Radial solutions with a vortex to an asymptotically linear elliptic equation

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Abstract. In this paper, we study a two-dimensional nonlinear elliptic equation:

$$-\Delta v(x) + V(x)v(x) = f(v(x)), x \in \mathbb{R}^2$$

where $V(x)$ is radial, $V(x)$ behaves like $O\left(\frac{1}{|x|^2}\right)$ near zero and the nonlinearity f is asymptotically linear at infinity. We show the existence of a nontrivial radial solution of (1.1) via the variational approach.

Mathematics Subject Classification (2000). Primary 35J20; Secondary 35Q55.

Keywords. asymptotically linear nonlinearity, vortex, variational method.

1. Introduction

We consider the following elliptic equation in $\mathbb{R}^2$:

$$-\Delta v(x) + V(x)v(x) = f(v(x)). \quad (1.1)$$

In this paper, we study the case where $V(x)$ is radial, $V(x)$ behaves like $O\left(\frac{1}{|x|^2}\right)$ near zero and the nonlinearity f is asymptotically linear at infinity. In this situation, we show the existence of a nontrivial radial solution of (1.1) which satisfies $v(x) \rightarrow 0$ as $|x| \rightarrow \infty$. As we will see later, a radial solution of (1.1) also satisfies $v(0) = 0$ when $V(x)$ behaves like $O\left(\frac{1}{|x|^2}\right)$ in the two-dimensional problem.

Since we are looking for a radially symmetric solution, problem (1.1) leads to the following ordinary differential equation:

$$-v''(r) - \frac{1}{r}v'(r) + V(r)v(r) = f(v(r)), \quad r \in (0, \infty). \quad (1.2)$$

To find a nontrivial (weak) solution of (1.1), we reduce (1.2) to another ordinary differential equation as in [20]. We put $v(r) = r^{-\frac{1}{2}}u(r)$. Then $u(r)$ satisfies

$$-u''(r) + \left(V(r) - \frac{1}{4r^2}\right)u(r) = r^{\frac{1}{2}}f(r^{-\frac{1}{2}}u(r)), \quad r \in (0, \infty). \quad (1.3)$$

We find a nontrivial solution of (1.3) via the variational approach.

For the nonlinearity f , we assume

- (f1) $f \in C(\mathbb{R}^+, \mathbb{R})$. $f(s) \equiv 0$ for $s \leq 0$.
- (f2) $\frac{f(s)}{s} \rightarrow 0$ as $s \rightarrow 0^+$.
- (f3) There exists $c_0 \in (0, \infty)$ such that $\frac{f(s)}{s} \rightarrow c_0$ as $s \rightarrow \infty$.
- (f4) $\frac{f(s)}{s}$ is a non-decreasing function.

A typical example which satisfies (f1)-(f4) is given by $f(s) = \frac{c_0 s^3}{1+s^2}$. For a potential $V(x)$, we assume

- (V1) $V(x) = V(|x|)$, $V(r) \in C(0, \infty)$.
- (V2) There exists $V_0 > 0$ such that

$$V(r) - \frac{1}{4r^2} \geq V_0 \quad \text{for all } r \in (0, \infty).$$

- (V3) There exist $\omega > \frac{1}{2}$, $r_0 > 0$ and $\tilde{V}(r) \in C[0, \infty)$ such that $V(r)$ can be written by the form $V(r) = \frac{\omega^2}{r^2} + \tilde{V}(r)$ for $0 < r < r_0$.

Typical examples of $V(r)$ are $V(r) = 1 + \frac{\omega^2}{r^2}$ or $V(r) = r^2 + \frac{\omega^2}{r^2}$ ($V_0 = 2\sqrt{\omega^2 - \frac{1}{4}}$ in this case).

To state our main results, we define a function space X by the closure of $C_0^\infty(0, \infty)$ with respect to the norm:

$$\|u\|_X^2 := \int_0^\infty (u')^2 + \left(V(r) - \frac{1}{4r^2}\right)u^2 dr.$$

We also define

$$\lambda := \inf_{u \in X \setminus \{0\}} Q(u), \quad Q(u) := \frac{\int_0^\infty (u')^2 + \left(V(r) - \frac{1}{4r^2}\right)u^2 dr}{\int_0^\infty u^2 dr}, \quad u \in X \setminus \{0\}.$$

From (V2), we have $Q(u) \geq V_0$ for all $u \in X \setminus \{0\}$. This implies λ is well-defined and $\lambda \geq V_0$.

Theorem 1.1. *Assume (f1)-(f4), (V1)-(V3) hold. Moreover we assume $c_0 > \lambda$ where c_0 is defined in (f3). Then (1.1) has a nontrivial non-negative radial solution.*

In the case $V(r) = r^2 + \frac{\omega^2}{r^2}$, we can explicitly compute corresponding eigenvalues and eigenfunctions of the one-dimensional Schrödinger operator: $-\frac{d^2}{dr^2} + r^2 + (\omega^2 - \frac{1}{4})\frac{1}{r^2}$. Then we obtain the following corollary.

Corollary 1.2. *Let $V(r) = r^2 + \frac{\omega^2}{r^2}$ and assume (f1)-(f4) hold. If $c_0 > 2(\omega + 1)$, then (1.1) has a nontrivial non-negative radial solution.*

Finally we can obtain the multiplicity result in the case $V(r) = r^2 + \frac{\omega^2}{r^2}$.

Theorem 1.3. *Let $V(r) = r^2 + \frac{\omega^2}{r^2}$. Suppose (f2)-(f4) and (f5): $f \in C(\mathbb{R}, \mathbb{R})$ and $f(s)$ is an odd function.*

Moreover we assume that $2(\omega - 1 + 2k) < c_0 < 2(\omega + 1 + 2k)$ for $k \in \mathbb{N}$. Then (1.1) has at least k distinct pairs of nontrivial radial solutions.

Let us comment on the derivation of problem (1.1). We consider a nonlinear Schrödinger equation:

$$i \frac{\partial \psi}{\partial t} = -\Delta \psi + K(x)\psi - f(\psi), \quad (1.4)$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$ and $K : \mathbb{R}^N \mapsto \mathbb{R}$. A lot of works has been devoted in finding standing waves, that is, solutions of the form: $\psi(t, x) = \exp(-iEt)v(x)$ where E is a real constant. Then $v(x)$ should satisfy problem (1.1) with $V(x) = K(x) - E$. Especially, problem (1.1) has been studied when $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and the nonlinearity f is asymptotically linear at infinity (see [3], [8], [12], [13] and references therein). In the physical point of view, asymptotically linear nonlinearity is said to have the saturation effect. This type of nonlinearities are important in the study of nonlinear optics (see [19] for related topics).

Recently in two-dimensional problem, another type of solutions of (1.4) has been studied. We consider a solution of (1.4) of the form: $\psi(t, x) = \exp(i\omega\theta + iEt)v(r)$ where (r, θ) is a polar coordinate in $\mathbb{R}^2$ and $\omega \in \mathbb{N}$. Then we can see that problem (1.4) reduces to the equation:

$$-\Delta v(x) + \left(K(x) + E + \frac{\omega^2}{|x|^2} \right) v(x) = f(v(x)), \quad x \in \mathbb{R}^2.$$

This kind of problems has been studied when the nonlinearity $f(s)$ is $|s|^{p-1}s$ or superlinear at infinity (see [2], [4], [5], [10]).

When $f(s) = -|s|^2s$, problem (1.1) appears in the study of the Gross-Pitaevskii theory of Bose-Einstein condensation. Especially a function $r^2 + \frac{\omega^2}{r^2}$ appears if the Bose-Gas is confined in a harmonic trap potential and rotates around the origin (see [16], [17]). In this case, a constant ω means an eigenvalue of the angular momentum and a solution of (1.1) is called vortex state or ω -vortex.

We should also mention that inverse square potentials are said to be Hardy type singular potentials in higher dimensional problems. Several papers are concerned with elliptic problems with this type of potentials (see e.g. [6], [7], [15], [18] and references therein).

The main features of this paper are the followings. Since the singularity $\frac{1}{|x|^2}$ is too strong in $\mathbb{R}^2$, we have to treat estimates near the origin carefully. Especially

to find a weak solution of (1.1) (including the origin), we need decay estimates near the origin. We will show that if $v(x)$ is a nontrivial radial solution of (1.1), then $v(x)$ satisfies

$$|v(x)| \leq C|x|^\omega, \quad |\nabla v(x)| \leq C|x|^{\omega-1}, \quad |x| \sim 0. \quad (1.5)$$

Then we will see that it is sufficient to find a nontrivial critical point of a functional which corresponds to (1.3) (see Proposition 3.2 below). Since we are looking for solutions which satisfy (1.1) including the origin, we are restricted to the radially symmetric case. It will be a future research to find a non-radial solution.

To obtain estimate (1.5), assumption (V3) (particularly $\omega > \frac{1}{2}$) plays an important role. In the original meaning, ω should be a natural number. However we can treat the case where ω is a general constant in the study of (1.1). The author believes that $\omega = \frac{1}{2}$ is a threshold for the existence of nontrivial solutions which satisfy (1.1) including the origin.

To prove Corollary 1.2 and Theorem 1.3, we have to compute eigenvalues and eigenfunctions of the corresponding Schrödinger operator. We will see that we can reduce the corresponding eigenvalue problem to a confluent hypergeometric equation when $V(r) = r^2 + \frac{\omega^2}{r^2}$. Making use of confluent hypergeometric functions, we explicitly compute eigenvalues.

This paper is organized as follows. In section 2, we study basic properties of a function space X and a corresponding Schrödinger operator. In section 3, we prove Theorem 1.1 by using the Mountain Pass Theorem. In section 4, we consider the case $V(r) = r^2 + \frac{\omega^2}{r^2}$. We prove Theorem 1.3 by using the Linking Theorem.

2. Preliminaries

2.1 Properties of the function space X

In this subsection, we study basic properties of a function space which we will use in this paper. We define a function space X by the closure of $C_0^\infty(0, \infty)$ with respect to the norm

$$\|u\|_X = \int_0^\infty (u')^2 + \left(V(r) - \frac{1}{4r^2}\right) u^2 dr.$$

We also define the inner product on X by

$$\langle u_1, u_2 \rangle_X := \int_0^\infty u_1' u_2' + \left(V(r) - \frac{1}{4r^2}\right) u_1 u_2 dr \quad \text{for } u_1, u_2 \in X.$$

From (V2), we have

$$\int_0^\infty (u')^2 + u^2 dr \leq \frac{1}{\min\{1, V_0\}} \|u\|_X^2, \text{ for } u \in X.$$

This implies if $u \in X$, then

$$u \in H_0^1(0, \infty) = \{u \in L^2(0, \infty); u' \in L^2(0, \infty), u(0) = 0\}.$$

By one-dimensional Sobolev inequality, we have

$$\|u\|_{L^\infty(0, \infty)} \leq C \|u\|_X, \quad \lim_{r \rightarrow \infty} u(r) = 0 \quad (2.1)$$

for some $C > 0$.

Now for $u \in X$, we put $v(r) = r^{-\frac{1}{2}}u(r)$. Then we can see that

$$\|u\|_X^2 = \frac{1}{2\pi} \int_{\mathbb{R}^2} |\nabla v|^2 + V(x)v^2 dx.$$

From (V2), we have $V(x) \geq V_0 + \frac{1}{4|x|^2}$. Thus it follows

$$\int_{\mathbb{R}^2} v^2 dx \leq \frac{2\pi}{V_0} \|u\|_X^2, \quad \int_{\mathbb{R}^2} \frac{v^2}{|x|^2} dx \leq 8\pi \|u\|_X^2.$$

This implies if $u \in X$, then $v(x) = |x|^{-\frac{1}{2}}u(x)$ satisfies

$$v \in H^1(\mathbb{R}^2), \quad v(x) = v(|x|) \text{ and } \int_{\mathbb{R}^2} V(x)v^2 dx < \infty.$$

Next we study the behavior of $u \in X$ near the origin.

Lemma 2.1. Assume (V1)-(V2) hold. Then $u(r)$ satisfies $\lim_{r \rightarrow 0^+} r^{-\frac{1}{2}}u(r) = 0$.

Proof. Since $v(r) = r^{-\frac{1}{2}}u(r)$, we show that $\lim_{r \rightarrow 0^+} v(r) = 0$. For $0 < a < b < 1$, we have

$$\begin{aligned} |v(b)^2 - v(a)^2| &= \left| \int_a^b \frac{d}{dr} (v(r))^2 dr \right| = \left| 2 \int_a^b v v' dr \right| \\ &\leq C \left(\int_a^b \frac{v^2}{r} dr \right)^{\frac{1}{2}} \left(\int_a^b r (v')^2 dr \right)^{\frac{1}{2}} \leq C \left(\int_{a \leq |x| \leq b} \frac{v^2}{|x|^2} dx \right)^{\frac{1}{2}} \left(\int_{a \leq |x| \leq b} |\nabla v|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \left(\int_{|x| \leq b} \frac{v^2}{|x|^2} dx \right)^{\frac{1}{2}} \left(\int_{|x| \leq b} |\nabla v|^2 dx \right)^{\frac{1}{2}} \text{ for some } C > 0. \end{aligned}$$

Since $|\nabla v|^2, \frac{v^2}{|x|^2} \in L^1(\mathbb{R}^2)$, it follows that $\{v(r)^2\}_{0 < r < 1}$ is a Cauchy-sequence as $r \rightarrow 0^+$. Thus there exists $\alpha \geq 0$ such that $\lim_{r \rightarrow 0^+} v(r)^2 = \alpha$. On the other hand, since $\int_{\mathbb{R}^2} \frac{v^2}{|x|^2} dx < \infty$, it follows that $\alpha = 0$. Thus we have

$$\lim_{r \rightarrow 0^+} r^{-\frac{1}{2}} u(r) = \lim_{r \rightarrow 0^+} v(r) = 0. \quad \square$$

2.2 Properties of one-dimensional Schrödinger operator L

In this subsection, we correct basic properties of an operator which corresponds to problem (1.3). We define one-dimensional Schrödinger operator L by

$$L := -\frac{d^2}{dr^2} + \left(V(r) - \frac{1}{4r^2} \right) \text{ in } L^2(0, \infty), D(L) = \{H_0^1(0, \infty); Lu \in L^2(0, \infty)\}.$$

For the self-adjointness of L , we have the following result.

Proposition 2.2. [14] (*Weyl's limit point-limit circle criterion*)

Suppose that a continuous real-valued function $W(r)$ on $(0, \infty)$ satisfies the followings. $W(r) \geq \frac{3}{4}r^{-2}$ near zero.

There exists $M(r) \in C^1(0, \infty)$ such that $M(r) \geq 0$ and

- (i) $W(r) \geq -M(r)$ for all $r \in (0, \infty)$.
- (ii) $\int_1^\infty (M(r))^{-\frac{1}{2}} dr = \infty$.
- (iii) $\frac{M'(r)}{(M(r))^{\frac{3}{2}}}$ is bounded near infinity.

Then $H := -\frac{d^2}{dr^2} + W(r)$ is essentially self-adjoint on $C_0^\infty(0, \infty)$.

Lemma 2.3. *Assume and (V1)-(V3) hold and ω (defined in (V3)) satisfies $\omega > 1$. Then L is an essentially self-adjoint operator on $C_0^\infty(0, \infty)$.*

Proof. First we observe that from (V3),

$$V(r) - \frac{1}{4r^2} - \frac{3}{4r^2} = \tilde{V}(r) + \frac{\omega^2 - 1}{r^2}, \quad 0 < r < r_0.$$

Since $\omega > 1$ and $\tilde{V}(r)$ is bounded from below near zero, we have $V(r) - \frac{1}{4r^2} \geq \frac{3}{4r^2}$ near zero.

Next we choose $M(r) = r$. Then from (V2), we have

$$V(r) - \frac{1}{4r^2} \geq V_0 \geq -r = -M(r) \text{ for all } r > 0.$$

Moreover,

$$\int_1^{\infty} r^{-\frac{1}{2}} dr = +\infty, \quad \frac{r'}{r^{\frac{3}{2}}} = r^{-\frac{3}{2}} \rightarrow 0 \text{ as } r \rightarrow \infty.$$

Thus by Proposition 2.2, L is essentially self-adjoint on $C_0^\infty(0, \infty)$. $\square$

Remark 2.4. (i) In general, let H be a self-adjoint operator that is bounded from below. It is well-known that

$$\inf_{u \in D(H), \|u\|=1} (u, Hu) = \inf \sigma(H).$$

Then we can see that $\lambda = \inf \sigma(L)$ if $\omega > 1$. When $\frac{1}{2} < \omega \leq 1$, L is symmetric, positive definite but not necessarily self-adjoint. In this case, we can not readily see whether $\lambda = \inf \sigma(L)$ or not. We refer [9] for related topics.

- (ii) Suppose $\omega > 1$ and $V(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then the spectrum of L consists of discrete spectrums $\{\mu_k\}_{k=0}^\infty$ and the corresponding eigenfunction $\{\phi_k\}$ is an orthogonal complete system on $L^2(0, \infty)$.
- (iii) If $\tilde{V}(r)$ in (V3) is non-negative near zero, the statement in Lemma 2.3 holds for $\omega = 1$. Especially if $V(r) = r^2 + \frac{\omega^2}{r^2}$ and $\omega \geq 1$, then L is self-adjoint.

3. Existence of a nontrivial solution

In this section, we prove Theorem 1.1. Now we are looking for a nontrivial solution of the problem:

$$-u'' + \left(V(r) - \frac{1}{4r^2} \right) u = r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u). \quad (3.1)$$

For $u \in X$, we define an energy functional

$$I(u) := \frac{1}{2} \|u\|_X^2 - \int_0^\infty r F(r^{-\frac{1}{2}} u) dr, \text{ where } F(s) = \int_0^s f(t) dt.$$

Before we proceed to the proof of Theorem 1.1, we correct basic properties of the nonlinearity $f(s)$.

Suppose (f1)-(f3) hold. Then for all $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$|f(s)| \leq \epsilon |s| + C_\epsilon |s|^{p-1}, \quad p \in (2, \infty) \text{ for } s \geq 0. \quad (3.2)$$

Especially it follows

$$|f(s)| \leq \epsilon |s| + C_\epsilon |s|^5, \quad |F(s)| \leq \frac{\epsilon}{2} |s|^2 + \frac{C_\epsilon}{6} |s|^6, \quad s \geq 0. \quad (3.3)$$

Next since $f(s)$ is asymptotically linear at infinity, there exists $M > 0$ such that

$$\frac{|f(s)|}{|s|} \leq M, \quad \frac{|F(s)|}{s^2} \leq \frac{M}{2} \text{ for all } s \geq 0. \quad (3.4)$$

Finally we assume (f4) additionally. Then

$$\frac{2F(s)}{s^2} \text{ is a non-decreasing function.} \quad (3.5)$$

First we show that if u is a critical point of $I(u)$, then $v(x) := |x|^{-\frac{1}{2}}u(x)$ is a weak solution of problem (1.1). Since $\frac{1}{|x|^2} \notin L^1(\mathbb{R}^2)$, it needs some work.

Lemma 3.1. *Assume (f1)-(f3), (V1)-(V3) hold. Suppose $u(r)$ is a nontrivial critical point of $I(u)$, then $u(r)$ satisfies*

$$|u(r)| \leq Cr^{\frac{1}{2}+\omega}, \quad |u'(r)| \leq Cr^{\omega-\frac{1}{2}} \text{ for } 0 < r < r_0$$

for some $C > 0$ (which depends of u).

Proof. Since u is a critical point of $I(u)$, it satisfies the following ordinary differential equation:

$$-u'' + \left(\omega^2 - \frac{1}{4}\right) \frac{u}{r^2} = r^{\frac{1}{2}}f(r^{-\frac{1}{2}}u) - \tilde{V}(r)u =: G(r), \quad 0 < r < r_0. \quad (3.6)$$

Now a general solution of the homogeneous linear equation $-u'' + (\omega^2 - \frac{1}{2})\frac{u}{r^2} = 0$ is given by $u(r) = Ar^{\frac{1}{2}-\omega} + Br^{\frac{1}{2}+\omega}$ where A and B are arbitrary constants. Then the Wronskian of a pair $r^{\frac{1}{2}+\omega}, r^{\frac{1}{2}-\omega}$ is given by $W\{r^{\frac{1}{2}+\omega}, r^{\frac{1}{2}-\omega}\} = 2\omega$. Thus for $0 < r < r_0$, a solution u of (3.6) can be written by the form:

$$u(r) = r^{\frac{1}{2}-\omega} \left(\frac{-1}{2\omega} \int_r^{r_0} t^{\frac{1}{2}+\omega} G(t) dt + A \right) + r^{\frac{1}{2}+\omega} \left(\frac{1}{2\omega} \int_r^{r_0} t^{\frac{1}{2}-\omega} G(t) dt + B \right).$$

Since $\omega > \frac{1}{2}$ and $u(0) = 0$, we have

$$\lim_{r \rightarrow 0^+} \left(\frac{-1}{2\omega} \int_r^{r_0} t^{\frac{1}{2}+\omega} G(t) dt + A \right) = 0, \text{ that is, } A = \frac{1}{2\omega} \int_0^{r_0} t^{\frac{1}{2}+\omega} G(t) dt.$$

Thus we obtain

$$u(r) = \frac{r^{\frac{1}{2}-\omega}}{2\omega} \int_0^r t^{\frac{1}{2}+\omega} G(t) dt + r^{\frac{1}{2}+\omega} \left(\frac{1}{2\omega} \int_r^{r_0} t^{\frac{1}{2}-\omega} G(t) dt + B \right), \quad 0 < r < r_0. \quad (3.7)$$

Since $G(r) = r^{\frac{1}{2}}f(r^{-\frac{1}{2}}u) - \tilde{V}(r)u$ and $\tilde{V}(r)$ is bounded in $[0, r_0]$, we have

$$|G(r)| \leq r^{\frac{1}{2}}|f(r^{-\frac{1}{2}}u)| + \sup_{0 \leq r \leq r_0} |\tilde{V}(r)||u|.$$

Using (3.2), we obtain

$$|G(r)| \leq r^{\frac{1}{2}}(\epsilon|r^{-\frac{1}{2}}u| + C_\epsilon|r^{-\frac{1}{2}}u|^{p-1}) + \sup_{0 \leq r \leq r_0} |\tilde{V}(r)||u|. \quad (3.8)$$

By Lemma 2.1, we have $r^{-\frac{1}{2}}u(r) \rightarrow 0$ as $r \rightarrow 0^+$. Thus there exists $C > 0$ such that

$$|r^{-\frac{1}{2}}u(r)| \leq C \text{ for } 0 < r < r_0.$$

Therefore from (3.8), it follows $|G(r)| \leq Cr^{\frac{1}{2}}$, $0 < r < r_0$. Then we have

$$\left| \int_0^r t^{\frac{1}{2}+\omega} G(t) dt \right| \leq \int_0^r C t^{\frac{1}{2}+\omega+\frac{1}{2}} dt = \frac{C}{\omega+2} r^{\omega+2}, \quad (3.9)$$

$$\left| \int_r^{r_0} t^{\frac{1}{2}-\omega} G(t) dt \right| \leq \int_r^{r_0} C t^{\frac{1}{2}-\omega+\frac{1}{2}} dt \leq Cr^{2-\omega} + C. \quad (3.10)$$

From (3.7), (3.9) and (3.10), we obtain

$$|u(r)| \leq C(r^{\frac{1}{2}+\omega} + r^{\frac{1}{2}+2}), \quad 0 < r < r_0.$$

If $\frac{1}{2} \leq \omega \leq 2$, we are done. Suppose $\omega > 2$. Then $u(r) \leq Cr^{\frac{1}{2}+2}$ for $0 < r < r_0$. Using this estimate in (3.8), we have $|G(r)| \leq Cr^{\frac{1}{2}+2}$. Then we obtain $|u(r)| \leq C(r^{\frac{1}{2}+\omega} + r^{\frac{1}{2}+4})$. Continuing these procedures, we have $|u(r)| \leq Cr^{\frac{1}{2}+\omega}$ for $0 < r < r_0$.

Differentiating (3.7), we obtain $|u'(r)| \leq Cr^{-\frac{1}{2}+\omega}$, $0 < r < r_0$ in a similar way. $\square$

Proposition 3.2. Assume (V1)-(V3) hold. Let $u(r)$ be a critical point of $I(u)$. Then $v(x) = |x|^{-\frac{1}{2}}u(x)$ is a weak solution of

$$-\Delta v + V(x)v = f(v) \text{ in } \mathbb{R}^2.$$

Proof. For $\phi \in C_0^\infty(\mathbb{R}^2)$, we show that

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \phi + V(x)v\phi dx = \int_{\mathbb{R}^2} f(v)\phi dx.$$

Now since $u(r)$ is a critical point of $I(u)$, $u(r)$ satisfies (3.1). Then $v(x) = |x|^{-\frac{1}{2}}u(|x|)$ satisfies

$$-\Delta v(x) + V(x)v(x) = f(v(x)) \text{ a.e. } x \in \mathbb{R}^2 \setminus \{|x| \geq \epsilon\}$$

for all $\epsilon > 0$. Multiplying ϕ and integrating over $|x| \geq \epsilon$, we have

$$\int_{|x| \geq \epsilon} (-\Delta v + V(x)v - f(v))\phi dx = 0. \quad (3.11)$$

Next it follows that

$$\begin{aligned} \int_{\mathbb{R}^2} \nabla v \cdot \nabla \phi dx &= \lim_{\epsilon \rightarrow 0} \left\{ - \int_{|x|=\epsilon} \nabla v \cdot \nu \phi dS - \int_{|x| \geq \epsilon} \Delta v \phi dx \right\}, \\ \int_{\mathbb{R}^2} V(x)v \phi dx &= \int_{|x| < \epsilon} V(x)v \phi dx + \int_{|x| \geq \epsilon} V(x)v \phi dx, \end{aligned}$$

where ν is an interior unit normal vector.

Now a direct computation shows that

$$\frac{\partial v}{\partial x_i} = \frac{x_i}{r} \left(u' - \frac{1}{2r} u \right) r^{-\frac{1}{2}}, \quad x \neq 0.$$

Then by Lemma 3.1, we have $|\nabla v \cdot \nu| \leq C\epsilon^{\omega-1}$ on $\{|x| = \epsilon\}$ if ϵ is small. Thus we obtain

$$\int_{|x|=\epsilon} \nabla v \cdot \nu \phi dS \leq C\epsilon^{\omega} \max_{|x|=\epsilon} |\phi(x)| \rightarrow 0 \text{ as } \epsilon \rightarrow 0$$

and hence

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \phi dx = \lim_{\epsilon \rightarrow 0} \int_{|x| \geq \epsilon} -\Delta v \phi dx.$$

Moreover from (V3), we have

$$\begin{aligned} \int_{|x| < \epsilon} V(x)v \phi dx &\leq \sup_{x \in \mathbb{R}^2} |\phi(x)| \int_{|x| < \epsilon} V(x)|v| dx \\ &\leq C \sup_{x \in \mathbb{R}^2} |\phi(x)| \int_0^\epsilon \left(\frac{\omega^2}{r^2} + \tilde{V}(r) \right) r^{\frac{1}{2}} |u| dr \leq C \sup_{x \in \mathbb{R}^2} |\phi(x)| \epsilon^{\omega} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned}$$

Thus taking a limit $\epsilon \rightarrow 0$ in (3.11), we obtain

$$\int_{\mathbb{R}^2} \nabla v \cdot \nabla \phi + V(x)v \phi dx = \int_{\mathbb{R}^2} f(v)\phi dx. \quad \square$$

To prove Theorem 1.1, it suffices to show the existence of a nontrivial critical point of $I(u)$ by Proposition 3.2. To this aim, we use the variant of Mountain Pass Theorem due to Bartolo, Benci and Fortunato [1].

Proposition 3.3. *Let E be a Hilbert space and consider a functional $J \in C(E, \mathbb{R})$ satisfying: $J(0)=0$,*

- (i) *There exists $\rho > 0$, $\alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in E$, $\|u\| = \rho$.*
- (ii) *There exists $e \in E$, $\|e\| > \rho$ such that $J(e) \leq 0$.*
- (iii) *For all $c > 0$, every bounded sequence $\{u_n\} \subset E$ such that $J(u_n) \rightarrow c$, $J'(u_n) \rightarrow 0$ in E^* as $n \rightarrow \infty$, has a convergent subsequence.*
- (iv) *For all $c > 0$, there exist δ , R , $\eta > 0$ such that if $J(u) \in [c - \delta, c + \delta]$, $\|u\| \geq R$, then $\|J'(u)\| \|u\| \geq \eta$.*

Then there exists $u \in E$ such that $J(u) \geq \alpha$ and $J'(u) = 0$.

Hereafter in this section, we establish (i)-(iv) under assumptions (f1)-(f4), (V1)-(V3).

Lemma 3.4. *Assume (f1)-(f3), (V1)-(V2) hold. Then there exists $\rho > 0$ and $\alpha > 0$ such that $I(u) \geq \alpha$ for all $u \in X$, $\|u\|_X = \rho$.*

Proof. Now from (3.3), we have

$$rF(r^{-\frac{1}{2}}u) \leq r \left(\frac{\epsilon}{2}(r^{-\frac{1}{2}}u)^2 + \frac{C_\epsilon}{6}(r^{-\frac{1}{2}}u)^6 \right) = \frac{\epsilon}{2}u^2 + \frac{C_\epsilon}{6} \frac{u^2}{r^2} u^4.$$

Thus from (2.1), we obtain

$$\int_0^\infty rF(r^{-\frac{1}{2}}u)dr \leq \frac{\epsilon}{2}\|u\|_{L^2}^2 + \frac{C_\epsilon}{6}\|u\|_{L^\infty}^4 \int_0^\infty \frac{u^2}{r^2}dr \leq \{C\epsilon + C\|u\|_X^4\}\|u\|_X^2$$

where C is a positive constant which is independent of u . Thus it follows

$$I(u) \geq \frac{1}{2}\|u\|_X^2 - \{C\epsilon + C\|u\|_X^4\}\|u\|_X^2.$$

Choosing ϵ and $\|u\|_X = \rho$ small, we obtain $I(u) \geq \alpha$ for some $\alpha > 0$. $\square$

Lemma 3.5. *Assume (f1)-(f3), (V1)-(V2) hold. Moreover suppose $c_0 > \lambda$. Then there exists $e \in X$, $\|e\|_X > \rho$ such that $I(e) \leq 0$.*

Proof. By the definition of λ and from $c_0 > \lambda$, there exists $u_0 \in X$ such that $\lambda \leq Q(u_0) < c_0$. Especially we have

$$\int_0^\infty (u'_0)^2 + \left(V(r) - \frac{1}{4r^2} - c_0 \right) u_0^2 dr < 0.$$

Now we claim that $\lim_{t \rightarrow \infty} \frac{I(tu_0)}{t^2} < 0$. To this aim, we show

$$\lim_{t \rightarrow \infty} \int_0^\infty \frac{rF(tr^{-\frac{1}{2}}u_0)}{t^2} dr = \frac{c_0}{2} \int_0^\infty u_0^2 dr.$$

Now if $u_0(r) \neq 0$, we have

$$\lim_{t \rightarrow \infty} \frac{rF(tr^{-\frac{1}{2}}u_0(r))}{t^2} = \lim_{t \rightarrow \infty} \frac{F(tr^{-\frac{1}{2}}u_0)}{(tr^{-\frac{1}{2}}u_0)^2} u_0^2 = \frac{c_0}{2} u_0^2.$$

Moreover from (3.4), we have

$$0 \leq \frac{rF(tr^{-\frac{1}{2}}u_0)}{t^2} \leq \frac{F(tr^{-\frac{1}{2}}u_0)}{(tr^{-\frac{1}{2}}u_0)^2} u_0^2 \leq \frac{M}{2} u_0^2 \text{ a.e. } r \in (0, \infty)$$

for all $t > 0$. Thus by Lebesgue's dominated convergence theorem,

$$\lim_{t \rightarrow \infty} \int_0^\infty \frac{rF(tr^{-\frac{1}{2}}u_0)}{t^2} dr = \frac{c_0}{2} \int_0^\infty u_0^2 dr$$

and

$$\lim_{t \rightarrow \infty} \frac{I(tu_0)}{t^2} = \frac{1}{2} \int_0^\infty (u'_0)^2 + \left(V(r) - \frac{1}{4r^2} - c_0 \right) u_0^2 dr < 0.$$

Putting $e = tu_0$ for sufficiently large t , we have $I(e) \leq 0$. $\square$

Lemma 3.6. Assume (f1)-(f3) hold. For all $c > 0$, let $\{u_n\} \subset X$ be a sequence such that $I(u_n) \rightarrow c$, $I'(u_n) \rightarrow 0$, $\|u_n\|_X$ is bounded. Then $\{u_n\}$ has a convergent subsequence.

Proof. Since $\|u_n\|_X$ is bounded, we may assume that $u_n \rightharpoonup u$ in X for some $u \in X$. Now for all $\phi \in X$, we have $I'(u_n)\phi \rightarrow 0$, $I'(u_n)u_n \rightarrow 0$. Thus we have

$$\begin{aligned} \int_0^\infty u'_n u' + \left(V(r) - \frac{1}{4r^2} \right) u_n u - r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) u dr &= o(1), \\ \|u_n\|_X^2 - \int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) u_n dr &= o(1). \end{aligned}$$

Since $u_n \rightharpoonup u$ in X , it follows

$$\int_0^\infty u'_n u' + \left(V(r) - \frac{1}{4r^2} \right) u_n u dr \rightarrow \|u\|_X^2.$$

Thus we have

$$\|u_n\|_X^2 - \|u\|_X^2 = \int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr + o(1).$$

Then it suffices to show that $\int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr \rightarrow 0$. We fix $R > 0$ arbitrary. Then from (3.4),

$$\begin{aligned} & \int_0^R \left| r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) \right| dr \\ & \leq \int_0^R \left| \frac{f(r^{-\frac{1}{2}} u_n)}{r^{-\frac{1}{2}} u_n} \|u_n\| |u_n - u| \right| dr \leq M \|u_n\|_X \left(\int_0^R |u_n - u|^2 dr \right)^{\frac{1}{2}}. \end{aligned}$$

Since $u_n \rightarrow u$ in $L^2_{loc}(0, \infty)$, we have

$$\lim_{n \rightarrow \infty} \int_0^R r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr = 0.$$

Next from (3.2), we have

$$\begin{aligned} & \int_R^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr \\ & \leq \epsilon \int_R^\infty |u_n| |u_n - u| dr + C_\epsilon \int_R^\infty r^{-\frac{1}{2}(p-2)} |u_n|^{p-1} |u_n - u| dr \\ & \leq \epsilon \|u_n\|_X (\|u_n\|_X + \|u\|_X) + \frac{C_\epsilon}{R^{\frac{1}{2}(p-2)}} \|u_n\|_X^{p-1} (\|u_n\|_X + \|u\|_X) \\ & \leq C\epsilon + \frac{C_\epsilon C}{R^{\frac{1}{2}(p-2)}} \end{aligned}$$

for some $C > 0$ independent of ϵ and R . Thus we obtain

$$\limsup_{n \rightarrow \infty} \left| \int_R^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr \right| \leq C\epsilon + \frac{C_\epsilon C}{R^{\frac{1}{2}(p-2)}}.$$

Since R is arbitrary and C_ϵ is independent of R , we have $\frac{C_\epsilon C}{R^{\frac{1}{2}(p-2)}} \rightarrow 0$ by taking a limit $R \rightarrow \infty$. Next passing a limit $\epsilon \rightarrow 0$, we obtain

$$\lim_{n \rightarrow \infty} \int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n)(u_n - u) dr = 0. \quad \square$$

Lemma 3.7. *Assume (f1)-(f4) hold. Moreover suppose $c_0 > \lambda$. For all $c > 0$, there exist $\delta, R, \eta > 0$ such that if $c - \delta \leq I(u) \leq c + \delta$ and $\|u\|_X \geq R$, then $\|I'(u)\| \|u\|_X \geq \eta$.*

Proof. We fix $c > 0$ arbitrary. For $n \in \mathbb{N}$, we put

$$\begin{aligned} m_n &:= \inf\{\|I'(u)\| \|u\|_X; u \in A_n\}, A_n \\ &= \left\{u \in X; I(u) \in \left[c - \frac{1}{n}, c + \frac{1}{n}\right], \|u\|_X \geq n\right\}. \end{aligned}$$

Here we define $m_n = +\infty$ if $A_n = \emptyset$. We show that $m_n > 0$ for some $n \in \mathbb{N}$.

Now we suppose by contradiction that $m_n = 0$ for every $n \in \mathbb{N}$. Then there exists $u_n \in A_n$ such that

$$I(u_n) \rightarrow c, \quad \|u_n\|_X \rightarrow \infty \text{ as } n \rightarrow \infty, \quad \|I'(u_n)\| \|u_n\|_X < \frac{1}{n}. \quad (3.12)$$

We divide the proof into four steps.

Step 1: For all $t > 0$ and $n \in \mathbb{N}$, it follows

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n) \text{ for all } t > 0, n \in \mathbb{N}. \quad (3.13)$$

From (3.12), we have

$$-\frac{1}{n} < \|u_n\|_X^2 - \int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) u_n dr < \frac{1}{n}. \quad (3.14)$$

We claim that for all $t > 0$ and $n \in \mathbb{N}$,

$$I(tu_n) \leq \frac{t^2}{2n} + \int_0^\infty \frac{1}{2} r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) u_n - rF(r^{-\frac{1}{2}} u_n) dr.$$

We put

$$G(t) := \frac{1}{2} t^2 r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) u_n - rF(tr^{-\frac{1}{2}} u_n).$$

Then

$$G'(t) = tr^{\frac{1}{2}}f(r^{-\frac{1}{2}}u_n)u_n - r^{\frac{1}{2}}u_nf(tr^{-\frac{1}{2}}u_n) = t \left(\frac{f(r^{-\frac{1}{2}}u_n)}{r^{-\frac{1}{2}}u_n} - \frac{f(tr^{-\frac{1}{2}}u_n)}{tr^{-\frac{1}{2}}u_n} \right) u_n^2.$$

Since $\frac{f(s)}{s}$ is a non-decreasing function, we have $G(t) \leq G(1)$ for all $t > 0$. Now

$$\begin{aligned} I(tu_n) &= \frac{t^2}{2} \|u_n\|_X^2 - \int_0^\infty rF(tr^{-\frac{1}{2}}u_n)dr < \frac{t^2}{2n} + \int_0^\infty G(t)dr \leq \frac{t^2}{2n} + \int_0^\infty G(1)dr \\ &= \frac{t^2}{2n} + \int_0^\infty \frac{1}{2} r^{\frac{1}{2}} f(r^{-\frac{1}{2}}u_n)u_n - rF(r^{-\frac{1}{2}}u_n)dr. \end{aligned} \quad (3.15)$$

On the other hand, from (3.14),

$$\begin{aligned} & -\frac{1}{2n} + \int_0^\infty \frac{1}{2} r^{\frac{1}{2}} f(r^{-\frac{1}{2}}u_n)u_n - rF(r^{-\frac{1}{2}}u_n)dr \\ & \leq I(u_n) = \frac{1}{2} \|u_n\|_X^2 - \int_0^\infty rF(r^{-\frac{1}{2}}u_n)dr. \end{aligned}$$

Thus

$$\int_0^\infty \frac{1}{2} r^{\frac{1}{2}} f(r^{-\frac{1}{2}}u_n)u_n - rF(r^{-\frac{1}{2}}u_n)dr \leq I(u_n) + \frac{1}{2n}. \quad (3.16)$$

Substituting (3.16) into (3.15), we obtain

$$I(tu_n) \leq \frac{1+t^2}{2n} + I(u_n) \text{ for all } t > 0, n \in \mathbb{N}.$$

Next we define $w_n := \frac{2c^{\frac{1}{2}}u_n}{\|u_n\|_X}$. Then we may assume that $w_n \rightharpoonup w$ in X for some $w \in X$.

Step 2: $w(r) \not\equiv 0$.

We suppose by contradiction that $w(r) \equiv 0$. Then for any $R > 0$, $w_n(r) \rightarrow 0$ a.e. $r \in (0, R)$. First we show that $\int_0^\infty rF(r^{-\frac{1}{2}}w_n)dr \rightarrow 0$ as $n \rightarrow \infty$. Now from (3.5), $\frac{2F(s)}{s^2}$ is a non-decreasing function. Since $\|w_n\|_{L^\infty} \leq M\|w_n\|_X = M2c^{\frac{1}{2}}$ for some $M > 0$, we have

$$\begin{aligned} \int_R^\infty rF(r^{-\frac{1}{2}}w_n)dr &= \int_R^\infty \frac{r(r^{-\frac{1}{2}}w_n)^2}{2} \times \frac{2F(r^{-\frac{1}{2}}w_n)}{(r^{-\frac{1}{2}}w_n)^2}dr \\ &\leq \int_R^\infty \frac{w_n^2}{2} \times \frac{2F(r^{-\frac{1}{2}}M2c^{\frac{1}{2}})}{(r^{-\frac{1}{2}}M2c^{\frac{1}{2}})^2}dr \leq C \sup_{r \in [R, \infty)} \frac{F(r^{-\frac{1}{2}}M2c^{\frac{1}{2}})}{(r^{-\frac{1}{2}}M2c^{\frac{1}{2}})^2}. \end{aligned}$$

From (f2), we have $\frac{F(s)}{s^2} \rightarrow 0$ as $s \rightarrow 0^+$. Thus for all $\epsilon > 0$, there exists $R_\epsilon > 0$ such that

$$\sup_{r \in [R_\epsilon, \infty)} \frac{F(r^{-\frac{1}{2}} M 2c^{\frac{1}{2}})}{(r^{-\frac{1}{2}} M 2c^{\frac{1}{2}})^2} < \epsilon.$$

On the other hand, from (3.4), it follows

$$\int_0^{R_\epsilon} r F(r^{-\frac{1}{2}} w_n) dr \leq C \int_0^{R_\epsilon} w_n^2 dr.$$

Thus

$$\limsup_{n \rightarrow \infty} \left| \int_0^\infty r F(r^{-\frac{1}{2}} w_n) dr \right| \leq \epsilon.$$

Since ϵ is arbitrary, we obtain $\int_0^\infty r F(r^{-\frac{1}{2}} w_n) dr \rightarrow 0$ as $n \rightarrow \infty$.

Now by the definition of w_n and the above estimate, we have

$$I(w_n) = \frac{1}{2} \|w_n\|_X^2 - \int_0^\infty r F(r^{-\frac{1}{2}} w_n) dr = 2c + o(1).$$

We put $t_n = \frac{2\sqrt{c}}{\|u_n\|}$. Then $t_n \rightarrow 0$ as $n \rightarrow \infty$. Moreover from (3.13),

$$2c + o(1) = I(w_n) = I(t_n u_n) \leq \frac{1 + t_n^2}{2n} + I(u_n) = c + o(1).$$

This is a contradiction. Thus we have $w(r) \not\equiv 0$.

Step 3: $w(r)$ satisfies

$$\int_0^\infty w' \phi' + \left(V(r) - \frac{1}{4r^2} \right) w \phi - c_0 w \phi dr = 0 \text{ for all } \phi \in X. \quad (3.17)$$

First we show that $w(r) \geq 0$. In fact, since $I'(u_n) \rightarrow 0$, we have $I'(u_n) w_n^- \rightarrow 0$ where $w_n^-(r) = \max\{0, -w_n(r)\}$. Then from $f(s) \equiv 0$ for $s \leq 0$, we obtain $\|w_n^-\|_X^2 \rightarrow 0$ and hence $w(r) \geq 0$ a.e. $r \in (0, \infty)$. Now we fix $\phi \in X$ arbitrary. Then

$$|I'(u_n) \phi| \leq \|I'(u_n)\| \|u_n\|_X \times \frac{\|\phi\|_X}{\|u_n\|_X} \leq \frac{\|\phi\|_X}{n \|u_n\|_X}.$$

On the other hand since

$$I'(u_n) \phi = \langle u_n, \phi \rangle_X - \int_0^\infty r^{\frac{1}{2}} f(r^{-\frac{1}{2}} u_n) \phi dr,$$

we have

$$\frac{2c^{\frac{1}{2}}I'(u_n)\phi}{\|u_n\|} = \langle w_n, \phi \rangle_X - 2c^{\frac{1}{2}} \int_0^\infty \frac{r^{\frac{1}{2}}f(r^{-\frac{1}{2}}u_n)\phi}{\|u_n\|_X} dr. \quad (3.18)$$

We claim that

$$\int_0^\infty \frac{2c^{\frac{1}{2}}r^{\frac{1}{2}}f(r^{-\frac{1}{2}}u_n)}{\|u_n\|_X} \phi dr \rightarrow \int_0^\infty c_0 w \phi dr \quad (3.19)$$

as $n \rightarrow \infty$. Now we can see that

$$\frac{2c^{\frac{1}{2}}}{\|u_n\|_X} r^{\frac{1}{2}} f(r^{-\frac{1}{2}}u_n) = \frac{f(r^{-\frac{1}{2}}u_n)}{r^{-\frac{1}{2}}u_n} \times w_n.$$

Since $u_n(r) = \frac{\|u_n\|_X}{2c^{\frac{1}{2}}} w_n(r)$ and $w(r) \not\equiv 0$, we have $u_n(r) \rightarrow \infty$ a.e. $r \in (0, \infty)$ as $n \rightarrow \infty$. Now we may assume that $\phi \in C_0^\infty(0, \infty)$ and $\text{supp } \phi \subset (a, b)$ for some $0 < a < b < \infty$. Since $w_n(r) \rightarrow w(r)$ in $L^2(a, b)$, there exists $h(r) \in L^1(a, b)$ such that $|w_n(r)| \leq h(r)$ a.e. $r \in (a, b)$. Combining with (3.4), we can apply Lebesgue's dominated convergence theorem to obtain (3.19). Taking a limit in (3.18), we obtain (3.17).

Step 4: If $c_0 > \lambda$, then there is no nontrivial non-negative function which satisfies (3.17).

First we fix Λ such that $\lambda < \Lambda < c_0$. By the definition of λ , there exists $v \in X$ so that $Q(v) < \Lambda$. Then we may assume that $v \in C_0^\infty(0, \infty)$. We choose $0 < a < b < \infty$ so that $\text{supp } v \subset (a, b)$.

For $u \in H_0^1(a, b) \setminus \{0\}$, we define

$$\tilde{Q}(u) := \frac{\int_a^b (u')^2 + \left(V(r) - \frac{1}{4r^2}\right) u^2 dr}{\int_a^b u^2 dr}.$$

Since $V(r) - \frac{1}{4r^2}$ is bounded on $[a, b]$, $\tilde{\lambda} := \inf_{u \in H_0^1(a, b) \setminus \{0\}} \tilde{Q}(u)$ is attained. Then

$$\tilde{\lambda} \leq \tilde{Q}(v) \leq Q(v) < \Lambda < c_0. \quad (3.20)$$

On the other hand, let $\tilde{u} \in H_0^1(a, b)$ be a minimizer of $\tilde{Q}(u)$. Then we may assume that $\tilde{u}(r) \geq 0$ and $\tilde{u}$ satisfies

$$-\tilde{u}'' + \left(V(r) - \frac{1}{4r^2}\right) \tilde{u} = \tilde{\lambda} \tilde{u}, \quad r \in (a, b).$$

Moreover by Hopf's Lemma and Maximum Principle, it follows $\tilde{u}'(a) > 0$, $\tilde{u}'(b) < 0$ and $\tilde{u}(r) > 0$. Now

$$\begin{aligned} \tilde{\lambda} \int_a^b w \tilde{u} dr &= \int_a^b w \left(-\tilde{u}'' + \left(V(r) - \frac{1}{4r^2} \right) \tilde{u} \right) dr \\ &= \int_a^b w' \tilde{u}' + \left(V(r) - \frac{1}{4r^2} \right) w \tilde{u} dr - w(b) \tilde{u}'(b) + w(a) \tilde{u}'(a) \\ &\geq \int_a^b w' \tilde{u}' + \left(V(r) - \frac{1}{4r^2} \right) w \tilde{u} dr = \int_a^b \left(-w'' + \left(V(r) - \frac{1}{4r^2} \right) w \right) \tilde{u} dr \\ &\quad + w'(b) \tilde{u}(b) - w'(a) \tilde{u}(a) = c_0 \int_a^b w \tilde{u} dr. \end{aligned}$$

Since $w(r) \geq 0$ and $\tilde{u}(r) > 0$, this implies $\tilde{\lambda} \geq c_0$. This contradicts to (3.20).

By step 1-4, we obtain $m_n > 0$ for some $n \in \mathbb{N}$. $\square$

By Lemma 3.4-3.7, we can apply Proposition 3.3. Since $f(s) \equiv 0$ for $s \leq 0$, a nontrivial critical point $u(r)$ of $I(u)$ satisfies $u(r) \geq 0$ for all $r \geq 0$. By Maximum Principle, we have $u(r) > 0$ for all $r \in (0, \infty)$. Together with Proposition 3.2, we obtain Theorem 1.1.

4. The special case $V(r) = r^2 + \frac{\omega^2}{r^2}$

In this section, we consider the case $V(r) = r^2 + \frac{\omega^2}{r^2}$. First we consider the following eigenvalue problem:

$$-U'' + \left(r^2 + \left(\omega^2 - \frac{1}{4} \right) \frac{1}{r^2} \right) U = \mu U, \quad r \in (0, \infty) \quad (4.1)$$

with the boundary conditions $U(0) = 0$ and $U(r) \rightarrow 0$ as $r \rightarrow \infty$.

Using confluent hypergeometric functions, we can explicitly obtain eigenvalues and eigenfunctions of (4.1). For basic properties of confluent hypergeometric functions, we refer [11] chapter 9.

Lemma 4.1. *Eigenvalues μ_k and eigenfunctions $\psi_k(r)$ of problem (4.1) are given by*

$$\mu_k = 2(\omega + 1 + 2k), \quad k \in \mathbb{N} \cup \{0\}, \quad \phi_k(r) = e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \sum_{j=0}^k \frac{(-1)^j}{(\omega+1)_j} r^{2j},$$

where $(p)_j = \frac{\Gamma(p+j)}{\Gamma(p)} = p(p+1) \cdots (p+j-1)$, $(p)_0 = 1$.

Proof. For a solution $U(r)$ of (4.1), we put $U(r) = e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} z(r^2)$. A direct computation shows that a function $z(r)$ should satisfy the following equation:

$$rz'' + (\omega + 1 - r)z' - \left(\frac{2\omega + 2 - \mu}{4} \right) z = 0, \quad r \in (0, \infty).$$

We write $\alpha := \frac{2\omega+2-\mu}{4}$, $\gamma := 1+\omega$ for simplicity. Then $z(r)$ satisfies so-called confluent hypergeometric equation of the form:

$$rz'' + (\gamma - r)z' - \alpha z = 0. \quad (4.2)$$

Our purpose is to choose μ so that $U(r) = e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} z(r^2)$ satisfies the two boundary conditions. For this purpose, we have to give general solutions of (4.2).

If $\gamma \neq 0, -1, -2, \dots$, a function

$$\Phi(\alpha, \gamma; r) := \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(\gamma)_j} \frac{r^j}{j!}$$

satisfies (4.2). Function Φ is said to be a confluent hypergeometric function. Especially if $\alpha = -k$ for some $k \in \mathbb{N} \cup \{0\}$, then $(\alpha)_{k+1} = 0$. This implies

$$\Phi(-k, \gamma; r) = \sum_{j=0}^{\infty} \frac{(-k)_j}{(\gamma)_j} \frac{r^j}{j!} = \sum_{j=0}^k \frac{(-1)^j}{(\gamma)_j} \frac{k!}{(k-j)!j!} r^j.$$

We suppose now that $\alpha \neq -k$ for all $k \in \mathbb{N} \cup \{0\}$.

If $\gamma \notin \mathbb{N}$, then $r^{1-\gamma} \Phi(1 + \alpha - \gamma, 2 - \gamma; r)$ is also a solution of (4.2) and it is linearly independent of $\Phi(\alpha, \gamma; r)$.

If $\gamma = n_0 + 1$ for some $n_0 \in \mathbb{N} \cup \{0\}$, then the linearly independent solution is given by $\Psi(\alpha, n_0 + 1; r)$

$$\begin{aligned} &= \frac{(-1)^{n_0+1}}{\Gamma(\alpha - n_0)} \sum_{j=0}^{\infty} \frac{(\alpha)_j}{(n_0 + j)!} \frac{r^j}{j!} (\psi(\alpha + j) - \psi(1 + j) - \psi(n_0 + 1 + j) + \log r) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n_0-1} \frac{(-1)^j (n_0 - j - 1)! (\alpha - n_0)_j}{j!} r^{j-n_0}, \end{aligned}$$

where $\psi(p) = \frac{\Gamma'(p)}{\Gamma(p)}$ and we set $\sum_{j=0}^{-1}$ is zero. Function Ψ is called a confluent hypergeometric function of the second kind.

Then by the definitions, we can see that

$$\begin{aligned} \Phi(\alpha, \gamma; r) &= 1 + O(r), \quad r \sim 0, \\ r^{1-\gamma} \Phi(1 + \alpha - \gamma, 2 - \gamma; r) &= r^{1-\gamma} + o(r^{1-\gamma}), \quad r \sim 0, \\ \Psi(\alpha, n_0 + 1; r) &= \frac{(n_0 - 1)!}{\Gamma(\alpha)} r^{-n_0} + o(r^{-n_0}), \quad r \sim 0. \end{aligned}$$

Moreover it is known that

$$\begin{aligned}\Phi(\alpha, \gamma; r) &\sim e^r r^{-(\gamma-\alpha)}, \quad r \sim \infty, \\ r^{1-\gamma} \Phi(1+\alpha-\gamma, 2-\gamma; r) &\sim e^r r^{-(\gamma-\alpha)}, \quad r \sim \infty, \\ \Psi(\alpha, n_0+1; r) &\sim r^{-\alpha}, \quad r \sim \infty.\end{aligned}$$

Next we claim that if $\alpha \neq -k$ for all $k \in \mathbb{N} \cup \{0\}$, then there is no nontrivial solution of (4.1) which satisfies boundary conditions $U(0) = 0$ and $U(r) \rightarrow 0$ as $r \rightarrow \infty$.

We suppose $\alpha \neq -k$ for all $k \in \mathbb{N} \cup \{0\}$ and $\gamma = n_0 + 1$, $n_0 \in \mathbb{N} \cup \{0\}$. Then $U(r)$ can be written by the form:

$$U(r) = Ae^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \Phi(\alpha, \gamma; r^2) + Be^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \Psi(\alpha; \gamma; r^2)$$

where A, B are constants.

Since $r^{\omega+\frac{1}{2}} \Psi(\alpha; \gamma; r^2) \sim r^{-\omega+\frac{1}{2}}$, $r \sim 0$ and $\omega > \frac{1}{2}$, $U(0) = 0$ implies $B = 0$. On the other hand, since

$$e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \Phi(\alpha, \gamma; r^2) \sim e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} e^{r^2} r^{-2(\gamma-\alpha)} \rightarrow \infty \text{ as } r \rightarrow \infty,$$

$U(r) \rightarrow 0$ as $r \rightarrow \infty$ implies $A = 0$. Thus $U(r) \equiv 0$. Arguing similarly, we obtain the same conclusion in the case $\gamma \notin \mathbb{N} \cup \{0\}$.

From above, we obtain $\alpha = -k$, that is, $\mu = 2(\omega + 1 + 2k)$ for $k \in \mathbb{N} \cup \{0\}$. We define

$$\phi_k(r) := e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \sum_{j=0}^k \frac{(-1)^j}{(\gamma)_j} \frac{k!}{(k-j)!j!} r^{2j}.$$

Then we can see that $\phi_k(r)$ satisfies (4.1). Moreover $\phi_k(0) = 0$ and $\lim_{r \rightarrow \infty} \phi_k(r) = 0$. This implies $\phi_k(r)$ is an eigenfunction of (4.1) corresponding to the eigenvalue $\mu_k = 2(\omega + 1 + 2k)$. $\square$

When $V(r) = r^2 + \frac{\omega^2}{r^2}$, we can easily see that λ is achieved because the embedding $X \hookrightarrow L^2(0, \infty)$ is compact. Then by Lemma 4.1, we have $\lambda = 2(\omega + 1)$ for $V(r) = r^2 + \frac{\omega^2}{r^2}$. Thus we obtain Corollary 1.2.

Next we put $\psi_k(r) := \frac{\phi_k(r)}{\|\phi_k\|_{L^2(0, \infty)}}$. By Remark 2.4, we have already known the completeness of $\{\psi_k\}$ in the case $\omega \geq 1$. Our next purpose is to show that $\{\psi_k\}$ is an orthonormal complete system even if $\frac{1}{2} < \omega < 1$ by using the explicit expressions of $\{\psi_k\}$.

Lemma 4.2. $\{\psi_k(r)\}_{k=0}^{\infty}$ is an orthonormal complete system on $L^2(0, \infty)$.

Proof. Now we can easily see that $\int_0^\infty \psi_{j'} \psi_j dr = 0$ for $j' \neq j$. Next we assume that there exists $u \in L^2(0, \infty)$ such that $\int_0^\infty u(r) \psi_k(r) dr = 0$ for all $k \in \mathbb{N} \cup \{0\}$. We show that $u(r) = 0$ a.e. $r \in (0, \infty)$. This implies $\{\psi_k(r)\}$ is an orthonormal complete system.

Now by the definition of $\psi_k(r)$, it follows

$$\psi_k(r) = \psi_{k-1}(r) + e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} \frac{(-1)^k}{(\gamma)_k} r^{2k}.$$

Since $\int_0^\infty u \psi_k dr = 0$ for all $k \in \mathbb{N} \cup \{0\}$, we obtain

$$\int_0^\infty u(r) e^{-\frac{r^2}{2}} \frac{(-1)^k}{(1+\omega)_k} r^{\omega+\frac{1}{2}+2k} dr = 0 \text{ for all } k \in \mathbb{N} \cup \{0\}.$$

Especially for $k \in \mathbb{N} \cup \{0\}$,

$$\int_0^\infty u(r) e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} r^{2k} dr = 0. \quad (4.3)$$

Now we put $v(r) := u(r) e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}}$ and define $\tilde{v}(r) = v(r)$ for $r \geq 0$, $= v(-r)$ for $r < 0$. Then $\tilde{v}(r)$ is an even function, $\tilde{v} \in L^2(\mathbb{R})$ and from (4.3), it satisfies

$$\int_{-\infty}^\infty \tilde{v}(r) r^{2k} dr = 0 \text{ for all } k \in \mathbb{N} \cup \{0\}. \quad (4.4)$$

For $z \in \mathbb{C}$, we define $G(z) := \int_{\mathbb{R}} \tilde{v}(r) e^{-izr} dr$. Then $G(z)$ is well-defined for $z \in \mathbb{C}$ and analytic in z . For $k \in \mathbb{N}$, we have $G^{(k)}(0) = (-i)^k \int_{\mathbb{R}} \tilde{v}(r) r^k dr$. From (4.4), we have $G^{(2k)}(0) = 0$. On the other hand, since $\tilde{v}(r)$ is even and r^{2k+1} is odd, we have

$$\int_{\mathbb{R}} \tilde{v}(r) r^{2k+1} dr = 0 \text{ for all } k \in \mathbb{N} \cup \{0\}.$$

Hence $G^{(k)}(0) = 0$ for all $k \in \mathbb{N} \cup \{0\}$. Since $G(z)$ is analytic, this implies $G(z) \equiv 0$. On the other hand, $G(z)$ is a Fourier transform of $\frac{\tilde{v}(r)}{(2\pi)^{\frac{1}{2}}}$, we have $\tilde{v}(r) = 0$ a.e. $r \in \mathbb{R}$. Thus $u(r) e^{-\frac{r^2}{2}} r^{\omega+\frac{1}{2}} = 0$ and hence $u(r) = 0$ a.e. $r \in (0, \infty)$. $\square$

To prove Theorem 1.3, we use a variant of Linking Theorem due to Bartolo, Benci and Fortunato [1].

Proposition 4.3. *Let H be a Hilbert space and suppose that $J \in C^1(H, \mathbb{R})$ satisfies:*

- (i) $J(0) \geq 0$.
- (ii) J is even.
- (iii) For all $c > 0$, every bounded sequence $\{u_n\} \subset E$ such that $J(u_n) \rightarrow c$, $J'(u_n) \rightarrow 0$ in E^* as $n \rightarrow \infty$, has a convergent subsequence.
- (iv) For all $c \in (0, \infty)$, there exist $\sigma, R, \alpha > 0$ such that if $[c - \sigma, c + \sigma] \subset (0, \infty)$, $J(u) \in [c - \sigma, c + \sigma]$ and $\|u\| \geq R$, then $\|J'(u)\| \|u\| \geq \alpha$.
- (v) There exist two closed subspaces H^+, H^- with $\text{codim } H^+ < \infty$ and constants $J(0) < c_1 < c_2, \rho > 0$ such that

$$J(u) \geq c_1, u \in H^+ \cap \{u \in H; \|u\| = \rho\}, J(u) < c_2, u \in H^-.$$

If $\dim H^- \geq \text{codim } H^+$, then J has at least $n = \dim H^- - \text{codim } H^+$ distinct pairs of critical points whose critical values belong to $[c_1, c_2]$.

Proof of Theorem 1.3. We assume (f2)-(f5) and $2(\omega - 1 + 2k) < c_0 < 2(\omega + 1 + 2k)$ for $k \in \mathbb{N}$. By assumptions (f2) and (f5), (i), (ii) of Proposition 4.3 are clear. Moreover we have already proved (iii) by Lemma 3.6.

Next we show (iv). Arguing similarly as Lemma 3.7, we can see that there exists $w(r) \in X$, $w(r) \not\equiv 0$ such that

$$\int_0^\infty w' \phi' + \left(r^2 + \left(\omega^2 - \frac{1}{4} \right) \frac{1}{r^2} \right) w \phi dr = c_0 \int_0^\infty w \phi dr$$

for all $\phi \in X$. The difference to the proof of Lemma 3.7 is that we can not obtain $w(r) \geq 0$. However by Lemma 4.1, we have c_0 should satisfy $c_0 = 2(\omega + 1 + 2k_0)$ for $k_0 \in \mathbb{N} \cup \{0\}$. This contradicts to the assumption $2(\omega - 1 + 2k) < c_0 < 2(\omega + 1 + 2k)$, $k \in \mathbb{N}$. Thus (iv) is satisfied.

It remains to show (v). First we set suitable two closed subspaces. For eigenvalue μ_k of problem (4.1), we denote M_k by the corresponding eigenspace. For given $k \in \mathbb{N}$, we define

$$H^- := \oplus_{0 \leq j \leq k-1} M_j, H^+ := \overline{\text{span} \{\psi_j\}_{j \geq 0}}.$$

Then it follows that $\dim H^- = k$. Moreover by Lemma 4.2, we have $\text{codim } H^+ = 0$.

Now we can see that there exist $c_1, \rho > 0$ such that if $u \in H^+$, $\|u\|_X = \rho$, then $I(u) \geq c_1$. The proof is similar to that of Lemma 3.4.

Next we show that there exists $c_2 > c_1$ such that $I(u) < c_2$ for $u \in H^-$. First we claim that for all $\delta > 0$, there exists $R > 0$ such that $I(u) \leq \delta$, $u \in H^-$, $\|u\|_X \geq R$.

Now we fix $u \in H^-$ and consider tu for $t > 0$. Then $tu \in H^-$ and

$$I(tu) = \frac{t^2}{2} \|u\|_X^2 - \int_0^\infty r F(r^{-\frac{1}{2}} tu) dr.$$

By the proof of Lemma 3.5, we have

$$\frac{1}{t^2} \int_0^\infty r F(r^{-\frac{1}{2}} tu) dr \rightarrow \frac{c_0}{2} \int_0^\infty u^2 dr \text{ as } t \rightarrow \infty.$$

On the other hand, for $u \in H^-$, u can be written by the form $u = \sum_{j=0}^{k-1} c_j \psi_j$, $c_j \in \mathbb{R}$. Then

$$\|u\|_X^2 = \sum_{j=0}^{k-1} \mu_j c_j^2 \|\psi_j\|_{L^2(0,\infty)}^2 \leq \mu_{k-1} \|u\|_{L^2(0,\infty)}^2.$$

Thus we have

$$\frac{I(tu)}{t^2} \rightarrow \frac{1}{2}(\mu_k - c_0) \|u\|_{L^2(0,\infty)}^2 \text{ as } t \rightarrow \infty.$$

By the assumption $c_0 > 2(\omega - 1 + 2k) = \mu_{k-1}$, we have $\frac{I(t_0 u)}{t_0^2} < 0$ for sufficiently large t_0 . Putting $v = t_0 u$, we have $I(v) \leq \delta$ for all $\delta > 0$.

Next we fix $\delta > c_1$ arbitrary. Then there exists $R > 0$ such that if $u \in H^-$, $\|u\|_X \geq R$, then $I(u) \leq \delta$.

We put

$$K := \sup\{I(u); u \in H^-, \|u\|_X \leq R\}.$$

Then we can see that $K < \infty$. Putting $c_2 := \max\{\delta, K\}$, we have $I(u) \leq c_2$ for all $u \in H^-$. Thus (v) is satisfied.

From above, we can apply Proposition 4.3. Since $\dim H^- - \operatorname{codim} H^+ = k$, we obtain k distinct pairs of critical points.

Remark 4.4. Suppose $\omega > 1$ and $V(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then by Remark 2.4, the same conclusion as Theorem 1.3 holds if c_0 lies in the k -th spectral gap of L , that is, $\mu_{k-1} < c_0 < \mu_k$.

Acknowledgement

The author would like to thank the referees for reading the manuscript carefully.

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Received: 6 September 2007

Accepted: 14 January 2008

Published Online First 03 October 2008

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