,Id} = A_{r+1} \cup \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \min_{i=1,2} |x_i| < 1 \right\} \setminus \overline{A_r}$$

and

$$B_{r,g} = g(B_{r,Id}) \quad \text{for all } g \in G.$$

Let $\varphi_{r,Id} \in C^\infty(\mathbb{R}^3; [0, 1])$ such that

$$\varphi_{r,Id} = \begin{cases} 1 & \text{on } \mathbb{R}^3 \setminus B_{r,Id} \\ 0 & \text{on } \partial B_{r+\frac{1}{2},Id} \cup \left\{ x \in \mathbb{R}^3 \setminus B_{r+\frac{1}{2},Id} : \min_{i=1,2} |x_i| = 0 \right\}, \end{cases}$$

and $\varphi_{r,g}(x) = \varphi_{r,Id}(gx)$ for $x \in \mathbb{R}^3$ and $g \in G$. For $R > r + \frac{1}{2}$ and $\mathcal{U} = (U, V) \in \Lambda_{8,\varepsilon}(R_\varepsilon^{(8)})$, we put

$$\varphi_{r,g}U = W_{8,0} + \sum_{i=1}^8 W_{8,i},$$

where $\text{supp } W_{8,i}$ are mutually disjoint and for $1 \leq i, j \leq 8$, $W_{8,i+1} = \sigma_4(W_{8,i})$ for $1 \leq i \leq 7, i \neq 4$ and $W_{8,i+4} = \kappa W_{8,i}$ for $1 \leq i \leq 4$. Then by using $B_{r,g}$, $\varphi_{r,g}$ and $W_{8,i}$ instead of $B_{r,z}$, $\varphi_{r,z}$ and $W_\pm$, we can prove the assertion by the same argument as in the proof of Proposition 1. $\square$

Proposition 4. *There exists $M_0^{(8)} > 0$ such that for ε sufficiently large,*

$$\Phi(\mathcal{U}) \geq 8c_1 + c_2 - \beta M_0^{(8)} e^{-2\sqrt{\mu_2}|x|} \quad \text{for } \mathcal{U} \in \Lambda_{8,\varepsilon}(R_\varepsilon^{(8)}),$$

where $x \in \mathbb{R}^3$ such that $\mathcal{U}$ satisfies (4.2).

Proof. As in Lemma 6, we have that there exists $k_1^{(8)} > 0$ such that for $\varepsilon > 0$ sufficiently small, each $\mathcal{U} = (U, V) \in \mathcal{M}_8 \cap \Phi^{8c_1+c_2+\varepsilon}$ satisfies

$$\Phi_2(\mathcal{U}) \geq c_2 + k_1^{(8)} \|v\|_{\mu_1}^2,$$

where $v \in \{U_2\}^\perp$ such that $V = \alpha U_2 + v$ for some $\alpha \in \mathbb{R}$. The proof of Lemma 6 is valid for this case. Next for $r > 0$ and $x \in \partial B_1(0)$, we put

$$\begin{aligned}\Omega_0(r) &= \left\{ z = (z_1, z_2, z_3) \in \mathbb{R}^3 : \min_{i=1,2,3} |z_i| \leq r \right\}, \\ \Omega_1(r) &= \left\{ z \in \mathbb{R}^3 \setminus \Omega_0(r) : z_1, z_2, z_3 \geq 0 \right\}, \\ \Omega_{i,h}(r) &= h(\Omega_i(r)) = \{hz : z \in \Omega_i\}, \quad \text{for } i = 0, 1, \\ h \in G_8 &= \{1, \sigma_4, \sigma_4^2, \sigma_4^3, \kappa, \kappa\sigma_4, \kappa\sigma_4^2, \kappa\sigma_4^3\},\end{aligned}$$

and

$$a_{n,h}(u) = \int_{\Omega_{0,h}(n)} |u|_{\mu_1}^2 dx \quad \text{for each } n \geq 1, h \in G_8 \quad \text{and } u \in H.$$

For $n \geq 1$ and $h \in G_8$, we define $\psi_{n,h} \in C^\infty(\mathbb{R}^3, [0, 1])$ such that

$$\psi_{n,h}(z) = 1 \quad \text{on } \Omega_{1,h}(n+1) \quad \text{and} \quad \psi_{n,h}(z) = 0 \quad \text{on } \mathbb{R}^3 \setminus \Omega_{1,h}(n).$$

Then using $\Omega_{i,h}(r)$ and $\psi_{n,h}(z)$ instead of $\Omega_0(r, x)$, $\Omega_\pm(r, z)$ and $\psi_{n,x}^\pm$, we can see that all the computations in proof of Lemma 7 for $U_{1,x} + \psi_{n,x}^+(U_{1,-} + u)$ are valid for $U_{1,x_1} + \psi_{n,h}(\sum_{i=2}^8 U_{1,x_i} + u)$. That is we obtain by a parallel argument as in the proof of Lemma 7 that for each $\mathcal{U} \in \Lambda_{8,\varepsilon}(R_\varepsilon^{(8)})$,

$$\Phi_1(\mathcal{U}) \geq 32c_1 - \frac{1}{4} \int_{\mathbb{R}^3} \beta V^2 U^2 + O(e^{-2(\sqrt{\mu_1}-\rho)(|x|)})$$

where $x \in \mathbb{R}^3$ such that $\mathcal{U}$ has the form (4.2). Then the assertion follows again by the same argument as in the proof of Proposition 2. $\square$

Here we define a class $\Gamma_8(x)$, $x = (x_1, x_1, x_1) \in \mathbb{R}^3 \setminus \{0\}$ by

$$\Gamma_8(x) = \left\{ p \in C([0, 1], \mathcal{M}_2) : p(1) = \mathcal{N}\left(\sum_{h \in G} U_{1,hx}, U_2\right), p(0) = \mathcal{N}(U_1, U_2) \right\}.$$

Then by the same argument as in the proof of Lemma 5, we have that

Lemma 9. *For given $\varepsilon > 0$, there exists $\beta_\varepsilon^{(8)} \in (\beta_1^{(8)}, 0)$ such that for each $\beta \in (\beta_\varepsilon^{(8)}, 0)$ and for each $x = (x_1, x_1, x_1) \in \mathbb{R}^3 \setminus \{0\}$, there exists a path $p \in \Gamma_8(x)$ such that*

$$\max_{t \in [0, 1]} \Phi(p(t)) \leq 8c_1 + c_2 + \varepsilon$$

Then the proof of Theorem 1 for $i = 8$ carried out by the same way as that for the case $i = 2$, using Proposition 3, Proposition 4 and Lemma 9.

Theorem 2 is proved by the same argument for Theorem 1. We just need a minor modification of the setting. Let $k \geq 1$ and put

$$\tilde{b}_r(U) = \int_{\mathbb{R}^3 \setminus D_r} |U|_{\mu_1}^2 \quad \text{for } r > 0 \quad \text{and } U \in H. \quad (4.3)$$

and

$$\tilde{\Lambda}_{k,\varepsilon}(R) = \left\{ \mathcal{U} = (U, V) \in \Phi^{2c_1+c_2+\varepsilon} \cap \mathcal{M}_k : \tilde{b}_R(U) > kc_1 - \min \left\{ \frac{1}{2m_4^4}, c_1 \right\} \right\}$$

for $R, \varepsilon > 0$. Then by using $\tilde{b}_R$ instead of b_R , we have that for $\varepsilon > 0$ sufficiently small, there exists $\tilde{R}_\varepsilon^{(k)} > 0$ such that each $\mathcal{U} \in \tilde{\Lambda}_{k,\varepsilon}(\tilde{R}_\varepsilon^{(k)})$ has the form

$$\mathcal{U} = \left(\alpha \sum_{i=0}^{k-1} U_{1,\sigma_k^i x} + u, \beta U_2 + v \right) \quad (4.4)$$

where α and β are close to 1, u, v are small. From the definition of the space $\tilde{H}_k$ and $\tilde{b}_r$, one can see that

$$x \in \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_1, x_2 \in \mathbb{R}\} \quad \text{with } |x| \geq R_\varepsilon. \quad (4.5)$$

We also have, instead of Proposition 2, that there exists $\tilde{\beta}_1^{(k)} \in (-1, 0)$ and $\tilde{M}_0 > 0$ satisfying that for $\beta \in (\tilde{\beta}_1^{(k)}, 0)$ and for $\varepsilon > 0$ sufficiently small,

$$\Phi(\mathcal{U}) \geq kc_1 + c_2 - \beta \tilde{M}_0^{(k)} e^{-2\sqrt{\mu_1}|x|} \quad \text{for each } \mathcal{U} \in \tilde{\Lambda}_{k,\varepsilon}(\tilde{R}_\varepsilon^{(k)})$$

where $x \in \mathbb{R}^3$ such that $\mathcal{U}$ has the form (3.1).

Then the assertion of Theorem 2 follows by the same argument as in the proof of Theorem 1 for the case that $i = 2$.

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