Flows and Critical Points

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Abstract. We use flows and the cohomological index to adapt the method of sandwich pairs to better suit quasilinear elliptic boundary value problems.

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1. Introduction

The notion of sandwich pairs was introduced by Schechter [12] based upon the sandwich theorem for complementing subspaces by Silva [13] and Schechter [9, 10].

Definition 1. We say that a pair of subsets A, B of a Banach space E forms a sandwich pair if for any $G \in C^1(E, \mathbb{R})$,

$$-\infty < b_0 := \inf_B G \leq \sup_A G =: a_0 < +\infty \quad (1)$$

implies that there is a sequence $\{u_j\} \subset E$ and a $c \in [b_0, a_0]$ such that

$$G(u_j) \rightarrow c, \quad G'(u_j) \rightarrow 0. \quad (2)$$

The sandwich pairs used in the literature so far have been formed using the eigenspaces of a semilinear operator and are therefore unsuitable for dealing with quasilinear problems where there are no eigenspaces. The purpose of the present paper is to show how this method can be modified to apply to p -Laplacian problems of the form

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \quad (3)$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian of u , $p \in (1, \infty)$, and f is a Carathéodory function on $\Omega \times \mathbb{R}$ with subcritical growth, i.e.,

$$|f(x, t)| \leq C(|t|^{r-1} + 1) \quad \forall (x, t) \in \Omega \times \mathbb{R} \quad (4)$$

for some $r \in [1, p^*)$, where

$$p^* = \begin{cases} \frac{np}{n-p}, & p < n \\ \infty, & p \geq n \end{cases} \quad (5)$$

is the critical Sobolev exponent, and $C > 0$.

Solutions of (3) coincide with the critical points of the C^1 functional

$$G(u) = \int_{\Omega} |\nabla u|^p - p F(x, u), \quad (6)$$

where

$$F(x, t) = \int_0^t f(x, s) ds, \quad (7)$$

defined on the Sobolev space $W_0^{1,p}(\Omega)$. Sandwich pairs that produce Palais–Smale sequences for G were constructed in [7]. Here we modify the method to produce Cerami sequences, which give better results. The distinguishing feature of our current method is the use of flows to obtain critical values. This allows us more flexibility in dealing with minimax situations.

2. Flows

Let E be a Banach space and let Σ be the set of all continuous maps $\sigma = \sigma(t)$ from $E \times [0, 1]$ to E such that

1. $\sigma(0)$ is the identity map,
2. for each $t \in [0, 1]$, $\sigma(t)$ is a homeomorphism of E onto E .

Because of the special geometry of our problem, we shall require special types of flows. Let A_1, B_1 be a pair of disjoint nonempty closed symmetric subsets of the unit sphere S in E and let

$$A = \pi^{-1}(A_1) \cup \{0\}, \quad B = \pi^{-1}(B_1) \cup \{0\} \quad (8)$$

where $\pi : E \setminus \{0\} \rightarrow S$, $u \mapsto u/\|u\|$ is the radial projection onto S . For each $R > 1$, let $K = K(R) = e^3 + R$, and define

$$\tilde{B}_K = \{\lambda v : v \in B_1, 0 \leq \lambda \leq K\}.$$

Let

$$\tilde{\Sigma}_R = \left\{ \sigma \in \Sigma : \min_{t \in [0,1]} d(\sigma(t)(Ru), \tilde{B}_K) > 0, \max_{t \in [0,1]} \|\sigma(t)(\lambda u)\| \leq K, \right. \\ \left. u \in A_1, 0 \leq \lambda \leq R \right\}. \quad (9)$$

Note that $\sigma(t)u \equiv u$ is in $\tilde{\Sigma}_R$ for each $R > 1$. This follows from the fact that $K > R$ and consequently that

$$d(Ru, B) = d(Ru, \tilde{B}_K), \quad u \in A_1.$$

We shall prove

Theorem 2. *Assume that $d(RA_1, B) > e^3$,*

$$\sigma(1) A \cap \tilde{B}_K \neq \emptyset, \quad \sigma \in \tilde{\Sigma}_R, \quad (10)$$

and

$$-\infty < b_0 := \inf_B G \leq \sup_A G =: a_0 < +\infty. \quad (11)$$

Then there are a number $\tilde{c}$ satisfying $b_0 \leq \tilde{c} \leq a_0$ and a sequence $\{u_j\} \subset E$ satisfying

$$G(u_j) \rightarrow \tilde{c}, \quad (1 + \|u_j\|) G'(u_j) \rightarrow 0. \quad (12)$$

For our applications, we shall need conditions which will imply (10). This will be done in the next section.

3. Cohomological index

We recall the construction and some properties of the cohomological index of Fadell and Rabinowitz [5]. Writing the group $\mathbb{Z}_2$ multiplicatively as $\{1, -1\}$, a paracompact $\mathbb{Z}_2$ -space is a paracompact space X together with a mapping $\mu : \mathbb{Z}_2 \times X \rightarrow X$, called a $\mathbb{Z}_2$ -action on X , such that

$$\mu(1, x) = x, \quad -(-x) = x \quad \forall x \in X \quad (13)$$

where $-x := \mu(-1, x)$. The action is fixed-point free if

$$-x \neq x \quad \forall x \in X. \quad (14)$$

A subset A of X is invariant if

$$-A := \{-x : x \in A\} = A, \quad (15)$$

and a map $f : X \rightarrow X'$ between two paracompact $\mathbb{Z}_2$ -spaces is equivariant if

$$f(-x) = -f(x) \quad \forall x \in X. \quad (16)$$

Two spaces X and X' are equivalent if there is an equivariant homeomorphism $f : X \rightarrow X'$. We denote by $\mathcal{F}$ the set of all paracompact free $\mathbb{Z}_2$ -spaces, identifying equivalent ones.

A principal $\mathbb{Z}_2$ -bundle with paracompact base is a triple $\xi = (E, p, B)$ consisting of an $E \in \mathcal{F}$, called the total space, a paracompact space B , called the base space, and a map $p : E \rightarrow B$, called the bundle projection, such that there are

1. an open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of B ,
2. for each $\lambda \in \Lambda$, a homeomorphism $\varphi_\lambda : U_\lambda \times \mathbb{Z}_2 \rightarrow p^{-1}(U_\lambda)$ satisfying

$$\varphi_\lambda(b, -1) = -\varphi_\lambda(b, 1), \quad p \varphi_\lambda(b, \pm 1) = b \quad \forall b \in B. \quad (17)$$

Then each $p^{-1}(b)$, called a fiber, is some pair $\{e, -e\}$, $e \in E$. A bundle map $f : \xi \rightarrow \xi'$ consists of an equivariant map $f : E \rightarrow E'$ and a map $\bar{f} : B \rightarrow B'$ such that $p'f = \bar{f}p$, i.e., the diagram

$$\begin{array}{ccc} E & \xrightarrow{f} & E' \\ p \downarrow & & \downarrow p' \\ B & \xrightarrow{\bar{f}} & B' \end{array}$$

commutes. Two bundles ξ and ξ' are equivalent if there are bundle maps $f : \xi \rightarrow \xi'$ and $f' : \xi' \rightarrow \xi$ such that $f'f$ and ff' are the identity bundle maps on ξ and ξ' , respectively. We denote by $\text{Prin}_{\mathbb{Z}_2} B$ the set of principal $\mathbb{Z}_2$ -bundles over B and $\text{Prin}_{\mathbb{Z}_2}$ the set of all principal $\mathbb{Z}_2$ -bundles with paracompact base, identifying equivalent ones.

Each $X \in \mathcal{F}$ can be identified with a $\xi \in \text{Prin}_{\mathbb{Z}_2}$ as follows. Let $\bar{X} = X/\mathbb{Z}_2$ be the quotient space of $X \in \mathcal{F}$ with each x and $-x$ identified, called the orbit space of X , and $\pi : X \rightarrow \bar{X}$ the quotient map. Then

$$\mathcal{P} : \mathcal{F} \rightarrow \text{Prin}_{\mathbb{Z}_2}, \quad X \mapsto \xi := (X, \pi, \bar{X}) \quad (18)$$

is a one-to-one correspondence.

A map $f : B \rightarrow B'$ induces a bundle $f^* \xi' = (f^*(E'), p, B) \in \text{Prin}_{\mathbb{Z}_2}$, called the pullback, where

$$f^*(E') = \{(b, e') \in B \times E' : f(b) = p'(e')\}, \quad -(b, e') = (b, -e') \quad (19)$$

and

$$p(b, e') = b. \quad (20)$$

Homotopic maps induce equivalent bundles, so for each $B' \in \text{Prin}_{\mathbb{Z}_2}$, we have the mapping

$$\mathcal{T} : [B, B'] \rightarrow \text{Prin}_{\mathbb{Z}_2} B, \quad [f] \mapsto f^* \xi' \quad (21)$$

where $[B, B']$ is the set of homotopy classes of maps from B to B' . For the bundle $\xi' = (S^\infty, \pi, \mathbb{RP}^\infty)$, called the universal principal $\mathbb{Z}_2$ -bundle, where S^∞ is the unit sphere in $\mathbb{R}^\infty$, $\mathbb{RP}^\infty$ is the infinite dimensional real projective space, and π identifies antipodal points $\pm x$, $\mathcal{T}$ is a one-to-one correspondence (see Dold [3]). Thus for each $X \in \mathcal{F}$, there is a map $f : \bar{X} \rightarrow \mathbb{RP}^\infty$, unique up to homotopy and called the classifying map, such that

$$\mathcal{T}([f]) = \mathcal{P}(X). \quad (22)$$

Let $f^* : H^*(\mathbb{RP}^\infty) \rightarrow H^*(\bar{X})$ be the induced homomorphism of the Alexander-Spanier cohomology rings. The cohomological index of X is defined by

$$i(X) = \begin{cases} \sup \{k \geq 1 : f^*(\omega^{k-1}) \neq 0\}, & X \neq \emptyset \\ 0, & X = \emptyset \end{cases} \quad (23)$$

where $\omega \in H^1(\mathbb{RP}^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{RP}^\infty) = \mathbb{Z}_2[\omega]$. The index $i : \mathcal{F} \rightarrow \mathbb{N} \cup \{0, \infty\}$ has the usual properties of an index theory:

1. Definiteness: $i(X) = 0$ if and only if $X = \emptyset$.
2. Monotonicity: If $f : X \rightarrow Y$ is an equivariant map, in particular, if $X \subset Y$, then

$$i(X) \leq i(Y).$$

Thus, equality holds when f is an equivariant homeomorphism.

3. Subadditivity: If $X \in \mathcal{F}$ and A, B are closed invariant subsets of X such that $X = A \cup B$, then

$$i(A \cup B) \leq i(A) + i(B).$$

4. Continuity: If $X \in \mathcal{F}$ and A is a closed invariant subset of X , then there is a closed invariant neighborhood N of A in X such that

$$i(N) = i(A).$$

5. Neighborhood of zero: If U is a bounded symmetric neighborhood of 0 in a Banach space E , then

$$i(\partial U) = \dim E.$$

Now let E be a Banach space and let $\mathcal{A} \subset \mathcal{F}$ denote the class of symmetric subsets of $E \setminus \{0\}$. The suspension SA of a nonempty subset A of E is the quotient space of $A \times [-1, 1]$ with $A \times \{1\}$ and $A \times \{-1\}$ collapsed to different points, which can be realized in $E \oplus \mathbb{R}$ as the union of all line segments joining the two points $(0, \pm 1) \in E \oplus \mathbb{R}$ to points of $A \subset E$. The cohomological index also has the following important stability property: If $A \in \mathcal{A}$ is closed, then

$$i(SA) = i(A) + 1 \quad (24)$$

(see Fadell and Rabinowitz [5]). Let $S \in \mathcal{A}$ be the unit sphere in E and let π be the radial projection onto S . Our main theorem for this section is

Theorem 3. *Let A_1, B_1 be a pair of disjoint nonempty closed symmetric subsets of S such that*

$$i(A_1) = i(S \setminus B_1) < \infty \quad (25)$$

and let

$$A = \pi^{-1}(A_1) \cup \{0\}, \quad B = \pi^{-1}(B_1) \cup \{0\}. \quad (26)$$

Assume that

$$-\infty < b_0 := \inf_B G \leq \sup_A G =: a_0 < +\infty. \quad (27)$$

Then there are a number $\tilde{c}$ satisfying $b_0 \leq \tilde{c} \leq a_0$ and a sequence $\{u_j\} \subset E$ satisfying

$$G(u_j) \rightarrow \tilde{c}, \quad (1 + \|u_j\|) G'(u_j) \rightarrow 0. \quad (28)$$

In proving Theorem 3, we shall make use of the following considerations.

Theorem 4. *If (25) holds, then for each $R > 1$,*

$$\sigma(1)A \cap \tilde{B}_K \neq \emptyset, \quad \sigma \in \tilde{\Sigma}_R. \quad (29)$$

Proof. Suppose there is a $\sigma \in \widetilde{\Sigma}_R$ such that

$$\sigma(1) A \cap \widetilde{B}_K = \emptyset. \quad (30)$$

By the definition of $\widetilde{\Sigma}_R$, we have

$$\{\sigma(t)(RA_1) : t \in [0, 1]\} \cap \widetilde{B}_K = \emptyset. \quad (31)$$

Let

$$\Gamma(t)u = \begin{cases} (1 - 3t + 3Rt)u, & u \in A_1, 0 \leq t \leq 1/3 \\ \sigma(3t - 1)(Ru), & u \in A_1, 1/3 < t \leq 2/3 \\ \sigma(1)(3(1 - t)Ru), & u \in A_1, 2/3 < t \leq 1. \end{cases} \quad (32)$$

By (30) and (31),

$$\{\Gamma(t)A_1 : t \in [0, 1]\} \cap \widetilde{B}_K = \emptyset. \quad (33)$$

Moreover, since

$$\max_{t \in [0, 1]} \|\sigma(t)(\lambda u)\| \leq K, \quad u \in A_1, \quad 0 \leq \lambda \leq R \quad (34)$$

and

$$\|u\| > K, \quad u \in B \setminus \widetilde{B}_K, \quad (35)$$

we also have

$$\{\Gamma(t)A_1 : t \in [0, 1]\} \cap (B \setminus \widetilde{B}_K) = \emptyset. \quad (36)$$

So $\Gamma = \Gamma(t)$ is a continuous map from $A_1 \times [0, 1]$ to $E \setminus B$ such that $\Gamma(0)$ is the identity map on A_1 and $\Gamma(1)A_1$ is the single point $\sigma(1)(0) \in E$. Then

$$SA_1 \rightarrow S \setminus B_1, \quad (u, t) \mapsto \begin{cases} \pi(\Gamma(t)u), & u \in A_1, t \in [0, 1] \\ -\pi(\Gamma(-t)(-u)), & u \in A_1, t \in [-1, 0] \end{cases} \quad (37)$$

is an odd map and hence

$$i(S \setminus B_1) \geq i(SA_1) = i(A_1) + 1 \quad (38)$$

by the monotonicity of the index and (24), contradicting (25). $\square$

Theorem 5. Assume that $d(RA_1, B) > e^3$ and

$$\widetilde{c}_R := \inf_{\sigma \in \widetilde{\Sigma}_R} \sup_{u \in A} G(\sigma(1)u) \quad (39)$$

is finite. Then there is a sequence $\{u_j\} \subset E$ satisfying

$$G(u_j) \rightarrow \widetilde{c}_R, \quad \left(1 + d(u_j, \widetilde{B}_K)\right) G'(u_j) \rightarrow 0. \quad (40)$$

Theorem 5 will be proved in Section 5.

Proof of Theorem 2. Assumption (10) guarantees that $\widetilde{c}_R$ is finite. In fact, we have

$$b_0 \leq \widetilde{c}_R \leq a_0.$$

Then by Theorem 5, there is a sequence satisfying (40). Since $\|u\| \leq K + d(u, \widetilde{B}_K)$, the result follows. $\square$

Proof of Theorem 3. Take R so large that $d(RA_1, B) > e^3$. By (25) and Theorem 4, (10) holds, so the result follows from Theorem 2. $\square$

4. Ordinary differential equations

In proving Theorem 5 we shall make use of various extensions of Picard's theorem in a Banach space. Some are well known.

Lemma 6. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^T \gamma(s) ds, \quad (41)$$

where $t_0 < T$ and u_0 are given positive numbers. Then there is a unique solution of

$$u'(t) = \gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0 \quad (42)$$

which is positive in $[t_0, T)$ and depends continuously on u_0 .

Proof. One can separate variables to obtain

$$W(u) = \int_{u_0}^u \frac{d\tau}{\rho(\tau)} = \int_{t_0}^t \gamma(s) ds.$$

The function $W(u)$ is differentiable and increasing in $\mathbb{R}$, positive in $[u_0, \infty)$, depends continuously on u_0 and satisfies

$$W(u) \rightarrow L = \int_{u_0}^{\infty} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^T \gamma(s) ds, \quad \text{as } u \rightarrow \infty.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [u_0, \infty)$ such that

$$u = W^{-1} \left(\int_{t_0}^t \gamma(s) ds \right)$$

is the unique solution of (42), and it depends continuously on u_0 . $\square$

Lemma 7. *Let $\gamma(t)$ and $\rho(t)$ be continuous functions on $[0, \infty)$, with $\gamma(t)$ nonnegative and $\rho(t)$ positive. Assume that*

$$\int_m^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^T \gamma(s) ds, \quad (43)$$

where $t_0 < T$ and $m < u_0$ are given positive numbers. Then there is a unique solution of

$$u'(t) = -\gamma(t)\rho(u(t)), \quad t \in [t_0, T), \quad u(t_0) = u_0 \quad (44)$$

which is $\geq m$ in $[t_0, T)$ and depends continuously on u_0 .

Proof. One can separate variables to obtain

$$W(u) = \int_u^{u_0} \frac{d\tau}{\rho(\tau)} = \int_{t_0}^t \gamma(s) ds.$$

The function $W(u)$ is differentiable and decreasing in $\mathbb{R}$, positive in $[m, u_0]$, depends continuously on u_0 and satisfies

$$W(u) \rightarrow L = \int_m^{u_0} \frac{d\tau}{\rho(\tau)} > \int_{t_0}^T \gamma(s) ds, \quad \text{as } u \rightarrow m.$$

Thus, for each $t \in [t_0, T)$ there is a unique $u \in [m, u_0]$ such that

$$u = W^{-1} \left(\int_{t_0}^t \gamma(s) ds \right)$$

is the unique solution of (44), and it depends continuously on u_0 . $\square$

Theorem 8. Let $g(t, x)$ be a continuous map from $\mathbb{R} \times H$ to H , where H is a Banach space. Assume that for each point $(t_0, x_0) \in \mathbb{R} \times H$, there are constants $K, b > 0$ such that

$$\|g(t, x) - g(t, y)\| \leq K\|x - y\|, \quad |t - t_0| < b, \quad \|x - x_0\| < b, \quad \|y - x_0\| < b. \quad (45)$$

Assume also that

$$\|g(t, x)\| \leq \gamma(t)\rho(\|x\|), \quad x \in H, \quad t \in [t_0, \infty), \quad (46)$$

where $\gamma(t)$, $\rho(t)$ satisfy the hypotheses of Lemma 6 with $\rho(t)$ nondecreasing. Then for each $x_0 \in H$ and $t_0 > 0$ there is a unique solution $x(t)$ of the equation

$$\frac{dx(t)}{dt} = g(t, x(t)), \quad t \in [t_0, \infty), \quad x(t_0) = x_0. \quad (47)$$

Moreover, $x(t)$ depends continuously on x_0 and satisfies

$$\|x(t)\| \leq u(t), \quad t \in [t_0, \infty), \quad (48)$$

where $u(t)$ is the solution of (42) in that interval satisfying $u(t_0) = u_0 \geq \|x_0\|$.

We also have the following.

Theorem 9. Let ρ, γ satisfy the hypotheses of Lemma 7, with ρ locally Lipschitz continuous. Let $u(t)$ be the solution of (44), and let $h(t)$ be a continuous function satisfying

$$h(t) \geq h(s) - \int_s^t \gamma(r)\rho(h(r)) dr, \quad t_0 \leq s < t < T, \quad h(t_0) \geq u_0. \quad (49)$$

Then

$$u(t) \leq h(t), \quad t \in [t_0, T). \quad (50)$$

Proof. Assume that there is a point t_1 in the interval such that

$$h(t_1) < u(t_1).$$

Let

$$y(t) = u(t) - h(t), \quad t \in [t_0, T].$$

Then, $y(t_0) \leq 0$ and $y(t_1) > 0$. Let τ be the largest point $< t_1$ such that $y(\tau) = 0$. Then

$$y(t) > 0, \quad t \in (\tau, t_1]. \quad (51)$$

Moreover, by (44) and (49) we have

$$y(t) \leq - \int_{\tau}^t \gamma(s) [\rho(u(s)) - \rho(h(s))] ds \leq L \int_{\tau}^t y(s) ds, \quad (52)$$

where L is the Lipschitz constant for ρ at $u(\tau)$ times the maximum of γ in the interval. Let

$$w(t) = \int_{\tau}^t y(s) ds.$$

Then

$$[e^{-Lt}w(t)]' = e^{-Lt}[y(t) - Lw(t)] \leq 0, \quad t \in [\tau, t_1].$$

Consequently,

$$e^{-Lt}w(t) \leq e^{-L\tau}w(\tau) = 0, \quad t \in [\tau, t_1].$$

Hence,

$$y(t) \leq Lw(t) \leq 0, \quad t \in [\tau, t_1],$$

contradicting (51). This completes the proof. $\square$

5. Proof of Theorem 5

Let $\rho(r) = 1 + r$. If the theorem were not true, there would be a $\delta > 0$ such that

$$\rho(d(u, \tilde{B}_K)) \|G'(u)\| \geq 1 \quad (53)$$

would hold for all u in the set

$$Q = \{u \in E : \tilde{c}_R - 3\delta \leq G(u) \leq \tilde{c}_R + 3\delta\}. \quad (54)$$

Let

$$Q_0 = \{u \in Q : \tilde{c}_R - 2\delta \leq G(u) \leq \tilde{c}_R + 2\delta\} \quad (55)$$

$$Q_1 = \{u \in Q : \tilde{c}_R - \delta \leq G(u) \leq \tilde{c}_R + \delta\} \quad (56)$$

and

$$Q_2 = E \setminus Q_0, \quad \eta(u) = d(u, Q_2) / [d(u, Q_1) + d(u, Q_2)]. \quad (57)$$

It is easily checked that $\eta(u)$ is locally Lipschitz continuous on E and satisfies

$$\begin{cases} \eta(u) = 1, & u \in Q_1, \\ \eta(u) = 0, & u \in \overline{Q_2}, \\ \eta(u) \in (0, 1), & \text{otherwise.} \end{cases} \quad (58)$$

For any $\theta < 1$ there is a locally Lipschitz continuous map $Y(u)$ of $\hat{E} = \{u \in E : G'(u) \neq 0\}$ into itself such that

$$\|Y(u)\| \leq 1, \quad \theta \|G'(u)\| \leq (G'(u), Y(u)), \quad u \in \hat{E} \quad (59)$$

(cf., e.g., [11]). Let $\sigma(t)$ be the flow generated by

$$W(u) = -\eta(u)Y(u)\rho(d(u, \tilde{B}_K)). \quad (60)$$

Since $\|W(u)\| \leq \rho(d(u, \tilde{B}))$ and $W(u)$ is locally Lipschitz continuous, $\sigma(t)$ exists for all $t \in \mathbb{R}^+$ in view of Theorem 8. We also have

$$\begin{aligned} dG(\sigma(t)u)/dt &= (G'(\sigma), \sigma') \\ &= -\eta(\sigma)(G'(\sigma), Y(\sigma))\rho(d(\sigma, \tilde{B}_K)) \\ &\leq -\theta\eta(\sigma)\|G'(\sigma)\|\rho(d(\sigma, \tilde{B}_K)) \\ &= -\theta\eta(\sigma) \end{aligned} \quad (61)$$

in view of (53) and (59). Let T satisfy $2\delta/\theta < T < 3$, and suppose $u \in Q_1$ is such that there is a $t_1 \in [0, T]$ for which $\sigma(t_1)u \notin Q_1$. Then

$$G(\sigma(t_1)u) < \tilde{c}_R - \delta,$$

since we cannot have $G(\sigma(t_1)u) > \tilde{c}_R + \delta$ for $u \in Q_1$ by (61). But this implies

$$G(\sigma(T)u) < \tilde{c}_R - \delta. \quad (62)$$

On the other hand, if $\sigma(t)u \in Q_1$ for all $t \in [0, T]$, then

$$G(\sigma(T)u) \leq G(u) - \theta \int_0^T dt \leq \tilde{c}_R + \delta - \theta T < \tilde{c}_R - \delta$$

by (61). Thus, (62) holds for $u \in Q_1$.

We claim that $\sigma_1(t) = \sigma(tT) \in \tilde{\Sigma}_R$. To see this, note that since

$$\sigma(t)u - u = \int_0^t W(\sigma(\tau)u) d\tau, \quad (63)$$

we have

$$\|\sigma(t)u - \sigma(s)u\| \leq \int_s^t \rho(d(\sigma(r)u, \tilde{B}_K)) dr.$$

If $v \in \tilde{B}_K$, we have

$$h(s) = d(\sigma(s)u, \tilde{B}_K) \leq \|\sigma(s)u - v\| \leq \|\sigma(t)u - v\| + \int_s^t \rho(d(\sigma(r)u, \tilde{B})) dr.$$

This implies,

$$h(s) \leq h(t) + \int_s^t \rho(h(r)) dr. \quad (64)$$

Moreover, if we replace u by Ru , we see by Lemma 7 and Theorem 9, that $h(s)$ satisfies

$$m(s) \leq h(s), \quad 0 \leq s \leq T,$$

where $m(s)$ is given by

$$\int_{m(s)}^{m(0)} \frac{d\tau}{\rho(\tau)} = s.$$

Note that for $u \in A_1$, we have $e^3 m(T) \geq 1$. Thus

$$\min_{t \in [0,1]} d(\sigma_1(t)(Ru), \tilde{B}_K) > 0$$

when $u \in A_1$. Moreover, by Theorem 8, we have

$$\|x(t)\| = \|\sigma(t)(\lambda u) - \lambda u\| \leq e^3, \quad u \in A_1, \quad 0 \leq \lambda \leq R.$$

This follows from the fact that $x(0) = 0$ and

$$\|x(t)\| \leq \tilde{m}(t), \quad t \in [0, T],$$

where $\tilde{m}(t)$ satisfies

$$\int_0^{\tilde{m}(t)} \frac{ds}{1+s} = t, \quad t \in [0, T].$$

Thus, $\tilde{m}(t) \leq e^T \leq e^3$. Hence,

$$\|\sigma(t)(\lambda u)\| \leq K, \quad t \in [0, T], \quad u \in A_1, \quad 0 \leq \lambda \leq R.$$

This tells us that $\sigma_1 \in \tilde{\Sigma}_R$ and

$$G(\sigma_1(1)u) < \tilde{c}_R - \delta, \quad u \in A. \quad (65)$$

But this contradicts (11). Hence (53) cannot hold for u satisfying (54). The proof is complete.

6. Applications to the p -Laplacian

We shall now study problem (3). Consider the nonlinear eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (66)$$

Its eigenvalues coincide with the critical values of the C^1 functional

$$I(u) = \frac{1}{\int_{\Omega} |u|^p} \quad (67)$$

defined on the unit sphere S in $W_0^{1,p}(\Omega)$. Let $\tilde{\mathcal{F}}$ denote the class of symmetric subsets of S and set

$$\lambda_l = \inf_{\substack{M \in \tilde{\mathcal{F}} \\ i(M) \geq l}} \sup_{u \in M} I(u) \quad (68)$$

where i is the cohomological index. Then

$$0 < \lambda_1 < \lambda_2 \leq \cdots \rightarrow \infty \quad (69)$$

are eigenvalues of (66) (cf. Theorem 4.2.1 of Perera et al. [8]).

Setting $H(x, t) = pF(x, t) - tf(x, t)$, we shall prove

Theorem 10. *If $\lambda_l < \lambda_{l+1}$ and*

$$\lambda_l |t|^p - W(x) \leq pF(x, t) \leq \lambda_{l+1} |t|^p + W(x), \quad t \in \mathbb{R} \quad (70)$$

for some $l > 0$ and $W \in L^1(\Omega)$, and

$$H(x, t) \leq C(|t|^\alpha + 1), \quad \overline{H}(x) := \overline{\lim}_{|t| \rightarrow \infty} \frac{H(x, t)}{|t|^\alpha} < 0 \quad \text{a.e.} \quad (71)$$

for some α satisfying $0 < \alpha \leq p$, then (3) has a solution.

Theorem 11. *If $\lambda_l < \lambda_{l+1}$ and (70) holds for some $l > 0$ and $W \in L^1(\Omega)$, and*

$$H(x, t) \geq -C(|t|^\alpha + 1), \quad \underline{H}(x) := \underline{\lim}_{|t| \rightarrow \infty} \frac{H(x, t)}{|t|^\alpha} > 0 \quad \text{a.e.} \quad (72)$$

for some α satisfying $0 < \alpha \leq p$, then (3) has a solution.

Theorem 12. *If $\lambda_l < \lambda_{l+1}$ and (70) holds for some $l > 0$ and $W \in L^1(\Omega)$, and*

$$H(x, t) \leq W_1(x) \in L^1(\Omega), \quad H(x, t) \rightarrow -\infty \quad \text{as } |t| \rightarrow \infty, \quad (73)$$

then (3) has a solution.

Theorem 13. *If $\lambda_l < \lambda_{l+1}$ and (70) holds for some $l > 0$ and $W \in L^1(\Omega)$, and*

$$H(x, t) \geq -W_1(x) \in L^1(\Omega), \quad H(x, t) \rightarrow +\infty \quad \text{as } |t| \rightarrow \infty, \quad (74)$$

then (3) has a solution.

Similar resonance problems have been studied by Perera [6] when $f(x, t)/|t|^{p-2}t \rightarrow \alpha_\pm(x) \in L^\infty(\Omega)$ as $t \rightarrow \pm\infty$ and by Arcoya and Orsina [1], Bouchala and Drábek [2], and Drábek and Robinson [4] for the special case $\alpha_\pm = \text{const.}$

Proof of Theorem 10. Let

$$A_1 = \{u \in S : I(u) \leq \lambda_l\}, \quad B_1 = \{u \in S : I(u) \geq \lambda_{l+1}\}. \quad (75)$$

Then

$$i(A_1) = i(S \setminus B_1) = l \quad (76)$$

by Theorem 4.2.2 of Perera et al. [8]. Let A, B be as in Theorem 3 and let G be given by (6). Since

$$\int_\Omega |\nabla u|^p \geq \lambda_{l+1} \int_\Omega |u|^p, \quad u \in B \quad (77)$$

and

$$\int_\Omega |\nabla u|^p \leq \lambda_l \int_\Omega |u|^p, \quad u \in A, \quad (78)$$

(70) implies

$$-\int_\Omega W \leq \inf_B G \leq \sup_A G \leq \int_\Omega W, \quad (79)$$

so there is a sequence $\{u_j\} \subset W_0^{1,p}(\Omega)$ satisfying (28) by Theorem 3.

We claim that $\{u_j\}$ is bounded and hence has a convergent subsequence by a standard argument. If $\rho_j = \|u_j\| \rightarrow \infty$, a subsequence of $\tilde{u}_j = u_j/\rho_j$ converges to some $\tilde{u}$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω . Then

$$\int_{\Omega} \frac{H(x, u_j)}{\rho_j^\alpha} = \frac{G'(u_j) u_j/p - G(u_j)}{\rho_j^\alpha} \rightarrow 0 \quad (80)$$

by (12) and

$$\overline{\lim} \int_{\Omega} \frac{H(x, u_j)}{\rho_j^\alpha} \leq \int_{\Omega} \overline{\lim} \frac{H(x, u_j)}{|u_j|^\alpha} |\tilde{u}_j|^\alpha \leq \int_{\Omega} \overline{H}(x) |\tilde{u}|^\alpha \leq 0 \quad (81)$$

by (71). Since $\overline{H} < 0$ a.e., it follows that $\tilde{u} = 0$ a.e. Now passing to the limit in

$$1 - \frac{G(u_j)}{\rho_j^p} = \int_{\Omega} \frac{pF(x, u_j)}{\rho_j^p} \leq \int_{\Omega} \lambda_{l+1} |\tilde{u}_j|^p + \frac{W}{\rho_j^p} \quad (82)$$

gives

$$1 \leq \int_{\Omega} \lambda_{l+1} |\tilde{u}|^p,$$

contradicting the fact that $\tilde{u} = 0$ a.e. $\square$

Proof of Theorem 11. As in the proof of Theorem 10, there is a sequence $\{u_j\} \subset W_0^{1,p}(\Omega)$ satisfying (28). We claim that $\{u_j\}$ is bounded and hence a subsequence converges to a critical point of G . If $\rho_j = \|u_j\| \rightarrow \infty$, a subsequence of $\tilde{u}_j = u_j/\rho_j$ converges to some $\tilde{u}$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω . Then

$$\int_{\Omega} \frac{H(x, u_j)}{\rho_j^\alpha} = \frac{G'(u_j) u_j/p - G(u_j)}{\rho_j^\alpha} \rightarrow 0 \quad (83)$$

by (12) and

$$\underline{\lim} \int_{\Omega} \frac{H(x, u_j)}{\rho_j^\alpha} \geq \int_{\Omega} \underline{\lim} \frac{H(x, u_j)}{|u_j|^\alpha} |\tilde{u}_j|^\alpha \geq \int_{\Omega} \underline{H}(x) |\tilde{u}|^\alpha \geq 0 \quad (84)$$

by (72). Since $\underline{H} > 0$ a.e., it follows that $\tilde{u} = 0$ a.e. Now passing to the limit in

$$1 - \frac{G(u_j)}{\rho_j^p} = \int_{\Omega} \frac{pF(x, u_j)}{\rho_j^p} \leq \int_{\Omega} \lambda_{l+1} |\tilde{u}_j|^p + \frac{W}{\rho_j^p} \quad (85)$$

again gives

$$1 \leq \int_{\Omega} \lambda_{l+1} |\tilde{u}|^p,$$

contradicting the fact that $\tilde{u} = 0$ a.e. $\square$

Proof of Theorem 12. We follow the proof of Theorem 10. We can conclude that there is a sequence $\{u_j\} \subset W_0^{1,p}(\Omega)$ satisfying (28). Again we claim that $\{u_j\}$ is bounded, and hence a subsequence converges to a critical point of G . If $\rho_j =$

$\|u_j\| \rightarrow \infty$, a subsequence of $\tilde{u}_j = u_j/\rho_j$ converges to some $\tilde{u}$ weakly in $W_0^{1,p}(\Omega)$, strongly in $L^p(\Omega)$, and a.e. in Ω . Now

$$\int_{\Omega} H(x, u_j) dx = \int_{\Omega} \{G'(u_j) u_j/p - G(u_j)\} dx \rightarrow -\tilde{c}. \quad (86)$$

This implies

$$\left| \int_{\Omega} H(x, u_j) dx \right| \leq K. \quad (87)$$

As before, we show that $\tilde{u}(x) \neq 0$. Let Ω_0 be the subset of Ω on which $\tilde{u} \neq 0$. Then

$$|u_j(x)| = \rho_j |\tilde{u}_j(x)| \rightarrow \infty, \quad x \in \Omega_0. \quad (88)$$

If $\Omega_1 = \Omega \setminus \Omega_0$, then we have

$$\int_{\Omega} H(x, u_j) dx = \int_{\Omega_0} + \int_{\Omega_1} \leq \int_{\Omega_0} H(x, u_j) dx + \int_{\Omega_1} W_1(x) dx \rightarrow -\infty. \quad (89)$$

This contradicts (87), and we see that $\rho_j = \|u_j\|$ is bounded. $\square$

Proof of Theorem 13. In this case we have

$$\int_{\Omega} H(x, u_j) dx = \int_{\Omega_0} + \int_{\Omega_1} \geq \int_{\Omega_0} H(x, u_j) dx - \int_{\Omega_1} W_1(x) dx \rightarrow \infty, \quad (90)$$

which contradicts (87) as well. $\square$

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