Limiting behavior of solutions of subelliptic heat equations

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Abstract. We investigate the behavior, as $\varepsilon \rightarrow 0^+$, of $\varepsilon \log w^\varepsilon(t, x)$ where w^ε are solutions of a suitable family of subelliptic heat equations. Using the Large Deviation Principle, we show that the limiting behavior is described by the metric inf-convolution w.r.t. the associated Carnot-Carathéodory distance.

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1 Introduction

It is well-known that the limiting behavior, as $\varepsilon \rightarrow 0^+$, of the solutions of

$$\begin{cases} w_t^\varepsilon - \varepsilon \Delta w^\varepsilon = 0, & x \in \mathbb{R}^n, t > 0, \\ w^\varepsilon(0, x) = e^{-\frac{g(x)}{2\varepsilon}}, & x \in \mathbb{R}^n. \end{cases} \quad (1)$$

is described by the Hamilton-Jacobi-Cauchy problem

$$\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (2)$$

see, for example, [20] and [4] or [2]. More precisely, if $g : \mathbb{R}^n \rightarrow \mathbb{R}$ is a bounded and continuous function, the logarithmic transform of w^ε , i.e. $u^\varepsilon = -2\varepsilon \log w^\varepsilon$, converges, locally uniformly, as $\varepsilon \rightarrow 0^+$, to the unique viscosity solution u of (2). One way of proving this is to use both the representation of w^ε as the integral

convolution and the Hopf-Lax representation of the viscosity solution of (2) as the (euclidean) inf-convolution

$$g_t(x) = \inf_{y \in \mathbb{R}^n} \left[g(y) + \frac{|x - y|^2}{2t} \right],$$

and to apply the Large Deviation Principle (see [18]) in order to establish the validity of the limiting relation

$$\lim_{\varepsilon \rightarrow 0^+} -2\varepsilon \log w^\varepsilon = u.$$

The aim of this paper is to generalize the procedure described above in order to analyze the limiting behavior of some subelliptic diffusion equations in term of the Carnot-Carathéodory inf-convolutions. Let w^ε the solutions of

$$\begin{cases} w_t^\varepsilon - \varepsilon \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 w^\varepsilon}{\partial x_i \partial x_j} = 0, & x \in \mathbb{R}^n, t > 0, \\ w^\varepsilon(0, x) = e^{-\frac{g(x)}{2\varepsilon}}, & x \in \mathbb{R}^n, \end{cases} \quad (3)$$

where the matrix $A(x) = (a_{i,j}(x))_{i,j}$, for $i, j = 1, \dots, n$, is of the form

$$A(x) = \sigma^t(x) \sigma(x),$$

with $\sigma(x)$ $m \times n$ -matrix ($m \leq n$), satisfying the Hörmander condition.

Under this condition, a finite Carnot-Carathéodory (C-C) distance can be associated by control theory to the matrix σ (see [3]).

The inf-convolution of the initial datum g , with metric d as kernel, namely

$$g_t(x) = \inf_{y \in \mathbb{R}^n} \left[g(y) + \frac{d(x, y)^2}{2t} \right], \quad (4)$$

produces the viscosity solution of

$$\begin{cases} u_t + \frac{1}{2} |\sigma(x) Du|^2 = 0, & x \in \mathbb{R}^n, t > 0, \\ u(0, x) = g(x), & x \in \mathbb{R}^n. \end{cases} \quad (5)$$

Our main result is the following.

Theorem 1.1 *Let $g \in C(\mathbb{R}^n)$ bounded, d C-C distance associated to an Hörmander's matrix $\sigma(x)$ and g_t the inf-convolution defined by (4). If w^ε are the solutions of (3), then*

$$\lim_{\varepsilon \rightarrow 0^+} -2\varepsilon \log w^\varepsilon(t, x) = g_t(x), \quad (6)$$

locally uniformly on $[0, +\infty) \times \mathbb{R}^n$.

Observe that in the special case $A(x) = I$, then $d(x, y)$ reduces to the standard euclidean distance, thus recovering the well-known classical result, see Capuzzo Dolcetta [4] for a recent presentation.

2 Preliminary results about Carnot- Carathéodory inf-convolutions and the Large Deviation Principle

Let d a distance on $\mathbb{R}^n$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$, then a *metric inf-convolution* can be defined, for any $t > 0$, as (4). We look, in particular, at the C-C distances satisfying the Hörmander condition. We recall some notions about these, one can find more information in [3].

Fix $x \in \mathbb{R}^n$ and let $\sigma(x)$ a $m \times n$ -matrix with C^∞ -coefficients and $m \leq n$. Setting $X_1(x), \dots, X_m(x)$ the vector fields corresponding to the lines of $\sigma(x)$ (i.e. $\sigma(x)^t = [X_1(x), \dots, X_m(x)]$), we consider the control system

$$\dot{\gamma}(t) = \sum_{i=1}^m \alpha_i(t) X_i(\gamma(t)), \quad (7)$$

with $\alpha_1, \dots, \alpha_m$ measurable real control functions.

We say that an absolutely continuous curve $\gamma : [0, T] \rightarrow \mathbb{R}^n$ is *admissible* or also *σ -horizontal* if there exists $\alpha : [0, T] \rightarrow \mathbb{R}^m$ measurable function such that

$$\dot{\gamma}(t) = \sigma^t(\gamma(t))\alpha(t), \quad \text{a.e. } t \in [0, T].$$

For any admissible curve γ and any admissible coordinate-vector $\alpha(t)$, we define the *length* as

$$l(\gamma) = \int_0^T \|\alpha(t)\| dt, \quad (8)$$

where $\|\cdot\|$ is the standard euclidean norm in $\mathbb{R}^m$.

Remark 2.1 To get the uniqueness of the admissible coordinate-vector $\alpha(t)$, one can assume that the family of vector fields $X_1, \dots, X_m$ satisfies the following weak-linear-independent condition: for all point x , there exist $1 \leq k \leq m$ and $1 \leq j_1 < \dots < j_k \leq m$ such that

$$\text{rank}\{X_{j_1}(x), \dots, X_{j_k}(x)\} = k, \text{ and } X_j(x) = 0, \quad \forall j \notin \{j_1, \dots, j_k\}.$$

If this condition doesn't hold, the length of an admissible curve can be defined as the infimum of the integrals (8) over all the admissible coordinates $\alpha(t)$.

Nevertheless a such condition is very general. In fact it holds for any generic distribution (so in particular for the Carnot groups) and for any Grušin-type space.

Definition 2.1 The C-C distance associated to $\sigma(x)$ is

$$d(x, y) = \inf\{l(\gamma) \mid \gamma \text{ } \sigma\text{-horizontal curve joining } x \text{ to } y\}. \quad (9)$$

In general, the function (9) is a distance on whole $\mathbb{R}^n$ but it is not always a finite distance. In order to overcome this problem, it is introduced the Hörmander condition. We recall that a bracket between two vector fields X and Y acts, by derivation, on all the smooth real functions f , as

$$[X, Y]f = X(Yf) - Y(Xf).$$

Let $\mathcal{L}^0 = \{X_1, \dots, X_m\}$, $\mathcal{L}^1 = \{[X_i, X_j] \mid i, j = 1, \dots, m\}$ and

$$\mathcal{L}^k = \{[Y_i, Y_j] \mid Y_i \in \mathcal{L}^h, Y_j \in \mathcal{L}^l, h, l = 0, \dots, k-1\} \setminus \bigcup_{i=0}^{k-1} \mathcal{L}^i,$$

then the *Lie algebra* associated to the distribution spanned by $X_1, \dots, X_m$ is the set $\mathcal{L} = \bigcup_{k \in \mathbb{N}} \mathcal{L}^k$. We say that a matrix $\sigma(x)$ satisfies the *Hörmander condition*, if and only if, the associated Lie algebra spans whole of the tangent space, that in our case is $\mathbb{R}^n$, in any point.

If a matrix $\sigma(x)$ satisfies the Hörmander condition, the Chow's Theorem implies that the associated C-C distance (9) is finite for any pair of points and it induces on $\mathbb{R}^n$ the euclidean topology (see [3]).

Moreover, we say that a matrix satisfies the Hörmander condition with step $k \geq 1$, if and only if,

$$\bigcup_{i=1}^k \mathcal{L}^i(x) = \mathbb{R}^n,$$

in any point $x \in \mathbb{R}^n$. The Riemannian case is when $k = 1$.

Now we give some examples of Hörmander's matrixes.

Example 2.1 The matrix

$$\sigma(x) = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

satisfies the Hörmander condition with step 2 and it induces a sub-Riemannian geometry on $\mathbb{R}^2$, known as *Grushin plane*.

Example 2.2 Let

$$\sigma(x) = \begin{pmatrix} 1 & 0 & -2y \\ 0 & 1 & 2x \end{pmatrix},$$

that is an Hörmander's matrix (with step 2) and it is associated to the *1-dimensional Heisenberg group*.

Example 2.3 The following matrix satisfies the Hörmander condition with step 3 and the distribution associated to its lines is known as *Martinet distribution*:

$$\sigma(x) = \begin{pmatrix} 1 & 0 & -y^2 \\ 0 & 1 & 0 \end{pmatrix}.$$

For any C-C distance, with constant step $k \geq 1$, holds a locally euclidean estimate ([16]). In fact, for any $K \in \mathbb{R}^n$ compact, there exists a constant $C = C(K) > 0$ such that

$$\frac{1}{C}|x - y| \leq d(x, y) \leq C|x - y|^{\frac{1}{k}}, \quad \forall x, y \in K. \quad (10)$$

The metric inf-convolution (4) is a particular case of the Hopf-Lax function

$$u(t, x) = \inf_{y \in \mathbb{R}^n} \left[g(y) + t\Phi^* \left(\frac{d(x, y)}{t} \right) \right],$$

introduced by Manfredi-Stroffolini for the case of the Heisenberg group in [15] and generalized in [4, 5], see also [9]. We recall some properties proved in [9], rewritten directly for the inf-convolutions.

Theorem 2.1 *Let $g \in LSC(\mathbb{R}^n)$ (lower semicontinuous) and bounded and d C-C distance, satisfying the Hörmander condition with step $k \geq 1$, then the metric inf-convolution g_t , defined in (4), is such that*

- (i) $g_t \leq g$, for any $t > 0$,
- (ii) the infimum in (4) is attained in the closed Carnot-Carathéodory ball, centered in x and with radius $R(t) = 2t^{\frac{1}{2}} \|g\|_{\infty}^{\frac{1}{2}}$,
- (iii) g_t is locally d -Lipschitz in x for $t > 0$ and so, by estimate (10), is locally Hölder continuous with exponent $1/k$. Moreover, g_t is locally Lipschitz continuous in $t > 0$, for any $x \in \mathbb{R}^n$,
- (iv) g_t monotonously converges (in the lower weak Barles-Perthame's sense, [2]) to g , as $t \rightarrow 0^+$,
- (v) if $g(x) \leq -C(1 + d(0, x))$, for some constant $C > 0$, then

$$g_t(x) \leq -C'(1 + t + d(0, x)),$$

for any $x \in \mathbb{R}^n$ and $t > 0$, with $C' = \max\{C, \frac{1}{2}C^2\}$.

So the Carnot-Carathéodory inf-convolutions (4) are a monotonous Lipschitz approximation of the original function as in the euclidean case (see [1]). Moreover in [9] (Theorem 4.1) we have proved that $u(t, x) = g_t(x)$ solves (in the viscosity sense) the Cauchy problem (5). At last, when g is continuous, the solution of (5) is also unique (see [6]).

We recall that, if $\sigma(x)$ is an Hörmander's matrix, then the differential operator $L := \sum_{i,j} a_{i,j}(x) \partial_{x_i} \partial_{x_j}$ is hypoelliptic.

By theory of subelliptic heat equations (see [10, 12]), we know that there exists an heat kernel associated to L , which we indicate by $p(t, x, y)$, smooth, for $t > 0$,

in whole $\mathbb{R}^n \times \mathbb{R}^n$ and, moreover, there exists $M \in [1, +\infty)$ such that, for any $0 < t \leq 1$ and $x, y \in \mathbb{R}^n$, it holds

$$\frac{1}{M\mu(B^d(x, \sqrt{t}))} e^{-M\frac{d(x,y)^2}{t}} \leq p(t, x, y) \leq \frac{M}{\mu(B^d(x, \sqrt{t}))} e^{-\frac{d(x,y)^2}{Mt}}. \quad (11)$$

see [11]. At last, let $p^\varepsilon(t, x, y)$ the heat kernel associated to εL , the solution of (3) is given by

$$w^\varepsilon(t, x) = \int_{\mathbb{R}^n} e^{-\frac{g(x)}{2\varepsilon}}(y) p^\varepsilon(t, x, y) dy. \quad (12)$$

So, in order to prove Theorem 1.1, we need to investigate the limiting behavior of

$$u^\varepsilon(t, x) = -2\varepsilon \log \left(\int_{\mathbb{R}^n} e^{-\frac{g(y)}{2\varepsilon}} p^\varepsilon(t, x, y) dy \right). \quad (13)$$

As in [4], we want to apply a Laplace-Varadhan's type theorem, that is an application of the Large Deviation Principle.

Now we recall both of these results, for some information about the Large Deviation theory, one can see [18] or also [7, 8].

Definition 2.2 (Large Deviation Principle). Let P_ε a family of probability measures, defined on the borel sets of a complete and separable metric space X . A family P_ε satisfies the *Large Deviation Principle* (LDP) if there exists a function (called *rate function*) $I : X \rightarrow [0, +\infty]$ such that

- (i) $I \in LSC(X)$,
- (ii) for any $k < +\infty$, the sublevel set $\{y \in X \mid I(y) \leq k\}$ is compact,
- (iii) for any $A \subset X$ open set,

$$\liminf_{\varepsilon \rightarrow 0^+} \varepsilon \log P_\varepsilon(A) \geq - \inf_{y \in A} I(y),$$

- (iv) for any $C \subset X$ closed set,

$$\limsup_{\varepsilon \rightarrow 0^+} \varepsilon \log P_\varepsilon(C) \leq - \inf_{y \in C} I(y).$$

Theorem 2.2 Let X a complete and separable metric space and P_ε a family of probability measures satisfying the LDP with rate function I , then, for any $F \in C(X)$ bounded,

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \log \left(\int_X \exp \left[\frac{F(y)}{\varepsilon} \right] dP_\varepsilon(y) \right) = \sup_{y \in X} [F(y) - I(y)]. \quad (14)$$

Let $X = \mathbb{R}^n$ and fixed $x \in \mathbb{R}^n$ and $t > 0$. We can define, for any $B \subset \mathbb{R}^n$ borel set, the following probability measures

$$P_\varepsilon^{t,x}(B) = \int_B p^\varepsilon(t, x, y) dy. \quad (15)$$

If we show that previous family of probability measures (15) satisfies the LDP with rate function

$$I^{t,x}(y) = \frac{d(x, y)^2}{4t}, \quad (16)$$

by Theorem 2.2 with $F = -g/2$, it is immediate to get (6). The difficulty is to verify properties (iii) and (iv). If p^ε is the heat kernel associated to some uniformly elliptic operators there is a non-trivial proof of this fact in [20] (note that, in a such case, d is a Riemannian distance). Nevertheless, in the euclidean case, it is enough easy to get properties (iii) and (iv) (as unique limit) in the borel and bounded sets. In fact, setting $q = 1/\varepsilon$, by the convergence of the L^q -norm to the L^∞ -norm, as $q \rightarrow +\infty$, one can deduce directly that

$$\lim_{\varepsilon \rightarrow 0^+} \varepsilon \log \left((4\pi\varepsilon t)^{-\frac{n}{2}} \int_B e^{-\frac{|x-y|^2}{4\varepsilon t}} dy \right) = \log \left(\lim_{q \rightarrow +\infty} \left\| e^{-\frac{|x-\cdot|^2}{4t}} \right\|_{q,B} \right) = - \inf_{y \in B} \frac{|x-y|^2}{4t}.$$

This remark has given us the idea for an analytic proof which covers also the Carnot-Carathéodory case.

3 Proof of the main result

To apply LDP in order to get Theorem 1.1, we need to generalize to the hypoelliptic case the asymptotic estimates, proved in [20] for uniform elliptic operators. Next lemma is a key-point.

Lemma 3.1 *Let $p(t, x, y)$ the heat kernel associated to A , then*

$$p^\varepsilon(t, x, y) = p(\varepsilon t, x, y).$$

Proof. The result follows immediately from the uniqueness for the Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t} p^\varepsilon(t, x, y) - \varepsilon \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} p^\varepsilon(t, x, y) = 0, & x, y \in \mathbb{R}^n, t > 0, \\ p^\varepsilon(0, x, y) = \delta_x(y), & x, y \in \mathbb{R}^n. \end{cases} \quad (17)$$

In fact, since the coefficients of the equation don't depend on the time-variable, it is trivial that $p(\varepsilon t, x, y)$ satisfies the Cauchy problem (17). So, by the uniqueness (see [12]), we can conclude. $\square$

The second key-result is the following locally uniform limit, proved in [19] for uniformly elliptic operators and generalized to the hypoelliptic case in [14, 13].

Theorem 3.1 *Let $p(t, x, y)$ as in Lemma 3.1 and $d(x, y)$ the C-C distance defined in (9), then*

$$\lim_{\tau \rightarrow 0^+} 4\tau \log p(\tau, x, y) = -d(x, y)^2. \quad (18)$$

Moreover previous convergence is uniform in the bounded sets.

The idea is to use previous results in order to investigate the limiting behavior of $(P_\varepsilon^{t,x})^\varepsilon$ in the bounded sets and then deduce the behavior in the open and closed (unbounded) sets.

Proposition 3.1 *Let $p^\varepsilon(t, x, y)$ the heat kernel associated to the hypoelliptic operator $L^\varepsilon = \varepsilon \sum_{i,j} a_{i,j}(x) \frac{\partial^2}{\partial x_j \partial x_i}$, with $A(x) = (a_{i,j}(x))_{i,j=1}^n = \sigma^t(x) \sigma(x)$ and $\sigma(x)$ Hörmander's matrix. If $P_\varepsilon^{t,x}$ and $I^{t,x}$ are the family of probability measures and the rate function defined in (15) and (16), respectively, fix $t > 0$, $x \in \mathbb{R}^n$, then, for any $B \subset \mathbb{R}^n$ bounded set,*

$$\lim_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(B)]^\varepsilon = e^{-\inf_{y \in B} I^{t,x}(y)}. \quad (19)$$

Proof. We use the exponential form of the uniform limit (18), that is

$$\lim_{\tau \rightarrow 0^+} p(\tau, x, y)^\tau = e^{-\frac{d(x,y)^2}{4}}.$$

Let $q = \frac{1}{\tau}$ and $f_q(y) = p\left(\frac{1}{q}, x, y\right)^{\frac{1}{q}}$, we can write

$$\left(\int_B [p(\tau, x, y)^\tau]^{\frac{1}{\tau}} dy \right)^\tau = \left(\int_B \left[p\left(\frac{1}{q}, x, y\right)^{\frac{1}{q}} \right]^q dy \right)^{\frac{1}{q}} = \|f_q\|_{q,B},$$

where by $\|\cdot\|_{q,B}$ we indicate the usual L^q -norm in B , with $q \geq 1$. By Lemma 3.1 and setting $\tau = \varepsilon t$, we get

$$\lim_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(B)]^\varepsilon = \lim_{\tau \rightarrow 0^+} \left(\int_B p(\tau, x, y) dy \right)^{\frac{\tau}{t}} = \lim_{q \rightarrow +\infty} \|f_q\|_{q,B}^{\frac{1}{t}}. \quad (20)$$

Set $f(y) = e^{-\frac{d(x,y)^2}{4}}$, by the triangle inequality for the L^q -norm, we get

$$|\|f_q\|_{q,B} - \|f\|_{q,B}| \leq \|f_q - f\|_{q,B} \leq \|f_q - f\|_{\infty,B} [\mu(B)]^{\frac{1}{q}}.$$

As $q \rightarrow \infty$, the last member goes to zero and $\|f(y)\|_{q,B} \rightarrow \|f(y)\|_{\infty,B}$, since B is bounded. So we get

$$\lim_{q \rightarrow +\infty} \|f_q\|_{q,B}^{\frac{1}{q}} = \sup_{y \in B} |f(y)|^{\frac{1}{q}} = e^{-\inf_{y \in B} \frac{d(x,y)^2}{4t}}.$$

Hence, by (20), the convergence result (19) holds. $\square$

To get, by approximation, the corresponding estimate for the lower-limit in the open (unbounded) sets, is very easy.

Proposition 3.2 *Under assumptions of Proposition 3.1, then, for any open set $A \subset \mathbb{R}^n$,*

$$\liminf_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(A)]^\varepsilon \geq e^{-\inf_{y \in A} I^{t,x}(y)}. \quad (21)$$

Proof. Let $A_R := A \cap B_R(0)$. Since A_R is bounded, we can apply the limiting behavior proved in Proposition 3.1 and so

$$\liminf_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(A)]^\varepsilon \geq \liminf_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(A_R)]^\varepsilon = e^{-\inf_{y \in A_R} \frac{d(x,y)^2}{4t}}.$$

Taking the supremum for $R > 0$, we can immediately conclude that

$$\liminf_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(A)]^\varepsilon \geq \sup_{R > 0} e^{-\inf_{y \in A_R} \frac{d(x,y)^2}{4t}} \geq e^{-\inf_{y \in A} \frac{d(x,y)^2}{4t}}.$$

$\square$

To get the estimate for the upper-limit in the closed (unbounded) sets, is more complicate and, first, we need to investigate the limiting behavior outside large balls.

Lemma 3.2 *Let $\delta \in (0, 1)$, then there exists $R_\delta > 0$ such that*

$$\limsup_{\tau \rightarrow 0^+} \left(\int_{\mathbb{R}^n \setminus \overline{B_{R_\delta}^d}(x)} p(\tau, x, y) dy \right)^\tau < \delta.$$

Proof. Set $B_R^- = \mathbb{R}^n \setminus \overline{B_R^d}(x)$, by the Hörmander assumption, it is well-known (see [17]) that there exists $c > 0$ such that

$$B(x, c\tau^{\frac{k}{2}}) \subset B^d(x, \sqrt{\tau}),$$

where $k \geq 1$ is the step of the distribution associated to $X_1, \dots, X_m$, then $\mu(B^d(x, \sqrt{\tau}))^{-1} \leq (\omega_n c^n \tau^{\frac{nk}{2}})^{-1}$, with ω_n measure of the unit euclidean ball. By estimate (11) and setting $\lambda = M\omega_n^{-1}c^{-n} > 0$, we get

$$\limsup_{\tau \rightarrow 0^+} \left(\int_{B_R^-} p(\tau, x, y) dy \right)^\tau \leq \limsup_{\tau \rightarrow 0^+} \lambda^\tau \tau^{-\frac{nk}{2}\tau} \left(\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy \right)^\tau.$$

It is trivial that $\lim_{\tau \rightarrow 0^+} (\lambda \tau^{-\frac{nk}{2}})^\tau = 1$, so it remains to estimate

$$L_R = \limsup_{\tau \rightarrow 0^+} \left(\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy \right)^\tau.$$

Using the continuity of the logarithm function, we study $\log L_R$ and apply a version of the De l'Hôpital Theorem for the upper-limit. In fact, by the Cauchy Theorem, it is easy to show that

$$\limsup_{\tau \rightarrow 0^+} \frac{f(\tau)}{g(\tau)} \leq \limsup_{\tau \rightarrow 0^+} \frac{f'(\tau)}{g'(\tau)},$$

whenever f and g are continuous differentiable. Then

$$\log L_R = \limsup_{\tau \rightarrow 0^+} \frac{\log \left(\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy \right)}{\frac{1}{\tau}} \leq \limsup_{\tau \rightarrow 0^+} -\tau^2 \frac{\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} \frac{d^2(x,y)}{M\tau^2} dy}{\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy}$$

Since $y \in \mathbb{R}^n \setminus \overline{B_R^d}(x)$, then $d(x,y) \geq R$. Therefore we get

$$\log L_R \leq \limsup_{\tau \rightarrow 0^+} -\frac{R^2}{M} \frac{\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy}{\int_{B_R^-} e^{-\frac{d(x,y)^2}{M\tau}} dy} = -\frac{R^2}{M}.$$

We can conclude that, for any $R > 0$,

$$\limsup_{\tau \rightarrow 0^+} \left(\int_{B_R^-} p(\tau, x, y) dy \right)^\tau \leq e^{-\frac{R^2}{M}}.$$

Hence, for any $0 < \delta < 1$, we can choose $R_\delta > \sqrt{\frac{-\log \delta}{M}}$ so that $e^{-\frac{R_\delta^2}{M}} < \delta$ and this concludes the proof. $\square$

Proposition 3.3 *Under assumptions of Proposition 3.1, then, for any closed set $C \subset \mathbb{R}^n$,*

$$\limsup_{\varepsilon \rightarrow 0^+} [P_\varepsilon^{t,x}(C)]^\varepsilon \leq e^{-\inf_{y \in C} I^{t,x}(y)}. \quad (22)$$

Proof. As for the bounded sets, let $\tau = \varepsilon t$, instead of (22), we can show

$$\limsup_{\tau \rightarrow 0^+} \left(\int_C p(\tau, x, y) dy \right)^\tau \leq e^{-\inf_{y \in C} \frac{d(x,y)^2}{4}}. \quad (23)$$

Since $\tau \in (0, 1)$, for any $\delta \in (0, 1)$, we can decompose $C = C_\delta \cup C_\delta^-$, where $C_\delta = C \cap \overline{B}_{R_\delta}^d(x)$ and $C_\delta^- = C \setminus \overline{B}_{R_\delta}^d(x)$. In the bounded set C_δ we can apply Proposition 3.1 while in C_δ^- we can use Lemma 3.2, so

$$\begin{aligned} \limsup_{\tau \rightarrow 0^+} \left(\int_C p(\tau, x, y) dy \right)^\tau &\leq \lim_{\tau \rightarrow 0^+} \left(\int_{C_\delta} p(\tau, x, y) dy \right)^\tau + \\ \limsup_{\tau \rightarrow 0^+} \left(\int_{C_\delta^-} p(\tau, x, y) dy \right)^\tau &\leq e^{-\inf_{y \in C_\delta} \frac{d(x, y)^2}{4}} + \delta \leq e^{-\inf_{y \in C} \frac{d(x, y)^2}{4}} + \delta. \end{aligned}$$

Passing to the limit as $\delta \rightarrow 0^+$, we get estimate (23). $\square$

Finally we can give the proof of the main result.

Proof of Theorem 1.1 We remark that, since d is a C-C distance, properties (i) and (ii) of the Large Deviation Principle hold. In fact, we have already remarked that the Hörmander condition implies that d induces on $\mathbb{R}^n$ the euclidean topology. It means that d is continuous and the sublevels are compact sets.

Moreover, since the logarithm is a continuous and non decreasing function, Propositions 3.2 and 3.3 give properties (iii) and (iv) of the Large Deviation Principle. Applying the Large Deviation Theorem 2.2 with $F(y) = e^{-\frac{g}{2}(y)}$ we find the convergence result (6). $\square$

Remark 3.1 Note that this gives also an alternative proof for the result showed in [20].

Moreover, we want to remark that

$$\begin{cases} u_t^\varepsilon - \varepsilon \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} + \frac{1}{2} |\sigma(x) Du^\varepsilon|^2 = 0, & x \in \mathbb{R}^n, t > 0, \\ u^\varepsilon(0, x) = g(x), & x \in \mathbb{R}^n, \end{cases} \quad (24)$$

gives a second-order approximation of the Cauchy problem (5).

By the Hopf-Cole transform $w^\varepsilon = e^{-\frac{u^\varepsilon}{2\varepsilon}}$, we can linearize problem (24). In fact, setting $A(x) = \sigma^t(x)\sigma(x)$, we find

$$w_t^\varepsilon - \varepsilon \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 w^\varepsilon}{\partial x_i \partial x_j} = -\frac{w^\varepsilon}{2\varepsilon} \left(u_t^\varepsilon - \varepsilon \sum_{i,j=1}^n a_{i,j}(x) \frac{\partial^2 u^\varepsilon}{\partial x_i \partial x_j} - \frac{1}{2} A(x) Du^\varepsilon \cdot Du^\varepsilon \right).$$

Remarking that

$$|\sigma(x) Du^\varepsilon|^2 = \sigma(x) Du^\varepsilon \cdot \sigma(x) Du^\varepsilon = \sigma^t(x) \sigma(x) Du^\varepsilon \cdot Du^\varepsilon = A(x) Du^\varepsilon \cdot Du^\varepsilon,$$

we get that, if u^ε solves the Cauchy problem (24), then its Hopf-Cole transform w^ε solves exactly the Cauchy problem (3).

So it is natural that Theorem 1.1 holds, because it means the convergence of the solutions of the approximating problem (24), i.e. $u^\varepsilon = -2\varepsilon \log w^\varepsilon$, to the unique viscosity solution of the original Cauchy problem.

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