Multiplicity of Positive Solutions for a Class of Inhomogeneous Neumann Problems Involving the $p(x)$ -Laplacian

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Abstract. We study the existence and multiplicity of positive solutions for the inhomogeneous Neumann boundary value problems involving the $p(x)$ -Laplacian of the form

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda|u|^{p(x)-2}u = f(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbf{R}^N$, $p \in C^1(\overline{\Omega})$ and $p(x) > 1$ for $x \in \overline{\Omega}$, $\varphi \in C^{0,\gamma}(\partial\Omega)$ with $\gamma \in (0, 1)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$ on $\partial\Omega$. Using the sub-supersolution method and the variational method, under appropriate assumptions on f , we prove that, there exists $\lambda_* > 0$ such that the problem has at least two positive solutions if $\lambda > \lambda_*$, has at least one positive solution if $\lambda = \lambda_*$, and has no positive solution if $\lambda < \lambda_*$. To prove the result we establish a special strong comparison principle for the Neumann problems.

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1. Introduction

The purpose of this paper is to study the existence and multiplicity of positive solutions for the following inhomogeneous Neumann problem involving the $p(x)$ -Laplacian

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda|u|^{p(x)-2}u = f(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \quad (1.1_\lambda)$$

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where Ω is a bounded smooth domain in $\mathbf{R}^N$ with $C^{1,\beta}$ smooth boundary, η is the unit outward normal to $\partial\Omega$, $\lambda \in \mathbf{R}$, p , f and φ satisfy the following assumption (A):

Assumption (A). $p \in C^1(\overline{\Omega})$, $p_- := \inf_{\overline{\Omega}} p(x) > 1$, $f \in C(\overline{\Omega} \times \mathbf{R})$, $\varphi \in C^{0,\gamma}(\partial\Omega)$ with some $\gamma \in (0, 1)$, $\varphi \geq 0$ and $\varphi \not\equiv 0$ on $\partial\Omega$.

The operator $-\Delta_{p(x)}u := -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called $p(x)$ -Laplacian, which becomes p -Laplacian when $p(x) \equiv p$ (a constant). The study of various mathematical problems with variable exponent has been received considerable attention in recent years. These problems are interesting in applications (see e.g. [37, 42] and raise many difficult mathematical problems. We refer to [14, 38] for the overview and references of this subject, and to [2, 4, 16–20, 22, 31–33, 41] for the study of the $p(x)$ -Laplacian equations and the corresponding variational problems.

Many authors have studied the inhomogeneous (and nonlinear) Neumann problems involving the p -Laplacian, see e.g. [1, 13, 26, 34, 35, 40] and the references therein. In [13] and [1] the authors have studied the problem (1.1 $_{\lambda}$) in the cases of $p(x) \equiv p = 2$ and of $p(x) \equiv p > 1$ respectively. The aim of the present paper is to generalize the main results of [1, 13] to the $p(x)$ -Laplacian case. In [1, 13] only the case that $f(x, u) = |u|^{q-2}u$ is considered, where $p < q \leq p^*$. In the present paper the function $f(x, u)$ is of general type which includes $|u|^{q(x)-2}u$ as a special case. In studying the existence of positive solutions of (1.1 $_{\lambda}$) we do not restrict the growth condition of $f(x, u)$, but in studying the multiplicity of positive solutions of (1.1 $_{\lambda}$), in this paper, we shall restrict ourselves to the case that $f(x, u)$ satisfies the subcritical growth condition because the study of the $p(x)$ -Laplacian equations in the critical growth case is very difficult and requires some special preliminaries which are not ready up to the present.

In this paper we use the following notations:

$$\begin{aligned} p_- &= \inf_{\overline{\Omega}} p(x), \quad p_+ = \sup_{\overline{\Omega}} p(x), \\ p^*(x) &:= \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N, \end{cases} \\ F(x, t) &= \int_0^t f(x, s)ds, \\ \Lambda &= \{\lambda \in \mathbf{R} : \text{there exists at least a positive solution of (1.1}_{\lambda})\}, \\ \lambda_* &= \inf \Lambda. \end{aligned}$$

The main results of this paper are the following theorems in which we always suppose that the Assumption (A) holds.

Theorem 1.1. *Suppose that f satisfies the following conditions:*

$$f(x, t) \geq 0, \quad \forall x \in \Omega, \quad \forall t \geq 0, \quad (1.2)$$

and

for each $x \in \Omega$, $f(x, t)$ is nondecreasing with respect to $t \geq 0$. (1.3)

Then $\Lambda \neq \emptyset$, $\lambda_* \geq 0$ and $(\lambda_*, +\infty) \subset \Lambda$. Moreover, for every $\lambda > \lambda_*$, there exists a minimal positive solution u_λ of (1.1 $_\lambda$) such that $u_{\lambda_1} \leq u_{\lambda_2}$ if $\lambda_* < \lambda_2 < \lambda_1$.

Theorem 1.2. Suppose that (1.2) and (1.3) hold, and there exist positive constants M , c_1 and c_2 such that

$$f(x, t) \leq c_1 + c_2 t^{q(x)-1}, \quad \forall x \in \Omega, \quad \forall t \geq M, \quad (1.4)$$

where $q \in C(\overline{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \overline{\Omega}$. Then for each $\lambda \in (\lambda_*, +\infty)$, (1.1 $_\lambda$) has a positive solution u_λ which is a local minimizer of the energy functional associated with (1.1 $_\lambda$). Moreover, for $\lambda_* < \lambda_2 < \lambda_1$, there are the corresponding local minimizers u_{λ_1} and u_{λ_2} such that $u_{\lambda_1} < u_{\lambda_2}$ on $\overline{\Omega}$.

Theorem 1.3. Under the assumptions of Theorem 1.2, also suppose that there exist positive constants M and $\theta > p_+$ such that

$$0 < \theta F(x, t) \leq f(x, t), \quad \forall x \in \Omega, \quad \forall t \geq M. \quad (1.5)$$

Then for each $\lambda \in (\lambda_*, +\infty)$, (1.1 $_\lambda$) has at least two positive solutions u_λ and v_λ , where u_λ is a local minimizer of the energy functional and $u_\lambda \leq v_\lambda$.

Theorem 1.4.

(1) Suppose that f satisfies (1.2),

$$f(x, 0) \leq f(x, t) \quad \text{for } t > 0 \quad \text{and } x \in \Omega, \quad (1.6)$$

and the following conditions:

$$f(x, t) \leq c_1 + c_2 t^{r(x)-1}, \quad \forall x \in \Omega, \quad \forall t \geq M, \quad (1.7)$$

where M , c_1 and c_2 are positive constants, $r \in C(\overline{\Omega})$ and $1 \leq r(x) < p(x)$ for $x \in \overline{\Omega}$. Then $\lambda_* = 0$.

(2) If f satisfies (1.2)–(1.5), then $\lambda_* > 0$ and $\lambda_* \in \Lambda$.

The underlying idea for proving Theorems 1.1–1.4 is similar to the one of [1]. The special features of this class of problems considered in the present paper are that they involve the variable exponent. The $p(x)$ -Laplacian possesses more complicated nonlinearities than the p -Laplacian. For example, it is inhomogeneous. To prove Theorems 1.1–1.4 we use the theory of variable exponent Sobolev spaces, established first by Kováčik and Rákosník [27], and some research results obtained recently for the $p(x)$ -Laplacian equations. In particular we use the results of [18] on the global $C^{1,\alpha}$ regularity of the weak solutions for the $p(x)$ -Laplacian equations. The main method used in this paper is the sub-supersolution method for the Neumann problems involving the $p(x)$ -Laplacian, which is similar to that given in [17] for the Dirichlet problems involving the $p(x)$ -Laplacian. A main difficulty for proving Theorem 1.2 is that a special strong comparison principle is required. It is well known that, when $p \neq 2$, the strong comparison principles for the p -Laplacian equations are very complicated (see e.g. [11, 12, 23, 36]). In [17, 24, 25] the required

strong comparison principles for the Dirichlet problems have been established, however, they cannot be applied to the Neumann problems. To prove Theorem 1.2, we establish a special strong comparison principle for the Neumann problem (1.1_λ) (see Lemma 3.6 in Section 3), which is also valid for the homogeneous Neumann boundary value problems.

In Section 2 we give some preliminary knowledge. In Section 3 we give the proof of Theorems 1.1–1.4.

2. Preliminaries

Let Ω be a bounded smooth domain in $\mathbf{R}^N$, $p \in L^\infty(\Omega)$ and $p_- \geq 1$. Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω . Note that two measurable functions are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Define the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathbf{S}(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \left\{ \sigma > 0 : \int_{\Omega} \left| \frac{u(x)}{\sigma} \right|^{p(x)} dx \leq 1 \right\},$$

and the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \}$$

with the norm

$$\|u\| = \|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are all separable Banach spaces. When $p_- > 1$ these spaces are reflexive. We refer to [14, 15, 21, 27, 38, 39] for the elementary properties of these spaces. The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is a generalization of the standard Sobolev space $W^{1,p}(\Omega)$. Another generalization of the standard Sobolev space is the Orlicz–Sobolev space. For the study of quasilinear elliptic equations in Orlicz–Sobolev space setting we refer to [28] and the references therein.

Let $\lambda > 0$. Define for $u \in W^{1,p(x)}(\Omega)$,

$$\|u\|_\lambda = \inf \left\{ \sigma > 0 : \int_{\Omega} \left(\left| \frac{\nabla u}{\sigma} \right|^{p(x)} + \lambda \left| \frac{u}{\sigma} \right|^{p(x)} \right) dx \leq 1 \right\}.$$

Then $\|u\|_\lambda$ is a norm on $W^{1,p(x)}(\Omega)$ equivalent to $\|u\|_{W^{1,p(x)}(\Omega)}$.

By the definition of $\|u\|_\lambda$ we have the following

Proposition 2.1 (See [21, 27]). Put $\rho_\lambda(u) = \int_\Omega (|\nabla u|^{p(x)} + \lambda |u|^{p(x)}) dx$ for $\lambda > 0$ and $u \in W^{1,p(x)}(\Omega)$. We have:

- 1). $\|u\|_\lambda \geq 1 \implies \|u\|_\lambda^{p_-} \leq \rho_\lambda(u) \leq \|u\|_\lambda^{p_+}$;
- 2). $\|u\|_\lambda \leq 1 \implies \|u\|_\lambda^{p_+} \leq \rho_\lambda(u) \leq \|u\|_\lambda^{p_-}$;
- 3). $\|u_k\|_\lambda \rightarrow 0 \iff \rho_\lambda(u_k) \rightarrow 0$ (as $k \rightarrow \infty$);
- 4). $\|u_k\|_\lambda \rightarrow \infty \iff \rho_\lambda(u_k) \rightarrow \infty$ (as $k \rightarrow \infty$).

Proposition 2.2 (See [21]). Let $p \in C(\overline{\Omega})$. If $q \in C(\overline{\Omega})$ satisfies the condition

$$1 \leq q(x) < p^*(x), \quad \forall x \in \overline{\Omega}, \quad (2.1)$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

For $A \subset \overline{\Omega}$, denote $p_-(A) = \inf\{p(x) : x \in A\}$ and $p_+(A) = \sup\{p(x) : x \in A\}$. Put

$$p^\partial(x) := (p(x))^\partial := \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.3. Let $p \in C(\overline{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition

$$1 \leq q(x) < p^\partial(x), \quad \forall x \in \partial\Omega, \quad (2.2)$$

then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. In particular, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.

Proof. Let $u \in W^{1,p(x)}(\Omega)$. Given any $x \in \partial\Omega$, by (2.2), there is a relatively open neighborhood Ω_x of x in $\overline{\Omega}$ such that

$$q_+(\partial\Omega \cap \Omega_x) < (p_-(\Omega_x))^\partial.$$

Now $u|_{\Omega_x} \in W^{1,p(x)}(\Omega_x) \subset W^{1,p_-(\Omega_x)}(\Omega_x)$. By Sobolev trace embedding theorem (see e.g. [3]), $W^{1,p_-(\Omega_x)}(\Omega_x) \hookrightarrow L^{q_+(\partial\Omega \cap \Omega_x)}(\partial\Omega \cap \Omega_x)$, so $u|_{\Omega_x} \in L^{q_+(\partial\Omega \cap \Omega_x)}(\partial\Omega \cap \Omega_x) \subset L^{q(x)}(\partial\Omega \cap \Omega_x)$. Using the finite covering theorem for the compact set $\partial\Omega$, we can obtain $u \in L^{q(x)}(\partial\Omega)$. This shows that there is an embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$. Since each embedding $W^{1,p_-(\Omega_x)}(\Omega_x) \hookrightarrow L^{q_+(\partial\Omega \cap \Omega_x)}(\partial\Omega \cap \Omega_x)$ is compact, we can see that the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$ is compact. $\square$

Throughout the paper f and φ will be supposed to satisfy Assumption (A).

Definition 2.1.

- (1) $u \in W^{1,p(x)}(\Omega)$ is called a weak solution of the problem (1.1 $_\lambda$) if for all $v \in W^{1,p(x)}(\Omega)$,

$$\int_\Omega (|\nabla u|^{p(x)-2} \nabla u \nabla v + \lambda |u|^{p(x)-2} uv) dx - \int_{\partial\Omega} \varphi v d\sigma = \int_\Omega f(x, u) v dx. \quad (2.3)$$

- (2) $u \in W^{1,p(x)}(\Omega)$ is called a generalized solution of the equation

$$-\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + \lambda |u|^{p(x)-2} u = f(x, u) \quad \text{in } \Omega, \quad (2.4_\lambda)$$

if for all $v \in C_0^\infty(\Omega)$

$$\int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla v + \lambda |u|^{p(x)-2} uv) dx = \int_{\Omega} f(x, u) v dx. \quad (2.5)$$

Obviously, every weak solution of problem (1.1 _{λ}) is also a generalized solution of equation (2.4 _{λ}).

In this paper, we need the global regularity results for the weak solution of (1.1 _{λ}), which are stated in the following Proposition 2.4. For the $L^\infty(\Omega)$ and $C^{0,\alpha}(\overline{\Omega})$ regularity see [20], and for the $C^{1,\alpha}(\overline{\Omega})$ regularity see [18] in which the global $C^{1,\alpha}(\overline{\Omega})$ regularity results for the p -Laplacian equations obtained by Lieberman [29, 30] are generalized to the case of the $p(x)$ -Laplacian equations.

Proposition 2.4.

- (1) (see [20, 18]) If $p \in C(\overline{\Omega})$ and f satisfy the sub-critical growth condition (1.4), then $u \in L^\infty(\Omega)$ for every weak solution u of (1.1 _{λ}), and $|u|_{L^\infty(\Omega)}$ depends only on $\|u\|_\lambda$, $|\varphi|_{L^\infty(\partial\Omega)}$, p_- , p_+ , N , λ , q_+ , c_1 and c_2 .
- (2) (see [20, 18]) Let $u \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ be a weak solution of (1.1 _{λ}). If the function p is log-Hölder continuous on $\overline{\Omega}$, i.e., there is a positive constant H such that

$$|p(x) - p(y)| \leq \frac{H}{-\log|x-y|} \quad \text{for } x, y \in \overline{\Omega} \quad \text{with } |x-y| \leq \frac{1}{2}, \quad (2.6)$$

then $u \in C^{0,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$.

- (3) (see [18]) If in the statement (2), the condition (2.6) is replaced by that $p \in C^{0,\alpha_1}(\overline{\Omega})$, then $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$ depends only on $|u|_{L^\infty(\Omega)}$, $\|\varphi\|_{C^{0,\gamma}(\partial\Omega)}$, $\|p\|_{C^{0,\alpha_1}(\overline{\Omega})}$, p_- , p_+ , N , λ , q_+ , c_1 , c_2 and Ω .

For $u, v \in \mathbf{S}(\Omega)$, we write $u \leq v$ if $u(x) \leq v(x)$ for a.e. $x \in \Omega$. Define $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$.

Proposition 2.5 (See [22]). (A strong maximum principle) Suppose that $p \in C^1(\Omega)$, u is a generalized solution of equation (2.4 _{λ}), $u \geq 0$, $u \not\equiv 0$ and $f(x, u) \geq 0$ in Ω . Then $u > 0$ in Ω . In addition, if Ω satisfies the interior-ball condition and for some $x_0 \in \partial\Omega$, $u(x_0) = 0$ and $u \in C^1(\Omega \cup \{x_0\})$, then $\frac{\partial u}{\partial \eta}(x_0) < 0$.

Definition 2.2. $u \in W^{1,p(x)}(\Omega)$ is called a subsolution (resp. supersolution) of (1.1 _{λ}) if for all $v \in W^{1,p(x)}(\Omega)$ with $v \geq 0$,

$$\int_{\Omega} (|\nabla u|^{p(x)-2} \nabla u \nabla v + \lambda |u|^{p(x)-2} uv) dx - \int_{\partial\Omega} \varphi v d\sigma \leq (\text{resp. } \geq) \int_{\Omega} f(x, u) v dx.$$

Proposition 2.6 (See [17]). Let $\lambda > 0$ and $q \in C(\overline{\Omega})$ satisfy (2.1). Then for each $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + \lambda |u|^{p(x)-2} u = h(x) & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \quad (2.7_\lambda)$$

has a unique solution $u \in W^{1,p(x)}(\Omega)$.

Let $q \in C(\overline{\Omega})$ satisfy (2.1). For $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$, we denote by $K(h) = K_\lambda(h) = u$ the unique solution of (2.7 $_\lambda$). $K = K_\lambda$ is called the solution operator for (2.7 $_\lambda$).

Proposition 2.7.

- (1) (see [17]) The mapping $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \rightarrow W^{1,p(x)}(\Omega)$ is continuous and bounded. Moreover, the mapping $K : L^{\frac{q(x)}{q(x)-1}}(\Omega) \rightarrow L^{q(x)}(\Omega)$ is completely continuous since the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact.
- (2) (see [17]) If p is log-Hölder continuous on $\overline{\Omega}$, then the mapping $K : L^\infty(\Omega) \rightarrow C^{0,\alpha}(\overline{\Omega})$ is bounded, and hence the mapping $K : L^\infty(\Omega) \rightarrow C(\overline{\Omega})$ is completely continuous.
- (3) (see [17]) If p is Hölder continuous on $\overline{\Omega}$, then the mapping $K : L^\infty(\Omega) \rightarrow C^{1,\alpha}(\overline{\Omega})$ is bounded, and hence the mapping $K : L^\infty(\Omega) \rightarrow C^1(\overline{\Omega})$ is completely continuous.
- (4) K is an increasing operator, that is, $K(h_1) \leq K(h_2)$ if $h_1 \leq h_2$.

Proposition 2.8. If $h \in L^{\frac{q(x)}{q(x)-1}}(\Omega)$ and $h \geq 0$, where $q \in C(\overline{\Omega})$ satisfy (2.1), then $K(h) \geq 0$. If $p \in C^1(\Omega)$, $h \in L^\infty(\Omega)$ and $h \geq 0$, then $K(h) > 0$ on $\overline{\Omega}$.

Proof. Let $h \geq 0$ and $K(h) = u$. Then u is the unique global minimizer of the energy functional

$$I_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \lambda |u|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} \varphi u d\sigma - \int_\Omega h u dx, \quad \forall u \in W^{1,p(x)}(\Omega).$$

Setting $v(x) = |u(x)|$ we can see that $I_\lambda(v) \leq I_\lambda(u)$, which shows that $u = v$. Hence $K(h) = u \geq 0$. In the case that $p \in C^1(\Omega)$, $h \in L^\infty(\Omega)$ and $h \geq 0$, we have $K(h) = u \in C^1(\overline{\Omega})$, $u \geq 0$ and $u \not\equiv 0$. By Proposition 2.5 and noting that $\varphi \geq 0$, we obtain $u > 0$ on $\overline{\Omega}$. $\square$

In this paper we will use the following sub-supersolution principle, the proof of which is based on the well known fixed point theorem for the increasing operator on the order interval (see e.g. [5]) and is similar to that given in [17] for Dirichlet problems involving the $p(x)$ -Laplacian.

Theorem 2.1 (A sub-supersolution principle). Let Assumption (A) hold and $\lambda > 0$. Suppose that $u_0, v^0 \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, u_0 and v^0 are a subsolution and a supersolution of (1.1 $_\lambda$) respectively, and $u_0 \leq v^0$. If f satisfies the condition:

$$f(x, t) \text{ is nondecreasing in } t \in [\inf u_0(x), \sup v^0(x)], \quad (2.8)$$

then (1.1 $_\lambda$) has a minimal solution u_* and a maximal solution v^* in the order interval $[u_0, v^0]$, i.e., $u_0 \leq u_* \leq v^* \leq v^0$ and if u is any solution of (1.1 $_\lambda$) such that $u_0 \leq u \leq v^0$, then $u_* \leq u \leq v^*$.

The energy functional corresponding to (1.1_λ) is

$$J_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \lambda |u|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} \varphi u d\sigma - \int_\Omega F(x, u) dx, \quad \forall u \in W^{1,p(x)}(\Omega). \quad (2.9)$$

The critical points of J_λ are just the solutions of (1.1_λ). Many authors, for example, Chang [10], Brezis–Nirenberg [9] and Ambrosetti–Brezis–Cerami [6], have combined the sub-supersolution method with the variational method and studied successfully the semilinear elliptic problems, where a key lemma is that a local minimizer of the associated energy functional in the C^1 -topology is also a local minimizer in the H^1 -topology. Such lemma have been extended to the case of the p -Laplacian equations (see [8, 25]) and also to the case of the $p(x)$ -Laplacian equations (see [17, Theorem 3.1]). The Theorem 3.1 of [17] concerns the Dirichlet problems, but the method for proving the theorem is also valid for the Neumann problems. Thus we have the following.

Theorem 2.2 (See [17] Theorem 3.1). *Let $\lambda > 0$ and Assumption (A) and (1.4) hold. If $u \in C^1(\overline{\Omega})$ is a local minimizer of J_λ in the $C^1(\overline{\Omega})$ -topology, then u is also a local minimizer of J_λ in the $W^{1,p(x)}(\Omega)$ -topology.*

3. Proof of theorems

In this section we shall prove Theorems 1.1–1.4. We always suppose that the Assumption (A) holds. Since only the positive solutions are considered, without loss of generality, we can assume that

$$f(x, t) = f(x, 0) \quad \text{for } t < 0 \quad \text{and } x \in \overline{\Omega},$$

otherwise we may replace $f(x, t)$ by $f^{(+)}(x, t)$, where

$$f^{(+)}(x, t) = \begin{cases} f(x, t) & \text{if } t \geq 0, \\ f(x, 0) & \text{if } t < 0. \end{cases}$$

The proof of Theorem 1.1 consists of the following four Lemmata 3.1–3.4.

Lemma 3.1. *Let (1.2) hold. Then $\lambda \notin \Lambda$ if $\lambda \leq 0$*

Proof. Let $\lambda \in \Lambda$ and u be a positive solution of (1.1_λ). Taking $v \equiv 1$ as a test function in (2.3) yields

$$\lambda \int_\Omega u^{p(x)-1} dx = \int_\Omega f(x, u) dx + \int_{\partial\Omega} \varphi d\sigma, \quad (3.1)$$

which implies $\lambda > 0$ because the value of the right side in (3.1) is positive. $\square$

Lemma 3.2. *Let (1.2) and (1.3) hold. Then $\Lambda \neq \emptyset$.*

Proof. By Propositions 2.6, 2.8 and 2.4.(3), the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{p(x)-2}u = 0 & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \quad (3.2)$$

has a unique positive solution $w_1 \in C^1(\overline{\Omega})$ and $w_1(x) \geq \varepsilon > 0$ for $x \in \overline{\Omega}$. We can assume $\varepsilon \leq 1$. Put $d = \sup\{f(x, w_1(x)) : x \in \overline{\Omega}\}$, $M_1 = \frac{d}{\varepsilon^{p_+-1}}$ and $\lambda_1 = 1 + M_1$. Then

$$\begin{aligned} -\Delta_{p(x)}w_1 + \lambda_1 w_1^{p(x)-1} &= -\Delta_{p(x)}w_1 + w_1^{p(x)-1} + M_1 w_1^{p(x)-1} \\ &\geq M_1 \varepsilon^{p_+-1} = d \geq f(x, w_1(x)). \end{aligned}$$

This shows that w_1 is a supersolution of the problem (1.1_{λ_1}) . Obviously 0 is a subsolution of (1.1_{λ_1}) . By Theorem 2.1, (1.1_{λ_1}) has a solution u_{λ_1} such that $0 \leq u_{\lambda_1} \leq w_1$. By Proposition 2.8, $u_{\lambda_1} > 0$ on $\overline{\Omega}$. So $\lambda_1 \in \Lambda$ and $\Lambda \neq \emptyset$. $\square$

Lemma 3.3. *Let (1.2) and (1.3) hold. If $\lambda_0 \in \Lambda$, then $\lambda \in \Lambda$ for all $\lambda > \lambda_0$.*

Proof. Let $\lambda_0 \in \Lambda$ and $\lambda > \lambda_0$. Let u_{λ_0} be a positive solution of (1.1_{λ_0}) . Then u_{λ_0} is a supersolution of (1.1_λ) . We know that 0 is a subsolution of (1.1_λ) . By Theorem 2.1, (1.1_λ) has a solution u_λ such that $0 \leq u_\lambda \leq u_{\lambda_0}$. By Proposition 2.8, $u_\lambda > 0$ on $\overline{\Omega}$. Thus $\lambda \in \Lambda$. $\square$

Lemma 3.4. *Let (1.2) and (1.3) hold. Then for every $\lambda > \lambda_*$, there exists a minimal positive solution u_λ of (1.1_λ) such that $u_{\lambda_1} \leq u_{\lambda_2}$ if $\lambda_* < \lambda_2 < \lambda_1$.*

Proof. Let $\lambda > \lambda_*$. Noting that 0 is a subsolution of (1.1_λ) and 0 is not a solution of (1.1_λ) , we can see that there exists a minimal positive solution u_λ of (1.1_λ) . Let $\lambda_* < \lambda_2 < \lambda_1$ and let u_{λ_1} and u_{λ_2} are the minimal positive solutions of (1.1_{λ_1}) and (1.1_{λ_2}) respectively. Since u_{λ_2} is a supersolution of (1.1_{λ_1}) , we can see that $u_{\lambda_1} \leq u_{\lambda_2}$. $\square$

The proof of Theorem 1.1 is complete. Let us now turn to the proof of Theorem 1.2.

Lemma 3.5. *Let (1.2) and (1.3) hold. Let $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_2 < \lambda < \lambda_1$. Suppose that u_{λ_1} and u_{λ_2} are the positive solutions of (1.1_{λ_1}) and (1.1_{λ_2}) respectively and $u_{\lambda_1} \leq u_{\lambda_2}$. Then there exists a positive solution v_λ of (1.1_λ) such that $u_{\lambda_1} \leq v_\lambda \leq u_{\lambda_2}$ and v_λ is a global minimizer of the restriction of J_λ to the order interval $[u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(x)}(\Omega)$*

Proof. Define $\tilde{f} : \overline{\Omega} \times \mathbf{R} \rightarrow \mathbf{R}$ by

$$\tilde{f}(x, t) = \begin{cases} f(x, u_{\lambda_1}(x)), & \text{if } t < u_{\lambda_1}(x) \\ f(x, t), & \text{if } u_{\lambda_1}(x) \leq t \leq u_{\lambda_2}(x) \\ f(x, u_{\lambda_2}(x)), & \text{if } t > u_{\lambda_2}(x). \end{cases}$$

Define $\tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) ds$ and for all $u \in W^{1,p(x)}(\Omega)$,

$$\tilde{J}_\lambda(u) = \int_\Omega \frac{|\nabla u|^{p(x)} + \lambda |u|^{p(x)}}{p(x)} dx - \int_{\partial\Omega} \varphi u d\sigma - \int_\Omega \tilde{F}(x, u) dx.$$

It is easy to see that the global minimum of $\tilde{J}$ on $W^{1,p(x)}(\Omega)$ is achieved at some $v_\lambda \in W^{1,p(x)}(\Omega)$. Thus v_λ is a solution of the following problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2} \nabla u) + \lambda |u|^{p(x)-2} u = \tilde{f}(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \quad (3.3)$$

and $v_\lambda \in C^1(\overline{\Omega})$. Noting that

$$f(x, u_{\lambda_1}) = \tilde{f}(x, u_{\lambda_1}) \leq \tilde{f}(x, v_\lambda) \leq \tilde{f}(x, u_{\lambda_2}) = f(x, u_{\lambda_2})$$

and $\lambda_2 < \lambda < \lambda_1$, by Proposition 2.7.(4), we obtain that $u_{\lambda_1} \leq v_\lambda \leq u_{\lambda_2}$. So $\tilde{f}(x, v_\lambda) = f(x, v_\lambda)$, and v_λ is a positive solution of (1.1 $_{\lambda}$). It is easy to see that there exists a constant c such that $J_\lambda(u) = \tilde{J}_\lambda(u) + c$ for $u \in [u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(x)}(\Omega)$. Hence v_λ is a global minimizer of $J_\lambda|_{[u_{\lambda_1}, u_{\lambda_2}] \cap W^{1,p(x)}(\Omega)}$. $\square$

A key lemma of this paper is the following strong comparison principle.

Lemma 3.6 (A strong comparison principle). *Let (1.2) and (1.3) hold. Let $\lambda_1, \lambda_2 \in \Lambda$ and $\lambda_2 < \lambda_1$. Suppose that u_{λ_1} and u_{λ_2} are the positive solutions of (1.1 $_{\lambda_1}$) and (1.1 $_{\lambda_2}$) respectively and $u_{\lambda_1} \leq u_{\lambda_2}$. Then $u_{\lambda_1} < u_{\lambda_2}$ on $\overline{\Omega}$.*

Proof. Since $u_{\lambda_1}, u_{\lambda_2} \in C^1(\overline{\Omega})$ and $u_{\lambda_1} > 0$ on $\overline{\Omega}$, there exist the positive constants $b_1 \leq 1$ and b_2 such that

$$b_1 \leq u_{\lambda_1} \leq u_{\lambda_2} \leq b_2 \quad \text{on } \overline{\Omega}.$$

For $\varepsilon \in (0, \frac{b_1}{2})$, setting $v_\varepsilon = u_{\lambda_2} - \varepsilon$, then

$$\begin{aligned} -\operatorname{div}(|\nabla v_\varepsilon|^{p(x)-2} \nabla v_\varepsilon) + \lambda_1 v_\varepsilon^{p(x)-1} &= -\operatorname{div}(|\nabla u_{\lambda_2}|^{p(x)-2} \nabla u_{\lambda_2}) + \lambda_2 v_\varepsilon^{p(x)-1} \\ &\quad + (\lambda_1 - \lambda_2) v_\varepsilon^{p(x)-1} \\ &\geq -\operatorname{div}(|\nabla u_{\lambda_2}|^{p(x)-2} \nabla u_{\lambda_2}) + \lambda_2 u_{\lambda_2}^{p(x)-1} \\ &\quad - \lambda_2 (u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}) \\ &\quad + (\lambda_1 - \lambda_2) \left(\frac{b_1}{2}\right)^{p_+-1} \\ &= f(x, u_{\lambda_2}) - \lambda_2 (u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}) \\ &\quad + (\lambda_1 - \lambda_2) \left(\frac{b_1}{2}\right)^{p_+-1}. \end{aligned}$$

Taking an $\varepsilon > 0$ sufficiently small such that

$$\lambda_2 (u_{\lambda_2}^{p(x)-1} - v_\varepsilon^{p(x)-1}) < (\lambda_1 - \lambda_2) \left(\frac{b_1}{2}\right)^{p_+-1} \quad \text{for } x \in \overline{\Omega},$$

then

$$-\operatorname{div}(|\nabla v_\varepsilon|^{p(x)-2}\nabla v_\varepsilon) + \lambda_1 v_\varepsilon^{p(x)-1} = g(x) \geq f(x, u_{\lambda_2}),$$

consequently, v_ε is a solution of the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda_1 |u|^{p(x)-2}u = g(x) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases}$$

where $g(x) \geq f(x, u_{\lambda_2})$. With other words, $v_\varepsilon = K_{\lambda_1}(g)$, where K_{λ_1} is the solution operator of (2.7_{λ_1}) . Since $u_{\lambda_1} = K_{\lambda_1}(h)$, where $h(x) = f(x, u_{\lambda_1}) \leq f(x, u_{\lambda_2}) \leq g(x)$, noting that K_{λ_1} is increasing, we have $v_\varepsilon \geq u_{\lambda_1}$, that is, $u_{\lambda_2} - \varepsilon \geq u_{\lambda_1}$ on $\overline{\Omega}$. $\square$

Proof of Theorem 1.2. Let (1.2), (1.3) and (1.4) hold. Let $\lambda > \lambda_*$. Take $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 < \lambda < \lambda_1$ and let $u_{\lambda_1} \leq v_\lambda \leq u_{\lambda_2}$ be as in Lemma 3.5. By Lemma 3.6, $u_{\lambda_1} < v_\lambda < u_{\lambda_2}$ on $\overline{\Omega}$. Thus there is a C^0 -neighborhood U of v_λ such that $U \subset [u_{\lambda_1}, u_{\lambda_2}]$, consequently v_λ is a local minimizer of J_λ in the C^0 -topology, and of course, also in the C^1 -topology. By Theorem 2.2, v_λ is also a local minimizer of J_λ in the $W^{1,p(x)}(\Omega)$ -topology. Moreover, it is easy to see that, for $\lambda_* < \lambda_2 < \lambda_1$, there are u_{λ_1} and u_{λ_2} , being respectively a local minimizer of J_{λ_1} and J_{λ_2} , such that $0 < u_{\lambda_1} \leq u_{\lambda_2}$. Then, by Lemma 3.6, $u_{\lambda_1} < u_{\lambda_2}$ on $\overline{\Omega}$. $\square$

Remark 3.1. In [1] the Neumann problems (1.1_λ) with $p(x) \equiv p$ have been considered. In the proof of Proposition 3.1 of [1], the authors of [1] think the assertion that v_λ is a local minimizer of J_λ in the C^1 -topology can be obtained by the Proposition 2.1 in Guedda–Veron [24] (see [1, p. 1454]), this is a carelessness because the strong comparison principle established by Guedda–Veron [24] is appropriate for the Dirichlet problems but is not appropriate for the Neumann problems (1.1_λ) . Our Lemma 3.6 ensured the validity of the assertion of [1] Proposition 3.1.

Remark 3.2. It is easy to see that our Lemma 3.6 is also valid for the homogeneous Neumann boundary value problems.

Proof of Theorem 1.3. Let (1.2)–(1.5) hold. Let $\lambda > \lambda_*$. Take $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_2 < \lambda < \lambda_1$ and let $u_{\lambda_1} \leq u_\lambda \leq u_{\lambda_2}$ be the above-mentioned positive solutions of (1.1_{λ_1}) , (1.1_λ) and (1.1_{λ_2}) respectively and u_λ a local minimizer of J_λ in the $W^{1,p(x)}(\Omega)$ -topology. Define

$$\widetilde{f}_\lambda(x, t) = \begin{cases} f(x, t), & \text{if } t > u_\lambda(x), \\ f(x, u_\lambda(x)), & \text{if } t \leq u_\lambda(x), \end{cases}$$

and $\widetilde{F}_\lambda(x, t) = \int_0^t \widetilde{f}_\lambda(x, s) ds$. Consider the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + \lambda |u|^{p(x)-2}u = \widetilde{f}_\lambda(x, u) & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} = \varphi & \text{on } \partial\Omega, \end{cases} \quad (3.4_\lambda)$$

and denote by $\widetilde{J}_\lambda$ the energy functional corresponding to (3.4_λ) . By the definition of $\widetilde{f}_\lambda$, we have $\widetilde{f}_\lambda(x, u(x)) \geq f(x, u_\lambda(x))$ for every $u \in W^{1,p(x)}(\Omega)$. Hence, for each

solution u of $(3.4)_\lambda$, we have that $u \geq u_\lambda$, consequently $\tilde{f}_\lambda(x, u) = f(x, u)$ and u is also a solution of $(1.1)_\lambda$. It is easy to see that u_{λ_1} and u_{λ_2} are a subsolution and a supersolution of $(3.4)_\lambda$ respectively. By Theorems 2.1 and 1.2, there exists $u_\lambda^* \in [u_{\lambda_1}, u_{\lambda_2}] \cap C^1(\bar{\Omega})$ such that u_λ^* is a solution of $(3.4)_\lambda$ and is a local minimizer of $\tilde{J}_\lambda$ in the C^1 -topology. As was noted above, we know that $u_\lambda^* \geq u_\lambda$ and u_λ^* is also a solution of $(1.1)_\lambda$. If $u_\lambda^* \neq u_\lambda$, then the assertion of Theorem 1.3 already holds, hence we can assume that $u_\lambda^* = u_\lambda$. Now u_λ is a local minimizer of $\tilde{J}_\lambda$ in the C^1 -topology, and so also in the $W^{1,p(x)}(\Omega)$ -topology. We can assume that u_λ is a strictly local minimizer of $\tilde{J}_\lambda$ in the $W^{1,p(x)}(\Omega)$ -topology, otherwise we have obtained the assertion of Theorem 1.3. It is easy to verify that, under the assumptions of Theorem 1.3, $\tilde{J}_\lambda \in C^1(W^{1,p(x)}(\Omega), \mathbf{R})$ and $\tilde{J}_\lambda$ satisfies the $(P.S)$ condition (see e.g. [19]). It follows from the condition (1.4) that $\inf\{\tilde{J}_\lambda(u) : u \in W^{1,p(x)}(\Omega)\} = -\infty$. Using the mountain pass lemma (see [7]), we know that $(3.4)_\lambda$ has a solution v_λ such that $v_\lambda \neq u_\lambda$. v_λ , as a solution of $(3.4)_\lambda$, must satisfy $v_\lambda \geq u_\lambda$, and v_λ is also a solution of $(1.1)_\lambda$. The proof of Theorem 1.3 is complete. $\square$

Remark 3.3. In the proof of Theorem 1.3 the mountain pass lemma due to Ambrosetti and Rabinowitz [7] is used. For the applications of the mountain pass lemma to the Dirichlet problems with variable exponent we refer to [16, 17, 19, 32]. In [19] and [16] the existence of a mountain pass type solution for the $p(x)$ -Laplacian equation and the $p(x)$ -Laplacian systems are obtained respectively. The common point between the main result of [32] and our Theorem 1.3 is that, using the mountain pass lemma for the energy functional J having a positive minimizer u , it is proved that J has the second positive critical point v which is of mountain pass type. The big difference between the main result of [32] and our Theorem 1.3 is that, in the former the energy functional is coercive and $v \leq u$ but in the latter the energy functional is not coercive and $v \geq u$. The underlying ideas of the present paper are similar to that developed in [17], only in the present paper the inhomogeneous Neumann problems are considered and hence a special strong comparison principle, Lemma 3.6, is needed. As was mentioned in the introduction, Theorem 1.3 is a generalization of the corresponding result of [1] to the variable exponent case.

Proof of Theorem 1.4.(1). Let f satisfy (1.2), (1.6) and (1.7). For given any $\lambda > 0$, consider the energy functional J_λ defined by (2.9). By (1.7) and noting that $r(x) < p(x)$ for $x \in \bar{\Omega}$, there is a positive constant M_1 such that

$$|F(x, t)| \leq \frac{\lambda}{2p^+} |t|^{p(x)}, \quad \forall x \in \Omega, \quad \forall |t| \geq M_1. \quad (3.5)$$

For $u \in W^{1,p(x)}(\Omega)$ with $\|u\|_\lambda \geq 1$, we have that

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega (|\nabla u|^{p(x)} + \lambda |u|^{p(x)}) dx - \int_{\partial\Omega} \varphi u d\sigma \\ &\quad - \frac{\lambda}{2p^+} \int_\Omega |u|^{p(x)} dx - c_1 \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{p^+} \int_{\Omega} |\nabla u|^{p(x)} dx + \frac{\lambda}{2p^+} \int_{\Omega} |u|^{p(x)} dx - c_2 \|u\|_{\lambda} - c_1 \\
&\geq \frac{1}{2p^+} \|u\|_{\lambda}^{p^-} - c_2 \|u\|_{\lambda} - c_1,
\end{aligned}$$

where c_1 and c_2 are positive constants. This shows that $J_{\lambda}(u) \rightarrow +\infty$ as $\|u\|_{\lambda} \rightarrow \infty$, that is, J_{λ} is coercive. The condition (1.7) also implies that J_{λ} is weakly sequentially lower semi-continuous. Thus J_{λ} has a global minimizer u_0 . Put $v_0(x) = |u_0(x)|$ for $x \in \bar{\Omega}$. It is easy to see that $J_{\lambda}(v_0) \leq J_{\lambda}(u_0)$, consequently, v_0 is a global minimizer of J_{λ} and is a positive solution of (1.1 $_{\lambda}$). This shows that $\lambda \in \Lambda$ for all $\lambda > 0$. Hence $\lambda_* = 0$ and the statement (1) is proved. $\square$

To prove Theorem 1.4.(2) we give the following two lemmata.

Lemma 3.7. *If f satisfies (1.2), (1.3) and there exists a positive constant $M_1 \geq 1$ such that*

$$f(x, t) \geq t^{p(x)-1} \quad \text{for } x \in \Omega \quad \text{and } t \geq M_1, \quad (3.6)$$

Then

$$\lambda_* \geq \min \left\{ \frac{b}{M_1^{p^+-1} |\Omega|}, 1 \right\}, \quad (3.7)$$

where $b = \int_{\partial\Omega} \varphi d\sigma$.

Proof. Let $\lambda \in \Lambda$ and u be a positive solution of (1.1 $_{\lambda}$). Then, as was noted earlier, (3.1) holds, that is

$$\lambda \int_{\Omega} u^{p(x)-1} dx - \int_{\Omega} f(x, u) dx = b. \quad (3.8)$$

By the assumption for φ , $b > 0$. To prove (3.7), arguing by contradiction, assume that there is $\lambda \in \Lambda$ such that $\lambda < 1$ and $\lambda < \frac{b}{M_1^{p^+-1} |\Omega|}$. Setting

$$\Omega_1 = \{x \in \Omega : u(x) < M_1\} \quad \text{and} \quad \Omega_2 = \{x \in \Omega : u(x) \geq M_1\},$$

then

$$\begin{aligned}
\lambda \int_{\Omega} u^{p(x)-1} dx - \int_{\Omega} f(x, u) dx &\leq \lambda \int_{\Omega_1} u^{p(x)-1} dx + \int_{\Omega_2} u^{p(x)-1} dx - \int_{\Omega_2} f(x, u) dx \\
&\leq \lambda \int_{\Omega_1} u^{p(x)-1} dx \leq \lambda M_1^{p^+-1} |\Omega| < b,
\end{aligned}$$

which contradicts (3.8). $\square$

Remark 3.4. Obviously, (1.5) implies (3.6).

Lemma 3.8. *Let (1.2) and (1.3) hold. Then for each $\lambda > \lambda_*$, (1.1 $_{\lambda}$) has a positive solution u_{λ} such that $J_{\lambda}(u_{\lambda}) \leq 0$.*

Proof. Let $\lambda > \lambda_*$. Take $\lambda_2 \in (\lambda_*, \lambda)$ and let u_{λ_2} be a positive solution of $(1.1)_{\lambda_2}$. then u_{λ_2} is a supersolution of $(1.1)_\lambda$. We know that 0 is a subsolution of $(1.1)_\lambda$. Analogously to the proof of Lemma 3.5, we can prove that $(1.1)_\lambda$ has a positive solution $u_\lambda \in [0, u_{\lambda_2}]$ such that $J_\lambda(u_\lambda) = \inf\{J_\lambda(u) : u \in [0, u_{\lambda_2}]\}$. So $J_\lambda(u_\lambda) \leq J_\lambda(0) = 0$. $\square$

Remark 3.5. In fact, in Lemma 3.8 we have the assertion that $J_\lambda(u_\lambda) < 0$. To see this, putting $v_t \equiv t$ on $\overline{\Omega}$, where t is a positive constant, then for sufficiently small $t > 0$, we have that $v_t \in [0, u_{\lambda_2}]$ and

$$\begin{aligned} J_\lambda(v_t) &= \int_{\Omega} \frac{\lambda t^{p(x)}}{p(x)} dx - \int_{\partial\Omega} \varphi t d\sigma - \int_{\Omega} F(x, t) dx \\ &\leq t^{p_-} \int_{\Omega} \frac{\lambda}{p(x)} dx - t \int_{\partial\Omega} \varphi d\sigma < 0, \end{aligned}$$

thus $J_\lambda(u_\lambda) \leq J_\lambda(v_t) < 0$.

Proof of Theorem 1.4.(2). Let (1.2)–(1.5) hold. By Lemma 3.7 and Remark 3.4, $\lambda_* > 0$. Let $\lambda_n > \lambda_*$ and $\lambda_n \rightarrow \lambda_*$ as $n \rightarrow \infty$. By Lemma 3.8, for each n , $(1.1)_{\lambda_n}$ has a positive solution u_{λ_n} such that $J_{\lambda_n}(u_{\lambda_n}) \leq 0$, that is

$$\int_{\Omega} \frac{|\nabla u_{\lambda_n}|^{p(x)} + \lambda_n |u_{\lambda_n}|^{p(x)}}{p(x)} dx \leq \int_{\partial\Omega} \varphi u_{\lambda_n} d\sigma + \int_{\Omega} F(x, u_{\lambda_n}) dx.$$

Since u_{λ_n} is a solution of $(1.1)_{\lambda_n}$, we have that

$$\int_{\Omega} (|\nabla u_{\lambda_n}|^{p(x)} + \lambda_n |u_{\lambda_n}|^{p(x)}) dx - \int_{\partial\Omega} \varphi u_{\lambda_n} d\sigma = \int_{\Omega} f(x, u_{\lambda_n}) u_{\lambda_n} dx.$$

It follows from (1.5) that there exists a positive constant c_1 such that

$$\int_{\Omega} F(x, u_{\lambda_n}) dx \leq c_1 + \frac{1}{\theta} \int_{\Omega} f(x, u_{\lambda_n}) u_{\lambda_n} dx.$$

Thus we have that

$$\begin{aligned} \frac{1}{p_+} \int_{\Omega} (|\nabla u_{\lambda_n}|^{p(x)} + \lambda_n |u_{\lambda_n}|^{p(x)}) dx &\leq \int_{\Omega} \frac{|\nabla u_{\lambda_n}|^{p(x)} + \lambda_n |u_{\lambda_n}|^{p(x)}}{p(x)} dx \\ &\leq \int_{\partial\Omega} \varphi u_{\lambda_n} d\sigma + c_1 + \frac{1}{\theta} \int_{\Omega} f(x, u_{\lambda_n}) u_{\lambda_n} dx \\ &\leq \int_{\partial\Omega} \varphi u_{\lambda_n} d\sigma + c_1 + \frac{1}{\theta} \int_{\Omega} (|\nabla u_{\lambda_n}|^{p(x)} \\ &\quad + \lambda_n |u_{\lambda_n}|^{p(x)}) dx, \end{aligned}$$

and consequently,

$$\begin{aligned} \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \|u_{\lambda_n}\|_{\lambda_n}^{p_-} &\leq \left(\frac{1}{p_+} - \frac{1}{\theta} \right) \int_{\Omega} (|\nabla u_{\lambda_n}|^{p(x)} + \lambda_n |u_{\lambda_n}|^{p(x)}) dx + c_2 \\ &\leq c_3 \|u_{\lambda_n}\|_{\lambda_n} + c_2 + c_1, \end{aligned}$$

where the positive constants c_1 , c_2 and c_3 are independent of n . This shows that $\{\|u_{\lambda_n}\|_{\lambda_n}\}$ is bounded. Noting that $\lambda_n \rightarrow \lambda_* > 0$, we have that $\{\|u_{\lambda_n}\|\}$ is

bounded. Without loss of generality we can assume that $u_{\lambda_n} \rightharpoonup u_*$ in $W^{1,p(x)}(\Omega)$ and $u_{\lambda_n}(x) \rightarrow u_*(x)$ for a.e. $x \in \Omega$. By (1.4) and the $L^\infty(\Omega)$ -regularity results of [20], the boundedness of $\{\|u_{\lambda_n}\|\}$ implies the boundedness of $\{|u_{\lambda_n}|_{L^\infty(\Omega)}\}$. By the $C^{1,\alpha}(\bar{\Omega})$ -regularity results of [18], the boundedness of $\{|u_{\lambda_n}|_{L^\infty(\Omega)}\}$ implies the boundedness of $\{\|u_{\lambda_n}\|_{C^{1,\alpha}(\bar{\Omega})}\}$, where $\alpha \in (0, 1)$ is a constant. Thus we have $u_{\lambda_n} \rightarrow u_*$ in $C^1(\bar{\Omega})$. For every $v \in W^{1,p(x)}(\Omega)$, since u_{λ_n} is a solution of (1.1) $_{\lambda_n}$, we have that, for each n ,

$$\int_{\Omega} (|\nabla u_{\lambda_n}|^{p(x)-2} \nabla u_{\lambda_n} \nabla v + \lambda_n |u_{\lambda_n}|^{p(x)-2} u_{\lambda_n} v) dx - \int_{\partial\Omega} \varphi v d\sigma = \int_{\Omega} f(x, u_{\lambda_n}) v dx.$$

Passing the limit of above equality as $n \rightarrow \infty$, yields

$$\int_{\Omega} (|\nabla u_*|^{p(x)-2} \nabla u_* \nabla v + \lambda_* |u_*|^{p(x)-2} u_* v) dx - \int_{\partial\Omega} \varphi v d\sigma = \int_{\Omega} f(x, u_*) v dx,$$

which shows that u_* is a solution of (1.1) $_{\lambda_*}$. Obviously $u_* \geq 0$ and $u_* \not\equiv 0$. Hence u_* is a positive solution of (1.1) $_{\lambda_*}$ and $\lambda_* \in \Lambda$. $\square$

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