Indirect Boundary Stabilization of a System of Schrödinger Equations with Variable Coefficients

Ilhem Hamchi and Salah Eddine Rebiai

Abstract. We consider a coupled system of two *complex* Schrödinger equations with *variable coefficients*. The boundary feedback appears only in one of the equations. The aim of this paper is to prove that we can apply the Riemann geometric approach developed to study the problems of *direct stabilization* for *real* hyperbolic equations (see [22]) and show that the sufficiently smooth solutions decays polynomially at infinity, by adapting the ideas of Alabau in [2] used to obtain *indirect stabilization* results for a system of two coupled *real* wave equations with *constant coefficients*.

Mathematics Subject Classification (2000). 93D15, 35Q40, 42B15.

Keywords. Riemannian geometry, Indirect damping, Multiplier method.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}^*$, be an open bounded domain with C^2 boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 \neq \emptyset$ and $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$.

$$\mathcal{A}y = - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial y}{\partial x_j} \right)$$

is a second order differential operator with real coefficients $a_{ij} = a_{ji}$ of class C^1 and satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(x) \zeta_i \zeta_j \geq a_0 \sum_{i=1}^n \zeta_i^2 \quad \text{for all } x \in \Omega \quad \text{and} \quad \zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{R}^n,$$

for some positive constant $a_0 > 0$. $v = (v_1, \dots, v_n)$ is the outward unit normal to Γ , $\frac{\partial y}{\partial v_{\mathcal{A}}} = \sum_{i,j=1}^n a_{ij}(x) \frac{\partial y}{\partial x_j} v_i$ denotes the co-normal derivative with respect to $\mathcal{A}$.

a and b are two functions in $L^\infty(\overline{\Omega})$ such that for some constants $a_*, b_* > 0$, we have

$$a_* \leq a(x) \quad \text{for all } x \in \Omega \quad \text{and} \quad b_* \leq \operatorname{Re}(b(x)) = b(x) \quad \text{for all } x \in \Gamma.$$

Given $T > 0$, let $Q = \Omega \times]0, T[$, $\Sigma = \Gamma \times]0, T[$ and $\Sigma_l = \Gamma_l \times]0, T[$ ($l = 0, 1$). In Ω , we consider the following coupled *complex valued* Schrödinger equations with *variable coefficients*

$$iy_t + \mathcal{A}y + az = 0 \quad \text{in } Q, \quad (1)$$

$$iz_t + \mathcal{A}z + ay = 0 \quad \text{in } Q, \quad (2)$$

$$y = 0 \quad \text{on } \Sigma_0, \quad \frac{\partial y}{\partial \nu_{\mathcal{A}}} + by_t = 0 \quad \text{on } \Sigma_1 \quad \text{and} \quad z = 0 \quad \text{on } \Sigma. \quad (3)$$

$$y(0) = y_0 \quad \text{and} \quad z(0) = z_0 \quad \text{in } \Omega. \quad (4)$$

This type of system describes a particle in a box with two levels internal and subject to a wave resonant laser with the transition between the two energy levels [20].

For the system (1)–(4), we can remark that a boundary feedback is acting on one end only (no damping acting on z on Σ_1).

The problem of proving exact controllability and uniform decay rates for solutions of one Schrödinger equation with *constant coefficients* has been treated by several authors, we can mention [17, 18]. Recently, *Riemann geometric methods* have emerged as a powerful tool to obtain continuous observability inequalities and *direct stabilization* for various classes of PDEs with *variable coefficients*, see for instance [12, 13, 15, 19, 22, 24–26]. Another question was considered in the literature: the problem of *indirect stabilization*, this problem was first studied by Russell [21] who introduced the terminology of *indirect damping*, since the first equation can be regarded as a stabilizer for the second one. Recently, Aassila [1] proved, using spectral theory, that the solution of the following coupled wave equations

$$y_{tt} - \Delta y + \lambda(y - z) = 0 \quad \text{in } Q,$$

$$z_{tt} - \Delta z + \lambda(z - y) = 0 \quad \text{in } Q,$$

$$y = z = 0 \quad \text{on } \Sigma_0,$$

$$\frac{\partial y}{\partial \nu} + \mu y + y_t = 0 \quad \text{and} \quad z = 0 \quad \text{on } \Sigma_1,$$

$$y(0) = y_0, \quad z(0) = z_0 \quad \text{and} \quad y_t(0) = y_1, z_t(0) = z_1 \quad \text{in } \Omega$$

is strongly stable for arbitrary n and is not uniformly exponentially stable when Ω is an interval ($n = 1$). Alabau [2] studied the *indirect boundary stabilization* of coupled *real hyperbolic* systems. She has proved that the feedback of the first equation is sufficient to stabilize polynomially the total system. So, she has established a polynomial decay lemma for a nonincreasing functional satisfying a generalized integral inequality. In [3], the authors have studied the problem of *indirect stabilization* of two coupled second order evolution equations via the zero order terms

by a feedback acting in the whole domain. These results have been extended, using the piecewise multiplier, to several cases: wave-wave, Petrowsky–Petrowsky coupling, for locally distributed action [7]. We note that, in all these works, the *coupling coefficient* is considered as a *positive constant* which is sufficiently small.

The goal of this paper is to prove that we can apply the *Riemann geometric* approach to the coupled *complex* Schrödinger equations with *variables coefficients* (here we construct a suitable Riemannian metric on $\mathbb{C}^n$, see Section 3 below) and show that we can obtain the *indirect boundary stabilization* of this system, by adapting the method of Alabau developed in the context of coupled *real* wave equations with *constant coefficients*. We note here that the *coupling coefficient* a is considered as a *function* with $\|a\|_{L^\infty(\Omega)}$ sufficiently small. To obtain our result we need some geometric assumptions, two examples are presented to verify these assumptions.

Our paper is organized as follow: in Section 2, we shall present some notations and results on the Riemannian geometry in $\mathbb{R}^n$ generated by the principal part $\mathcal{A}$. In Section 3, we construct a suitable Riemannian metric on $\mathbb{C}^n$ and prove some formulas we need. In Section 4, we give some examples, where the geometric assumption is illustrated. In Section 5, we discuss briefly the well posedness of the system (1)–(4). Section 6 is devoted to the proof of the indirect boundary stabilization result.

2. Riemannian metric on $\mathbb{R}^n$ generated by the principal part $\mathcal{A}$

2.1. Notations

Let $A(x)$ and $B(x)$ be, respectively, the coefficient matrix and its inverse:

$$A(x) = (a_{ij}(x))_{i,j=1,\dots,n} \quad \text{and} \quad B(x) = (g_{ij}(x))_{i,j=1,\dots,n}.$$

Let $\mathbb{R}^n$ have the usual topology and $x = (x_1, x_2, \dots, x_n)$ be the natural coordinates system. For each $x \in \mathbb{R}^n$, we define the inner product on the tangent space $\mathbb{R}_x^n = \mathbb{R}^n$ setting, for all $X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}$

$$g(X, Y) = \langle X, Y \rangle_g = \sum_{i,j=1}^n g_{ij}(x) \alpha_i \beta_j.$$

The corresponding norm is

$$\|X\|_g = \langle X, X \rangle_g^{1/2},$$

and $(\mathbb{R}^n, g)$ becomes a Riemannian manifold with Riemannian metric g .

Remark 2.1. When $A = \text{Identity}$, we find the usual dot product over $\mathbb{R}^n$: $X \cdot Y = \sum_{i=1}^n \alpha_i \beta_i$, for all $X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}$, $Y = \sum_{i=1}^n \beta_i \frac{\partial}{\partial x_i}$ and the norm is defined by $|X| = (X \cdot X)^{1/2}$.

For a *real valued function* f of $C^1(\Omega)$ and $X = \sum_{i=1}^n \alpha_i(x) \frac{\partial}{\partial x_i}$, we define the gradient of f and the divergence of X in the Euclidean metric by

$$\operatorname{div}_0(X) = \sum_{i=1}^n \frac{\partial \alpha_i(x)}{\partial x_i}, \quad \nabla_0 f = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_i}.$$

We define the gradient $\nabla_g f$ of f in the Riemannian metric g , via Riesz representation theorem, by

$$X(f) = \langle \nabla_g f, X \rangle_g,$$

where X is any vector field on the manifold $(\mathbb{R}^n, g)$. Denote the Levi-Civita connection in the Riemannian metric g by D . The covariant differential Dh of a vector field $h = \sum_{i=1}^n h_i(x) \frac{\partial}{\partial x_i}$ is defined by

$$Dh(Y, X) = \langle D_X h, Y \rangle_g,$$

where $D_X h$ is the covariant derivative of h with respect to X .

2.2. Further relationships

The following lemma provides further relationships (see [24]).

Lemma 2.2. *Let f, f_1, f_2 be real valued functions of $C^1(\overline{\Omega})$ and $h = \sum_{i=1}^n h_i(x) \frac{\partial}{\partial x_i}$ a vector field on $(\mathbb{R}^n, g)$. Then*

1.

$$h(f) = \langle \nabla_g f, h \rangle_g = \nabla_0 f \cdot h.$$

2.

$$\langle \nabla_g f_1, \nabla_g f_2 \rangle_g = \nabla_0 f_1 \cdot A(x) \nabla_0 f_2.$$

3. *A Green's formula. If $f_1, f_2 \in H^2(\Omega)$ then*

$$\int_{\Omega} \mathcal{A} f_1 f_2 = \int_{\Omega} \langle \nabla_g f_1, \nabla_g f_2 \rangle_g - \int_{\Gamma} \frac{\partial f_1}{\partial \nu_{\mathcal{A}}} f_2.$$

4. *An identity. If $f \in C^1(\overline{\Omega})$ then*

$$\left\langle \nabla_g f, \nabla_g (h(f)) \right\rangle_g = Dh(\nabla_g f, \nabla_g f) + \frac{1}{2} \operatorname{div}_0 (h \|\nabla_g f\|_g^2) + \frac{1}{2} \|\nabla_g f\|_g^2 \operatorname{div}_0 h. \quad (5)$$

3. Riemannian metric on $\mathbb{C}^n$

Using the inner product $\langle \cdot, \cdot \rangle_g$ on $\mathbb{R}^n$ we can define an inner product on $\mathbb{C}^n = \mathbb{C}_x^n$ (we take the same symbol g) by for all $Z_1, Z_2 \in \mathbb{C}^n$

$$\begin{aligned} \langle Z_1, Z_2 \rangle_g &= \langle \operatorname{Re} Z_1, \operatorname{Re} Z_2 \rangle_g + \langle \operatorname{Im} Z_1, \operatorname{Im} Z_2 \rangle_g \\ &\quad - i(\langle \operatorname{Re} Z_1, \operatorname{Im} Z_2 \rangle_g - \langle \operatorname{Im} Z_1, \operatorname{Re} Z_2 \rangle_g), \end{aligned}$$

so the norm is

$$\|Z\|_g^2 = \langle Z, Z \rangle_g = \|\operatorname{Re} Z\|_g^2 + \|\operatorname{Im} Z\|_g^2 \quad \text{for all } Z \in \mathbb{C}^n.$$

Let f be a *complex valued function* and h be a vector field on $(\mathbb{R}^n, g)$. We put

$$h(f) := h(\operatorname{Re} f) + ih(\operatorname{Im} f) \quad \text{and} \quad \nabla_g f := \nabla_g \operatorname{Re} f + i\nabla_g \operatorname{Im} f.$$

Main formulas

We give the counterpart of the Green's formula and the identity (5) for *complex valued functions*:

Lemma 3.1. 1. Let f_1, f_2 be complex valued functions in $H^2(\Omega)$. Then

$$\operatorname{Re} \int_{\Omega} \mathcal{A} f_1 \overline{f_2} = \operatorname{Re} \int_{\Omega} \langle \nabla_g f_1, \nabla_g f_2 \rangle_g - \operatorname{Re} \int_{\Gamma} \frac{\partial f_1}{\partial \nu_{\mathcal{A}}} \overline{f_2}.$$

2. Let f be a complex valued function in $C^1(\overline{\Omega})$ and h a vector field on $(\mathbb{R}^n, g)$. Then

$$\begin{aligned} \operatorname{Re} \left\langle \nabla_g f, \nabla_g (h(f)) \right\rangle_g &= Dh(\nabla_g \operatorname{Re} f, \nabla_g \operatorname{Re} f) + Dh(\nabla_g \operatorname{Im} f, \nabla_g \operatorname{Im} f) \\ &\quad + \frac{1}{2} \operatorname{div}_0 (h \|\nabla_g f\|_g^2) - \frac{1}{2} \|\nabla_g f\|_g^2 \operatorname{div}_0 h. \end{aligned} \quad (6)$$

Proof. 1. Using Green's formula we obtain

$$\begin{aligned} \operatorname{Re} \int_{\Omega} \mathcal{A} f_1 \overline{f_2} &= \operatorname{Re} \int_{\Omega} \mathcal{A} (\operatorname{Re} f_1 + i \operatorname{Im} f_1) (\operatorname{Re} f_2 - i \operatorname{Im} f_2) \\ &= \int_{\Omega} \mathcal{A} \operatorname{Re} f_1 \operatorname{Re} f_2 + \int_{\Omega} \mathcal{A} \operatorname{Im} f_1 \operatorname{Im} f_2 \\ &= \int_{\Omega} \langle \nabla_g \operatorname{Re} f_1, \nabla_g \operatorname{Re} f_2 \rangle_g + \int_{\Omega} \langle \nabla_g \operatorname{Im} f_1, \nabla_g \operatorname{Im} f_2 \rangle_g \\ &\quad - \left(\int_{\Gamma} \frac{\partial \operatorname{Re} f_1}{\partial \nu_{\mathcal{A}}} \operatorname{Re} f_2 - \int_{\Gamma} \frac{\partial \operatorname{Im} f_1}{\partial \nu_{\mathcal{A}}} \operatorname{Im} \overline{f_2} \right). \end{aligned}$$

The proof of desired formula is complete.

2. It is sufficient to observe that

$$\operatorname{Re} \left\langle \nabla_g f, \nabla_g (h(f)) \right\rangle_g = \left\langle \nabla_g \operatorname{Re} f, \nabla_g (h(\operatorname{Re} f)) \right\rangle_g + \left\langle \nabla_g \operatorname{Im} f, \nabla_g (h(\operatorname{Im} f)) \right\rangle_g,$$

so, (6) follows from (5). $\square$

Remark 3.2. We can see that there exist two positive constants α and β such that for all f of $H^1(\Omega)$ we have $\alpha \int_{\Omega} |\nabla_0 f|^2 \leq \int_{\Omega} \|\nabla_g f\|_g^2 = \int_{\Omega} \|\operatorname{Re} \nabla_g f\|_g^2 + \int_{\Omega} \|\operatorname{Im} \nabla_g f\|_g^2 \leq \beta \int_{\Omega} |\nabla_0 f|^2$.

In all this paper, C_1 and C_2 are the positive constants such that $\int_{\Omega} |f|^2 \leq C_1^2 \int_{\Omega} \|\nabla_g f\|_g^2$ and $\int_{\Gamma_1} |f|^2 \leq C_2^2 \int_{\Omega} \|\nabla_g f\|_g^2$ for all functions f of $H_{\Gamma_0}^1(\Omega)$. C is a generic positive constant which does not depend on the initial data.

4. Geometric assumptions and examples

Assume that there exists a real vector field $h \in [C^1(\overline{\Omega})]^n$ on the Riemannian manifold $(\mathbb{R}^n, g)$, a constant $m_0 > 0$ such that

$$Dh(X, X) \geq m_0 \|X\|_g^2, \quad \text{for all } X \in \mathbb{R}_x^n, \quad (7)$$

and

$$2m_0 > C_1 C_h, \quad (8)$$

where $C_h = \sup_{x \in \Omega} \|\nabla_g(\operatorname{div}_0 h)\|_g$.

We assume that Γ_0, Γ_1 are taken as

$$\Gamma_0 = \{x \in \Gamma \mid h \cdot \nu \leq 0\} \quad \text{and} \quad \Gamma_1 = \{x \in \Gamma \mid h \cdot \nu \geq h_0 > 0\}.$$

We now give two examples that verify assumptions (7) and (8).

Example 4.1. If (a_{ij}) is a positive, symmetric constant matrix, then we may take $h = \sum_{i=1}^n (x_i - x_i^0) \frac{\partial}{\partial x_i}$, $x_0 \in \mathbb{R}^n$ to satisfy (7) (see [24]). Moreover, in this case the condition (8) holds true as well: $\operatorname{div}_0 h = \sum_{i=1}^n \frac{\partial}{\partial x_i} (x_i - x_i^0) = n$, so $C_h = 0$.

Example 4.2. In [11] the authors have proved that the geometric condition (7), derived by Yao in term of the Riemannian geometry method, is equivalent to the following analytical condition given by Wyler [23] for boundary stabilization of wave equations with variable coefficients:

$$\begin{cases} (p_{ij}) \text{ is a uniformly positive definite matrix in } \Omega, \text{ where} \\ p_{ij} = p_{ji} = \sum_{k=1}^{k=n} a_{ik} \frac{\partial h_j}{\partial x_k} + \sum_{k=1}^{k=n} a_{jk} \frac{\partial h_i}{\partial x_k} - \nabla_0 a_{ij} \cdot h. \end{cases}$$

If $h(x) = \sum_{i=1}^n h_i(x) \frac{\partial}{\partial x_i}$ is a vector field in $\mathbb{R}^n$ such that $\operatorname{div}_0 h$ is a constant (8) is verified. If (a_{ij}) is a matrix defined by

$$a_{ij}(x_1, x_2, \dots, x_n) = \begin{cases} 0 & i \neq j, \\ f_i(x_i) & i = j \end{cases}$$

where, for all $i = 1, \dots, n$, $f_i: \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 satisfying the condition

$$\min f_i > 0$$

and

$$\begin{aligned} \min \left(2f_i \frac{\partial h_i}{\partial x_i} - \frac{\partial f_i}{\partial x_i} h_i \right) &> 0, \quad i = j, \\ f_i \frac{\partial h_j}{\partial x_i} + f_j \frac{\partial h_i}{\partial x_j} &= 0, \quad i \neq j, \end{aligned}$$

(p_{ij}) is uniformly positive definite matrix in Ω .

As an example of such vector and matrix we can take $h_i = x_i - x_i^0$, $x_0 \in \mathbb{R}^n$ and $f_i(x_i) = (x_i - x_i^0)^2 + \beta$ where $\beta \in \mathbb{R}_+^*$.

Remark 4.1. We note that assumption (8) has been used in [14] to study the exact controllability of *real* wave equation with *constant coefficients*.

5. Existence and regularity of solutions

Let $\mathbb{A}: D(\mathbb{A}) \subset H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega) \rightarrow H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$ be the operator defined by

$$\mathbb{A}(u_1, u_2) = (\mathcal{A}u_1 + iau_2, \mathcal{A}u_2 + iau_1),$$

where

$$D = D(\mathbb{A}) = \left\{ (u_1, u_2) \in H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega) \mid (\mathcal{A}u_1, \mathcal{A}u_2) \in H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega) \right. \\ \left. \text{and } \frac{\partial u_1}{\partial \nu_{\mathcal{A}}} + ib\mathcal{A}u_1|_{\Gamma_1} = 0 \right\}.$$

Using the idea in [9] we can show that (1)–(4) is equivalent to

$$\begin{cases} (y, z)_t = \mathbb{A}(y, z) \\ (y(0), z(0)) = (y_0, z_0). \end{cases} \quad (P)$$

Then we have the following result:

Theorem 5.1. *Let $N \geq 1$, if $(y_0, z_0) \in D(\mathbb{A}^N)$ the system (1)–(4) has a unique solution $(y, z) \in C^{N-j}([0, +\infty); D(\mathbb{A}^j))$ for $j = 0, \dots, N$. Here $D(\mathbb{A}^1) = D(\mathbb{A})$ and $D(\mathbb{A}^N) = \{(u_1, u_2) \in D(\mathbb{A}^{N-1}) \mid \mathbb{A}(u_1, u_2) \in D(\mathbb{A}^{N-1})\}$ for $N \geq 2$.*

Proof. It is sufficient to prove that $(-\mathbb{A})$ is a maximal monotone operator in the space $H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$ (see Theorem VII4 and Theorem VII5 in [8]).

First, we must show that given $(v_1, v_2) \in H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$, there exists a solution $(u_1, u_2) \in D(\mathbb{A})$ such that $[I - \mathbb{A}](u_1, u_2) = (v_1, v_2)$. This yields a corresponding elliptic problem

$$\begin{aligned} u_1 - i\mathcal{A}u_1 - iau_2 &= v_1 & \text{in } \Omega, \\ u_2 - i\mathcal{A}u_2 - iau_1 &= v_2 & \text{in } \Omega, \\ u_1 &= 0 & \text{on } \Gamma_0 \quad \text{and} \quad u_2 = 0 & \text{on } \Gamma, \\ \frac{\partial u_1}{\partial \nu} + b(u_1 - v_1) &= 0 & \text{on } \Gamma_1, \end{aligned}$$

and elliptic theory gives the desired conclusion. To prove that $(-\mathbb{A})$ is a monotone operator in $H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$, we have some difficulty to do it directly. To overcome this difficulty we use some results in semigroup theory: First, we can easily obtain, making use of the Faedo–Galerkin method (see [4, 5, 10, 16]), that if $(y_0, z_0) \in H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$ the system (1)–(4) has a unique solution $(y, z) \in C([0, +\infty); H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega))$ and that if $(y_0, z_0) \in D$ the system (1)–(4) has a unique solution $(y, z) \in C([0, +\infty); D) \cap C^1([0, +\infty); H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega))$. For all $t \geq 0$, we consider the mapping

$$T(t): H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega) \rightarrow H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$$

defined by

$$T(t)(y_0, z_0) = (y(t), z(t)),$$

where (y, z) is the solution of (P) (or (1)–(4)).

It is clear that $(T(t))_{t \geq 0}$ is a C_0 -semigroup on $H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$ and (see Lemma 6.1 below)

$$\begin{aligned} \|T(t)(y_0, z_0)\|_{H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)}^2 &= \|(y, z)\|_{H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)}^2 = 2E(t) \\ &\leq 2E(0) = \|(y_0, z_0)\|_{H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)}^2. \end{aligned}$$

Then $(T(t))_{t \geq 0}$ is a C_0 -semigroup of contraction.

Denote by $\mathbb{B}$ the infinitesimal generator of $(T(t))_{t \geq 0}$. By Theorem 8, Chapter 3 in [10], $\mathbb{B}$ is a maximal monotone operator in $H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$.

Let $(y_0, z_0) \in D(\mathbb{A})$ then

$$\frac{d}{dt}T(t)(y_0, z_0) = (y, z)_t = \mathbb{A}(y, z) = \mathbb{A}T(t)(y_0, z_0),$$

which implies that $[\frac{d}{dt}T(t)(y_0, z_0)]_{t=0} = \mathbb{A}(y_0, z_0)$ and therefore $\mathbb{A} \subset \mathbb{B}$. This implies that $\mathbb{A}$ is a monotone operator in $H_{\Gamma_0}^1(\Omega) \times H_0^1(\Omega)$. $\square$

6. Indirect boundary stabilization result

Consider the total energy E of (1)–(4) defined by

$$E(t) = E(y(t), z(t)) = E_1(y(t)) + E_2(z(t)) + \operatorname{Re} \int_{\Omega} ay\bar{z},$$

where

$$E_1(t) = E_1(y(t)) = \frac{1}{2} \left(\int_{\Omega} \|\nabla_g y\|_g^2 \right) \quad \text{and} \quad E_2(t) = E_2(z(t)) = \frac{1}{2} \left(\int_{\Omega} \|\nabla_g z\|_g^2 \right).$$

We can see that E is equivalent to $E_1 + E_2$ when we take $\|a\|_{L^\infty(\Omega)}$ sufficiently small.

The dissipative property of the solution of (1)–(4) is given by the following lemma:

Lemma 6.1. *We have for all $t \geq 0$, $E'(t) = - \int_{\Gamma_1} b|y_t|^2 \leq 0$.*

Proof. We multiply both side (1) by $\bar{y}_t$, integrate over Ω , take the real part, use Lemma 3.1, and finally use the boundary condition (3), to find

$$\operatorname{Re} \int_{\Omega} \langle \nabla_g y, \nabla_g y_t \rangle_g + \int_{\Gamma_1} b|y_t|^2 + \operatorname{Re} \int_{\Omega} az\bar{y}_t = 0.$$

We obtain a similar identity for z

$$\operatorname{Re} \int_{\Omega} \langle \nabla_g z, \nabla_g z_t \rangle_g + \operatorname{Re} \int_{\Omega} ay\bar{z}_t = 0.$$

But

$$E'(t) = \operatorname{Re} \int_{\Omega} \langle \nabla_g y, \nabla_g y_t \rangle_g + \operatorname{Re} \int_{\Omega} \langle \nabla_g z, \nabla_g z_t \rangle_g + \operatorname{Re} \int_{\Omega} a(y\bar{z})_t,$$

and we find the result. $\square$

Remark 6.2. We deduce from Lemma 6.1 that

$$E(T) \leq E(0) \quad (9)$$

and

$$\int_{\Sigma_1} |y_t|^2 \leq CE(0). \quad (10)$$

Our main result is

Theorem 6.3. *Let $N \geq 1$. For any initial data $(y_0, z_0) \in D(A^N)$, the energy E of the solution of system (1)–(4) decays polynomially:*

$$E(y(t), z(t)) \leq \frac{C}{t^N} \sum_{p=0}^{p=N} E(y^{(p)}(0), z^{(p)}(0)) \quad \text{for all } t > 0.$$

Proof. To prove our result we estimate $\int_0^T E_1(t)$ and $\int_0^T E_2(t)$ then, after summing up these two estimates, we conclude applying the Theorem 3.1 in [2] with $K = 1$.

Step 1. We prove an estimate useful to estimate the term $\int_0^T E_1(t)$.

For fixed t , we consider w the solution of the problem

$$\mathcal{A}w = 0 \quad \text{in } \Omega, \quad w = y \quad \text{on } \Gamma.$$

Using elliptic regularity (Lemma 2.1 in [6]), we can see that

$$\int_{\Omega} |w|^2 \leq C \int_{\Gamma_1} |y|^2 \leq CE_1(t).$$

If we use this inequality with the derivatives, integrate over $[0, T]$ and use (10) we obtain

$$\int_Q |w_t|^2 \leq C \int_{\Sigma_1} |y_t|^2 \leq CE(0). \quad (11)$$

On the other hand, we have

$$\operatorname{Re} \int_{\Omega} \mathcal{A}w \bar{z} = 0,$$

then

$$\operatorname{Re} \int_{\Omega} \langle \nabla_g w, \nabla_g z \rangle_g = 0. \quad (12)$$

Multiplying the conjugate of (2) by $y - w$, integrating over Q and taking the real part

$$\operatorname{Im} \int_Q \bar{z}_t(y - w) + \operatorname{Re} \int_Q \mathcal{A}\bar{z}(y - w) + \int_Q a|y|^2 - \operatorname{Re} \int_Q a\bar{y}w = 0,$$

then, by Lemma 3.1, (12) and the integration by parts, we find

$$\begin{aligned} \operatorname{Im} \int_{\Omega} \bar{z}(y - w)|_0^T - \operatorname{Im} \int_Q \bar{z}(y_t - w_t) \\ + \operatorname{Re} \int_Q \langle \nabla_g z, \nabla_g y \rangle_g + \int_Q a|y|^2 - \operatorname{Re} \int_Q a\bar{y}w = 0. \end{aligned}$$

If we multiply (1) by $(-\bar{z})$, integrate over Q , take the real part and sum the result with the last identity, we find

$$\int_Q a|z|^2 = \operatorname{Im} \int_\Omega \bar{z}(y-w) \Big|_0^T + \operatorname{Im} \int_Q \bar{z}w_t + \int_Q a|y|^2 - \operatorname{Re} \int_Q a\bar{y}w.$$

But

$$\left| \int_\Omega \bar{z}(y-w) \right| \leq C \left(\int_\Omega |y|^2 + \int_\Omega |z|^2 + \int_\Omega |w|^2 \right) \leq CE(0),$$

then, for all $\varepsilon > 0$, we have

$$\int_Q a|z|^2 \leq CE(0) + \frac{\varepsilon}{a_*2} \int_Q a|z|^2 + \frac{1}{2\varepsilon} \int_Q |w_t|^2 + C\|a\|_{L^\infty(\Omega)} \int_0^T E_1(t).$$

Using (11) and choosing $\varepsilon = a_*$ we find

$$\int_Q a|z|^2 \leq CE(0) + C\|a\|_{L^\infty(\Omega)} \int_0^T E_1(t). \quad (13)$$

Step 2. An estimate of the term $\int_0^T E_1(t)$.

Multiplying (1) by $2h(\bar{y}) + \operatorname{div}_0 h\bar{y}$, integrating over Q and taking the real part, we obtain

$$\begin{aligned} 0 &= \operatorname{Im} \int_Q y_t(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) - \operatorname{Re} \int_Q \mathcal{A}y(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ &\quad - \operatorname{Re} \int_Q az(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}). \end{aligned} \quad (14)$$

By integration by parts, we have

$$\int_Q y_t h(\bar{y}) = \int_\Omega y h(\bar{y}) \Big|_0^T - \int_Q y h(\bar{y}_t).$$

We invoke the standard divergence identity

$$\int_\Omega s \cdot \nabla_0 \psi = \int_\Gamma \psi s \cdot \nu - \int_\Omega \psi \operatorname{div}_0 s,$$

to obtain

$$\begin{aligned} \int_Q y_t h(\bar{y}) &= \int_\Omega y h(\bar{y}) \Big|_0^T - \int_\Sigma h \cdot \nu y \bar{y}_t + \int_Q \bar{y}_t \operatorname{div}_0(hy) \\ &= \int_\Omega y h(\bar{y}) \Big|_0^T - \int_\Sigma h \cdot \nu y \bar{y}_t + \int_Q \bar{y}_t \operatorname{div}_0 hy + \int_Q \bar{y}_t h(y). \end{aligned}$$

Then

$$2 \operatorname{Im} \int_Q y_t h(\bar{y}) = \operatorname{Im} \int_\Omega y h(\bar{y}) \Big|_0^T - \operatorname{Im} \int_\Sigma h \cdot \nu y \bar{y}_t - \operatorname{Im} \int_Q y_t \operatorname{div}_0 h\bar{y},$$

so

$$\operatorname{Im} \int_Q y_t(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) = \operatorname{Im} \int_\Omega y h(\bar{y}) \Big|_0^T - \operatorname{Im} \int_{\Sigma_1} h \cdot \nu y \bar{y}_t. \quad (15)$$

If we use Lemma 3.1, we find

$$\begin{aligned} & \operatorname{Re} \int_Q \mathcal{A}y(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ &= -\operatorname{Re} \int_{\Sigma} \frac{\partial y}{\partial v_{\mathcal{A}}} (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ & \quad + 2 \int_Q \operatorname{Re} \langle \nabla_g y, \nabla_g(h(y)) \rangle_g + \int_Q \operatorname{Re} \langle \nabla_g y, \nabla_g(\operatorname{div}_0 h y) \rangle_g. \end{aligned}$$

Indeed, from the identity (6) we obtain

$$\begin{aligned} & \operatorname{Re} \int_Q \mathcal{A}y(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ &= -\operatorname{Re} \int_{\Sigma} \frac{\partial y}{\partial v_{\mathcal{A}}} (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) + 2 \int_Q Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y) \\ & \quad + 2 \int_Q Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y) + \int_Q h \cdot \nabla_0(\|\nabla_g y\|_g^2) \\ & \quad + \operatorname{Re} \int_Q \langle \nabla_g y, \nabla_g(\operatorname{div}_0 h) \rangle_g \bar{y} + \int_Q \operatorname{div}_0 h \|\nabla_g y\|_g^2. \end{aligned}$$

Using the divergence identity, we find

$$\begin{aligned} & \operatorname{Re} \int_Q \mathcal{A}y(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ &= -\operatorname{Re} \int_{\Sigma} \frac{\partial y}{\partial v_{\mathcal{A}}} (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\ & \quad + 2 \int_Q Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y) + 2 \int_Q Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y) \\ & \quad + \int_{\Sigma} h \cdot v \|\nabla_g y\|_g^2 + \operatorname{Re} \int_Q \langle \nabla_g y, \nabla_g(\operatorname{div}_0 h) \rangle_g \bar{y}. \end{aligned}$$

Since, $\operatorname{Re} y = \operatorname{Im} y = 0$ on Γ_0 , then we have (see [24])

$$h(\operatorname{Re} y) = \frac{h \cdot v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial \operatorname{Re} y}{\partial v_{\mathcal{A}}} \quad \text{and} \quad \|\nabla_g \operatorname{Re} y\|_g^2 = \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left(\frac{\partial \operatorname{Re} y}{\partial v_{\mathcal{A}}} \right)^2$$

and

$$h(\operatorname{Im} y) = \frac{h \cdot v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial \operatorname{Im} y}{\partial v_{\mathcal{A}}} \quad \text{and} \quad \|\nabla_g \operatorname{Im} y\|_g^2 = \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left(\frac{\partial \operatorname{Im} y}{\partial v_{\mathcal{A}}} \right)^2.$$

So

$$h(y) = \frac{h \cdot v}{\|v_{\mathcal{A}}(x)\|_g^2} \frac{\partial y}{\partial v_{\mathcal{A}}} \quad \text{and} \quad \|\nabla_g y\|_g^2 = \frac{1}{\|v_{\mathcal{A}}(x)\|_g^2} \left| \frac{\partial y}{\partial v_{\mathcal{A}}} \right|^2.$$

Then

$$\begin{aligned}
& -\operatorname{Re} \int_Q \mathcal{A}y(2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) \\
& = \int_{\Sigma_0} \frac{h \cdot v}{\|v_{\mathcal{A}}(x)\|_g^2} \left| \frac{\partial y}{\partial v_{\mathcal{A}}} \right|^2 - \int_{\Sigma_1} h \cdot v \|\nabla_g y\|_g^2 \\
& \quad + \operatorname{Re} \int_{\Sigma_1} \frac{\partial y}{\partial v_{\mathcal{A}}} (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}) - 2 \int_Q Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y) \\
& \quad - 2 \int_Q Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y) - \operatorname{Re} \int_Q \langle \nabla_g y, \nabla_g (\operatorname{div}_0 h) \rangle_g \bar{y}. \quad (16)
\end{aligned}$$

Finally, we insert (15) and (16) in (14) to obtain

$$\begin{aligned}
& 2 \left(\int_Q Dh(\nabla_g \operatorname{Re} y, \nabla_g \operatorname{Re} y) + \int_Q Dh(\nabla_g \operatorname{Im} y, \nabla_g \operatorname{Im} y) \right) \\
& \quad = I_{\Omega} + I_{\Sigma_0} + I_{\Sigma_1} + I_Q, \quad (17)
\end{aligned}$$

where

$$\begin{aligned}
I_{\Omega} &= \operatorname{Im} \int_{\Omega} y h(\bar{y})|_0^T, \\
I_{\Sigma_0} &= \int_{\Sigma_0} \frac{h \cdot v}{\|v_{\mathcal{A}}(x)\|_g^2} \left| \frac{\partial y}{\partial v_{\mathcal{A}}} \right|^2, \\
I_{\Sigma_1} &= -\operatorname{Im} \int_{\Sigma_1} h \cdot v y \bar{y}_t - \int_{\Sigma_1} h \cdot v \|\nabla_g y\|_g^2 - \operatorname{Re} \int_{\Sigma_1} b y_t (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}), \\
I_Q &= -\operatorname{Re} \int_Q \langle \nabla_g y, \nabla_g (\operatorname{div}_0 h) \rangle_g \bar{y} - \operatorname{Re} \int_Q a z (2h(\bar{y}) + \operatorname{div}_0 h\bar{y}).
\end{aligned}$$

We can see that by (9)

$$I_{\Omega} \leq CE(0).$$

We have

$$I_{\Sigma_0} \leq 0.$$

We have for all $\varepsilon > 0$

$$\begin{aligned}
I_{\Sigma_1} &\leq \frac{C}{\varepsilon} \int_{\Sigma_1} |y_t|^2 + C\varepsilon \int_Q \|\nabla_g y\|_g^2 + (C\varepsilon - h_0) \int_{\Sigma_1} \|\nabla_g y\|_g^2 \\
&\stackrel{(10)}{\leq} CE(0) + C\varepsilon \int_0^T E_1(t) + (C\varepsilon - h_0) \int_{\Sigma_1} \|\nabla_g y\|_g^2.
\end{aligned}$$

We also have, for all $\eta > 0$, by (13)

$$I_Q \leq CE(0) + \left(\eta C_h^2 + \frac{C_1^2}{\eta} \right) \int_0^T E_1(t) + C\|a\|_{L^\infty(\Omega)} \int_0^T E_1(t).$$

Inserting the above estimates for I_Ω , I_{Σ_l} ($l = 0, 1$) and I_Q in (17), using (7) and (8), and choosing $\eta = \frac{C_1}{C_h}$, ε and $\|a\|_{L^\infty(\Omega)}$ sufficiently small, we find

$$\int_0^T E_1(t) \leq CE(0). \quad (18)$$

Step 3. An estimate of the term $\int_0^T E_2(t)$.

First we have, by (13) and (18),

$$\int_Q |z|^2 \leq CE(0) = E(y(0), z(0)). \quad (19)$$

If we use this inequality with the derivatives, we obtain

$$\int_Q |z_t|^2 \leq CE(y_t(0), z_t(0)). \quad (20)$$

On the other hand, if we multiply (2) by $\bar{z}$, integrate over Q , take the real part and we use Lemma 3.1, we find

$$\int_Q \|\nabla_g z\|_g^2 = \operatorname{Im} \int_Q z_t \bar{z} - \operatorname{Re} \int_Q ay \bar{z}.$$

If we use Cauchy Schwarz, (18), (19) and (20) we find

$$\int_0^T E_2(t) \leq C \left(E(y(0), z(0)) + E(y_t(0), z_t(0)) \right).$$

Step 4. We can now conclude the result of Theorem 6.3. We have, for all $T > 0$

$$\begin{aligned} \int_0^T E(y(t), z(t)) &= \int_0^T E_1(y(t)) + \int_0^T E_2(z(t)) + \operatorname{Re} \int_Q ay \bar{z} \\ &\leq C \left(E(y(0), z(0)) + E(y_t(0), z_t(0)) \right). \end{aligned}$$

The desired conclusion follows from Theorem 3.1 in [2] with $K = 1$. $\square$

Acknowledgements

The authors would like to thank the referee for giving us constructive remarks which resulted in this final version of our work and they would like to thank professor Pierre Rouchon (Ecole des Mines de Paris, Centre Automatique et Systèmes) for the related physical model.

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Ilhem Hamchi and Salah Eddine Rebiai
Department of Mathematics
University of Batna
05000, Algeria
e-mail: hamchi_ilhem@yahoo.fr
rebiai@hotmail.fr

Received: 1 December 2006.

Revised: 28 April 2008.

Accepted: 12 May 2008.