Stability of Travelling Fronts of the Fisher-KPP Equation in $\mathbb{R}^N$

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Abstract. Recently, Hamel and Nadirashvili [8] proved that there are infinite-dimensional manifolds of (nonplanar) travelling fronts for the Fisher-KPP equation in $\mathbb{R}^N$. In this paper we study the stability of these travelling fronts of the Fisher-KPP equation in $\mathbb{R}^N$. We prove that all (planar or nonplanar) of the travelling fronts are stable under the condition that the perturbations have compact supports. We also deal with the case of the perturbations without compact supports and prove the stability of the travelling fronts under perturbations that decay exponentially with an appropriate rate.

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1. Introduction and main results

In this paper we consider the stability of the travelling fronts of the following semilinear parabolic equation

$$u_t = \Delta u + f(u), \quad 0 < u(x, t) < 1, \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R}, \quad (1.1)$$

where the nonlinearity f is a concave function of class C^2 in $[0, 1]$ and satisfies

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f'(1) < 0, \quad f(u) > 0, \quad \forall 0 < u < 1. \quad (1.2)$$

A typical example of such a function f is the quadratic nonlinearity $f(u) = u(1-u)$ considered by Fisher [4] and Kolmogorov, Petrovsky and Piscunov [10]. Fisher-KPP equation (1.1) arises in various biological models, such as population dynamics and gene developments (see Aronson and Weinberger [1], Fife [3] and Murray [12] for details).

By a travelling wave for the equation (1.1), we understand a solution $u(x, t)$ such that

$$u(x, t + \tau) = u(x + c_0 \tau \nu_0, t), \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}, \quad \forall \tau \in \mathbb{R}, \quad (1.3)$$

for some direction $\nu_0 \in S^{N-1}$ and some speed $c_0 > 0$ (up to a change $\nu_0 \rightarrow -\nu_0$, one can always assume $c_0 > 0$). Such a wave is propagating in the direction $-\nu_0$ with the speed c_0 . The function u can then be written as

$$u(x, t) = \phi(x + c_0 t \nu_0), \quad (1.4)$$

where ϕ is uniquely defined by $\phi(x) = u(x, 0)$ for all $x \in \mathbb{R}^N$. The function ϕ is such that $0 < \phi(x) < 1$ for all $x \in \mathbb{R}^N$ and it satisfies the elliptic equation

$$\Delta \phi - c_0 \nu_0 \nabla \phi + f(\phi) = 0 \quad \text{in } \mathbb{R}^N. \quad (1.5)$$

Conversely, any solution $0 < \phi < 1$ of (1.5) gives rise to a travelling wave $u(x, t) = \phi(x + c_0 t \nu_0)$ for (1.1), which propagates in the direction $-\nu_0$ with the speed c_0 . It is well known that the equation (1.1) has, in dimension N , an $N + 1$ dimensional manifold of planar travelling waves, namely

$$u_{\nu, c, h}(x, t) = \varphi_c(x \cdot \nu + ct + h)$$

where ν belong to S^{N-1} which is the unit sphere in $\mathbb{R}^N$, $h \in \mathbb{R}$ and $c \in [c^*, +\infty)$ with

$$c^* = 2\sqrt{f'(0)}.$$

Such a front $\varphi_c(x \cdot \nu + ct + h)$ propagates in the direction $-\nu$ with the speed c . Especially in space dimension $N = 1$, there are two dimensional manifolds of travelling waves:

$$u_{c, h}^+(x, t) = \varphi_c(x + ct + h) \quad \text{and} \quad u_{c, h}^-(x, t) = \varphi_c(-x + ct + h).$$

For any $c \geq c^*$, the function φ_c satisfies

$$\begin{cases} \varphi_c'' - c\varphi_c' + f(\varphi_c) = 0, & \text{in } \mathbb{R}, \\ \varphi_c(-\infty) = 0, \quad \varphi_c(+\infty) = 1. \end{cases}$$

Such a solution φ_c is increasing and unique up to translation. Furthermore, up to shifts, the function φ_c satisfies

$$\forall c > c^*, \quad \varphi_c(s) \sim e^{\lambda_c s} \quad \text{as } s \rightarrow -\infty,$$

where $\lambda_c = \frac{c - \sqrt{c^2 - c^{*2}}}{2} > 0$. While for the travelling waves with minimal speed $c^* = 2\sqrt{f'(0)}$, it is known that, up to shifts,

$$\varphi_{c^*}(s) \sim |s|e^{\lambda^* s} \quad \text{as } s \rightarrow -\infty,$$

where $\lambda^* = \lambda_{c^*} = \frac{c^*}{2}$ (see for instance Bramson [2] and Sattinger [15]).

Recently, it was proved by Hamel and Nadirashvili [8, 9] that there exists an infinite-dimensional manifold of entire solutions (which are defined for all time and for all point $x \in \mathbb{R}^N$) for the Fisher-KPP equation (1.1). In particular, there are infinite-dimensional manifolds of (nonplanar) travelling fronts satisfying (1.3). Before introducing these interesting results, we need to give some notations. Let

$$B(0, c^*) = B(0, 2\sqrt{f'(0)}) = \{x \in \mathbb{R}^N, |x| < c^*\}$$

be the open ball of $\mathbb{R}^N$ with center 0 and radius c^* . The set of all the unit vectors in $\mathbb{R}^N$ is denoted by S^{N-1} . Define the sets

$$X = S^{N-1} \times [c^*, +\infty), \quad \hat{X} = S^{N-1} \times (c^*, +\infty)$$

equipped with the topology induced by the Euclidean structure of $\mathbb{R}^N$. For any $(\nu_0, c_0) \in S^{N-1} \times [c^*, +\infty)$, define

$$S_{(\nu_0, c_0)} = \{(\nu, c) \in X, c_0 \nu_0 \cdot \nu = c\} = \{S(c_0 \nu_0 / 2, c_0 / 2) \setminus B(0, c^*)\},$$

where $S(c_0 \nu_0 / 2, c_0 / 2)$ is the sphere with center $c_0 \nu_0 / 2$ and radius $c_0 / 2$. Let $\mathcal{M}$ be the set of all nonnegative and nonzero Radon-measures μ on X ($0 < \mu(X) < +\infty$), such that the restriction μ^* of μ on the sphere $S^{N-1} \times \{c^*\}$ can be written as a finite sum of Dirac masses:

$$\mu^* = \sum_{1 \leq i \leq k} m_i \delta_{(\nu_i, c^*)}$$

for some integer $k = k(\mu) \geq 0$, positive real numbers $m_i = m_i(\mu)$ and some directions $\nu_i = \nu_i(\mu) \in S^{N-1}$. For any $\mu \in \mathcal{M}$, we denote $\hat{\mu}$ the restriction of μ on the set $\hat{X}$. Then, we have $\mu = \mu^* + \hat{\mu}$. Let $\mathcal{M}_{TW}$ be the subset of $\mathcal{M}$ defined by

$$\mathcal{M}_{TW} = \{\mu \in \mathcal{M}; \exists (\nu_0, c_0) \in S^{N-1} \times [c^*, +\infty), \mu \text{ is concentrated on } S_{(\nu_0, c_0)}\}.$$

Theorem 1.1 (Hamel and Nadirashvili [8]).

- (1) Let u be a travelling wave for (1.1) and assume that u satisfies (1.3), namely, u propagates in the direction $-\nu_0$ with speed c_0 . Then,
 - (1-a) $c_0 \geq c^*$;
 - (1-b) the function ϕ defined by (1.4) is increasing in each direction $\nu \in S^{N-1}$ such that $\nu \cdot \nu_0 > \cos(\arcsin(\frac{c^*}{c_0}))$, namely, ν belongs to the open cone directed by ν_0 with angle $\arcsin(\frac{c^*}{c_0})$. Furthermore, for each such ν , one has

$$\lim_{s \rightarrow -\infty} \phi(a + s\nu) = 0, \quad \lim_{s \rightarrow +\infty} \phi(a + s\nu) = 1, \quad \forall a \in \mathbb{R}^N. \quad (1.6)$$

- (1-c) if $c_0 = c^*$, then u is a planar travelling wave with speed c^* , namely, $u(x, t) = \varphi_{c^*}(x \cdot \nu_0 + c^*t + h)$ for some $h \in \mathbb{R}$. In other words, if $0 < \varphi < 1$ is a solution of (1.5) for $c_0 = c^*$ and for some $\nu_0 \in S^{N-1}$, then $\varphi(x) = \varphi_{c^*}(x \cdot \nu_0 + h)$ for some $h \in \mathbb{R}$.
- (2) (2-a) In dimension $N \geq 2$, there exists an infinite-dimensional manifold of travelling waves for (1.1). Namely, the restriction of the map $\mu \mapsto u_\mu$ on $\mathcal{M}_{TW}$ ranges in the set of travelling waves for (1.1), and it is one-to-one on $\mathcal{M}_{TW}$ and continuous on $\mathcal{M}_{TW} \cap \hat{\mathcal{M}}$. If

$$\mu = \mu^* + \hat{\mu} \in \mathcal{M}$$

is concentrated on $S_{(\nu_0, c_0)}$ for some (ν_0, c_0) , then u_μ is a travelling wave satisfying (1.4).

- (2-b) In dimension $N \geq 2$, for each $c_0 > c^*$ and for each $\nu_0 \in S^{N-1}$, there exists an infinite-dimensional manifold of solutions $\phi(x)$, $0 < \phi < 1$, of the elliptic equation (1.5).
- (2-c) Let $u(x, t)$ be a travelling wave of (1.1) satisfying (1.3). If u is of the type u_μ for some $\mu \in \mathcal{M}$, then μ is concentrated on $S_{(\nu_0, c_0)}$.

Many works have been devoted to the question of the behavior for large time and the convergence to travelling waves for the solutions of the Cauchy problem for the Fisher-KPP equation (1.1), especially in dimension 1. See for example Kolmogorov, Petrovsky and Piscunov [10], Bramson [2], Sattinger [15, 16] and Gailay [5]. For the higher dimensional case, Mallordy and Roquejoffre [11] studied this subject in an infinite cylinder. But, as far as we know, there are no stability results about the nonplanar travelling fronts found by Hamel and Nadinashvili (notice that there are infinite-dimensional manifolds of such travelling fronts). That is just what we want to investigate in the present paper.

The first step to attack this problem is to consider the case of perturbations with compact supports. Physically speaking, it means that the initial value is fixed far away from the origin, and only the tip of the front (the tip of the flame in combustion problems) is perturbed. In this context, the following stability result is obtained.

Theorem 1.2. *For a given measure $\mu \in \mathcal{M}_{TW}$, let $u_\mu = \phi_{c_0}$ be the travelling front of the equation (1.1). That is, $\phi_{c_0}(x)$ solves the elliptic equation (1.5). Let $u(x, t)$ be the solution of the following Cauchy problem*

$$\begin{cases} u_t = \Delta u + f(u), & x \in \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.7)$$

where $0 \leq u_0(x) \leq 1$ is a uniformly continuous function defined in $\mathbb{R}^N$. If $u_0(x) - \phi_{c_0}(x)$ has compact support in $\mathbb{R}^N$, then

$$\sup_{x \in \mathbb{R}^N} |u(x - c_0 \nu_0 t, t) - \phi_{c_0}(x)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

We also deal with the case that the perturbations do not have compact support. Before stating the main result in this case, we first reformulate the question for the sake of convenience. Without loss of generality, we can restrict ourselves to the case of $\nu_0 = e_N = (0, \dots, 0, 1)$ up to a rotation of the frame. Then the equation (1.5) reads

$$\Delta \phi - c_0 \frac{\partial \phi}{\partial x_N} + f(\phi) = 0. \quad (1.8)$$

In order to prove that the solution $u(x - c_0 e_N t, t)$ of the Cauchy problem (1.7) tends to the solution $\phi_{c_0}(x)$ of the equation (1.8) as the time t tends to infinity, we can consider the following Cauchy problem

$$\begin{cases} u_t - \Delta u + c_0 \frac{\partial u}{\partial x_N} - f(u) = 0, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (1.9)$$

Here, and in what follows, $0 \leq u_0(x) \leq 1$ is always assumed to be a uniformly continuous function defined in $\mathbb{R}^N$. Then, we need to prove that the solution $u(x, t)$ of the above Cauchy problem (1.9) tends to the solution $\phi_{c_0}(x)$ of the equation (1.8) as time t tends to infinity.

Lemma 1.3 (Hamel [6]). *Let $N \geq 2$, $0 < \phi(x) < 1$ be a solution of (1.8) for some $c \geq c^*$. Then for each $\lambda \in (0, 1)$, the level set $\{x \in \mathbb{R}^N, \phi(x) = \lambda\} = \{x = (x_1, \dots, x_{N-1}, x_N) \in \mathbb{R}^N, x_N = g_\lambda(x_1, \dots, x_{N-1})\}$ is well defined and $\phi(x)$ satisfies*

$$\liminf_{x_N - g_\lambda(x_1, \dots, x_{N-1}) \rightarrow +\infty} \phi(x) = 1, \quad \limsup_{x_N - g_\lambda(x_1, \dots, x_{N-1}) \rightarrow -\infty} \phi(x) = 0. \quad (1.10)$$

Thanks to the above lemma, for each $\lambda \in (0, 1)$ we can define a set

$$\mathcal{C}_l^- := \{x = (x_1, \dots, x_N) \in \mathbb{R}^N; x_N \leq g_\lambda(x_1, \dots, x_{N-1}) + l\}, \quad (1.11)$$

which will be used in the proof of the main result in this paper.

In the following theorem, we consider initial perturbations which belong to some functional spaces with exponential weights in the spirit of the pioneering papers of Sattinger [15, 16].

Theorem 1.4. *Assume $N \geq 2$, $\mu = \mu^* + \hat{\mu} \in \mathcal{M}$ is concentrated on $S_{(e_N, c_0)}$ for some $c_0 > c^*$, and $\hat{\mu}(X) \neq 0$. For the given measure μ , let $u_\mu = \phi_{c_0}$ be the travelling front of the equation (1.1). That is, $\phi_{c_0}(x)$ solves the elliptic equation (1.8). Let $u(x, t)$ be the solution of the Cauchy problem (1.9). Suppose the initial value $u_0(x) \in (0, 1)$ satisfies the following two conditions:*

(H1) *there exists a positive constant C such that*

$$|u_0(x) - \phi_{c_0}(x)| \leq C \int_{\tilde{S}} e^{\lambda^* x \cdot \nu} d\hat{\mu}(\nu, c), \quad \forall x \in \mathbb{R}^N,$$

where $\tilde{S}$ is a subset of $S_{(e_N, c_0)}$ such that $\tilde{S} \cap \{(\nu, c) \in X; c^ < c \leq c^* + \rho\} = \emptyset$, for some $\rho > 0$, $\hat{\mu}(\tilde{S}) \neq 0$, and for any given $l \in \mathbb{R}$, there exists a positive constant $M_l < +\infty$ such that $\int_{\tilde{S}} e^{\lambda^* x \cdot \nu} d\hat{\mu}(\nu, c) \leq M_l$ for all $x \in \mathcal{C}_l^-$ (defined by (1.11)).*

(H2) $\liminf_{l \rightarrow +\infty} \inf_{x \in \mathcal{C}_l^+} u_0(x) = \sigma > 0$, where $\mathcal{C}_l^+ = \mathbb{R}^N \setminus \mathcal{C}_l^-$.

Then there exist constants $M > 0$ and $\gamma < 0$ such that

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - \phi_{c_0}(x)| \leq M e^{\lambda^* \gamma t}, \quad \forall t \geq 0.$$

Remark 1.5. For the assumption (H1) in the above theorem, we mean that for the given measure μ , there exists a set $\tilde{S}$ satisfying the conditions such that (H1) holds. We can give a simple example to see that $\tilde{S}$ is not an empty set. For example, if $\mu = m_1 \delta_{(\nu_1, c_1)} + m_2 \delta_{(\nu_2, c_2)}$ with $c_i = c_0 \nu_i \nu_0$ and $c_i > c^*$, $i = 1, 2$, then $\tilde{S} = \{(\nu_1, c_1), (\nu_2, c_2)\}$ is a subset of $S_{(e_N, c_0)}$ satisfying all of the assumptions in (H1). In this case, the assumption (H1) means the existence of a constant C , such that

$$|u_0(x) - \phi_{c_0}(x)| \leq C \left(m_1 e^{\lambda^* x \cdot \nu_1} + m_2 e^{\lambda^* x \cdot \nu_2} \right), \quad \forall x \in \mathbb{R}^N.$$

For this example, it is known from [6] that

$$\liminf_{\max\{x \cdot \nu_1, x \cdot \nu_2\} \rightarrow +\infty} \phi_{c_0}(x) = 1, \quad \limsup_{\max\{x \cdot \nu_1, x \cdot \nu_2\} \rightarrow -\infty} \phi_{c_0}(x) = 0.$$

Then, the assumption (H2) reads

$$\liminf_{l \rightarrow +\infty} \inf_{\max\{x \cdot \nu_1, x \cdot \nu_2\} \geq l} u_0(x) > 0.$$

Physically speaking, this means that the initial value decays to the front fast enough in the region where it approaches 0, and it is not too small in the region where the front approaches 1.

Remark 1.6. If we denote $h(x) = \int_{\tilde{S}} e^{\lambda^* x \cdot \nu} d\hat{\mu}(\nu, c)$ and define a Banach space B by the norm $\|\cdot\|_B = \|\cdot h^{-1}(x)\|_{L^\infty(\mathbb{R}^N)} + \|\cdot\|_{L^\infty(\mathbb{R}^N)}$, then, under the same assumptions of the above theorem, we can reformulate the result in the following sense

$$\|u(x, t) - \phi_{c_0}(x)\|_B \leq M e^{\lambda^* \gamma t}, \quad \forall t \geq 0,$$

which can be viewed as a similar result to the one dimensional case given by Sattinger [15, 16].

We would like to mention here that our results are the first stability results about the nonplanar KPP fronts in the whole space. For the case of perturbations with compact supports, we employ the maximum principle and the properties of the travelling fronts given by Hamel and Nadirashvili [8] to prove the stability result. For the case of the perturbations without compact supports, the method of the proof is motivated by Mallordy and Roquejoffre [11]. Although the previous papers give us a light to deal with the present cases, we can see that the nontrivial technical point in our proofs is to prove the uniform convergence in the whole space.

In the past decades, many papers have been published on the subject of the stability of the travelling waves of the semilinear parabolic equation (1.1) with various types of nonlinearities f (see for example [7, 13, 14] and the references therein). Especially for the following two cases: one is so-called “ignition temperature” type which can be derived from the combustion theory in the framework of the thermo-diffusive approximation (actually, in combustion models, the nonlinearities f may be less smooth, and even discontinuous). The other one is “bistable” case which arises in genetics or population dynamics. A typical example if such a nonlinearity f is cubic nonlinearity $f(s) = s(1-s)(s-\theta)$.

This paper is organized as follows. In Section 2, we consider the case of perturbations with compact supports and give the proof of the Theorem 1.2. In Section 3, we deal with the case of perturbations without compact supports. We first consider the stability of the travelling waves given by a special kind of measure μ , and give a stability result under certain conditions. Then, following the idea of the special case, we give the proof of our second main result Theorem 1.4.

2. The case of perturbations with compact supports (Theorem 1.2)

In this section, we deal with the case of perturbations with compact supports and give the proof of the Theorem 1.2 based on the method of maximum principle and the properties of the travelling waves given by Hamel and Nadirashvili [8].

We first give the following lemma which will be used in the proofs of our main theorems.

Lemma 2.1. *If $f(s)$ is a concave function of class C^2 in $[0, 1]$ and satisfies the assumption (1.2), and $f(s)$ is defined by 0 for all $s > 1$, then*

i) $f(s)$ is a sub-additive function. That is

$$f(s+t) \leq f(s) + f(t), \quad \forall 0 \leq s, \quad t \leq 1;$$

ii) $f(s) \leq f'(0)s, \forall s \in [0, 1];$

iii) $f'(s) \leq f'(0), \forall s \in [0, 1].$

Proof. All of the conclusions in this lemma can be proved by the definition of concave functions with $f(0) = 0$ directly and we omit the details here. $\square$

Proof of the Theorem 1.2. We first consider the case $c_0 > c^*$. Since $u_0(x) - \phi_{c_0}(x)$ has compact support, we know that there exists a ball $B_R(0)$ in $\mathbb{R}^N$ with center 0 and radius R such that $\text{supp}(u_0(x) - \phi_{c_0}(x)) \subset B_R(0)$. Noticing that both $u_0(x)$ and $\phi_{c_0}(x)$ take value in $(0, 1)$, we have

$$\phi_{c_0}(x) - \chi_{B_R(0)}(x) \leq u_0(x) \leq \phi_{c_0}(x) + \chi_{B_R(0)}(x), \quad \forall x \in \mathbb{R}^N, \quad (2.1)$$

where $\chi_{B_R(0)}(x)$ is a function defined by

$$\chi_{B_R(0)}(x) = \begin{cases} 1 & \text{if } x \in B_R(0), \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_R(0). \end{cases}$$

Let $v(x, t)$ solves the following Cauchy problem

$$\begin{cases} v_t = \Delta v + f(v), & x \in \mathbb{R}^N, \quad t > 0, \\ v(x, 0) = \chi_{B_R(0)}(x), & x \in \mathbb{R}^N. \end{cases} \quad (2.2)$$

Noticing the sub-additivity of the function f (see Lemma 2.1), we have

$$\begin{aligned} & \frac{\partial}{\partial t} (\phi_{c_0}(x + c_0 \nu_0 t) + v(x, t)) - \Delta (\phi_{c_0}(x + c_0 \nu_0 t) + v(x, t)) \\ & \quad - f(\phi_{c_0}(x + c_0 \nu_0 t) + v(x, t)) \\ & \geq \frac{\partial}{\partial t} (\phi_{c_0}(x + c_0 \nu_0 t) + v(x, t)) - \Delta (\phi_{c_0}(x + c_0 \nu_0 t) + v(x, t)) \\ & \quad - f(\phi_{c_0}(x + c_0 \nu_0 t)) - f(v(x, t)) \\ & = 0, \end{aligned}$$

which combined with the initial value condition (2.1) and the maximum principle gives that

$$u(x, t) \leq \phi_{c_0}(x + c_0 \nu_0 t) + v(x, t), \quad \forall x \in \mathbb{R}^N, \quad t \geq 0.$$

FIGURE 1. $B_{c^*t}(x'_0 + c_0\nu_0t)$.

A similar consideration shows that

$$\phi_{c_0}(x + c_0\nu_0t) - v(x, t) \leq u(x, t), \quad \forall x \in \mathbb{R}^N, \quad t \geq 0.$$

If we take a moving frame $x' = x + c_0\nu_0t$, then we have for all $x' \in \mathbb{R}^N$

$$\phi_{c_0}(x') - v(x' - c_0\nu_0t, t) \leq u(x' - c_0\nu_0t, t) \leq \phi_{c_0}(x') + v(x' - c_0\nu_0t, t). \quad (2.3)$$

$\forall \epsilon > 0$, we define a set

$$\Omega_\epsilon := \{x' \in \mathbb{R}^N; 1 - \epsilon \leq \phi_{c_0}(x') < 1\}.$$

It follows from (1.6) and the continuity of the solution ϕ_{c_0} that there exists a point $x'_0 \in \mathbb{R}^N$ such that $\phi_{c_0}(x'_0) = 1 - \epsilon$. By Theorem 1.1, we know that ϕ_{c_0} is increasing in each unit direction $\nu \in S^{N-1}$ such that $\nu \cdot \nu_0 > \cos(\arcsin(c^*/c_0))$. Then we have

$$\phi_{c_0}(x') \geq 1 - \epsilon, \quad \forall x' \in B_{c^*t}(x'_0 + c_0\nu_0t),$$

where $B_{c^*t}(x'_0 + c_0\nu_0t)$ is a ball in $\mathbb{R}^N$ with center $x'_0 + c_0\nu_0t$ and radius c^*t (see Figure 1). This fact implies that

$$B_{c^*t}(x'_0 + c_0\nu_0t) \subset \Omega_\epsilon, \quad \forall t \geq 0. \quad (2.4)$$

Then for any given $x' \in \mathbb{R}^N \setminus \Omega_\epsilon$, we have $x' \in \mathbb{R}^N \setminus B_{c^*t}(x'_0 + c_0\nu_0t)$. That is to say, $x \in \mathbb{R}^N \setminus B_{c^*t}(x'_0)$. Obviously, there exists a positive constant T_R such that $B_R(0) \subset B_{c^*t}(x'_0)$ holds for all $t \geq T_R$. Consider the following Cauchy problem

$$\begin{cases} w_t = \Delta w + f'(0)w, & x \in \mathbb{R}^N, \quad t > 0, \\ w(x, 0) = \chi_{B_R(0)}(x) & x \in \mathbb{R}^N. \end{cases} \quad (2.5)$$

A simple calculation shows that

$$\begin{aligned} w(x, t) &= \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \chi_{B_R(0)}(y) dy \\ &= \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{B_R(0)} e^{-\frac{|x-y|^2}{4t}} dy \end{aligned} \quad (2.6)$$

for all $x \in \mathbb{R}^N$, $t > 0$. Notice that $f(s) \leq f'(0)s$, $\forall s \in [0, 1]$. Then, using maximum principle on the problems (2.2) and (2.5) gives that

$$v(x, t) \leq w(x, t), \quad \forall x \in \mathbb{R}^N, \quad t \geq 0. \quad (2.7)$$

Particularly, for all $x \in \mathbb{R}^N \setminus B_{c^*t}(x'_0)$ and $t > T_R$, noticing that $c^* = 2\sqrt{f'(0)}$ and $B_R(0) \subset B_{c^*t}(x'_0)$, we have

$$\begin{aligned} 0 \leq v(x, t) &\leq w(x, t) = \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} \int_{B_R(0)} e^{-\frac{|x-y|^2}{4t}} dy \\ &\leq \frac{e^{f'(0)t}}{(4\pi t)^{N/2}} e^{-\frac{|c^*t - |x'_0| - R|^2}{4t}} |B_R(0)| \\ &= \frac{1}{(4\pi t)^{N/2}} e^{-\frac{||x'_0| + R|^2}{4t} + \frac{c^*(|x'_0| + R)}{2}} |B_R(0)|, \end{aligned} \quad (2.8)$$

where $|B_R(0)|$ is the Lebesgue measure of the ball $B_R(0)$ in $\mathbb{R}^N$, and $|x'_0|$ is the norm of x'_0 in $\mathbb{R}^N$. Then, we have

$$\lim_{t \rightarrow +\infty} \sup_{x \in \mathbb{R}^N \setminus B_{c^*t}(x'_0)} v(x, t) = 0,$$

i.e.

$$\lim_{t \rightarrow +\infty} \sup_{x' \in \mathbb{R}^N \setminus B_{c^*t}(x'_0 + c_0\nu_0 t)} v(x' - c_0\nu_0 t, t) = 0,$$

which combined with (2.3) and (2.4) gives

$$\sup_{x' \in \mathbb{R}^N \setminus \Omega_\epsilon} |u(x' - c_0\nu_0 t, t) - \phi_{c_0}(x')| \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \quad (2.9)$$

It remains to prove that the above result holds also for all $x' \in \Omega_\epsilon$. To this end, we define a function $\tilde{u}(x', t) = 1 - u(x' - c_0\nu_0 t, t)$, $\forall x' \in \Omega_\epsilon$. Then the function $\tilde{u}(x', t)$ satisfies the following problem

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} - c_0\nu_0 \nabla \tilde{u} - f(1 - \tilde{u}), & x' \in \Omega_\epsilon, \quad t > 0, \\ \tilde{u}(x', 0) = 1 - u_0(x'), & x' \in \Omega_\epsilon, \\ \tilde{u}(x', t) = 1 - u(x' - c_0\nu_0 t, t), & x' \in \partial\Omega_\epsilon, \quad t \geq 0. \end{cases} \quad (2.10)$$

By (2.9), we conclude that for any given $\epsilon > 0$, there exists a positive constant T_ϵ such that

$$-\epsilon \leq u(x' - c_0\nu_0 t, t) - \phi_{c_0}(x') \leq \epsilon, \quad \forall x' \in \mathbb{R}^N \setminus \Omega_\epsilon, \quad t \geq T_\epsilon.$$

It follows from the continuity of the solutions ϕ_{c_0} and u that the above inequality holds also for all $x' \in \partial\Omega_\epsilon$, $t \geq T_\epsilon$. By the definition of Ω_ϵ and the continuity of the solution ϕ_{c_0} , we have

$$1 - \epsilon \leq \phi_{c_0}(x') < 1, \quad \forall x' \in \partial\Omega_\epsilon.$$

Thus

$$1 - 2\epsilon \leq u(x' - c_0\nu_0 t, t) < 1, \quad \forall x' \in \partial\Omega_\epsilon \quad t \geq T_\epsilon. \quad (2.11)$$

It follows from (2.6) and (2.7) that

$$\begin{aligned} 0 \leq v(x, T_\epsilon) &\leq w(x, T_\epsilon) = \frac{e^{f'(0)T_\epsilon}}{(4\pi T_\epsilon)^{N/2}} \int_{B_R(0)} e^{-\frac{|x-y|^2}{4T_\epsilon}} dy \\ &\leq \frac{e^{f'(0)T_\epsilon}}{(4\pi T_\epsilon)^{N/2}} e^{-\frac{|l-R|^2}{4T_\epsilon}} |B_R(0)| \end{aligned}$$

holds for all $x \in \mathbb{R}^N \setminus B_l(0)$ with $l > R$. Then there exists a positive constant $R_\epsilon > R$ such that

$$0 \leq v(x, T_\epsilon) \leq \epsilon, \quad \forall x \in \mathbb{R}^N \setminus B_{R_\epsilon}(0),$$

which combined with (2.3) gives that

$$-\epsilon \leq u(x' - c_0\nu_0 T_\epsilon, T_\epsilon) - \phi_{c_0}(x') \leq \epsilon, \quad \forall x' \in \mathbb{R}^N \setminus B_{R_\epsilon}(c_0\nu_0 T_\epsilon).$$

It follows from the definition of the set Ω_ϵ gives that

$$1 - 2\epsilon \leq u(x' - c_0\nu_0 T_\epsilon, T_\epsilon) < 1, \quad \forall x' \in \Omega_\epsilon \cap (\mathbb{R}^N \setminus B_{R_\epsilon}(c_0\nu_0 T_\epsilon)).$$

By the continuity and positivity of u , we have

$$\delta(\epsilon) = \inf_{x' \in B_{R_\epsilon}(c_0\nu_0 T_\epsilon) \cap \Omega_\epsilon} u(x' - c_0\nu_0 T_\epsilon, T_\epsilon) > 0.$$

Define $\delta_0(\epsilon) = \min\{\delta(\epsilon), 1 - 2\epsilon\}$. Then we have

$$0 < \delta_0(\epsilon) \leq u(x' - c_0\nu_0 T_\epsilon, T_\epsilon) < 1, \quad \forall x' \in \Omega_\epsilon. \quad (2.12)$$

Notice that $u(x' - c_0\nu_0 t, t)$ satisfies the following equation

$$u_t - \Delta u + c_0\nu_0 \nabla u = f(u) \geq 0,$$

which combined with (2.11), (2.12) and the maximum principle implies that

$$\delta_0(\epsilon) \leq u(x' - c_0\nu_0 t, t) < 1, \quad \forall x' \in \Omega_\epsilon, \quad t \geq T_\epsilon. \quad (2.13)$$

Define

$$\sigma = \inf_{0 \leq s \leq 1} \frac{f(s)}{s(1-s)} > 0. \quad (2.14)$$

It follows from (2.10), (2.13) and (2.14) that

$$\begin{aligned} \tilde{u}_t &= \Delta \tilde{u} - c_0\nu_0 \nabla \tilde{u} - f(u) \\ &\leq \Delta \tilde{u} - c_0\nu_0 \nabla \tilde{u} - \sigma u \tilde{u}, \\ &\leq \Delta \tilde{u} - c_0\nu_0 \nabla \tilde{u} - \sigma \delta_0(\epsilon) \tilde{u} \end{aligned}$$

holds for all $x' \in \Omega_\epsilon, t > T_\epsilon$. Then $\tilde{u}(x', t)$ satisfies

$$\begin{cases} \tilde{u}_t \leq \Delta \tilde{u} - c_0 \nu_0 \nabla \tilde{u} - \sigma \delta_0(\epsilon) \tilde{u}, & x' \in \Omega_\epsilon, \quad t > T_\epsilon, \\ \tilde{u}(x', T_\epsilon) = 1 - u(x' - c_0 \nu_0 T_\epsilon, T_\epsilon) \leq 1, & x' \in \Omega_\epsilon, \\ \tilde{u}(x', t) = 1 - u(x' - c_0 \nu_0 t, t) \leq 2\epsilon, & x' \in \partial \Omega_\epsilon, \quad t \geq T_\epsilon. \end{cases} \quad (2.15)$$

We consider a function $\psi(t)$ which solves the following O.D.E

$$\begin{cases} \psi'(t) + \sigma \delta_0(\epsilon) \psi(t) = 2\sigma \delta_0(\epsilon) \epsilon, & t > T_\epsilon \\ \psi(T_\epsilon) = 1. \end{cases} \quad (2.16)$$

Then the maximum principle implies that

$$\tilde{u}(x', t) \leq \psi(t) = 2\epsilon + (1 - 2\epsilon)e^{\sigma \delta_0(\epsilon)(T_\epsilon - t)}, \quad \forall x' \in \Omega_\epsilon, \quad t \geq T_\epsilon.$$

There exists a constant $T \geq T_\epsilon$ such that

$$\tilde{u}(x', t) \leq \psi(t) \leq 3\epsilon, \quad \forall x' \in \Omega_\epsilon, \quad t \geq T.$$

By the definition of $\tilde{u}$, we have

$$1 - 3\epsilon \leq u(x' - c_0 \nu_0 t, t) < 1, \quad \forall x' \in \Omega_\epsilon, \quad t \geq T.$$

Then, by the definition of the set Ω_ϵ , we conclude

$$\sup_{x' \in \Omega_\epsilon} |u(x' - c_0 \nu_0 t, t) - \phi_{c_0}(x')| \leq 3\epsilon, \quad \forall t \geq T,$$

which combined with (2.9) implies that

$$\sup_{x' \in \mathbb{R}^N} |u(x' - c_0 \nu_0 t, t) - \phi_{c_0}(x')| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

Let us now deal with the case $c_0 = c^*$. From the Theorem 1.1, we know that $\phi_{c^*}(x + c^* \nu_0 t)$ is a planar travelling wave with speed c^* in this case, namely, $\phi_{c^*}(x + c^* \nu_0 t) = \phi_{c^*}(x \cdot \nu_0 + c^* t + h)$ for some $h \in \mathbb{R}$. In fact, we can also prove the result by the same way. Since $u_0(x) - \phi_{c^*}(x \cdot \nu_0 + h)$ has compact support, we know that there exists a constant $R \in \mathbb{R}$ such that $\text{supp}(u_0(x) - \phi_{c^*}(x \cdot \nu_0 + h)) \subset \{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}$. Noticing that both $u_0(x)$ and $\phi_{c^*}(x \cdot \nu_0 + h)$ take value in $(0, 1)$, we can conclude that

$$\begin{aligned} \phi_{c^*}(x + h) - \chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x) &\leq u_0(x) \\ &\leq \phi_{c^*}(x \cdot \nu_0 + h) + \chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x), \end{aligned} \quad (2.17)$$

holds for all $x \in \mathbb{R}^N$, where $\chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x)$ is a function defined by

$$\chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x) = \begin{cases} 1 & \text{if } x \in \{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}, \\ 0 & \text{if } x \in \{x \in \mathbb{R}^N; x \cdot \nu_0 > R\}. \end{cases}$$

Let $v(x \cdot \nu_0, t)$ solves the Cauchy problem (2.2) with initial value $v(x \cdot \nu_0, 0) = \chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x)$, $\forall x \in \mathbb{R}^N$. Then by the same arguments as in the previous case we have

$$\phi_{c^*}(x \cdot \nu_0 + c_0 t + h) - v(x \cdot \nu_0, t) \leq u(x, t) \leq \phi_{c^*}(x \cdot \nu_0 + c_0 t + h) + v(x \cdot \nu_0, t)$$

holds for all $x \in \mathbb{R}^N, t \geq 0$. If we take a moving frame $x' = x + c^* \nu_0 t$, then $x' \cdot \nu_0 = x \cdot \nu_0 + c^* t$. In this moving frame, the above inequalities read as

$$\begin{aligned} \phi_{c^*}(x' \cdot \nu_0 + h) - v(x' - c_0 \nu_0 t, t) &\leq u(x' - c_0 \nu_0 t, t) \\ &\leq \phi_{c^*}(x' \cdot \nu_0 + h) + v(x' - c_0 \nu_0 t, t). \end{aligned} \quad (2.18)$$

$\forall \epsilon > 0$, we define a set

$$\Omega_\epsilon := \{x' \in \mathbb{R}^N; 1 - \epsilon \leq \phi_{c^*}(x' \cdot \nu_0 + h) < 1\}.$$

It follows from (1.6) and the continuity of the solution ϕ_{c^*} that there exists a point $x'_0 \in \mathbb{R}^N$ such that $\phi_{c_0}(x'_0 \cdot \nu_0 + h) = 1 - \epsilon$. By Theorem 1.1, we know that $\phi_{c^*}(x'_0 \cdot \nu_0 + h)$ is increasing in each unit direction $\nu \in S^{N-1}$ such that $\nu \cdot \nu_0 > 0$. Then we have

$$\phi_{c_0}(x'_0 \cdot \nu_0 + h) \geq 1 - \epsilon, \quad \forall x' \in \{x' \in \mathbb{R}^N; x' \cdot \nu_0 \geq x'_0 \cdot \nu_0\}.$$

This fact implies that

$$\{x' \in \mathbb{R}^N; x' \cdot \nu_0 \geq x'_0 \cdot \nu_0\} \subset \Omega_\epsilon, \quad \forall t \geq 0. \quad (2.19)$$

Then for any given $x' \in \mathbb{R}^N \setminus \Omega_\epsilon$, we have $x' \in \{x' \in \mathbb{R}^N; x' \cdot \nu_0 < x'_0 \cdot \nu_0\}$. That is to say, $x \in \{x \in \mathbb{R}^N; x \cdot \nu_0 \leq x'_0 \cdot \nu_0 - c^* t\}$. Obviously, there exists a positive constant T_R such that $x \cdot \nu_0 \leq x'_0 \cdot \nu_0 - c^* t \leq -R$ holds for all $t \geq T_R$. Let $w(x \cdot \nu_0, t)$ solves the Cauchy problem (2.5) with initial value $w(x \cdot \nu_0, 0) = \chi_{\{x \in \mathbb{R}^N; x \cdot \nu_0 \leq R\}}(x)$, $\forall x \in \mathbb{R}^N$. Then, by a similar arguments as in the previous case we can inform that

$$\begin{aligned} 0 \leq v(x \cdot \nu_0, t) &\leq w(x \cdot \nu_0, t) = \frac{e^{f'(0)t}}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{|x \cdot \nu_0 - y|^2}{4t}} \chi_{\{y \in \mathbb{R}; |y| \leq R\}}(y) dy \\ &= \frac{e^{f'(0)t}}{\sqrt{4\pi t}} \int_{-R}^R e^{-\frac{|x \cdot \nu_0 - y|^2}{4t}} dy \\ &\leq 2R \frac{e^{f'(0)t}}{\sqrt{4\pi t}} e^{-\frac{|x'_0 \cdot \nu_0 - c^* t - R|^2}{4t}} \\ &= 2R \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x'_0 \cdot \nu_0 - R|^2}{4t} + \frac{c^*(x'_0 \cdot \nu_0 - R)}{2}}. \end{aligned}$$

holds for all $x \in \{x \in \mathbb{R}^N; x \cdot \nu_0 \leq x'_0 \cdot \nu_0 - c^* t\}$ and $t \geq T_R$. Then, we have

$$\lim_{t \rightarrow +\infty} \sup_{x \in \{x \in \mathbb{R}^N; x \cdot \nu_0 \leq x'_0 \cdot \nu_0 - c^* t\}} v(x \cdot \nu_0, t) = 0,$$

i.e.

$$\lim_{t \rightarrow +\infty} \sup_{x' \in \{x' \in \mathbb{R}^N; x' \cdot \nu_0 \leq x'_0 \cdot \nu_0\}} v(x' - c_0 \nu_0 t, t) = 0,$$

which combined with (2.18) and (2.19) gives

$$\sup_{x' \in \mathbb{R}^N \setminus \Omega_\epsilon} |u(x' - c_0 \nu_0 t, t) - \phi_{c^*}(x' \cdot \nu_0 + h)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty. \quad (2.20)$$

It remains to prove that the above result holds also for all $x' \in \Omega_\epsilon$. To this end, we define a function $\tilde{u}(x', t) = 1 - u(x' - c_0 \nu_0 t, t)$, $\forall x' \in \Omega_\epsilon$. Then the function

$\tilde{u}(x', t)$ satisfies the problem (2.10). By a complete similar arguments with the case $c_0 > c^*$, we conclude that (2.13) still holds in the present case. Then $\tilde{u}(x', t)$ satisfies the problem (2.15). Then, by the same way as in the case $c_0 > c^*$ we can prove that

$$\sup_{x' \in \Omega_\epsilon} |u(x' - c_0 \nu_0 t, t) - \phi_{c^*}(x' \cdot \nu_0 + h)| \leq 3\epsilon, \quad \forall t \geq T,$$

which combined with (2.20) gives that

$$\sup_{x' \in \mathbb{R}^N} |u(x' - c_0 \nu_0 t, t) - \phi_{c^*}(x' \cdot \nu_0 + h)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

The proof is completed. $\square$

3. The case of perturbations without compact supports (Theorem 1.4)

In this section, we consider the stability of the travelling waves when the perturbations do not have compact supports. We first focus ourselves on the travelling waves given by a special kind of measure μ , that is, $\mu = \mu^* + \hat{\mu} \in \mathcal{M}_{TW}$, where $\mu^* = \sum_{1 \leq i \leq k} m_i \delta_{(\nu_i, c_i^*)}$, $\hat{\mu} = \bar{m}_1 \delta_{(\nu_1, c_1)} + \bar{m}_2 \delta_{(\nu_2, c_2)} + \tilde{\mu} \in \hat{\mathcal{M}}_{TW}$ and $c_i > c^*$, $i = 1, 2$.

Lemma 3.1. *Consider two solutions u and v of the equation*

$$\partial_t u = \Delta u - c_0 \frac{\partial u}{\partial x_N} + f(u), \quad x \in \mathbb{R}^N, \quad t > 0,$$

and suppose that the initial data $u_0(x)$ and $v_0(x)$ take values in $(0, 1)$ and satisfy the following two conditions:

- i) $u_0(x) \leq v_0(x)$ in $\mathbb{R}^N$;
- ii) $|u_0(x) - v_0(x)| = \mathcal{O}(e^{\lambda^* x \cdot \nu_1} + e^{\lambda^* x \cdot \nu_2})$, as $x \in \mathcal{C}_l^-$ and $l \rightarrow -\infty$, where $\mathcal{C}_l^-$ (see Figure 2) is defined by

$$\mathcal{C}_l^- = \{x \in \mathbb{R}^N; x \cdot \nu_i \leq l, i = 1, 2\}, \quad (3.1)$$

ν_1 and ν_2 are the unit vectors in S^{N-1} such that there exist c_1 and c_2 with $c_i > c^*$, $i = 1, 2$ and (ν_1, c_1) and (ν_2, c_2) are in the support of μ .

Then for all $t \geq 0$ and $x \in \mathbb{R}^N$, we have

$$0 \leq v(x, t) - u(x, t) \leq Ch(x, t), \quad (3.2)$$

where

$$h(x, t) = e^{\lambda^*(x \cdot \nu_1 - (c_1 - c^*)t)} + e^{\lambda^*(x \cdot \nu_2 - (c_2 - c^*)t)}, \quad (3.3)$$

$\lambda^* = c^*/2$, and C is a positive constant.

FIGURE 2. $\mathcal{C}_l^-$ and $\mathcal{C}_l^+$.

Proof. The left hand side of the inequality (3.2) follows from the maximum principle. Now we are in the position to prove the right hand side of the inequality (3.2). By the Lemma 2.1, we inform that there exists a function w such that

$$\left(\partial_t - \Delta + c_0 \frac{\partial}{\partial x_N} \right) (v - u) = f(v) - f(u) = f'(w)(v - u) \leq f'(0)(v - u).$$

$\lambda^* = c^*/2$ solves the following equation

$$-(\lambda^*)^2 + c^* \lambda^* - f'(0) = 0.$$

Then, we have

$$-\lambda^*(c_i - c^*) = (\lambda^*)^2 - \lambda^* c_1 + f'(0), \quad i = 1, 2.$$

Thus

$$\begin{aligned} \frac{\partial}{\partial t} h(x, t) &= -\lambda^*(c_1 - c^*) e^{\lambda^*(x \cdot \nu_1 - (c_1 - c^*)t)} - \lambda^*(c_2 - c^*) e^{\lambda^*(x \cdot \nu_2 - (c_2 - c^*)t)} \\ &= ((\lambda^*)^2 - \lambda^* c_1 + f'(0)) e^{\lambda^*(x \cdot \nu_1 - (c_1 - c^*)t)} \\ &\quad + ((\lambda^*)^2 - \lambda^* c_2 + f'(0)) e^{\lambda^*(x \cdot \nu_2 - (c_2 - c^*)t)} \\ &= ((\lambda^*)^2 - \lambda^* c_0 e_N \cdot \nu_1 + f'(0)) e^{\lambda^*(x \cdot \nu_1 - (c_1 - c^*)t)} \\ &\quad + ((\lambda^*)^2 - \lambda^* c_0 e_N \cdot \nu_2 + f'(0)) e^{\lambda^*(x \cdot \nu_2 - (c_2 - c^*)t)} \\ &= (\Delta - c_0 \nu_0 \nabla + f'(0)) h(x, t). \end{aligned}$$

From the initial value condition *ii*), we can choose a constant $C > 0$ such that

$$v_0(x) - u_0(x) \leq Ch(x, 0), \quad \forall x \in \mathbb{R}^N.$$

The right hand side of the inequality (3.2) then follows from the maximum principle. The proof of this lemma is completed. $\square$

Theorem 3.2. Assume $N \geq 2$, $\mu = \mu^* + \hat{\mu} \in \mathcal{M}_{TW}$, where $\hat{\mu} = \bar{m}_1 \delta_{(\nu_1, c_1)} + \bar{m}_2 \delta_{(\nu_2, c_2)} + \tilde{\mu} \in \hat{\mathcal{M}}_{TW}$ and $c_i > c^*$, $i = 1, 2$. Denote the solution of the elliptic equation (1.8) by $\phi_\mu = \phi_{c_0}$ and let $u(x, t)$ be the solution of the following Cauchy problem

$$\begin{cases} u_t - \Delta u + c_0 \frac{\partial u}{\partial x_N} - f(u) = 0, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \quad (3.4)$$

Suppose the initial value $u_0 \in (0, 1)$ satisfies the following two conditions:

- i) $|u_0(x) - \phi_{c_0}(x)| = \mathcal{O}(e^{\lambda^* x \cdot \nu_1} + e^{\lambda^* x \cdot \nu_2})$ as $x \in \mathcal{C}_l^-$ and $l \rightarrow -\infty$, where $\mathcal{C}_l^-$ is defined by (3.1);
- ii) there exists a positive constant $r \in \mathbb{R}$ such that

$$\phi_{c_0}(x - re_N) \leq u_0(x), \quad \forall x \in \mathbb{R}^N.$$

Then there exist constants $\hat{C}, l > 0$ and $\gamma < 0$ such that

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - \phi_{c_0}(x)| \leq \hat{C} e^{\lambda^* (l + \gamma t)}, \quad \forall t \geq 0.$$

Proof. Define $\underline{u}_0(x) = \min \{u_0(x), \phi_{c_0}(x)\}$. It follows from the first initial value condition and the boundedness of $u_0(x)$ and $\phi_{c_0}(x)$ that there exists a constant C such that

$$0 \leq \phi_{c_0}(x) - \underline{u}_0(x) \leq |\phi_{c_0}(x) - u_0(x)| \leq Ch(x, 0), \quad \forall x \in \mathbb{R}^N,$$

where $h(x, t)$ is defined by (3.3). We denote by $\underline{u}(x, t)$ the solution of the Cauchy problem (3.4) with initial value $\underline{u}_0(x)$, then from the Lemma 3.1, we have

$$0 \leq \phi_{c_0}(x) - \underline{u}(x, t) \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

Similarly, we define $\bar{u}_0(x) = \max \{u_0(x), \phi_{c_0}(x)\}$. Then there exists a constant C such that

$$0 \leq \bar{u}_0(x) - \phi_{c_0}(x) \leq |\phi_{c_0}(x) - u_0(x)| \leq Ch(x, 0), \quad \forall x \in \mathbb{R}^N.$$

We denote by $\bar{u}(x, t)$ the solution of the Cauchy problem (3.4) with initial value $\bar{u}_0(x)$, then from the Lemma 3.1, we have

$$0 \leq \bar{u}(x, t) - \phi_{c_0}(x) \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

It follows from the maximum principle that

$$\phi_{c_0}(x) - Ch(x, t) \leq \underline{u}(x, t) \leq u(x, t) \leq \bar{u}(x, t) \leq \phi_{c_0}(x) + Ch(x, t)$$

holds for all $t \geq 0$ and $x \in \mathbb{R}^N$. That is

$$|u(x, t) - \phi_{c_0}(x)| \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

From the above inequality, we can inform that for any given constant l

$$\begin{aligned}
 \sup_{x \in \mathcal{C}_l^-} |u(x, t) - \phi_{c_0}(x)| &\leq C \sup_{x \in \mathcal{C}_l^-} h(x, t) \\
 &= C \sup_{x \in \mathcal{C}_l^-} \sum_{i=1,2} e^{\lambda^*(x \cdot \nu_i - (c_i - c^*)t)} \\
 &\leq C \sum_{i=1,2} e^{\lambda^*(l - (c_i - c^*)t)}.
 \end{aligned} \tag{3.5}$$

Then it remains to show that there exists a real constant l such that $u(x, t)$ tends to $\phi_{c_0}(x)$ uniformly with respect to $x \in \mathcal{C}_l^+ = \mathbb{R}^N \setminus \mathcal{C}_l^-$ as t tends to infinity. First, we claim that

$$\liminf_{l \rightarrow +\infty} \inf_{x \in \mathcal{C}_l^+} \phi_{c_0}(x) = 1. \tag{3.6}$$

By (3.5) in [8], we know the following lower bound of the solution u_μ of the equation (1.1)

$$\begin{aligned}
 u_\mu(x, t) &\geq \max \left(\max_{1 \leq i \leq k} \phi_{c^*}(x \cdot \nu_i + c^*t + c^* \ln m_i), \right. \\
 &\quad \left. \int_{\hat{X}} \phi_c(x \cdot \nu + ct + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu} \right). \\
 &\geq \int_{\hat{X}} \phi_c(x \cdot \nu + ct + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu},
 \end{aligned}$$

where $\hat{M} = \hat{\mu}(X)$. By the notation of (1.4), we have

$$\phi_{c_0}(x + c_0 t \nu_0) = u_\mu(x, t) \geq \int_{\hat{X}} \phi_c(x \cdot \nu + ct + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu}$$

Recalling $(\nu, c) \in S_{(\nu_0, c_0)}$, we have $c_0 \nu_0 \cdot \nu = c$. Then from the above inequality and the fact that $\hat{\mu} = \bar{m}_1 \delta_{(\nu_1, c_1)} + \bar{m}_2 \delta_{(\nu_2, c_2)} + \tilde{\mu}$, we have

$$\begin{aligned}
 \phi_{c_0}(x) &\geq \int_{\hat{X}} \phi_c(x \cdot \nu + c \ln \hat{M}) \frac{1}{\hat{M}} d\hat{\mu} \\
 &\geq \frac{\bar{m}_1}{\hat{M}} \phi_{c_1}(x \cdot \nu + c_1 \ln \hat{M}) + \frac{\bar{m}_2}{\hat{M}} \phi_{c_2}(x \cdot \nu + c_2 \ln \hat{M}).
 \end{aligned}$$

Then we have

$$\liminf_{l \rightarrow +\infty} \inf_{x \in \mathcal{C}_l^+} \phi_{c_0}(x) \geq \min \left(\frac{\bar{m}_1}{\hat{M}}, \frac{\bar{m}_2}{\hat{M}} \right) > 0. \tag{3.7}$$

Assume now that there exists $\varepsilon > 0$ and a sequence $x_n \in \mathcal{C}_{l_n}^+$ such that

$$\phi_{c_0}(x_n) \leq 1 - \varepsilon, \quad l_n \rightarrow +\infty.$$

Up to extraction of some subsequence, the functions $\phi_n(x) = \phi_{c_0}(x + x_n)$ converge in $C_{loc}^2(\mathbb{R}^N)$ to a solution ϕ_∞ of the equation (1.5) such that $\phi_\infty(0) \leq 1 - \varepsilon$. On

the other hand, the inequality (3.7) and the fact that $x_n \in \mathcal{C}_{l_n}^+$ and $l_n \rightarrow +\infty$ imply that

$$m_\infty := \inf_{\mathbb{R}^N} \phi_\infty \geq \min \left(\frac{\bar{m}_1}{\hat{M}}, \frac{\bar{m}_2}{\hat{M}} \right) > 0,$$

whence

$$\min \left(\frac{\bar{m}_1}{\hat{M}}, \frac{\bar{m}_2}{\hat{M}} \right) \leq m_\infty \leq 1 - \varepsilon < 1.$$

Let $x'_n \in \mathbb{R}^N$ be a sequence such that $u_\infty(x'_n) \rightarrow m_\infty$ as $n \rightarrow +\infty$. Up to extraction of some subsequence, the functions $\varphi_n(x) = \phi_\infty(x + x'_n)$ converge in $C_{loc}^2(\mathbb{R}^N)$ to a solution φ_∞ of the equation (1.5) such that

$$m_\infty = \varphi_\infty(0) = \min_{\mathbb{R}^N} \varphi_\infty,$$

which combined with the equation (1.5) gives that $f(m_\infty) \leq 0$, which contradicts the positivity of f on $[\min(\frac{\bar{m}_1}{\hat{M}}, \frac{\bar{m}_2}{\hat{M}}), 1)$. Thus the claim (3.6) holds. It follows from the second initial condition that there exists a positive constant $r \in \mathbb{R}$ such that

$$\phi_{c_0}(x - re_N) \leq u_0(x), \quad \forall x \in \mathbb{R}^N.$$

Then the maximum principle gives that

$$\phi_{c_0}(x - re_N) \leq u(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

By the claim (3.6), we can choose l large enough such that

$$1 - \epsilon \leq \phi_{c_0}(x - re_N) \leq u(x, t) \leq 1, \quad \forall t \geq 0, \quad x \in \mathcal{C}_l^+,$$

Obviously, we also have

$$1 - \epsilon \leq \phi_{c_0}(x) \leq 1, \quad \forall x \in \mathcal{C}_l^+.$$

For all $t \geq 0$, $x \in \mathcal{C}_l^+$, $v(x, t) = u(x, t) - \phi_{c_0}(x)$ satisfies the following equation

$$\begin{aligned} v_t - \Delta v + c_0 \nu_0 \nabla v - f(v + \phi_{c_0}) + f(\phi_{c_0}) &= v_t - \Delta v + c_0 \nu_0 \nabla v - f'(w)v \\ &= 0, \end{aligned}$$

where $w \in (1 - \epsilon, 1)$. We define a parabolic operator as follows

$$L = \frac{\partial}{\partial t} - \Delta + c_0 \frac{\partial}{\partial x_N} - f'(w).$$

Then we have $Lv = 0$. Define a function

$$\bar{v}(x, t) := \hat{C} e^{\lambda^*(l + \gamma t)}, \quad \forall t \geq 0, \quad x \in \mathbb{R}^N,$$

where $\gamma = \max\{-\min_{i=1,2}(c_i - c^*), f'(1 - \epsilon)/\lambda^*\} < 0$, $\hat{C} = \max\{\epsilon/e^{\lambda^* l}, 2C\}$ and C is the constant defined in Lemma 3.1. For all $x \in \mathcal{C}_l^+$ and $t \geq 0$, we have

$$\begin{aligned} L\bar{v}(x, t) &= (\lambda^* \gamma - f'(w))\bar{v}(x, t) \\ &\geq (f'(1 - \epsilon) - f'(w))\bar{v}(x, t) \\ &\geq 0 = Lv. \end{aligned}$$

We also have

$$v(x, 0) = u(x, 0) - \phi_{c_0}(x) \leq \epsilon \leq \hat{C}e^{\lambda^* l} = \bar{v}(x, 0), \quad \forall x \in \mathcal{C}_l^+.$$

It follows from (3.5) that

$$\begin{aligned} v(x, t) &\leq Ch(x, t) \\ &\leq Ce^{\lambda^*(l-(c_1-c^*)t)} + Ce^{\lambda^*(l-(c_2-c^*)t)} \\ &\leq 2Ce^{\lambda^*(l-\gamma t)} \leq \bar{v}(x, t) \end{aligned}$$

holds for all $x \in \partial\mathcal{C}_l^+ = \partial\mathcal{C}_l^-$ and $t \geq 0$. Then the maximum principle gives that

$$v(x, t) \leq \bar{v}(x, t) = \hat{C}e^{\lambda^*(l+\gamma t)}, \quad \forall t \geq 0, \quad x \in \mathcal{C}_l^+. \quad (3.8)$$

By a similar consideration, we have

$$v(x, t) \geq \underline{v}(x, t) := -\hat{C}e^{\lambda^*(l+\gamma t)}, \quad \forall t \geq 0, \quad x \in \mathcal{C}_l^+.$$

Then we have

$$\sup_{x \in \mathcal{C}_l^+} |u(x, t) - \phi_{c_0}(x)| \leq \hat{C}e^{\lambda^*(l+\gamma t)}, \quad \forall t \geq 0,$$

which combined with (3.5) gives that

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - \phi_{c_0}(x)| \leq \hat{C}e^{\lambda^*(l+\gamma t)}, \quad \forall t \geq 0.$$

The proof of the theorem is completed. $\square$

Now we are in the position to give the proof of the Theorem 1.4.

Proof of the Theorem 1.4. We first define a function

$$h(x, t) = \int_{\tilde{S}} e^{\lambda^*(x \cdot \nu - (c - c^*)t)} d\hat{\mu}(\nu, c). \quad (3.9)$$

Let $\underline{u}_0(x) = \min\{u_0(x), \phi_{c_0}(x)\}$. Then by the assumption (H1), there exists a constant C such that

$$0 \leq \phi_{c_0}(x) - \underline{u}_0(x) \leq |\phi_{c_0}(x) - u_0(x)| \leq Ch(x, 0), \quad \forall x \in \mathbb{R}^N.$$

We denote by $\underline{u}(x, t)$ the solution of the Cauchy problem (1.9) with initial value $\underline{u}_0(x)$. It follows from the maximum principle that

$$\phi_{c_0}(x) - \underline{u}(x, t) \geq 0, \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

Since $\phi_{c_0}(x) - \underline{u}(x, t) \geq 0$ and f is concave, there exists a function w such that

$$\begin{aligned} \left(\partial_t - \Delta + c_0 \frac{\partial}{\partial x_N} \right) (\phi_{c_0}(x) - \underline{u}(x, t)) &= f(\phi_{c_0}(x)) - f(\underline{u}(x, t)) \\ &= f'(w)(\phi_{c_0}(x) - \underline{u}(x, t)) \\ &\leq f'(0)(\phi_{c_0}(x) - \underline{u}(x, t)). \end{aligned}$$

Since λ^* satisfies the equation $(\lambda^*)^2 - c^*\lambda^* + f'(0) = 0$, one has

$$-\lambda^*(c - c^*) = (\lambda^*)^2 - \lambda^*c + f'(0), \quad \forall c > c^*.$$

Thus

$$\begin{aligned}
 h_t &= - \int_{\tilde{S}} \lambda^* (c - c^*) e^{\lambda^* (x \cdot \nu - (c - c^*)t)} d\hat{\mu}(\nu, c) \\
 &= \int_{\tilde{S}_{(\nu_0, c_0)}} ((\lambda^*)^2 - \lambda^* c + f'(0)) e^{\lambda^* (x \cdot \nu - (c - c^*)t)} d\hat{\mu}(\nu, c) \\
 &= \int_{\tilde{S}_{(\nu_0, c_0)}} ((\lambda^*)^2 - \lambda^* c_0 e_N \cdot \nu + f'(0)) e^{\lambda^* (x \cdot \nu - (c - c^*)t)} d\hat{\mu}(\nu, c) \\
 &= \Delta h - c_0 \frac{\partial}{\partial x_N} h + f'(0) h.
 \end{aligned}$$

Then the maximum principle gives that

$$\phi_{c_0}(x) - \underline{u}(x, t) \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

Similarly, let $\bar{u}_0(x) = \max \{u_0(x), \phi_{c_0}(x)\}$. Then there exists a constant C such that

$$0 \leq \bar{u}_0(x) - \phi_{c_0}(x) \leq |\phi_{c_0}(x) - u_0(x)| \leq Ch(x, 0), \quad \forall x \in \mathbb{R}^N.$$

We denote by $\bar{u}(x, t)$ the solution of the Cauchy problem (1.9) with initial value $\bar{u}_0(x)$. Then, we have

$$0 \leq \bar{u}(x, t) - \phi_{c_0}(x) \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N.$$

The maximum principle gives that

$$\phi_{c_0}(x) - Ch(x, t) \leq \underline{u}(x, t) \leq u(x, t) \leq \bar{u}(x, t) \leq \phi_{c_0}(x) + Ch(x, t),$$

for all $t \geq 0$ and $x \in \mathbb{R}^N$. That is

$$|u(x, t) - \phi_{c_0}(x)| \leq Ch(x, t), \quad \forall t \geq 0, \quad x \in \mathbb{R}^N. \quad (3.10)$$

Then for any given real constant l , we have

$$\begin{aligned}
 \sup_{x \in \mathcal{C}_l^-} |u(x, t) - \phi_{c_0}(x)| &\leq C \sup_{x \in \mathcal{C}_l^-} h(x, t) \\
 &= C \sup_{x \in \mathcal{C}_l^-} \int_{\tilde{S}} e^{\lambda^* (x \cdot \nu - (c - c^*)t)} d\hat{\mu}(\nu, c) \\
 &\leq C e^{-\lambda^* \rho t} \sup_{x \in \mathcal{C}_l^-} \int_{\tilde{S}} e^{\lambda^* x \cdot \nu} d\hat{\mu}(\nu, c) \\
 &\leq C M_l e^{-\lambda^* \rho t}.
 \end{aligned} \quad (3.11)$$

In what follows, we investigate the long time behavior of the solutions in the domain $\mathcal{C}_l^+ = \mathbb{R}^N \setminus \mathcal{C}_l^-$. By the assumption (H2), there exist a constant $l_0 > 0$ such that $u_0(x) \geq \sigma/2$, for all $x \in \mathcal{C}_{l_0-1}^+$. It follows from the Lemma 1.3 that for any given $\epsilon > 0$, there exists a constant $l_\epsilon > l_0$ such that

$$1 - \frac{\epsilon}{2} \leq \phi_{c_0}(x) < 1, \quad \forall x \in \mathcal{C}_{l_\epsilon}^+. \quad (3.12)$$

By the inequality (3.11), we know that there exists a positive constant T_ϵ such that

$$|u(x, t) - \phi_{c_0}(x)| \leq \frac{\epsilon}{2}, \quad \forall t \geq T_\epsilon, \quad x \in \mathcal{C}_{l_\epsilon}^-. \quad (3.13)$$

It follows from (3.12) and (3.13) that

$$1 - \epsilon \leq u(x, t) < 1, \quad \forall t \geq T_\epsilon, \quad x \in \partial\mathcal{C}_{l_\epsilon}^+ = \partial\mathcal{C}_{l_\epsilon}^-. \quad (3.14)$$

Since $l_\epsilon > l_0$, we also have

$$u_0(x) \geq \frac{\sigma}{2}, \quad \forall x \in \mathcal{C}_{l_{\epsilon-1}}^-. \quad (3.15)$$

For any given $x_0 \in \partial\mathcal{C}_{l_\epsilon}^+$, there exists a positive constant r_{x_0} such that $B(x_0, r_{x_0}) \subset \mathcal{C}_{l_{\epsilon-1}}^+$, where $B(x_0, r_{x_0})$ is a ball with center x_0 and radius r_{x_0} . Let $v(x, t)$ solve the following initial boundary value problem

$$\begin{cases} v_t = \Delta v - c_0 \frac{\partial v}{\partial x_N} + f(v), & 0 < t < T_\epsilon, \quad x \in B(x_0, r_{x_0}), \\ v(x, t) = 0, & 0 \leq t \leq T_\epsilon, \quad x \in \partial B(x_0, r_{x_0}), \\ v(x, 0) = v_0(x), & x \in B(x_0, r_{x_0}), \end{cases}$$

where the initial value $v_0(x)$ satisfies $0 \leq v_0(x) \leq \sigma/2$ for all $x \in B(x_0, r_{x_0})$ and $v_0(x) = 0$ for all $x \in \partial B(x_0, r_{x_0})$. Then we have

$$u_0(x) \geq v_0(x), \quad \forall x \in B(x_0, r_{x_0}).$$

It follows from (3.15) and the positivity of $u_0(x)$ for all $x \in \mathbb{R}^N$ that

$$u(x, t) \geq v(x, t) = 0, \quad \forall x \in \partial B(x_0, r_{x_0}), \quad 0 \leq t \leq T_\epsilon.$$

Then the maximum principle implies that

$$u(x, t) \geq v(x, t), \quad \forall 0 \leq t \leq T_\epsilon, \quad x \in B(x_0, r_{x_0}). \quad (3.16)$$

On the other hand, since x_0 is an interior point of $B(x_0, r_{x_0})$, the strong maximum principle implies that there exist a positive constant τ_{x_0} such that $v_0(x_0, t) \geq \tau_{x_0}$ holds for all $0 \leq t \leq T_\epsilon$. Then by (3.16) and the arbitrariness of the x_0 , we conclude that there exists a positive constant τ such that

$$u(x, t) \geq \tau, \quad \forall 0 \leq t \leq T_\epsilon, \quad x \in \partial\mathcal{C}_{l_\epsilon}^-.$$

Obviously, there exists a concave function $f_\epsilon(x)$ (see Figure 3) such that

$$\begin{aligned} f_\epsilon(x) &\leq f(x), \quad \forall 0 \leq x \leq 1, \quad \text{and} \\ f_\epsilon(0) &= f_\epsilon(1 - \epsilon) = 0, \quad f'_\epsilon(1 - \epsilon) \leq 0 \leq f'_\epsilon(0). \end{aligned}$$

We consider the following O.D.E

$$\begin{cases} \zeta'(t) = f_\epsilon(\zeta(t)), & t > 0, \\ \zeta(0) = \lambda, \end{cases}$$

where $\lambda = \min\{\sigma/2, \tau/e^{f'_\epsilon(0)T_\epsilon}\}$. It follows from the assumptions of the function f_ϵ that

$$\zeta'(t) = f_\epsilon(\zeta(t)) \leq f'_\epsilon(0)\zeta(t).$$

FIGURE 3. f and f_ϵ .

Then we have

$$\zeta(t) \leq \lambda e^{f'_\epsilon(0)T_\epsilon} \leq \tau \leq u(x, t), \quad 0 \leq t \leq T_\epsilon, \quad x \in \partial\mathcal{C}_{l_\epsilon}^+. \quad (3.17)$$

Obviously, the solution $\zeta(t)$ of the above O.D.E is increasing and satisfies

$$\lim_{t \rightarrow +\infty} \zeta(t) = 1 - \epsilon, \quad (3.18)$$

which combined with (3.14) gives that

$$\zeta(t) \leq 1 - \epsilon \leq u(x, t), \quad t \geq T_\epsilon, \quad x \in \partial\mathcal{C}_{l_\epsilon}^+.$$

The above inequality combined with (3.17) gives that

$$\zeta(t) \leq u(x, t), \quad t \geq 0, \quad x \in \partial\mathcal{C}_{l_\epsilon}^+. \quad (3.19)$$

By (3.15) we also have

$$\zeta(0) \leq u(x, 0), \quad \forall x \in \mathcal{C}_{l_\epsilon}^+. \quad (3.20)$$

Then it follows from the maximum principle that

$$u(x, t) \geq \zeta(t), \quad \forall t \geq 0, \quad x \in \mathcal{C}_{l_\epsilon}^+.$$

By (3.18), we conclude that there exists a positive constant $T > T_\epsilon$ such that

$$u(x, t) \geq 1 - 2\epsilon, \quad \forall t \geq T, \quad x \in \mathcal{C}_{l_\epsilon}^+,$$

which combined with (3.12) gives that

$$|u(x, t) - \phi_{c_0}(x)| \leq \epsilon, \quad \forall t \geq T, \quad x \in \mathcal{C}_{l_\epsilon}^+. \quad (3.21)$$

For all $t \geq T$, $x \in \mathcal{C}_{l_\epsilon}^+$, $v(x, t) = u(x, t) - \phi_{c_0}(x)$ satisfies the following equation

$$\begin{aligned} Lv &= v_t - \Delta v + c_0 \frac{\partial v}{\partial x_N} - f(v + \phi_{c_0}) + f(\phi_{c_0}) \\ &= v_t - \Delta v + c_0 \frac{\partial v}{\partial x_N} - f'(w)v = 0, \end{aligned}$$

where $w \in (1 - \epsilon, 1)$. Define a function

$$\bar{v}(x, t) := \hat{C} M_{l_\epsilon} e^{\lambda^* \gamma t}, \quad \forall t \geq T, \quad x \in \mathbb{R}^N,$$

where $\gamma = \max \{-\rho, f'(1 - \epsilon)/\lambda^*\}$ is negative and $\hat{C} = \max \left\{ \frac{\epsilon}{M_{l_\epsilon} e^{\lambda^* \gamma T}}, C \right\}$, C is the constant in (3.11). For all $x \in \mathcal{C}_{l_\epsilon}^+$ and $t \geq 0$, we have

$$L\bar{v} = (\lambda^* \gamma - f'(w))\bar{v} \geq (f'(1 - \epsilon) - f'(w))\bar{v} \geq 0 = Lv.$$

From (3.21), we have

$$v(x, t)|_{t=T} = u(x, t)|_{t=T} - \phi_{c_0}(x) \leq \epsilon \leq \hat{C} M_{l_\epsilon} e^{\lambda^* \gamma T} = \bar{v}(x, t)|_{t=T}.$$

From (3.11), we have

$$v(x, t) \leq C M_{l_\epsilon} e^{-\lambda^* \rho t} \leq C M_{l_\epsilon} e^{\lambda^* \gamma t} \leq \bar{v}(x, t)$$

holds for all $x \in \partial \mathcal{C}_{l_\epsilon}^+ = \partial \mathcal{C}_{l_\epsilon}^-$ and $t \geq T$. It follows from the maximum principle that

$$v(x, t) \leq \bar{v}(x, t) = \hat{C} M_{l_\epsilon} e^{\lambda^* \gamma t}, \quad \forall t \geq T, \quad x \in \mathcal{C}_{l_\epsilon}^+. \quad (3.22)$$

By a similar consideration, we have

$$v(x, t) \geq \underline{v}(x, t) := -\hat{C} M_{l_\epsilon} e^{\lambda^* \gamma t}, \quad \forall t \geq T, \quad x \in \mathcal{C}_{l_\epsilon}^+.$$

Then we have

$$\sup_{x \in \mathcal{C}_l^+} |u(x, t) - \phi_{c_0}(x)| \leq \sup_{x \in \mathcal{C}_l^+} |v(x, t)| \leq \hat{C} M_{l_\epsilon} e^{\lambda^* \gamma t}, \quad \forall t \geq T,$$

which combined with (3.11) gives that

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - \phi_{c_0}(x)| \leq \hat{C} M_{l_\epsilon} e^{\lambda^* \gamma t}, \quad \forall t \geq T.$$

Since u and ϕ_{c_0} are globally bounded, we conclude that there exists a constant M such that

$$\sup_{x \in \mathbb{R}^N} |u(x, t) - \phi_{c_0}(x)| \leq M e^{\lambda^* \gamma t}, \quad \forall t \geq 0.$$

The proof of this theorem is completed. $\square$

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References

- [1] D. G. Aronson and H. F. Weinberger, *Multidimensional nonlinear diffusions arising in population genetics*, Adv. Math., **30** (1978), 33–76.
- [2] M. Bramson, *Convergence of solutions of the Kolmogorov equation to travelling waves*, Memoirs Amer. Math. Soc., **44** (1983).
- [3] P. C. Fife, *Mathematical Aspects of Reacting and Diffusing Systems*, Lecture Notes in Biomathematics, 28, Springer-Verlag, Berlin-New York, 1979.
- [4] R. A. Fisher, *The advance of advantageous genes*, Ann. Eugenics, **7** (1937), 335–369.
- [5] Th. Gallay, *Local stability of critical fronts in nonlinear parabolic partial differential equations*, Nonlinearity, **7** (1994), 741–764.
- [6] F. Hamel, *Conical fronts and more general curved fronts for homoneneous equations in R^N* , Notes de cours en Ecole d’Eté de l’Institut Fourier, 2005, <http://www-fourier.ujf-grenoble.fr/>
- [7] F. Hamel, R. Monneau and J.-M. Roquejoffre, *Stability of conical fronts in a combustion model*, Ann. Sci. Ecole Normale Supérieure, **37** (2004), 469–506.
- [8] F. Hamel and N. Nadirashvili, *Travelling fronts and entire solutions of the Fisher-KPP equation in $\mathbb{R}^N$* , Arch. Ration. Mech. Anal., **157** (2001), 91–163.
- [9] F. Hamel and N. Nadirashvili, *Entire solutions of the KPP equation*, Comm. Pure Appl. Math., **52** (1999), 1255–1276.
- [10] A. N. Kolmogorov, I. G. Petrovsky and N. S. Piskunov, *Etude de l’équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique*, Bulletin Université d’Etat à Moscou (Bjul. Moskowskogo Gos. Univ.), Série Internationale, Section A, **1** (1937), 1–26.
- [11] J. F. Mallordy and J.-M. Roquejoffre, *A parabolic equation of the KPP type in higher dimensions*, SIAM J. Math. Anal., **26** (1995), 1–20.
- [12] J. D. Murray, *Mathematical Biology*, Springer-Verlag, 1989.
- [13] J.-M. Roquejoffre, *Stability of travelling fronts in a curved flame model, Part II: Non-linear orbital stability*, Arch. Rat. Mech. Anal., **117** (1992), 119–153.
- [14] J.-M. Roquejoffre, *Convergence to travelling waves for solutions of a class of semi-linear parabolic equation*, J. Diff. Equations, **108** (1994), 262–295.
- [15] D. H. Sattinger, *Stability of waves of nonlinear parabolic systems*, Adv. Math., **22** (1976), 312–355.
- [16] D. H. Sattinger, *Weighted norms for the stability of traveling waves*, J. Diff. Equations, **25** (1977), 130–144.

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