Compactness results for $\mathcal{L}^\infty$ -solution operator of a linear evolution equation involving measures

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Abstract. In this paper we establish and prove some sufficient conditions for the compactness of $\mathcal{L}^\infty$ -solution operator $(\xi, g) \mapsto u$ associated to a linear nonhomogeneous evolution equation involving measures $du = (Au) dt + dg$, $u(a) = \xi$, where A generates a compact C_0 -semigroup in a Banach space X , $\xi \in X$ and $g \in BV([a, b]; X)$.

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1 Introduction

The goal of this paper is to prove some sufficient conditions for the compactness in $C([a, b]; X)$ of the $\mathcal{L}^\infty$ -solution operator $(\xi, g) \mapsto u$ associated to the nonhomogeneous Cauchy problem of the type :

$$\begin{cases} du = (Au) dt + dg \\ u(a) = \xi, \end{cases} \quad (1.1)$$

where $A : D(A) \subseteq X \rightarrow X$ generates a C_0 -semigroup of contractions in a real Banach space X , $\xi \in X$ and $g \in BV([a, b]; X)$. In the specific case when the $\mathcal{L}^\infty$ -solution operator applies $X \times BV([a, b]; X)$ into $L^p(a, b; X)$, the problem above was studied recently by Vrabie [16]. For the classical case when g is defined by a density, i.e. there exists $f \in L^1(a, b; X)$ such that $dg(s) = f(s) ds$, some

compactness results in $C([a, b]; X)$ for the mild-solution operator are due in Baras, Hassan, Veron [4] and Vrabie [17]. The corresponding nonlinear case has been considered by Baras [3] and Vrabie [12], [13]. This kind of compactness properties are useful in establishing existence results for both Cauchy and periodic problems (see Grosu [6], Gutman [7], Hirano [9], Vrabie [12] and Shioji [11], Vrabie [14]), as well for optimal control problems with state constraints (see Ahmed [1], Barbu and Precupanu [5], Vrabie [17] and the references therein).

We assume familiarity with the basic concepts and results concerning C_0 -semigroups and infinite-dimensional vector-valued functions of bounded variation and we refer to Barbu and Precupanu [5], Pazy [10] and Vrabie [17] for details. First, we recall for easy reference some results established in Vrabie [16], [17].

Let $\mathcal{D}([a, b])$ be the set of all partitions of the interval $[a, b]$. We recall that, if $g : [a, b] \rightarrow X$ then, for each $\Delta \in \mathcal{D}([a, b])$, $\Delta : a = t_0 < t_1 < \dots < t_k = b$, the number

$$Var_{\Delta}(g, [a, b]) = \sum_{i=0}^{k-1} \|g(t_{i+1}) - g(t_i)\|$$

is called the *variation of the function g relatively to the partition Δ* . If

$$\sup_{\Delta \in \mathcal{D}([a, b])} Var_{\Delta}(g, [a, b]) < \infty,$$

then g is said to be of *bounded variation*, and the number

$$Var(g, [a, b]) = \sup_{\Delta \in \mathcal{D}([a, b])} Var_{\Delta}(g, [a, b])$$

is called *the variation of the function g on the interval $[a, b]$* . We denote by $BV([a, b]; X)$ the vector space of all function of bounded variation from $[a, b]$ to X .

Let $g \in BV([a, b]; X)$ and let $\{S(t); t \geq 0\}$ be a C_0 -semigroup of contractions¹ in a Banach space X such that $\{S(t); t \geq 0\}$ is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology. Then, for each $t \in]a, b]$, there exists a unique element $\int_a^t S(t-s) dg(s) \in X$ such that

$$\int_a^t S(t-s) dg(s) = \lim_{\mu(\Delta) \downarrow 0} \sum_{i=0}^{k-1} S(t-\tau_i)(g(t_{i+1}) - g(t_i))$$

and it is called *the Riemann-Stieltjes integral on $[a, b]$ of the operator-valued function $\tau \mapsto S(t-\tau)$ with respect to the vector-valued function g* . See Vrabie [17], p. 205, p. 206 and Theorem 9.1.1, p. 208.

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¹All the results which will follows hold true also for the general case of C_0 -semigroups not necessarily of contractions. However, for simplicity reasons, we preferred to consider only C_0 -semigroups of contractions.

Remark 1.1 Since $\{S(t); t \geq 0\}$ is a C_0 -semigroup of contractions, whenever $\int_a^t S(t-s) dg(s) \in X$, we have :

$$\left\| \int_a^t S(t-s) dg(s) \right\| \leq \text{Var}(g, [a, t]),$$

for each $t \in]a, b]$.

Definition 1.1 Let $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup of contractions $\{S(t); t \geq 0\}$, which is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology. The function $u : [a, b] \rightarrow X$ given by

$$u(t) = S(t-a)\xi + \int_a^t S(t-s) dg(s) \quad (1.2)$$

for each $t \in [a, b]$ is called an $\mathcal{L}^\infty$ -solution on $[a, b]$ of the problem (1.1).

See Vrabie [17], Definition 9.1.1, p. 209. A similar concept was introduced by Amann [2] in the case in which A generates an analytic C_0 -semigroup, by using a “transposition argument”.

Remark 1.2 We notice that each $\mathcal{L}^\infty$ -solution u satisfies

$$\|u(t)\| \leq \|\xi\| + \text{Var}(g, [a, b]),$$

for each $t \in [a, b]$.

See Vrabie [17], Remark 9.4.1, p. 216.

Proposition 1.1 If $g \in BV([a, b]; X)$, then g is piecewise continuous on $[a, b]$, i.e there exists an at most countable subset E of $[a, b]$, such that g is continuous on $[a, b] \setminus E$ and, at each $t \in E \cap [a, b[$ (at each $s \in E \cap]a, b]$), there exists the one-sided limit $g(t+0)$ ($g(s-0)$).

See Vrabie [17], Proposition 1.4.2, p. 14.

Theorem 1.1 (Regularity of $\mathcal{L}^\infty$ -solutions) Let $g \in BV([a, b]; X)$ and $\xi \in X$. Let $A : D(A) \subseteq X \rightarrow X$ be the infinitesimal generator of a C_0 -semigroup of contractions which is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology and let u be the $\mathcal{L}^\infty$ -solution of (1.1) corresponding to ξ and g . Then, for each $t \in [a, b[$ and each $s \in]a, b]$, there exists $u(t+0)$ and $u(s-0)$ and

$$\begin{cases} u(t+0) - u(t) = g(t+0) - g(t) \\ u(s) - u(s-0) = g(s) - g(s-0). \end{cases}$$

So, u is continuous from the right (left) at $t \in [a, b[$ ($t \in]a, b]$) if and only if g is continuous from the right (left) at t . In particular, u is continuous at any point at which g is continuous and thus u is piecewise continuous on $[a, b]$.

See Vrabie [17], Theorem 9.2.1, p. 210.

2 Equicontinuous in variation families

In order to formulate our compactness results, we need to introduce the notion of equicontinuous in variation family.

Let X be a real Banach space, $[a, b] \subset \mathbb{R}$ and $B \subseteq [a, b]$ such that each $t \in B$ is an accumulation point for B . We denote by $C(B; X)$ the set of all function continuous from B to X . Also, we denote by $C_r(B; X)$ (respectively $C_l(B; X)$) the set of all function continuous to the right on $B \subseteq [a, b[$ (respectively continuous to the left on $B \subseteq]a, b]$).

Definition 2.1 A family $\mathcal{F}$ in $C_r(B; X)$ is *equicontinuous to the right at t* in $B \subseteq [a, b[$ if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $f \in \mathcal{F}$ and for each $h \in]0, \delta[$ with $t + h \in B$, we have $\|f(t + h) - f(t)\| < \varepsilon$.

The family $\mathcal{F}$ in $C_r(B; X)$ is *equicontinuous to the right on $B \subseteq [a, b[$* if it is equicontinuous to the right at each point $t \in B$, in the sense mentioned above.

Definition 2.2 A family $\mathcal{F}$ in $C_l(B; X)$ is *equicontinuous to the left at t* in $B \subseteq]a, b]$ if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $f \in \mathcal{F}$ and for each $h \in]0, \delta[$ with $t - h \in B$, we have $\|f(t) - f(t - h)\| < \varepsilon$.

The family $\mathcal{F}$ in $C_l(B; X)$ is *equicontinuous to the left on $B \subseteq]a, b]$* if it is equicontinuous to the left at each point $t \in B$, in the sense mentioned above.

Remark 2.1 If $g \in BV([a, b]; X)$, let us define $v_g : [a, b] \rightarrow \mathbb{R}$ by

$$v_g(t) = \begin{cases} \text{Var}(g, [a, t]) & \text{if } t \in]a, b] \\ 0 & \text{if } t = a. \end{cases} \quad (2.1)$$

Let us observe that :

- (i) for each $t \in [a, b]$, $v_g(t) \geq 0$;
- (ii) for each $(t, s) \in [a, b] \times [a, b]$, $t < s$, $v_g(s) = v_g(t) + \text{Var}(g, [t, s])$;
- (iii) the function v_g is nondecreasing on $[a, b]$.

We mention without proof the next useful result.

Proposition 2.1 Let $BV([a, b]; X)$ and let $v_g : [a, b] \rightarrow \mathbb{R}$ be defined by (2.1). Then the function g is continuous at $t \in [a, b]$ if and only if the function v_g is continuous at $t \in [a, b]$.

See Haaser and Sullivan [8], Proposition 4.5, p. 232.

Definition 2.3 A family $\mathcal{G}$ in $BV([a, b]; X) \cap C_r(B; X)$ is *equicontinuous to the right in variation at t* in $B \subseteq [a, b[$ if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $g \in \mathcal{G}$ and for each $h \in]0, \delta[$ with $t + h \in B$, we have

$$\text{Var}(g, [t, t + h]) < \varepsilon.$$

The family $\mathcal{G}$ in $BV([a, b]; X) \cap C_r(B; X)$ is *equicontinuous to the right in variation on $B \subseteq [a, b[$* if it is equicontinuous to the right in variation at each point $t \in B$, in the sense mentioned above.

Definition 2.4 A family $\mathcal{G}$ in $BV([a, b]; X) \cap C_l(B; X)$ is *equicontinuous to the left in variation at t in $B \subseteq]a, b]$* if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $g \in \mathcal{G}$ and for each $h \in]0, \delta[$ with $t - h \in B$, we have

$$\text{Var}(g, [t - h, t]) < \varepsilon.$$

The family $\mathcal{G}$ in $BV([a, b]; X) \cap C_l(B; X)$ is *equicontinuous to the left in variation on $B \subseteq]a, b]$* if it is equicontinuous to the left in variation at each point $t \in B$, in the sense mentioned above.

Remark 2.2 We observe that :

- (i) A family $\mathcal{G}$ in $BV([a, b]; X) \cap C_r(B; X)$ is equicontinuous to the right in variation at t in $B \subseteq [a, b[$ if and only if, for t in $B \subseteq [a, b[$, we have

$$\lim_{h \downarrow 0, (t+h) \in B} \text{Var}(g, [t, t+h]) = 0, \quad (2.2)$$

uniformly for $g \in \mathcal{G}$.

- (ii) A family $\mathcal{G}$ in $BV([a, b]; X) \cap C_l(B; X)$ is equicontinuous to the left in variation at t in $B \subseteq]a, b]$ if and only if, for t in $B \subseteq]a, b]$, we have

$$\lim_{h \downarrow 0, (t-h) \in B} \text{Var}(g, [t-h, t]) = 0, \quad (2.3)$$

uniformly for $g \in \mathcal{G}$.

Definition 2.5 A family $\mathcal{G}$ in $BV([a, b]; X) \cap C(B; X)$ is *equicontinuous in variation on $B \subseteq [a, b]$* if it is equicontinuous both to the right and to the left in variation on $B \cap]a, b[$; is *equicontinuous in variation at $a \in B$* if it is equicontinuous to the right in variation at a and *equicontinuous in variation at $b \in B$* if it is equicontinuous to the left in variation at b .

Remark 2.3 We observe that a family $\mathcal{G}$ in $BV([a, b]; X) \cap C(B; X)$ is equicontinuous in variation at t in $B \subseteq [a, b]$ if and only if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $g \in \mathcal{G}$ and for each $s \in B$ with $|t - s| < \delta$, we have

$$|v_g(t) - v_g(s)| < \varepsilon,$$

where $v_g : [a, b] \rightarrow \mathbb{R}$ is defined by (2.1).

Moreover, a family $\mathcal{G}$ in $BV([a, b]; X) \cap C(B; X)$ is equicontinuous in variation on $B \subseteq [a, b]$ if and only if it is equicontinuous in variation at each point $t \in B$, in the sense mentioned above.

Definition 2.6 A family $\mathcal{G}$ in $BV([a, b]; X) \cap C(B; X)$ is *uniformly equicontinuous in variation on B* if it is equicontinuous in variation on B and $\delta(\varepsilon, t)$ can be chosen independent of $t \in B$.

Remark 2.4 A family $\mathcal{G}$ in $BV([a, b]; X) \cap C([a, b], X)$ is *equicontinuous in variation on $[a, b]$* if and only if it is uniformly equicontinuous in variation on $[a, b]$.

See Remark 2.3, Definition 2.6 and Remark A.2.1, p. 296. in Vrabie [17].

Remark 2.5 If $\mathcal{G}$ in $BV([a, b]; X) \cap C_r(B; X)$ (in $BV([a, b]; X) \cap C_l(B; X)$) is equicontinuous to the right (left) in variation at t in $B \subseteq]a, b[$ (in $B \subseteq [a, b[$) then it is equicontinuous to the right (left) at t in $B \subseteq]a, b[$ (in $B \subseteq [a, b[$).

Example 2.1 Let $\mathcal{G} = \{g_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $g_n : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$g_n(t) = t^n,$$

for each $t \in [0, 1]$. Clearly, each function of $\mathcal{G}$ is of bounded variation because $Var(g_n, [0, 1]) = 1$, for each $n \in \mathbb{N}^*$.

We observe that the family $\mathcal{G}$ is equicontinuous to the right in variation at each t in $[0, 1[$, is equicontinuous to the left in variation at each t in $]0, 1[$, but is not equicontinuous to the left in variation at $t = 1$.

Definition 2.7 A family $\mathcal{F}$ in $L^p(a, b; X)$ is *uniformly integrable on $[a, b]$* if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that, for each $f \in \mathcal{F}$ and for each measurable subset E in $[a, b]$, whose Lebesgue measure $\mu(E) < \delta(\varepsilon)$, we have

$$\int_E \|f(t)\| dt < \varepsilon.$$

Remark 2.6 We observe that a family $\mathcal{F}$ in $L^p(a, b; X)$ is uniformly integrable on $[a, b]$ if and only if

$$\lim_{\mu(E) \downarrow 0} \int_E \|f(t)\| dt = 0, \quad (2.4)$$

uniformly for $f \in \mathcal{F}$.

Example 2.2 Let $r \in BV([a, b]; X) \cap C([a, b]; X)$ and $\mathcal{F}$ in $L^1(a, b; X)$ which is uniformly integrable on $[a, b]$. For each $f \in \mathcal{F}$, let us define a $g_f : [a, b] \rightarrow X$ by

$$g_f(t) = \int_a^t f(s) ds.$$

We know that $\{g_f; f \in \mathcal{F}\}$ is in $AC([a, b]; X)$, where $AC([a, b]; X)$ is the set of all function absolutely continuous from $[a, b]$ to X . Hence, since $r \in C([a, b]; X)$,

by Remark 2.2 and Remark 2.6, it follows that $\{r + g_f; f \in \mathcal{F}\}$ is equicontinuous to the right in variation on $[a, b[$ and equicontinuous to the left in variation on $]a, b]$. In fact, if $\mathcal{G} \subset BV([a, b]; X) \cap C([a, b]; X)$ is equicontinuous in variation on $[a, b]$, then $\mathcal{G} + \mathcal{F} = \{r + g_f; r \in \mathcal{G}, f \in \mathcal{F}\}$ is equicontinuous in variation on $[a, b]$.

Definition 2.8 A family $\mathcal{G}$ in $BV([a, b]; X)$ is of *equibounded variation* on $[a, b]$ if there exists $m_{\mathcal{G}} > 0$ such that, for each $g \in \mathcal{G}$, we have

$$Var(g, [a, b]) \leq m_{\mathcal{G}}.$$

We include (again without proof) the following simple result :

Proposition 2.2 *If a family $\mathcal{G}$ in $BV([a, b]; X) \cap C([a, b]; X)$ is equicontinuous in variation on $[a, b]$, then $\mathcal{G}$ is of equibounded variation on $[a, b]$.*

3 Compactness of $\mathcal{L}^\infty$ -solution operator; the case when the semigroup is compact

Our goal here is to prove the main result of this paper, i.e. a sufficient condition in order that the family of all $\mathcal{L}^\infty$ -solution of the problem (1.1) be relatively compact in $C([a, b]; X)$, when ξ ranges in a given subset in X and g ranges in a subset in $BV([a, b]; X) \cap C([a, b]; X)$ which is equicontinuous in variation on $[a, b]$.

Lemma 3.1 *Let X a real Banach space, let Γ a nonempty subset in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ and let μ an accumulation point of Γ . Let K a bounded subset in X and let $\{Q_\lambda; \lambda \in \Gamma\}$ be a family of compact operators from K to X . If*

$$\lim_{\lambda \rightarrow \mu} Q_\lambda x = Qx,$$

uniformly for $x \in K$, then K is relatively compact subset in X .

See Vrabie [17], Corollary A.1.2, p. 294.

Lemma 3.2 *Let X be an real Banach space. If $\{S(t); t \geq 0\}$ is a C_0 -semigroup of contractions of linear operators on X and K is a relatively compact subset in X then*

$$\lim_{h \downarrow 0} \|S(h)x - x\| = 0,$$

uniformly for $x \in K$.

In that follows, we shall assume that the C_0 -semigroup of contractions is compact and $g \in BV([a, b]; X)$, $\xi \in X$. Then the Cauchy problem (1.1) has a unique $\mathcal{L}^\infty$ -solution u (see Definition 1.1).

The main result in this paper is :

Theorem 3.1 *Let $A : D(A) \subseteq X \rightarrow X$ the infinitesimal generator of a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$. Let $\mathcal{D}$ be relatively compact in X and $\mathcal{G} = \{g_n; n \in \mathbb{N}\}$ a subset in $BV([a, b]; X)$ of equibounded variation on $[a, b]$. Let E be the smallest at most countable subset of $[a, b]$ such that, for each $n \in \mathbb{N}$, g_n is continuous on $[a, b] \setminus E$. For each $n \in \mathbb{N}$, let us denote by u_n the unique $\mathcal{L}^\infty$ -solution for the Cauchy problem (1.1) corresponding to $(\xi_n, g_n) \in \mathcal{D} \times \mathcal{G}$. Assume that there exists an at most countable subset F of $[a, b]$, $F \supseteq E$, such that $\mathcal{G}$ is equicontinuous in variation on $]a, b[\setminus F$. Then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and $u :]a, b[\setminus F \rightarrow X$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in]a, b[\setminus F$. In addition, the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous on $]a, b[\setminus F$. Moreover, if $\mathcal{G}$ is equicontinuous in variation to the right at a and equicontinuous in variation to the left at b , then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and $u : [a, b] \setminus F \rightarrow X$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in [a, b] \setminus F$. In addition, the limit function $u : [a, b] \setminus F \rightarrow X$ is continuous on $[a, b] \setminus F$.*

Proof. We divide the proof into several steps.

Step 1. We first prove that, for each $t \in]a, b[\setminus F$, the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(t); n \in \mathbb{N}\}$ is relatively compact in X . Let $t \in]a, b[\setminus F$ and let $h_k \downarrow 0$ such that $t - h_k > a$ for each $k \in \mathbb{N}$. Now, for each $k \in \mathbb{N}$, let us define $Q_k : \{u_n(t); n \in \mathbb{N}\} \rightarrow X$ by

$$Q_k(u_n(t)) = S(h_k)u_n(t - h_k).$$

Let us observe that, by hypothesis, there exists $m_{\mathcal{D}} > 0$ and $m_{\mathcal{G}} > 0$ such that

$$\|\xi\| \leq m_{\mathcal{D}} \text{ and } \text{Var}(g, [a, b]) \leq m_{\mathcal{G}},$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$. Then, by virtue of Remark 1.2, for $M = m_{\mathcal{D}} + m_{\mathcal{G}} > 0$ we have

$$\|u(t)\| \leq M,$$

for each $(\xi, g) \in \mathcal{D} \times \mathcal{G}$ and each $t \in [a, b]$. It follows that $\{u_n(t); n \in \mathbb{N}\}$ is uniformly bounded in X . Moreover, $\{S(t); t \geq 0\}$ is a compact C_0 -semigroup of contractions and then Q_k is a compact operator for each $k \in \mathbb{N}$. By virtue of Lemma 3.1, to prove that $\{u_n(t); n \in \mathbb{N}\}$ is relatively compact in X it suffices to show that

$$\lim_{k \rightarrow \infty} \|Q_k(u_n(t)) - u_n(t)\| = 0, \quad (3.1)$$

uniformly for $n \in \mathbb{N}$. A computational argument involving (1.2) and Remark 1.1 shows that, for $t \in]a, b] \setminus F$, we have

$$\begin{aligned} & \|Q_k(u_n(t)) - u_n(t)\| \\ &= \left\| \int_{t-h_k}^t S(t-s) dg_n(s) \right\| \leq \text{Var}(g_n, [t-h_k, t]). \end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Since $\mathcal{G}$ is equicontinuous in variation to the left at $t \in]a, b] \setminus F$, $b \in F$ then there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $n \in \mathbb{N}$ and each $h \in]0, \delta[$ with $t-h \in]a, b[\setminus F$, we have

$$\text{Var}(g_n, [t-h, t]) < \varepsilon.$$

But $h_k \downarrow 0$ and then there exists $k(\varepsilon, t) \in \mathbb{N}$ such that, for each $k \in \mathbb{N}$, $k \geq k(\varepsilon, t)$, we have $0 < h_k < \delta$. Taking into account the preceding inequalities, we obtain

$$\text{Var}(g_n, [t-h_k, t]) < \varepsilon,$$

this means the relation (3.1). If $\mathcal{G}$ is equicontinuous in variation to the left at b , by a similar reasoning we obtain that *the b -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(b); n \in \mathbb{N}\}$ is relatively compact in X .*

We note that *the a -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. the set $\{u_n(a); n \in \mathbb{N}\}$ is relatively compact in X* because $\mathcal{D}$ is relatively compact in X .

Step 2. In accordance with Theorem 1.1, we observe that, for each $n \in \mathbb{N}$, u_n is continuous function on $[a, b] \setminus F$. Next, let us prove that *the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the right on $]a, b[\setminus F$* . Indeed, let $t \in]a, b[\setminus F$ and let $h > 0$ arbitrary such that $t+h \in]a, b[\setminus F$. By (1.2) and Remark 1.1 we deduce that

$$\begin{aligned} & \|u_n(t+h) - u_n(t)\| \\ & \leq \|S(h)u_n(t) - u_n(t)\| + \left\| \int_a^{t+h} S(t+h-s) dg_n(s) \right\| \\ & \leq \|S(h)u_n(t) - u_n(t)\| + \text{Var}(g_n, [t, t+h]). \end{aligned}$$

But $\mathcal{G}$ is equicontinuous in variation to the right on $[a, b[\setminus F$, $a \in F$. Then, as for $t \in]a, b[\setminus F$, $\{u_n(t); n \in \mathbb{N}\}$ is relatively compact in X , the last inequality, along with Lemma 3.2, shows that $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the right on $]a, b[\setminus F$.

We mention that, if $\mathcal{D}$ is relatively compact in X and $\mathcal{G}$ is equicontinuous in variation to the right at a , we can prove like above that *the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the right at a .*

Step 3. In that follows, we shall prove that *the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the left on $]a, b[\setminus F$* . To this aim, let $t \in]a, b[\setminus F$ and let $h > 0$ arbitrary such that $t-h \in]a, b[\setminus F$. We observe that

$$\begin{aligned} & \|u_n(t-h) - u_n(t)\| \\ & \leq \|u_n(t-h) - S(h)u_n(t-h)\| + \|S(h)u_n(t-h) - u_n(t)\|. \end{aligned}$$

By (1.2) and Remark 1.1 we deduce that

$$\begin{aligned} & \|u_n(t) - S(h)u_n(t-h)\| \\ &= \left\| \int_{t-h}^t S(t-s) dg_n(s) \right\| \leq \text{Var}(g_n, [t-h, t]). \end{aligned}$$

Also by (1.2), we have

$$\begin{aligned} & \|u_n(t-h) - S(h)u_n(t-h)\| \\ & \leq \|S(t-h-a)\|_{\mathcal{L}(X)} \|S(h)\xi_n - \xi_n\| \\ & \quad + \left\| \int_a^{t-h} (S(t-s) - S(t-h-s)) dg_n(s) \right\|. \end{aligned}$$

Let $\varepsilon > 0$. By virtue of the equicontinuity to the left in variation of $\{g_n; n \in \mathbb{N}\}$ at $t \in]a, b[\setminus F$, it follows that there exists $\delta = \delta(\varepsilon, t) > 0$ such that, for each $n \in \mathbb{N}$ and each $h \in]0, \delta[$, with $t-h \in]a, b[\setminus F$, we have

$$\text{Var}(rg_n, [t-h, t]) < \varepsilon$$

and even

$$\text{Var}(g_n, [t-\delta, t]) < \varepsilon.$$

Therefore, we obtain that

$$\begin{aligned} & \left\| \int_a^{t-h} (S(t-s) - S(t-h-s)) dg_n(s) \right\| \\ & \leq \left\| \int_a^{t-\delta} S(t-\delta-s) (S(\delta) - S(\delta-h)) dg_n(s) \right\| \\ & \quad + \left\| \int_{t-\delta}^{t-h} (S(t-s) - S(t-h-s)) dg_n(s) \right\| \\ & \leq \|S(\delta) - S(\delta-h)\| \text{Var}(g_n, [a, b]) + 2\text{Var}(g_n, [t-\delta, t-h]) \\ & \leq \|S(\delta) - S(\delta-h)\| \text{Var}(g_n, [a, b]) + 2\text{Var}(g_n, [t-\delta, t]). \end{aligned}$$

Since the semigroup is continuous in the uniform operator topology from $]0, +\infty[$ to $\mathcal{L}(X)$, it follows that, for the very same $\varepsilon > 0$, there exists $\delta_1 = \delta_1(\varepsilon) > 0$ such that, for each $h \in]0, \delta_1[$, we have

$$\|S(\delta) - S(\delta-h)\| < \varepsilon.$$

Moreover, because $\mathcal{D}$ is relatively compact in X , by Lemma 3.2, it follows that, for the very same $\varepsilon > 0$, there exists $\delta_2 = \delta_2(\varepsilon) > 0$ such that, for each $n \in \mathbb{N}$ and $h \in]0, \delta_2[$, we have

$$\|S(h)\xi_n - \xi_n\| < \varepsilon.$$

Hence, for each $\varepsilon > 0$, there exists $\delta_3 = \delta_3(\varepsilon, t) > 0$, $\delta_3 = \min(\delta, \delta_1, \delta_2)$ such that, for each $h \in]0, \delta_3[$ with $t - h \in]a, b[\setminus F$ we have

$$\|u_n(t - h) - u_n(t)\| < (4 + m_{\mathcal{G}})\varepsilon.$$

So, the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the left on $]a, b[\setminus F$.

We mention that, if $\mathcal{G}$ is equicontinuous in variation to the left at b , then the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the left at b . The proof of this statement it follows the very same way as that of the proof of equicontinuity to the left on $]a, b[\setminus F$ for the family $\{u_n; n \in \mathbb{N}\}$.

Step 4. Next, we shall find a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ and a function $u :]a, b[\setminus F \rightarrow X$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in]a, b[\setminus F$.

We observe that there exists $\{t_p; p \in \mathbb{N}\}$ dense in $]a, b[\setminus F$. But, for each $p \in \mathbb{N}$, $\{u_n(t_p); n \in \mathbb{N}\}$ is relatively compact subset in X . Then, from a diagonal process, we find a subsequence $(u_{n_n})_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, such that $(u_{n_n}(t_p))_{n \in \mathbb{N}}$ is convergent sequence in X , for each $p \in \mathbb{N}$, that is $\lim_{n \rightarrow \infty} u_{n_n}(t_p) = u(t_p)$, for each $p \in \mathbb{N}$.

Let $t \in]a, b[\setminus F$. We shall prove that $(u_{n_n}(t))_{n \in \mathbb{N}}$ is a Cauchy sequence in X . Let $\varepsilon > 0$. From the equicontinuity to the right of $\{u_{n_n}; n \in \mathbb{N}\}$ at $t \in]a, b[\setminus F$ there exists $\eta = \eta(\varepsilon, t) > 0$ such that for a $t_{p_0} \in \{t_p; p \in \mathbb{N}\}$, with $t_{p_0} \in]t, t + \eta[$ (we get $t_{p_0} = t_{p_0}(\eta)$ from density), we have

$$\|u_{n_n}(t) - u_{n_n}(t_{p_0})\| < \varepsilon.$$

Moreover, since the sequence $(u_{n_n}(t_{p_0}))_{n \in \mathbb{N}}$ is convergent in X , for the same $\varepsilon > 0$ there exists $p_1 \in \mathbb{N}$ such that for each $n \in \mathbb{N}$, and each $m \in \mathbb{N}$ with $n \geq p_1$ and $m \geq p_1$, we have

$$\|u_{n_n}(t_{p_0}) - u_{m_m}(t_{p_0})\| < \varepsilon.$$

Then, for each $n \in \mathbb{N}$, and each $m \in \mathbb{N}$ with $n \geq p_1$ and $m \geq p_1$, we have

$$\begin{aligned} & \|u_{n_n}(t) - u_{m_m}(t)\| \\ &= \|u_{n_n}(t) - u_{n_n}(t_{p_0})\| + \|u_{n_n}(t_{p_0}) - u_{m_m}(t_{p_0})\| \\ & \quad + \|u_{m_m}(t_{p_0}) - u_{m_m}(t)\| < 3\varepsilon. \end{aligned}$$

We conclude that $(u_{n_n}(t))_{n \in \mathbb{N}}$ is a Cauchy sequence in X , and it follows then there exists $u(t) \in X$ such that $\lim_{n \rightarrow \infty} u_{n_n}(t) = u(t)$.

Since t is arbitrary in $]a, b[\setminus F$, we find the subsequence $(u_{n_n})_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and the function $u :]a, b[\setminus F \rightarrow X$ such that $\lim_{n \rightarrow \infty} u_{n_n}(t) = u(t)$ pointwise for $t \in]a, b[\setminus F$.

We mention that, if $\mathcal{D}$ is relatively compact in X and $\mathcal{G}$ is equicontinuous in variation to the right at a , by the same way, we find the subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and the function $u :]a, b[\setminus F \rightarrow X$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise

for $t \in [a, b[\setminus F$. If, in addition, $\mathcal{G}$ is equicontinuous in variation to the left at b , we find the subsequence $(u_{n_n})_{n \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and the function $u : [a, b] \setminus F \rightarrow X$ such that $\lim_{n \rightarrow \infty} u_{n_n}(t) = u(t)$ pointwise for $t \in [a, b] \setminus F$. The proof of the last statement is based on the equicontinuity to the left of $\{u_{n_n}; n \in \mathbb{N}\}$ at b .

Step 5. Finally, we prove that the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous on $]a, b[\setminus F$. First, we prove that the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous to the right on $]a, b[\setminus F$. Let $t \in]a, b[\setminus F$. Let $\varepsilon > 0$. From the equicontinuity to the right of $\{u_{n_n}; n \in \mathbb{N}\}$ at $t \in]a, b[\setminus F$, there exists $\lambda = \lambda(\varepsilon, t) > 0$ such that, for each $n \in \mathbb{N}$ and each $h \in]0, \lambda[$, with $t + h \in]a, b[\setminus F$, we have

$$\|u_{n_n}(t + h) - u_{n_n}(t)\| < \varepsilon.$$

But $\lim_{n \rightarrow \infty} u_{n_n}(t) = u(t)$ and we find $n_1(\varepsilon) \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $n \geq n_1$, we have

$$\|u_{n_n}(t) - u(t)\| < \varepsilon.$$

Also, since $\lim_{n \rightarrow \infty} u_{n_n}(t + h) = u(t + h)$, there exists $n_2(\varepsilon) \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $n \geq n_2$, we have

$$\|u_{n_n}(t + h) - u(t + h)\| < \varepsilon.$$

Let us fix $\nu = \max(n_1, n_2)$ and we obtain

$$\begin{aligned} & \|u(t + h) - u(t)\| \\ & \leq \|u(t + h) - u_{\nu}(t + h)\| + \|u_{\nu}(t + h) - u_{\nu}(t)\| + \|u_{\nu}(t) - u(t)\| < 3\varepsilon. \end{aligned}$$

This means that the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous to the right on $]a, b[\setminus F$.

We mention that, if $\mathcal{D}$ is relatively compact in X and $\mathcal{G}$ is equicontinuous in variation to the right at a , then the limit function $u : [a, b[\setminus F \rightarrow X$ is continuous to the right on $[a, b[\setminus F$.

Next, we prove that the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous to the left on $]a, b[\setminus F$. Let $t \in]a, b[\setminus F$. Let $\varepsilon > 0$. From the equicontinuity to the left of $\{u_{n_n}; n \in \mathbb{N}\}$ at $t \in]a, b[\setminus F$, there exists $\mu = \mu(\varepsilon, t) > 0$ such that, for each $n \in \mathbb{N}$ and each $h \in]0, \mu[$, with $t - h \in]a, b[\setminus F$, we have

$$\|u_{n_n}(t - h) - u_{n_n}(t)\| < \varepsilon.$$

But $\lim_{n \rightarrow \infty} u_{n_n}(t) = u(t)$ and we find $n_3(\varepsilon) \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $n \geq n_3$, we have

$$\|u_{n_n}(t) - u(t)\| < \varepsilon.$$

Also, since $\lim_{n \rightarrow \infty} u_{n_n}(t-h) = u(t-h)$, there exists $n_4(\varepsilon) \in \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $n \geq n_4$, we have

$$\|u_{n_n}(t-h) - u(t-h)\| < \varepsilon.$$

Let us fix $\nu = \max(n_3, n_4)$ and we obtain

$$\begin{aligned} & \|u(t-h) - u(t)\| \\ & \leq \|u(t-h) - u_{\nu_\nu}(t-h)\| + \|u_{\nu_\nu}(t-h) - u_{\nu_\nu}(t)\| + \|u_{\nu_\nu}(t) - u(t)\| < 3\varepsilon. \end{aligned}$$

This means that the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous to the left on $]a, b[\setminus F$.

We mention that, if $\mathcal{G}$ is equicontinuous in variation to the left at b , then the limit function $u :]a, b[\setminus F \rightarrow X$ is continuous to the left on $]a, b[\setminus F$, and this completes the proof. $\square$

Corollary 3.1 *Let $A : D(A) \subseteq X \rightarrow X$ the infinitesimal generator of a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$. Let $\mathcal{D}$ be relatively compact in X and $\mathcal{G} = \{g_n; n \in \mathbb{N}\}$ a subset in $BV([a, b]; X) \cap C([a, b], X)$. For each $n \in \mathbb{N}$, let us denote by u_n the unique $\mathcal{L}^\infty$ -solution for the Cauchy problem (1.1) corresponding to $(\xi_n, g_n) \in \mathcal{D} \times \mathcal{G}$. If $\mathcal{G}$ is equicontinuous in variation on $[a, b]$, then there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$, and $u : [a, b] \rightarrow X$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in [a, b]$. Moreover, the limit function $u : [a, b] \rightarrow X$ is continuous on $[a, b]$.*

Remark 3.1 If in Theorem 3.1 we suppose that $\mathcal{D}$ is only bounded in X and the rest of hypotheses hold true, then the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. the set $\{u_n(t); n \in \mathbb{N}\}$, is relatively compact in X only for $t \in]a, b[\setminus F$, and the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the right on $]a, b[\setminus F$ but is not equicontinuous to the left on $]a, b[\setminus F$. It follows that the limit function $u :]a, b[\setminus F \rightarrow X$ is only continuous to the right on $]a, b[\setminus F$.

Also, if in Corollary 3.1 we suppose that D is only bounded in X and the rest of hypotheses hold true, then the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(t); n \in \mathbb{N}\}$, is relatively compact subset in X only for $t \in]a, b[$, and the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous to the right on $]a, b[$ but is not equicontinuous to the left on $]a, b[$. It follows that the limit function $u :]a, b[\rightarrow X$ is only continuous to the right on $]a, b[$.

Remark 3.2 If in Theorem 3.1 we assume that $\mathcal{G}$ is equicontinuous in variation only to the left on $]a, b[\setminus F$ and the rest of hypotheses hold true, then, for each $t \in [a, b] \setminus F$, the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(t); n \in \mathbb{N}\}$, is relatively compact subset in X and the family $\{u_n; n \in \mathbb{N}\}$ is equicontinuous only to the left at t in $]a, b[\setminus F$. We mention that the a -cross-section of $\{u_n; n \in \mathbb{N}\}$ is relatively compact in X because $\mathcal{D}$ is relatively compact in X . It follows that the limit

function u is defined on $[a, b] \setminus F$ and, moreover, it is continuous to the left on $]a, b[\setminus F$.

Also, if in Theorem 3.1 we assume that $\mathcal{G}$ is equicontinuous in variation only to the right on $[a, b[\setminus F$ and the rest of hypotheses hold true, then we cannot prove that, for t in $[a, b[\setminus F$, the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(t); n \in \mathbb{N}\}$, is relatively compact subset in X . Hence, we have to compensate for the lack of equicontinuity in variation to the left of $\mathcal{G}$ by some extra-assumption on $\{u_n; n \in \mathbb{N}\}$. A suitable assumption on $\{u_n; n \in \mathbb{N}\}$ is the equicontinuity to the right on $[a, b[\setminus F$. In this case, even if $\mathcal{D}$ is only bounded, we can prove that for each t in $[a, b[\setminus F$, the t -cross-section of $\{u_n; n \in \mathbb{N}\}$, i.e. $\{u_n(t); n \in \mathbb{N}\}$, is relatively compact subset in X and, also, that the limit function u is defined on $[a, b[\setminus F$ and, moreover, it is continuous to the right on $[a, b[\setminus F$.

Next, if $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$, then the latter is continuous from $]0, \infty[$ to $\mathcal{L}(X)$ in the uniform operator topology and therefore, for each $(\xi, g) \in X \times BV([a, b]; X) \cap C([a, b]; X)$, the problem (1.1) has a unique $\mathcal{L}^\infty$ -solution $u \in C([a, b]; X)$ (see Definition 1.1 and Theorem 1.1). So, we define the $\mathcal{L}^\infty$ -solution operator $Q : X \times BV([a, b]; X) \cap C([a, b]; X) \rightarrow C([a, b]; X)$ by $Q(\xi, g) = u$.

In that follows, we prove a more powerful compactness result, partially covered by Corollary 3.1.

Theorem 3.2 *Let $A : D(A) \subseteq X \rightarrow X$ the infinitesimal generator of a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$. Let $\mathcal{D}$ be relatively compact in X and $\mathcal{G}$ be a subset in $BV([a, b]; X) \cap C([a, b]; X)$. If $\mathcal{G}$ is equicontinuous in variation on $[a, b]$, then $Q(\mathcal{D}, \mathcal{G})$ is relatively compact in $C([a, b]; X)$.*

Remark 3.3 We mention that, in Corollary 3.1 and Theorem 3.2, since $\mathcal{G}$ is equicontinuous in variation on $[a, b]$, it is of equibounded variation on $[a, b]$. See Proposition 2.2.

Proof. Like in the proof of Theorem 3.1 we obtain that for each $t \in [a, b]$, the t -cross-section of $Q(\mathcal{D}, \mathcal{G})$, i.e. $Q(\mathcal{D}, \mathcal{G})(t) = \{Q(\xi, g)(t); (\xi, g) \in \mathcal{D} \times \mathcal{G}\}$, is relatively compact in X (see Step 1). Next, we obtain that the family $Q(\mathcal{D}, \mathcal{G})$ is equicontinuous on $[a, b]$ (see Step 2 and Step 3). By Arzelà-Ascoli's Theorem we conclude that $Q(\mathcal{D}, \mathcal{F})$ is relatively compact subset in $C([a, b]; X)$.

Now, let us consider the nonhomogeneous Cauchy problem

$$\begin{cases} u' = Au + f \\ u(a) = \xi, \end{cases} \quad (3.2)$$

where $A : D(A) \subseteq X \rightarrow X$ is the infinitesimal generator of a C_0 -semigroup of contractions $\{S(t); t \geq 0\}$, $\xi \in X$ in a Banach space X , $\xi \in X$ and $f \in$

$L^1(a, b; X)$. We recall that the function $u : [a, b] \rightarrow X$ given by

$$u(t) = S(t-a)\xi + \int_a^t S(t-s)f(s)ds \quad (3.3)$$

for each $t \in [a, b]$ is called an C^0 -solution (or mild solution) on $[a, b]$ of the problem (3.2). See Vrabie [17], Definition 8.1.3, p. 185.

For each (ξ, f) in $X \times L^1(a, b; X)$, the Cauchy problem (3.2) has a unique mild-solution u in $C([a, b]; X)$ and we can define the mild-solution operator by $T : X \times L^1(a, b; X) \rightarrow C([a, b]; X)$, $T(\xi, f) = u$.

Remark 3.4 Let $\mathcal{F}$ be a family in $L^1(a, b; X)$ which is uniformly integrable on $[a, b]$. For each $f \in \mathcal{F}$, we define a function $g_f : [a, b] \rightarrow X$ by

$$g_f(t) = \int_a^t f(s)ds.$$

By Example 2.2, we have that $\mathcal{G} = \{g_f; f \in \mathcal{F}\}$ is equicontinuous in variation on $[a, b]$. From Theorem 3.1 it then follows:

Theorem 3.3 (Baras-Hassan-Veron) *Let $A : D(A) \subseteq X \rightarrow X$ the infinitesimal generator of a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$. Let $\xi \in X$, $\mathcal{D} = \{\xi\}$ and $\mathcal{F}$ be a uniformly integrable subset in $L^1(a, b; X)$. Then $T(\mathcal{D}, \mathcal{F})$ is relatively compact in $C([a, b]; X)$.*

See Baras, Hassan, Veron [4], Théorème 1 and Vrabie [17], Theorem 8.4.2, p. 196.

Finally, we conclude with the following examples inspired from Baras, Hassan, Veron [4].

Example 3.1 Let $X = \mathbb{R}$ and $A : \mathbb{R} \rightarrow \mathbb{R}$ be the operator defined by $Au = 0$ for each $u \in \mathbb{R}$. Obviously, A generates a compact C_0 -semigroup of contractions, i.e. the identity of $\mathbb{R}$. Let $\mathcal{D} = \{0\}$ (relatively compact in $\mathbb{R}$) and let $\mathcal{G} = \{g_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $g_n : [0, 2] \rightarrow \mathbb{R}$ is defined by

$$g_n(t) = \begin{cases} t^n & \text{if } t \in [0, 1[\\ 1 & \text{if } t \in [1, 2]. \end{cases}$$

Clearly, $\mathcal{G}$ is of equibounded variation on $[0, 2]$ and, for each $n \in \mathbb{N}^*$, g_n is continuous function on $[0, 2]$. A standard calculation shows that the family of $\mathcal{L}^\infty$ -solution of the problem (1.1) corresponding for $\mathcal{D} = \{0\}$ and $\mathcal{G} = \{g_n; n \in \mathbb{N}^*\}$ is $\{u_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $u_n : [0, 2] \rightarrow \mathbb{R}$ is defined by

$$u_n(t) = \begin{cases} t^n & \text{if } t \in [0, 1[\\ 1 & \text{if } t \in [1, 2]. \end{cases}$$

We mention that we have not the result of Theorem 3.3 for the corresponding problem Cauchy (3.2). Indeed, let $\mathcal{F} = \{f_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $f_n : [0, 2] \rightarrow \mathbb{R}$ is defined a.e. by

$$f_n(t) = \begin{cases} nt^n & \text{if } t \in]0, 1[\\ 0 & \text{if } t \in]1, 2[. \end{cases}$$

We observe that $\mathcal{F}$ lacks uniform integrability in $t = 1$. We can apply Théorème 1 in Baras, Hassan, Veron [4] (see also Theorem 9.4.2, p. 219 in Vrabie [17]), and we obtain that, for each $p \in [1, +\infty[$, $\{u_n; n \in \mathbb{N}^*\}$ is relatively compact subset in $L^p(0, 2; \mathbb{R})$, i.e. there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}^*}$ of $(u_n)_{n \in \mathbb{N}^*}$, and $u \in L^p(0, 2; \mathbb{R})$ such that $\lim_{k \rightarrow \infty} u_{n_k} = u$ in $L^p(0, 2; \mathbb{R})$. Therefore, there exists a subsequence $(u_{n_{k_p}})_{p \in \mathbb{N}^*}$ of $(u_{n_k})_{k \in \mathbb{N}^*}$, such that $\lim_{p \rightarrow \infty} u_{n_{k_p}} = u$ a.e. on $[0, 2]$.

But $\mathcal{G}$ is equicontinuous to the right in variation on $[0, 2]$ and equicontinuous to the left in variation on $]0, 2] \setminus \{1\}$. Then, by Theorem 3.1, it follows that there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}^*}$ of $(u_n)_{n \in \mathbb{N}^*}$ and a function $u : [0, 2] \setminus \{1\} \rightarrow \mathbb{R}$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in [0, 2] \setminus \{1\}$. Moreover, the limit function $u : [0, 2] \setminus \{1\} \rightarrow \mathbb{R}$ is continuous on $[0, 2] \setminus \{1\}$.

Example 3.2 Let Ω be a bounded domain in $\mathbb{R}^k$, $k \in \mathbb{N}^*$, with sufficiently smooth boundary Γ . Take the real Banach space $X = L^1(\Omega)$ and let us define the operator $A : D(A) \subseteq L^1(\Omega) \rightarrow L^1(\Omega)$ by

$$\begin{cases} D(A) = \{u \in W_0^{1,1}(\Omega); \Delta u \in L^1(\Omega)\} \\ Au = \Delta u, \text{ for each } u \in D(A), \end{cases}$$

where Δ is the Laplace operator. It is known that A generates a compact C_0 -semigroup of contractions $\{S(t); t \geq 0\}$ on X . See for instance Vrabie [17], Theorem 7.2.7, p. 160. Let $\xi \in L^1(\Omega)$, $\mathcal{D} = \{\xi\}$ and let $\tilde{\mathcal{G}} = \{\tilde{g}_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $\tilde{g}_n : [0, 2] \rightarrow L^1(\Omega)$ is defined by

$$\tilde{g}_n(t)(x) = g_n(t) \bar{g}(x),$$

while g_n is as in Example 3.1 and $\bar{g} \in L^1(\Omega)$ is a given function with $\|\bar{g}\|_{L^1(\Omega)} = 1$. We observe that $\tilde{\mathcal{G}}$ is of equibounded variation on $[0, 2]$ and, for each $n \in \mathbb{N}^*$, $\tilde{g}_n$ is continuous function on $[0, 2]$.

For each $n \in \mathbb{N}^*$, let us denote by u_n the unique $\mathcal{L}^\infty$ -solution for the Cauchy problem (1.1) corresponding to $(\xi, \tilde{g}_n) \in \mathcal{D} \times \tilde{\mathcal{G}}$. See Definition 1.1.

As in Example 3.1, Theorem 3.3 is not applicable here. Indeed, let $\tilde{\mathcal{F}} = \{\tilde{f}_n; n \in \mathbb{N}^*\}$, where, for each $n \in \mathbb{N}^*$, $\tilde{f}_n : [0, 2] \rightarrow L^1(\Omega)$ is defined a.e. by

$$\tilde{f}_n(t)(x) = f_n(t) \bar{g}(x),$$

while f_n is as in Example 3.1. We observe that $\tilde{\mathcal{F}}$ lacks uniform integrability in $t = 1$. On the other hand, again as in the Example 3.1, we can apply Théorème 1 in Baras, Hassan, Veron [4] (see also Theorem 9.4.2, p. 219 in Vrabie [17]), and we obtain that, for each $p \in [1, +\infty[$, $\{u_n; n \in \mathbb{N}^*\}$ is relatively compact subset in $L^p(0, 2; L^1(\Omega))$, i.e. there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}^*}$ of $(u_n)_{n \in \mathbb{N}^*}$, and $u \in L^p(0, 2; L^1(\Omega))$ such that $\lim_{k \rightarrow \infty} u_{n_k} = u$ in $L^p(0, 2; L^1(\Omega))$. Therefore, there exists a subsequence $(u_{n_{k_p}})_{p \in \mathbb{N}^*}$ of $(u_{n_k})_{k \in \mathbb{N}^*}$, such that $\lim_{p \rightarrow \infty} u_{n_{k_p}} = u$ a.e. on $[0, 2]$.

But a standard calculation shows that $\tilde{\mathcal{G}}$ is equicontinuous to the right in variation on $[0, 2[$ and equicontinuous to the left in variation on $]0, 2] \setminus \{1\}$. Then, by Theorem 3.1, it follows that there exists a subsequence $(u_{n_k})_{k \in \mathbb{N}}$ of $(u_n)_{n \in \mathbb{N}}$ and a function $u : [0, 2] \setminus \{1\} \rightarrow L^1(\Omega)$ such that $\lim_{k \rightarrow \infty} u_{n_k}(t) = u(t)$ pointwise for $t \in [0, 2] \setminus \{1\}$. Moreover, the limit function $u : [0, 2] \setminus \{1\} \rightarrow L^1(\Omega)$ is continuous on $[0, 2] \setminus \{1\}$.

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