The effect of Harnack inequality on the existence and nonexistence results for quasi-linear parabolic equations related to Caffarelli-Kohn-Nirenberg inequalities

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Abstract. In this work we study the problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \lambda \frac{u^\alpha}{|x|^{p(\gamma+1)}} + f \text{ in } \Omega \times (0, T), \\ u \geq 0 \text{ in } \Omega \times (0, T) \text{ and } u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 3$) is a bounded regular domain such that $0 \in \Omega$, $\alpha \geq p - 1$, $-\infty < \gamma < \frac{N-p}{p}$, $\lambda > 0$, $f \in L^1(\Omega \times (0, T))$ and $u_0 \in L^1(\Omega)$ are positive functions. The main points under analysis are some nonexistence results and complete blow-up in the case $p > 2$ and $\gamma + 1 > 0$ and some examples of existence for $(\gamma + 1) > 0$ and $1 < p < 2$. These results are interesting as they prove the role of Harnack inequality in this kind of problems and allow to understand better the blow-up behavior.

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1 Introduction

This paper deals with the following quasilinear parabolic equation

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \lambda \frac{u^\alpha}{|x|^{p(\gamma+1)}} + f & \text{in } \Omega_T \equiv \Omega \times (0, T), \\ u \geq 0 & \text{in } \Omega_T \equiv \Omega \times (0, T) \text{ and } u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (2)$$

where Ω is a bounded regular domain such that $0 \in \Omega$, $\alpha \geq p - 1$, $\lambda > 0$, f and u_0 are positive functions such that $f \in L^1(\Omega \times (0, T))$ and $u_0 \in L^1(\Omega)$.

The cases $p = 2$, $\gamma = 0$ and $p = 2$, $(1 + \gamma) > 0$ have been studied in [9] and [1] respectively. In both cases the complete instantaneous blow-up is proved. In [1] such result appears related to the parabolic Harnack-type inequality and to the best constant in the Hardy-Sobolev inequality. In [12] the existence of solutions is proved in the case $(1 + \gamma) \leq 0$ and $1 < p < N$. We will present here some results in the complementary range. We point out that the corresponding Harnack-type inequality for this kind of operator is obtained in [3].

We first define the *weighted Sobolev space* $\mathcal{D}_\gamma^{1,p}(\Omega)$, which is the completion of $C^\infty(\Omega)$, with respect to the norm

$$\|\phi\|_{p,\gamma} = \left(\int_\Omega (|\phi|^p + |\nabla \phi|^p)|x|^{-p\gamma} dx \right)^{1/p}$$

and we denote by $\mathcal{D}_{0,\gamma}^{1,p}(\Omega)$ the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{p,\gamma}$. The corresponding evolution spaces are denoted by $L^r((0, T); \mathcal{D}_{0,\gamma}^{1,p}(\Omega))$ for different values of r .

Notice that problem (2) is related to the classical Hardy-Sobolev inequality in the case $\gamma = 0$. Using Caffarelli-Kohn-Nirenberg inequalities, see [10], we have the next result that we will use systematically in this paper.

Theorem 1.1 (Hardy-Sobolev inequality) *Let $1 < p < N$ and $-\infty < \gamma < \frac{N-p}{p}$. Then for all $u \in \mathcal{D}_{0,\gamma}^{1,p}(\Omega)$, we have*

$$\begin{aligned} \int_{\mathbb{R}^N} |u|^p |x|^{-p(\gamma+1)} dx &\leq \Lambda_{N,p,\gamma}^{-1} \int_{\mathbb{R}^N} |\nabla u|^p |x|^{-p\gamma} dx, \quad \Lambda_{N,p,\gamma} \\ &= \left(\frac{N - p(\gamma + 1)}{p} \right)^p. \end{aligned} \quad (3)$$

Moreover $\Lambda_{N,p,\gamma}^{-1}$ is optimal and it is not achieved.

To precise the sense in which we understand the solutions we introduce the following concepts.

Definition 1.2 We say that $u \in \mathcal{T}_{0,\gamma}^{1,p}(\Omega_T)$, if $T_k(u) \in L^p((0, T); \mathcal{D}_{0,\gamma}^{1,p}(\Omega))$ for all $k > 0$.

Definition 1.3 Assume that f and u_0 are measurable functions such that $f \in L^1(\Omega \times (0, T))$ and $u_0 \in L^1(\Omega)$. We say that $u \in C([0, T]; L^1(\Omega))$ is an entropy solution to problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma} |\nabla u|^{p-2} \nabla u) = f & \text{in } \Omega_T, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (4)$$

if $u \in \mathcal{T}_{0,\gamma}^{1,p}(\Omega_T)$ and

$$\begin{aligned} & \int_{\Omega} \Theta_k(u - v)(T) dx + \int_0^T \langle v_t, T_k(u - v) \rangle dt \\ & + \int_0^T \int_{\Omega} |\nabla u|^{p-2} \langle \nabla u, \nabla(T_k(u - v)) \rangle |x|^{-p\gamma} dx dt \\ & = \int_{\Omega} \Theta_k(u_0 - v(0)) dx + \int_0^T \int_{\Omega} f T_k(u - v) dx dt, \end{aligned} \quad (5)$$

for all $k > 0$ and $v \in L^p((0, T), \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty(\Omega_T) \cap C([0, T]; L^1(\Omega))$ such that $v_t \in L^{p'}((0, T); \mathcal{D}_{-\gamma}^{-1,p'}(\Omega))$, $p' = p/(p-1)$, and where Θ_k is given by

$$\Theta_k(s) = \int_0^s T_k(t) dt. \quad (6)$$

Notice that $\int_{\Omega} u_t T_k(u) dx = \frac{d}{dt} \left(\int_{\Omega} \Theta(u) dx \right)$ or $\int_{\Omega} \Theta_k(u(t)) dx - \int_{\Omega} \Theta_k(u(0)) dx = \int_0^t \int_{\Omega} u_t T_k(u) dx dt$.

Definition 1.4 We say that $u \in C([0, T]; L^1(\Omega))$ is an entropy solution to problem (2) if $F(x, t) \equiv \frac{u^\alpha}{|x|^{p(\gamma+1)}} + f \in L^1(\Omega_T)$ and moreover u is an entropy solution in the sense of Definition 1.3 with data F and u_0 .

Using the same argument as in [5] (see also [18]) one can easily prove the following result.

Theorem 1.5 Let Ω be a bounded regular domain that contains 0, $f \in L^1(\Omega \times (0, T))$, $u_0 \in L^1(\Omega)$, then problem (4) has one and only one entropy solution.

And, as a consequence, we have the following comparison principle

Lemma 1.6 *Let u, v be an entropy solutions of (4) with data f, u_0 and g, v_0 respectively. Suppose that $0 \leq g \leq f$ and $v_0 \leq u_0$, then $0 \leq u \leq v$.*

The paper is organized as follows. In Section 2 we prove the nonexistence result in the case where $p > 2, \gamma + 1 > 0$ and $\alpha > p - 1$, namely Theorem 2.3. As a consequence we obtain the result of the complete blow-up, Theorem 2.6. In the subsection 2.2 we study the case $\alpha = p - 1$ and $\lambda > \Lambda_{N,p,\gamma}$. Notice that in this case we also extend the results of nonexistence in [17] for the case $\gamma = 0$ by a different method of proof.

In section 2.3, we prove some nonexistence results for the case $1 < p < 2$ by adding some positive constant to the right hand side of the equation, namely studying the problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}} + c \text{ in } \Omega_T, \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega, \end{cases} \quad (7)$$

where $\alpha > 1$ and $c > 0$.

To handle the lack of *selfadjointness* in problem (2) it is worthy to point out that we will use a general version of Picone's type inequality that holds for functions in Sobolev spaces, see [2].

Section 3 is devoted to study some cases of existence in two directions:

- i) If $(\gamma + 1) \leq 0$ and for all $p > 1$. We prove the existence of a supersolution that only depends on the time, therefore modulo an iteration argument we get the existence result at least for t small.
- ii) In the case $1 < p < 2$ and $(\gamma + 1) > 0$, we assume that $p - 1 < \alpha < \frac{N - p(\gamma + 1)}{N}$. We find a singular supersolution that allows us to prove existence of a global solution, which must be understood conveniently according with the size of α in the prescribed interval.

These results prove the crucial role that plays the Harnack inequality in this theory: in the case *i*) the lack of Harnack inequality is motivated by the condition $(\gamma + 1) \leq 0$, if $p \geq 2$, while in the case *ii*) is motivated by the fact that $1 < p < 2$.

Notation. We will denote

$$\oint_{B_\rho(x_0)} f(x) dx \equiv \frac{1}{|B_\rho|} \int_{B_\rho(x_0)} f(x) dx.$$

2 Nonexistence results of entropy solutions: Blow-up

2.1 The case $p > 2$ and $\alpha > p - 1$

In this section we deal with the following problem,

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}} & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (8)$$

We suppose that $u_0 \in L^1(\Omega)$ satisfies the following hypothesis

$$(H1) \quad u_0 \not\equiv 0 \quad \text{and} \quad u_0 > 0 \quad \text{in} \quad B_\delta(0).$$

A difficulty in Problem (8) is the lack of *selfadjointness* that could be overcome by the following general version of Picone's type inequality (see [2] for details).

Theorem 2.1 *Let $u \in \mathcal{D}_{0,\gamma}^{1,p}(\Omega)$, $v \in \mathcal{D}_{0,\gamma}^{1,p}(\Omega)$ and $-\operatorname{div}(|x|^{-p\gamma}|\nabla v|^{p-2}\nabla v) = \nu$ a Radon positive bounded measure with $v|_{\partial\Omega} = 0$ and $v \not\equiv 0$. Then,*

$$\int_{\Omega} |\nabla u|^p |x|^{-p\gamma} dx \geq \int_{\Omega} \frac{|u|^p}{v^{p-1}} \left(-\operatorname{div}(|x|^{-p\gamma}|\nabla v|^{p-2}\nabla v) \right) dx. \quad (9)$$

We also use the following weak Harnack inequality whose proof can be found in [3]

Theorem 2.2 *Let u be a non-negative weak solution to problem*

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega_T, \\ u \geq 0 & \text{in } \Omega_T. \end{cases} \quad (10)$$

Let $(x_0, t_0) \in \Omega_T$ be fixed and assume that $u(x_0, t_0) > 0$, then there exist $C_1, C_2 > 1$ such that

$$u(x_0, t_0) \leq C_1 \inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta), \quad \text{where } \theta = \frac{C_2 \rho^{p(\gamma+1)}}{u^{p-2}(x_0, t_0)}. \quad (11)$$

As a consequence, there exists $B(N, p, \gamma) > 1$ such that for all $(x_0, t_0) \in \Omega_T$, for all θ , and for all $\rho > 0$ verifying $Q_{4\rho}(\theta) \equiv B_{4\rho}(x_0) \times (t_0 - \theta, t_0) \subset \Omega_T$ we have

$$\begin{aligned} \int_{B_\rho(x_0)} u(x, t_0) dx &\leq B \left[\left(\frac{\rho^{p(\gamma+1)}}{\theta} \right)^{\frac{1}{p-2}} \right. \\ &\quad \left. + \left(\frac{\theta}{\rho^{p(\gamma+1)}} \right)^{\frac{N}{p(\gamma+1)}} \left(\inf_{B_\rho(x_0)} u(\cdot, t_0 + \theta) \right)^{\lambda_\gamma/p(\gamma+1)} \right], \end{aligned} \quad (12)$$

where $\lambda_\gamma = N(p-2) + p(\gamma+1)$.

Notice that in the case $\Omega \equiv \mathbb{R}^N$, $p > 2$, $(1 + \gamma) > 0$, using similar computations as in the case $\gamma = 0$, we have the localized fundamental solution for the equation $w_t - \operatorname{div}(|x|^{-p\gamma}|\nabla w|^{p-2}\nabla w) = 0$,

$$P_{k,\rho}(x,t) = \frac{k\rho^N}{S^{\frac{1}{\lambda_\gamma}}(t)} \left[1 - \left(\frac{|x|}{S^{\frac{1}{\lambda_\gamma}}(t)} \right)^{\frac{p(\gamma+1)}{p-1}} \right]^{\frac{p-1}{p-2}}_+, \quad (13)$$

where $S(t) = \lambda_\gamma \left(\frac{p(\gamma+1)}{p-2} \right)^{p-1} k^{p-2} \rho^{N(p-2)} (t - \tilde{t}) + \rho^{\lambda_\gamma}$, for $t \geq \tilde{t}$ and k, ρ are positive constants. Notice that $P_{k,\rho} \leq k$ and for $t = \tilde{t}$, $P_{k,\rho}$ vanishes outside the ball $B_\rho(0)$. For $\tilde{t} \leq t \leq t_1$, the support of $P_{k,\rho}$ is given by

$$D = \{|x| < S^{\frac{1}{\lambda_\gamma}}(t)\} \times [\tilde{t}, t_1],$$

which is contained in the cylinder $Q \equiv B_{S^{\frac{1}{\lambda_\gamma}}(t_1)}(0) \times [\tilde{t}, t_1]$.

The main result of this section is the following theorem.

Theorem 2.3 *Assume that $p > 2$, $u_0 \geq 0$ and that (H1) holds, then equation (8) has no entropy positive solution.*

Proof. Using truncating arguments and by comparison we can assume that $u_0 \in L^\infty(\Omega)$. Suppose that (8) has a nonzero positive entropy solution u . Define

$$a_n(x) = \min \left\{ n, \frac{1}{|x|^{p(\gamma+1)}} \right\} \quad \text{and} \quad g_n(x,t) = \inf \{n, u^\alpha\}. \quad (14)$$

Let $f_n(x,t) = a_n(x)g_n(x,t)$; then $f_n(x,t) \in L^\infty(\Omega_T)$ and $f_n \uparrow f(x,t) = \frac{u^\alpha}{|x|^{p(\gamma+1)}}$ in $L^1(\Omega_T)$. Let u_n be the unique solution to problem

$$\begin{cases} (u_n)_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) = f_n(x,t) & \text{in } \Omega_T, \\ u_n = 0 & \text{in } \partial\Omega \times (0,T), \\ u_n(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (15)$$

Since $\{f_n\}$ is an increasing sequence, then $\{u_n\}$ is also increasing and therefore we conclude that the sequence $\{u_n\}$ converges to u in $C([0,T]; L^1(\Omega))$ and almost everywhere. Since $u \not\equiv 0$, then $u_n \not\equiv 0$.

Since $p > 2$ and u_0 satisfies (H1), then using the weak Harnack inequality (see [3]) we get the existence of $c_1, \eta > 0$ such that

$$u_1 \geq c_1 \text{ a.e. } B_\eta \times (\tau, T),$$

for some $0 < \tau < T$ and $B_\eta = B_\eta(0)$. Therefore we conclude that

$$u \geq u_n \geq c_1 \text{ a.e. } B_\eta \times (\tau, T).$$

Let $v(x) = c \left(\log \frac{1}{|x|} - \log \frac{1}{|\eta|} \right)$, then we can chose $c = c(\eta, p, \varepsilon)$ such that

$$\begin{cases} -(\operatorname{div}(|x|^{-p\gamma}|\nabla v|^{p-2}\nabla v) = \frac{\varepsilon}{|x|^{p(\gamma+1)}} \text{ in } B_\eta(0), \\ v|_{\partial B_\eta(0)} = 0. \end{cases} \quad (16)$$

By setting $w(x, t) = (t - \tau)v(x)$, we obtain that w satisfies the equation,

$$\begin{cases} w_t - (\operatorname{div}(|x|^{-p\gamma}|\nabla w|^{p-2}\nabla w) = v(x) + (t - \tau)^{p-1} \frac{\varepsilon}{|x|^{p(\gamma+1)}} \text{ in } B_\eta(0) \times (\tau, T), \\ w|_{\partial B_\eta(0)} = 0. \end{cases} \quad (17)$$

Notice that $\lim_{t \rightarrow \tau} \int_{B_\eta(0)} w(x, t) \phi(x) dx = 0$, for all $\phi \in C_0^\infty(B_\eta(0))$. Then w is in particular an entropy positive solution to problem (17). It is obvious that

$$w_t - \operatorname{div}(|x|^{-p\gamma}|\nabla w|^{p-2}\nabla w) \leq v(x) + T^{p-1} \frac{\varepsilon}{|x|^{p(\gamma+1)}} \text{ and } \lim_{|x| \rightarrow 0} |x|^{p(\gamma+1)} v(x) = 0.$$

Then we chose $\varepsilon < c_1$ and η such that $w_t - \operatorname{div}(|x|^{-p\gamma}|\nabla w|^{p-2}\nabla w) \leq \frac{\varepsilon}{|x|^{p(\gamma+1)}}$ in $B_\eta(0) \times (\tau, T)$.

Let w_n be the unique solution to problem

$$\begin{cases} (w_n)_t - \operatorname{div}(|x|^{-p\gamma}|\nabla w_n|^{p-2}\nabla w_n) = c_1 a_n(x) \text{ in } B_\eta(0) \times (\tau, T), \\ (w_n)|_{\partial B_\eta(0)} = 0, \\ w_n(x, \tau) = 0. \end{cases} \quad (18)$$

By comparison,

$$w(x, t) \leq w_1(x, t) = \lim_{n \rightarrow \infty} w_n(x, t).$$

Since $u_n(x, t) \geq u_1(x, t) \geq c_1$, using the comparison principle we conclude that $u_n(x, t) \geq w_n(x, t)$ for all $(x, t) \in (\tau, T) \times B_\eta(0)$.

Taking into account that $\lim_{|x| \rightarrow 0} w(x, t) = \infty$ for all $t > \tau$, we get

$$\lim_{|x| \rightarrow 0} \lim_{n \rightarrow \infty} u_n(x, t) = \infty \quad \forall t \in (\tau, T).$$

Hence $\lim_{|x| \rightarrow 0} u(x, t) = \infty$ uniformly for $t \in (t_1, t_2)$ where $\tau < t_1 < t_2 < T$, in the sense that for all $c > 0$ there exist $n_0 \in \mathbb{N}$ and $\eta_1 > 0$ such that $u_n(x, t) \geq c$ for all $x \in B_{\eta_1}(0)$ and for all $t \in (t_1, t_2)$.

Hereafter we consider τ and T fixed to have $u_n(x, t) \geq c$ in $(x, t) \in B_{\eta_1}(0) \times (t_1, t_2)$. We take $\Lambda_{N,p,\gamma}$, the optimal constant in the Hardy-Sobolev inequality

(see Theorem 1.1). Since $p > 2$ there exists $c > 1$ such that

$$\begin{cases} \left(\frac{(\alpha-(p-1))}{p-1} \right)^{p-1} c^{\alpha-(p-1)} - \frac{1}{t_2 - t_1} > \Lambda_{N,p,\gamma}, \\ \left(\frac{(\alpha-(p-1))}{p-1} \right)^{p-1} \frac{2}{p(2c)^p \left(2^{\frac{\alpha-(p-1)}{p-1}} - 1 \right)^{p-1}} << 1. \end{cases} \quad (19)$$

In a similar way as in [9], we define the function,

$$\phi(s) = \begin{cases} \frac{p-1}{\alpha-(p-1)} \left(\frac{1}{c^{\frac{\alpha-(p-1)}{p-1}}} - \frac{1}{s^{\frac{\alpha-(p-1)}{p-1}}} \right) & \text{if } s > c, \\ \frac{1}{c^{\frac{\alpha}{p-1}}} (s - c) & \text{if } s \leq c. \end{cases} \quad (20)$$

Notice that $\phi \in C^1(\mathbb{R})$ is a concave function, $\phi'(s) = \frac{1}{s^{\frac{\alpha}{p-1}}}$ and $0 < \phi'(s) < a$ for all $s > c$. Since $\phi(s)$ is a regular function for $s > c$, and $u_n > \bar{c}$ for all $n \geq n_0 \in \mathbb{N}$, there results that

$$\begin{aligned} -\operatorname{div}(|x|^{-p\gamma}|\nabla(\phi(u_n))|^{p-2}\nabla(\phi(u_n))) &= \\ \phi'(u_n)^{p-1}(-\operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) - (p-1)|x|^{-p\gamma}\phi''(u_n)(\phi'(u_n))^{p-2}|\nabla u_n|^p). \end{aligned}$$

Moreover, using the fact that $u_n(x) > c$ in $B_\eta(0)$, there results that $\phi''(u_n) < 0$, hence

$$\begin{aligned} -\operatorname{div}(|x|^{-p\gamma}|\nabla(\phi(u_n))|^{p-2}\nabla(\phi(u_n))) &\geq \phi'(u_n)^{p-1}(-\operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n)) \\ &\geq (\phi'(u_n))^{p-1}f_n - (\phi'(u_n))^{p-1}(u_n)_t. \end{aligned}$$

Let $B_{\bar{\eta}}(0) \subset\subset B_\eta(0)$ and consider $\psi \in C_0^\infty(B_{\bar{\eta}}(0))$. Using Theorem 2.1 we obtain

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{-(\operatorname{div}(|x|^{-p\gamma}|\nabla(\phi(u_n))|^{p-2}\nabla(\phi(u_n))))}{\phi(u_n)^{p-1}}.$$

Hence

$$\begin{aligned} \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx &\geq \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{\phi'(u_n)^{p-1}}{(\phi(u_n))^{p-1}} dx \\ &\quad - \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx. \end{aligned} \quad (21)$$

By integration in t we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx dt &\geq \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{(\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx dt \\ &\quad - \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx dt. \end{aligned}$$

Since $x \in B_{\bar{\eta}}(0)$, by choosing n very large, we can assume that $u_n(x) > 2c$. Hence

$$\frac{1}{(\phi(u_n))^{p-1}} \geq \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} c^{\alpha-(p-1)}.$$

On the other hand $\phi'(u_n)^{p-1} f_n \rightarrow \phi'(u)^{p-1} \frac{u^\alpha}{|x|^{p(\gamma+1)}} \text{ a.e. in } \Omega$. Since $\phi'(u_n) \leq a$ for some positive constant a and for all $n \geq n_0$, we obtain that $\phi'(u_n)^{p-1} f_n \leq a^{p-1} f$. Therefore by the *dominated convergence theorem* we conclude that

$$\phi'(u_n)^{p-1} f_n \rightarrow \phi'(u)^{p-1} \frac{u^\alpha}{|x|^{p(\gamma+1)}} = \frac{1}{|x|^{p(\gamma+1)}} \text{ in } L^1(B_{\bar{\eta}}(0)).$$

Using Fatou's Lemma we obtain

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{(\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} &\geq (t_2 - t_1) \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} \\ &\quad c^{\alpha-(p-1)} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} + o(1). \end{aligned}$$

To estimate the part where we have the derivative with respect to t , we define

$$\begin{aligned} I(t_2, t_1)(x) &= \int_{t_1}^{t_2} \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dt \\ &= \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} \int_{t_1}^{t_2} \frac{c^{\alpha-(p-1)} (u_n)_t dt}{(u_n)^{p-1} \left((u_n)^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}}. \end{aligned}$$

Since

$$\frac{(u_n)_t dt}{(u_n)^{p-1} \left((u_n)^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}} = \frac{d}{dt} F(u_n(t)),$$

where

$$F(s) = \int_{2c}^s \frac{dy}{y^{p-1} \left(y^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}},$$

we obtain

$$\begin{aligned} I(t_2, t_1)(x) &= \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} c^{\alpha-(p-1)} \int_{t_1}^{t_2} \frac{d}{dt} F(u_n(t)) \\ &= \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} c^{\alpha-(p-1)} (F(u_n(t_2)) - F(u_n(t_1))). \end{aligned}$$

Since $p > 2$ and $c \gg 1$, we obtain that

$$\begin{aligned} F(s) &\leq \frac{1}{c^{\alpha-(p-1)} \left(2^{\frac{\alpha-(p-1)}{p-1}} - 1\right)^{p-1}} \int_{2c}^{\infty} \frac{1}{y^{p-1}} dy \\ &\leq \frac{1}{c^{\alpha-(p-1)} \left(2^{\frac{\alpha-(p-1)}{p-1}} - 1\right)^{p-1}} \frac{1}{p(2c)^p}. \end{aligned}$$

Hence

$$\begin{aligned} |I(t_1, t_2)(x)| &\leq \left(\frac{\alpha - (p-1)}{p-1}\right)^{p-1} c^{\alpha-(p-1)} (F(u_n(t_2)) + F(u_n(t_1))) \\ &\leq \left(\frac{\alpha - (p-1)}{p-1}\right)^{p-1} \frac{2}{p(2c)^p \left(2^{\frac{\alpha-(p-1)}{p-1}} - 1\right)^{p-1}}. \end{aligned}$$

Moreover, using the fact that

$$\int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} = \int_{B_{\bar{\eta}}(0)} |\psi|^p I(t_2, t_1)(x) dx,$$

we get

$$\begin{aligned} &\left| \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{\phi(u_n)^{p-1}} dx dt \right| \\ &\leq \left(\frac{\alpha - (p-1)}{p-1}\right)^{p-1} \frac{2}{p(2c)^p \left(2^{\frac{\alpha-(p-1)}{p-1}} - 1\right)^{p-1}} \int_{B_{\bar{\eta}}(0)} |\psi|^p dx. \end{aligned}$$

Since $\bar{\eta} \ll 1$, then by (19) we conclude that

$$\left| \int_{t_1}^{t_2} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{\phi(u_n)^{p-1}} dx dt \right| \leq \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx.$$

Therefore,

$$\begin{aligned} &\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \\ &\geq \left(\frac{\alpha - (p-1)}{p-1}\right)^{p-1} c^{\alpha-(p-1)} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx - \frac{1}{t_2 - t_1} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx, \end{aligned}$$

and then

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq A \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} \text{ for all } \psi \in \mathcal{C}_0^\infty(B_{\bar{\eta}}(0)),$$

where

$$A = \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} c^{\alpha-(p-1)} - \frac{1}{t_2 - t_1}.$$

Using again (19) we obtain that $A > \Lambda_{N,p,\gamma}$, a contradiction with the optimality of $\Lambda_{N,p,\gamma}$. Hence the result follows. $\square$

Corollary 2.4 *Suppose that $f \in L^1(\Omega \times (0, T))$ and $u_0 \in L^1(\Omega)$ are nonnegative functions. Assume that one of the following condition holds*

1. u_0 satisfies (H1),
2. $f > 0$ in $B_r(0) \times (0, \tau) \subset \Omega_T$,

then problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma} |\nabla u|^{p-2} \nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}} + f, & u \geq 0 \text{ in } \Omega_T, \\ u = 0 & \text{in } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (22)$$

has no positive entropy solution.

As consequence of the non-existence results studied above we can drive some precise information on the behavior of the approximate problems, precisely we will prove that such solutions blow-up in a strong sense that we will state later.

Lemma 2.5 *Let u_n be the minimal positive solution to problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(|x|^{-p\gamma} |\nabla u_n|^{p-2} \nabla u_n) = a_n(x) g_n(u_n) & \text{in } \Omega_T, \\ u_n \geq 0 & \text{in } \Omega_T \text{ and } u_n|_{\partial\Omega} = 0, \\ u_n(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (23)$$

where $a_n(x) = \min \left\{ n, \frac{1}{|x|^{p(\gamma+1)}} \right\}$ and $g_n(s) = \min\{n, s^\alpha\}$, $\alpha > (p-1)$. Assume that $u_0 \in L^\infty(\Omega)$ and that (H1) holds. Then for all $r > 0$ such that $B_{4r}(0) \subset \Omega$ and for all $t \in (0, T)$ we have

$$\int_{B_r(0)} u_n(x, t) dx \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (24)$$

Proof. We argue by contradiction. Assume the existence of $\tau > 0$, $r_0 > 0$ such that

$$\int_{B_{r_0}(0)} u_n(x, \tau) dx \leq C \text{ for all } n.$$

Without loss the generality we can assume that $B_{4r_0}(0) \subset \Omega$.

Since $\{u_n\}$ is an increasing sequence, we get the existence of a measurable function $u(x, \tau) \in L^1(B_{r_0})$ such that $u_n(x, \tau) \uparrow u(x, \tau)$ for all $x \in B_{r_0}(0)$. Since $u_0 \neq 0$ in $B_{r_0}(0)$ then $u \neq 0$ and $u < \infty$ a.e., in particular $\inf_{B_{r_0}(0)} u_n(x, \tau) < C$, for all n .

Then using estimate (12) we obtain that

$$\int_{B_{r_0}(0)} u_n(x, t) dx \leq C(r_0, \tau), \quad \text{for all } 0 < t \leq \tau.$$

Hence we are able to define $u(x, t)$, the almost everywhere and L^1 limit of $u_n(x, t)$ for $(x, t) \in B_{r_0} \times (0, \tau)$ and moreover

$$\int_{t_1}^{t_2} \int_{B_{r_0}(0)} u(x, t) dx dt < \infty, \quad (t_1, t_2) \subset (0, \tau).$$

Since $u_1(x, t) > 0$ for $x \in B_r(0)$, then using the regularity of u_1 we get the existence of $c_1 > 0$ such that $u_1 \geq c_1$ in $B_r(0) \times (\frac{\tau}{4}, \tau)$.

In a similar way to the proof of Theorem 2.3, if we consider $w(x, t) = c(t - \frac{\tau}{4}) \left(\log \frac{1}{|x|} - \log \frac{1}{|\eta|} \right)$, then we obtain that $u(x, t) \geq w(x, t)$ for all $(x, t) \in B_r(0) \times (\frac{\tau}{4}, \tau)$ and then

$$\lim_{|x| \rightarrow 0} u(x, t) = \lim_{|x| \rightarrow 0} \lim_{n \rightarrow \infty} u_n(x, t) = \infty \text{ uniformly for } t \in \left(\frac{\tau}{2}, \tau \right).$$

Hence, for all $C > 0$ there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ we have $u_n \geq C$ in $B_\eta(0) \times (t_1, t_2)$. In particular, we can chose C such that

$$C^{\alpha-(p-1)} - \frac{1}{(t_2 - t_1)C^{p-2}} > \Lambda_{N,p,\gamma}. \quad (25)$$

Consider $\phi \in C_0^\infty(B_r(0))$ with $r < \eta$; using Theorem 2.1 we obtain the following estimate,

$$\int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx \geq \int_{B_r(0)} \frac{-\operatorname{div}(|x|^{-p\gamma} |\nabla u_n|^{p-2} \nabla u_n)}{u_n^{p-1}} \phi^p dx.$$

Using (23) we get

$$\int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx \geq \int_{B_r(0)} \frac{a_n(x) g_n(u_n)}{u_n^{p-1}} \phi^p dx - \int_{B_r(0)} \frac{\partial_t u_n}{u_n^{p-1}} \phi^p dx.$$

By integration with respect to t we conclude that

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx dt &\geq \int_{t_1}^{t_2} \int_{B_r(0)} \frac{a_n(x) g_n(u_n)}{u_n^{p-1}} \phi^p dx dt \\ &\quad + \frac{1}{p-2} \int_{B_r(0)} \left(\frac{1}{u_n^{p-2}(x, t_2)} - \frac{1}{u_n^{p-2}(x, t_1)} \right) \phi^p dx, \end{aligned}$$

hence

$$\begin{aligned} \int_{t_1}^{t_2} \int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx dt &\geq \int_{t_1}^{t_2} \int_{B_r(0)} \frac{a_n(x)g_n(u_n)}{u_n^{p-1}} \phi^p dx dt \\ &\quad - \frac{1}{p-2} \int_{B_r(0)} \frac{\phi^p}{u_n^{p-2}(x, t_1)} dx. \end{aligned}$$

Since $\{u_n\}$ is increasing we obtain that $-\frac{\phi^p}{u_n(x, t_1)^{p-2}} \geq -\frac{\phi^p}{C^{p-2}}$, and then,

$$\int_{t_1}^{t_2} \int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx dt + \int_{B_r(0)} \frac{\phi^p}{C^{p-2}} dx \geq \int_{t_1}^{t_2} \int_{B_r(0)} \frac{a_n(x)g_n(u_n)}{u_n^{p-1}} dx dt.$$

Notice that for $(x, t) \in B_r(0) \times (t_1, t_2)$, we have $\frac{a_n(x)g_n(u_n)}{u_n^{p-1}} \rightarrow \frac{u^{\alpha-(p-1)}}{|x|^{p(\gamma+1)}} \geq \frac{C^{\alpha-(p-1)}}{|x|^{p(\gamma+1)}}$ where C satisfies (25). Hence we conclude that

$$\begin{aligned} &\int_{B_r(0)} |\nabla \phi|^p |x|^{-p\gamma} dx dt \\ &\geq \left(C^{\alpha-(p-1)} - \frac{1}{(t_2 - t_1)C^{p-2}} \right) \int_{B_r(0)} \frac{|\phi|^p}{|x|^{p(\gamma+1)}} dx, \text{ for all } \phi \in \mathcal{C}_0^\infty(B_r(0)). \end{aligned}$$

Taking into account the condition (25), we get a contradiction with the optimality of $\Lambda_{N,p,\gamma}$.

Hence the result follows. $\square$

We can state and prove the main result of this section.

Theorem 2.6 *Assume u_n be the unique positive solution to problem (23) where u_0 a bounded function that satisfies (H1): then, for all $(x_0, t_0) \in \Omega_T$, we have $\lim_{n \rightarrow \infty} u_n(x_0, t_0) = \infty$.*

Proof. Consider $(x_0, t) \in \Omega_T$, such that $x_0 \in B_r(0)$, with $B_{4r}(0) \subset \Omega$ for some $r > 0$. Then using Harnack inequality Theorem 2.2, we get

$$u_n(x_0, t) \geq c_1 + c_2 \int_{B_r(0)} u_n(x, t - \varepsilon) dx,$$

where ε will be chosen in such a way that $t - \varepsilon > 0$. Using again Lemma 2.5 we obtain that

$$\int_{B_r(0)} u_n(x, t - \varepsilon) dx \rightarrow \infty, \text{ hence } \lim_{n \rightarrow \infty} u_n(x_0, t) = \infty.$$

Assume that $x_0 \notin B_r(0)$; if $B_r(0) \cap B_r(x_0)$ is not empty, then there exists $y \in \Omega$ such that $B_\eta(y) \subset B_r(0) \cap B_r(x_0)$. Therefore using again Harnack inequality and the result in Lemma 2.5 to conclude that

$$\begin{aligned} u_n(x_0, t) &\geq c_1 + c_2 \int_{B_r(x_0)} u_n(x, t - \varepsilon) dx \geq c_1 + c_2 \int_{\bar{B}_\eta(y)} u_n(x, t - \varepsilon) dx \\ &\geq c_1 + c_2 \inf_{x \in \bar{B}_\eta(y)} u_n(x, t - \varepsilon) \geq c_1 + c_2 u_n(x_1, t - \varepsilon). \end{aligned}$$

Since $x_1 \in B_r(0)$, $u_n(x_1, t - \varepsilon) \rightarrow \infty$, then we obtain that $u_n(x_0, t - \varepsilon) \rightarrow \infty$ for $n \rightarrow \infty$.

We consider now the general case where x_0 is arbitrary in Ω . Since $d(x, B_r(0)) < \infty$, by iterating the above argument in a finite number of steps we reach the blow-up region and we obtain the global blow-up result. $\square$

Remark 2.7 If the support of the initial data does not contain the origin, the result in Theorem 2.6 is an open problem.

2.2 The case $p > 2$ and $\alpha = p - 1$

Using the same argument as in Theorem 2.3 we give a simple proof of the nonexistence result given in [17] for a more general class of problems. More precisely we have the following Theorem.

Theorem 2.8 Assume that u_0 satisfies (H1) and that $\lambda > \Lambda_{N,p,\gamma}$, then problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma} |\nabla u|^{p-2} \nabla u) = \lambda \frac{u^{p-1}}{|x|^{p(\gamma+1)}}, & u \geq 0 \text{ in } \Omega_T, \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), \end{cases} \quad (26)$$

has no positive entropy solution.

Proof. We argue by contradiction. Suppose that $u \geq 0$ is an entropy solution to problem (26); then we define u_n to be the unique positive solution to problem

$$\begin{cases} (u_n)_t - \operatorname{div}(|x|^{-p\gamma} |\nabla u_n|^{p-2} \nabla u_n) = \lambda a_n(x) g_n(u) \text{ in } \Omega_T, \\ u_n = 0 \text{ in } \partial\Omega \times (0, T), \text{ and } u_n(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \quad (27)$$

where a_n and g_n are given by (14) with $\alpha = 1$. Since $p > 2$, $u_0 \geq 0$ and as in the case $\alpha > 1$, we can prove that for all $T > 0$, there exists $0 < \tau < T$ such that

$$\lim_{|x| \rightarrow 0} \lim_{n \rightarrow \infty} u_n(x, t) = \infty \quad \forall t \in (\tau, T),$$

namely, $\lim_{|x| \rightarrow 0} u(x, t) = \infty$ for all $t \in (\tau, T)$.

Therefore, for all $c > 0$ there exist $n_0 \in \mathbb{N}$ and $\eta_1 > 0$ such that $u_n(x, t) \geq c$ for all $(x, t) \in B_{\eta_1}(0) \times (\tau, T)$ and for all $n > n_0$. Since $\lambda > \Lambda_{N,p,\gamma}$, then for τ and T fixed, we can chose c such that

$$\lambda - \frac{1}{(p-2)(T-\tau)} \frac{1}{c^{p-2}} > \Lambda_{N,p,\gamma}. \quad (28)$$

Let $B_{\bar{\eta}}(0) \subset\subset B_{\eta}(0)$ and consider $\psi \in C_0^\infty(B_{\bar{\eta}}(0))$. Using Theorem 2.1, we obtain that

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{-\operatorname{div}(|x|^{-p\gamma} |\nabla u_n|^{p-2} \nabla u_n)}{u_n^{p-1}} dx$$

and by the definition of u_n we obtain,

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \lambda \int_{B_{\bar{\eta}}(0)} |\psi|^p a_n(x) \frac{g_n(u)}{u_n^{p-1}} dx - \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t}{u_n^{p-1}} dx. \quad (29)$$

Integrating in time,

$$\begin{aligned} \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx dt &\geq \lambda \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\psi|^p a_n(x) \frac{g_n(u)}{u_n^{p-1}} dx dt \\ &\quad - \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t}{u_n^{p-1}} dx dt, \end{aligned}$$

namely,

$$\begin{aligned} \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx dt &\geq \lambda \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\psi|^p a_n(x) \frac{g_n(u)}{u_n^{p-1}} dx dt \\ &\quad - \frac{1}{p-2} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{u_n^{p-1}(x, \tau)} dx dt. \end{aligned}$$

Since $x \in B_{\bar{\eta}}(0)$, then $u_n(x) > 2\bar{c}$. Hence we get

$$g_n(u) a_n(x) \uparrow \frac{u^{p-1}}{|x|^{p(\gamma+1)}} \in L^1(B_{\bar{\eta}}(0)) \quad \text{and} \quad \frac{|\psi|^p}{u^{p-1}} \in L^\infty(B_{\bar{\eta}}(0)).$$

Therefore using Lebesgue Theorem we conclude that

$$\lambda \int_{\tau}^T \int_{B_{\bar{\eta}}(0)} |\psi|^p a_n(x) \frac{g_n(u)}{u_n^{p-1}} dx dt \rightarrow \lambda(T-\tau) \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx.$$

Since $\bar{\eta} \ll 1$, we obtain

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \left(\lambda - \frac{1}{(p-2)(T-\tau)} \frac{1}{c^{p-2}} \right) \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx.$$

Notice that by (28), we have $\lambda - \frac{1}{(p-2)(T-\tau)} \frac{1}{c^{p-2}} > \Lambda_{N,p,\gamma}$ and then we reach a contradiction with the optimality of $\Lambda_{N,p,\gamma}$. Hence the result follows. $\square$

We have a similar blow-up result as in Theorem 2.6.

Theorem 2.9 *Let u_n be the minimal positive solution to problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) = \lambda a_n(x)g_n(u_n) \text{ in } \Omega_T, \\ u_n \geq 0 \text{ in } \Omega \text{ and } u_n|_{\partial\Omega} = 0, \\ u_n(x, 0) = u_0(x), \end{cases} \quad (30)$$

with $u_0(x) > 0$ in $B_r(0)$, then for all $(x_0, t_0) \in \Omega_T$, we have $\lim_{n \rightarrow \infty} u_n(x_0, t_0) = \infty$.

We omit the details of the proof which is exactly the same that the one of Theorem 2.6.

2.3 Further nonexistence results

In this section we deal with problem,

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}} + c \text{ in } \Omega_T, \\ u \geq 0 \text{ and } u = 0 \text{ in } \partial\Omega \times (0, T), \\ u(x, 0) = u_0 \geq 0 \text{ in } \Omega, \end{cases} \quad (31)$$

where $1 < p$, $c > 0$, $(1 + \gamma) > 0$ and $\alpha > \max\{p - 1, 1\}$. The case $p > 2$ and u_0 satisfying (H1) has been studied above, even if $c = 0$. The main feature now is that here we are able to consider $1 < p < 2$ and initial data without the assumption (H1). Notice that if $1 < p < 2$, the homogeneous parabolic operator doesn't satisfy a Harnack inequality and this is the reason for which we study the perturbed problem with a constant. To simplify the typing we will assume, without loss of generality, that $u_0 \equiv 0$. The main result of this section is the following.

Theorem 2.10 *Assume that $p > 1$, $\alpha > \max\{p - 1, 1\}$, $u_0 \equiv 0$ and $c > 0$. Then problem (31) has no positive entropy solution*

Proof. We use the same argument as in the proof of Theorem 2.3. Assume that problem (31) has an entropy solution $u \geq 0$. Defining $f(x, t) = \frac{u^\alpha}{|x|^{p(\gamma+1)}}$ and $f_n(x, t) = \min\{n, f(x, t)\}$, we obtain that $f_n \in L^\infty(\Omega_T)$ and $f_n \uparrow f$ in $L^1(\Omega_T)$. Let u_n be the solution to problem

$$\begin{cases} (u_n)_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) = f_n + c \text{ in } \Omega \times (0, T), \\ u_n = 0 \text{ in } \partial\Omega \times (0, T), \\ u_n(x, 0) = 0 \text{ in } \Omega. \end{cases} \quad (32)$$

Since $c > 0$, then $u_n \not\equiv 0$ and u_n converges to u in $C([0, T]; L^1(\Omega))$. Notice that the sequence $\{u_n\}$ is increasing in n .

Let z be the solution to the elliptic problem

$$-\operatorname{div}(|x|^{-p\gamma}|\nabla z|^{p-2}\nabla z) = c_1 \text{ in } B_\sigma(0) \text{ and } z|_{\partial B_\sigma(0)} = 0.$$

If we define $v(x, t) = tz(x)$, then

$$\begin{cases} v_t - \operatorname{div}(|x|^{-p\gamma}|\nabla v|^{p-2}\nabla v) = z(x) + t^{p-1}c_1 \text{ in } B_\sigma(0) \times (0, T), \\ v(x, t) = 0 \text{ on } \partial B_\sigma(0) \times (0, T), \\ v(x, 0) = 0 \text{ in } B_\sigma(0). \end{cases} \quad (33)$$

By choosing t_0 , σ and c_1 such that $z(x) + t^{p-1}c_1 \leq c$ for all $t \in (0, t_0)$ and for all $|x| \leq \eta$, we obtain that u_n is a supersolution to problem (33). Using the comparison principle we obtain that $u_1 \geq v$ in $B_\eta(0) \times (0, t_0)$. As a consequence we get the existence of $(t_1, t_2) \subset (0, t_0)$, $\varepsilon > 0$ and $\eta < \sigma$ such that

$$u_n \geq u_1 \geq \varepsilon \text{ in } B_\eta(0) \times (t_1, t_2)$$

and by the fact that $u \geq u_n$ we conclude that

$$u \geq \varepsilon \text{ in } B_\eta \times (\tau, T).$$

Therefore, Since $u > \varepsilon$ in the cylinder $B_\eta \times (\tau, T)$, then using the same comparison function as in the in the proof of Theorem 2.3, we obtain that $u(x, t) \rightarrow \infty$ as $|x| \rightarrow 0$ uniformly in $t \in (t_1, t_2)$.

Since $\{u_n\}$ is an increasing sequence and $u_n \uparrow u$ in $C([0, T]; L^1(\Omega))$, then for all $t \in (t_1, t_2)$ we have $u_n(x, t) \rightarrow \infty$ as $|x| \rightarrow 0$ and $n \rightarrow \infty$. Hence, for all $c > 0$, there exist $n_0 \in \mathbb{N}$ and $\eta_1 > 0$ such that $u_n(x, t) \geq c$ for all $x \in B_{\eta_1}(0)$, uniformly in $t \in (\tau, T)$.

Let $(t_3, t_4) \subset (t_1, t_2)$, and choose c such that

$$\begin{cases} c - \frac{1}{t_4 - t_3} > \Lambda_{N,p,\gamma}, \\ \left(\frac{\alpha - (p-1)}{p-1}\right)^{p-1} \frac{2^{3-p}}{(\alpha - 1)c^{p-1}} < 1. \end{cases} \quad (34)$$

From Theorem 2.1 with ϕ defined by (20), we obtain that for all $\psi \in C_0^\infty(B_{\bar{\eta}}(0))$, with $B_{\bar{\eta}}(0) \subset \subset B_\eta(0)$,

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{-\operatorname{div}(|x|^{-p\gamma}|\nabla \phi(u_n)|^{p-2}\nabla \phi(u_n))}{\phi(u_n)^{p-1}} dx,$$

hence

$$\begin{aligned} \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx &\geq \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{\phi'(u_n)^{p-1}}{\phi(u_n)^{p-1}} dx \\ &\quad - \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{\phi(u_n)^{p-1}} dx. \end{aligned} \quad (35)$$

Thus integrating in time we have the inequality,

$$\begin{aligned} \int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx dt &\geq \int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{\phi'(u_n)^{p-1}}{\phi(u_n)^{p-1}} dx dt \\ &\quad - \int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{\phi(u_n)^{p-1}} dx dt. \end{aligned}$$

As in the proof of Theorem 2.3, we conclude

$$\int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p f_n \frac{(\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx dt \geq (t_4 - t_3) c \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx + o(1).$$

To estimate the term involving the t -derivative we consider

$$\begin{aligned} I(t_4, t_3)(x) &= \int_{t_3}^{t_4} \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dt \\ &= \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} \int_{t_3}^{t_4} \frac{c^{\alpha-(p-1)} (u_n)_t dt}{(u_n)^{p-1} \left((u_n)^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}}. \end{aligned}$$

Notice that

$$\frac{(u_n)_t dt}{(u_n)^{p-1} \left((u_n)^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}} = \frac{d}{dt} F(u_n(t)),$$

where

$$F(s) = \int_{2c}^s \frac{dy}{y^{p-1} \left(y^{\frac{\alpha-(p-1)}{p-1}} - c^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}}.$$

Moreover, using the fact that

$$\begin{aligned} F(s) &= \int_{2c}^s \frac{dy}{y^\alpha \left(1 - \left(\frac{c}{y} \right)^{\frac{\alpha-(p-1)}{p-1}} \right)^{p-1}} \leq \frac{2^{\alpha-(p-1)}}{\left(2^{\frac{\alpha-(p-1)}{p-1}} - 1 \right)^{p-1}} \int_{2c}^s \frac{dy}{y^\alpha} \\ &\leq \frac{1}{\alpha-1} \frac{2^{2-p}}{\left(2^{\frac{\alpha-(p-1)}{p-1}} - 1 \right)^{p-1}} \frac{1}{c^\alpha}, \end{aligned}$$

we get

$$|I(t_4, t_3)(x)| \leq \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} \frac{2^{3-p}}{(\alpha-1)c^{p-1}}.$$

Since

$$\int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} = \int_{B_{\bar{\eta}}(0)} |\psi|^p I(t_4, t_3)(x) dx,$$

we conclude

$$\begin{aligned} & \left| \int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx dt \right| \\ & \leq \left(\frac{\alpha - (p-1)}{p-1} \right)^{p-1} \frac{2^{3-p}}{(\alpha-1)c^{p-1}} \int_{B_{\bar{\eta}}(0)} |\psi|^p dx. \end{aligned}$$

On the other hand, since $\bar{\eta} \ll 1$ and by using (34) we obtain

$$\left| \int_{t_3}^{t_4} \int_{B_{\bar{\eta}}(0)} |\psi|^p \frac{(u_n)_t (\phi'(u_n))^{p-1}}{(\phi(u_n))^{p-1}} dx dt \right| \leq \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx.$$

Hence

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq c \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx - \frac{1}{t_4 - t_3} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx.$$

Then it follows that

$$\int_{B_{\bar{\eta}}(0)} |\nabla \psi|^p |x|^{-p\gamma} dx \geq \tilde{C} \int_{B_{\bar{\eta}}(0)} \frac{|\psi|^p}{|x|^{p(\gamma+1)}} dx \text{ for all } \psi \in \mathcal{C}_0^\infty(B_{\bar{\eta}}(0))$$

where

$$\tilde{C} \equiv c - \frac{1}{t_4 - t_3}.$$

As (34) holds, we obtain that $\tilde{C} > \Lambda_{N,p,\gamma}$, which is a contradiction with the optimality of $\Lambda_{N,p,\gamma}$. $\square$

3 Some existence results

Notice that, in the case $\alpha = p-1$, if the constant λ is strictly less than the critical constant $\Lambda_{N,p,\gamma}$, then problem (26) has a nonnegative global solution for all data in $L^2(\Omega)$. We refer to [17] and [12] for a complete discussion. We study now some other interesting cases that show the influence of the Harnack inequality. In the first subsection the Harnack inequality does not hold since $1 < p < 2$ while in the second subsection is motivated by the fact that $(1 + \gamma) < 0$.

3.1 The case $1 < p < 2$, $(\gamma + 1) > 0$

If $1 < p < 2$, we have the following existence result.

Theorem 3.1 *Let u_0 be a positive function such that $u_0 \in L^2(\Omega)$. Assume that $1 < p < 2$ and $0 < \gamma + 1 < \frac{N(2-p)}{p}$. Then if α satisfies $p - 1 < \alpha < \frac{N-p(\gamma+1)}{N}$, problem*

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}}, u \geq 0 \text{ in } \Omega_T, \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega. \end{cases} \quad (36)$$

has a global distributional solution u such that $u^{\frac{p+s-1}{p}} \in L^p([0, T]; \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty([0, T]; L^{\frac{(s+1)p}{p+s-1}}(\Omega))$ for all $T > 0$ where s is chosen such that $p - 1 < \alpha < \frac{N-p(\gamma+1)(s+1)}{N}$.

Proof. By the assumptions on p and γ we obtain that $\frac{N-p(\gamma+1)}{N} > p - 1$, so that the interval $p - 1 < \alpha < \frac{N-p(\gamma+1)}{N}$ is not empty. Hence we get the existence of $s > 0$ such that $p - 1 < \alpha < \frac{N-p(\gamma+1)(s+1)}{N}$. Let u_n be the minimal solution to problem

$$\begin{cases} u_t^n - \operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) = T_n\left(\frac{1}{|x|^{p(\gamma+1)}}\right) u_n^\alpha \text{ in } \Omega_T, \\ u_n = 0 \text{ in } \partial\Omega \times (0, T), \\ u_n(x, 0) = u_0(x) \text{ in } \Omega. \end{cases} \quad (37)$$

Using u_n^s as a test function in (37), we obtain that

$$\begin{aligned} & \frac{1}{s+1} \int_{\Omega} u_n^{s+1}(x, T) dx + s \left(\frac{p}{p+s-1} \right)^p \int_0^T \int_{\Omega} |\nabla \left(u_n^{\frac{p+s-1}{p}} \right)|^p |x|^{-p\gamma} dx dt \\ &= \frac{1}{s+1} \int_{\Omega} u_0^{s+1} dx + \int_0^T \int_{\Omega} T_n \left(\frac{1}{|x|^{p(\gamma+1)}} \right) u_n^{\alpha+s} dx dt. \end{aligned}$$

Using Hölder and Young inequalities we get

$$\begin{aligned} & \frac{1}{s+1} \int_{\Omega} u_n^{s+1}(x, T) dx + s \left(\frac{p}{p+s-1} \right)^p \int_0^T \int_{\Omega} |\nabla \left(u_n^{\frac{p+s-1}{p}} \right)|^p |x|^{-p\gamma} dx dt \\ & \leq \frac{1}{s+1} \int_{\Omega} u_0^{s+1} dx + \int_0^T \left(\int_{\Omega} \frac{dx}{|x|^{\frac{p(\gamma+1)(s+1)}{1-\alpha}}} \right)^{\frac{1-\alpha}{s+1}} \left(\int_{\Omega} u_n^{s+1} dx \right)^{\frac{\alpha+s}{s+1}} dt \\ & \leq \frac{1}{s+1} \int_{\Omega} u_0^{s+1} dx + \frac{\alpha+s}{s+1} \int_0^T \int_{\Omega} u_n^{s+1} dx dt + \frac{1-\alpha}{s+1} \int_0^T \int_{\Omega} \frac{1}{|x|^{\frac{p(\gamma+1)(s+1)}{1-\alpha}}} dx dt. \end{aligned}$$

Since $\frac{p(\gamma+1)(s+1)}{1-\alpha} < N$, then $\int_{\Omega} \frac{1}{|x|^{\frac{p(\gamma+1)(s+1)}{1-\alpha}}} dx = A < \infty$.

Let

$$y_n(t) = \int_{\Omega} u_n^{s+1}(x, t) dx, \text{ and } h(t) = \int_{\Omega} u_0^{s+1} dx + (1-\alpha)At.$$

Therefore we obtain

$$y_n(T) \leq \frac{\alpha+s}{s+1} \int_0^T y_n(t) dt + h(T).$$

Using Gronwall's lemma we conclude that

$$y_n(t) \leq h(t) + e^{\frac{\alpha+s}{s+1}t} \int_0^t h(\sigma) e^{\frac{\alpha+s}{s+1}\tau} d\tau.$$

Hence u_n is a global solution to (37). Notice that the last estimate implies that the sequence $\{u_n^{\frac{p+s-1}{p}}\}$ is uniformly bounded in $L^p([0, T]; \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty([0, T]; L^{\frac{(s+1)p}{p+s-1}}(\Omega))$ for all $T > 0$. Since $\{u_n\}$ is an increasing sequence, we conclude that it converges everywhere and weakly to u such that $u^{\frac{p+s-1}{p}} \in L^p([0, T]; \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty([0, T]; L^{\frac{(s+1)p}{p+s-1}}(\Omega))$. Using the monotonicity of $\{u_n\}$ and the convergence result by Boccardo-Murat in [8] we can prove that u is a global distributional solution to (37). $\square$

Remark 3.2 If in addition we assume that $p-1 < \alpha < \frac{N-2p(\gamma+1)}{N}$, then problem (37) has a global solution $u \in L^p([0, T]; \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty([0, T]; L^2(\Omega))$, namely u is a solution in the energy sense.

Assume that $p-1 < \alpha < 1$ and let $u_0(x)$ be such that $u_0(x) \leq c_0 \left(\frac{1}{|x|^{p(\gamma+1)}} \right)^{\frac{1}{1-\alpha}}$.

If $w(x, t) = \left(\frac{t+c_0}{|x|^{p(\gamma+1)}} \right)^{\frac{1}{1-\alpha}}$, then by a direct computation we obtain that w is a supersolution to problem (36) at least in the sense of distributions in $\mathcal{D}'(\Omega \setminus \{0\} \times (0, T))$. Then using an iteration argument we get the existence of a function $0 \leq u \leq w$ such that u is a solution to problem (36) at least in the sense of distributions in $\mathcal{D}'(\Omega \setminus \{0\} \times (0, T))$. If in addition we assume that $\gamma+1 < \frac{(2-p)N}{p}$ and $p-1 < \alpha < \frac{N-p^2(\gamma+1)}{N-p(\gamma+1)}$, then u is an entropy solution to (36).

In a general way, without any restriction on $\alpha \in (p-1, 1)$, we are able to prove the next result.

Theorem 3.3 Assume that $1 < p < 2$ and that $\frac{(2-p)N}{2p} < \gamma + 1 < \frac{N}{p}$. If $p - 1 < \alpha < 1$, then problem (36) has a distributional solution u away from the origin. Moreover $u \in \mathcal{M}$ defined by

$$\mathcal{M} \equiv \{u \in L^p(0, T; \mathcal{D}_{0, \gamma-\beta}^{1,p}(\Omega)) \cap C^0([0, T]; L_{-\frac{\beta p}{2}}^2(\Omega)) : u' \in L^p(0, T; \mathcal{D}_{0, -\theta}^{-1,p'}(\Omega))\}$$

with $\beta > \frac{2(\gamma+1)}{1-\alpha} - \frac{N}{p}$ and $\theta = \gamma + \beta(p-1)$. Notice that the above conditions on γ and α are satisfied in the particular case where $\frac{N(2-p)}{p} \leq \gamma + 1 < \frac{N}{p}$ and $\frac{N-p(\gamma+1)}{N} \leq \alpha < 1$, which the complement of the existence case quoted in Theorem 3.1.

Proof. Let u_n be the minimal solution to problem (37) Using $|x|^{p\beta}u_n$ as a test function in (37) we obtain that

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_n^2(x, T) |x|^{p\beta} dx + \int_0^T \int_{\Omega} |\nabla u_n|^p |x|^{p(\beta-\gamma)} dx dt \\ & \leq \frac{1}{2} \int_{\Omega} u_0^2 |x|^{p\beta} dx + p\beta \int_0^T \int_{\Omega} |\nabla u_n|^{p-1} |x|^{p(\beta-\gamma)-1} dx dt \\ & \quad + \int_0^T \int_{\Omega} u_n^{\alpha+1} |x|^{p(\beta-\gamma-1)} dx dt. \end{aligned}$$

Using Hölder and Young inequalities we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} u_n^2(x, T) |x|^{p\beta} dx + c_1 \int_0^T \int_{\Omega} |\nabla u_n|^p |x|^{p(\beta-\gamma)} dx dt \\ & \leq \frac{1}{2} \int_{\Omega} u_0^2 |x|^{p\beta} dx + c_2 \int_0^T \int_{\Omega} u_n^2(x, t) |x|^{p\beta} dx dt \\ & \quad + c_3 T \left(\int_{\Omega} \left(|x|^{p(\beta-\gamma-1)} + |x|^{p(\beta-\frac{2(\gamma+1)}{1-\alpha})} \right) dx \right). \end{aligned}$$

Under the assumptions on α and β we conclude that the last integral is finite. Then using Gronwall's lemma we obtain that

$$u_n \text{ is bounded in } L^p(0, T; \mathcal{D}_{0, \gamma-\beta}^{1,p}(\Omega)) \cap C^0([0, T]; L_{-\frac{\beta p}{2}}^2(\Omega)).$$

Using equation (37) we obtain in the same way that

$$u'_n \text{ is bounded in } L^p(0, T; \mathcal{D}_{0, -\theta}^{-1,p'}(\Omega)).$$

Then the result follows using the argument of Boccardo and Murat in [8]. $\square$

The converse of Theorem (2.10) is the following existence result.

Theorem 3.4 *If the assumptions of Theorem 3.1 hold, then for all $c \geq 0$ problem*

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}} + c, u \geq 0 \text{ in } \Omega_T, \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega. \end{cases} \quad (38)$$

has a global distributional solution u such that $u^{\frac{p+s-1}{p}} \in L^p([0, T]; \mathcal{D}_{0,\gamma}^{1,p}(\Omega)) \cap L^\infty([0, T]; L^{\frac{(s+1)p}{p+s-1}}(\Omega))$ for all $T > 0$ where s is chosen such that $p-1 < \alpha < \frac{N-p(\gamma+1)(s+1)}{N}$.

Proof. Let $s > 0$ be such that $p-1 < \alpha < \frac{N-p(\gamma+1)(s+1)}{N}$ and consider u_n the minimal solution to problem

$$\begin{cases} u_t^n - \operatorname{div}(|x|^{-p\gamma}|\nabla u_n|^{p-2}\nabla u_n) = T_n\left(\frac{1}{|x|^{p(\gamma+1)}}\right)u_n^\alpha + c \text{ in } \Omega_T, \\ u_n = 0 \text{ in } \partial\Omega \times (0, T), \\ u_n(x, 0) = u_0(x) \text{ in } \Omega. \end{cases} \quad (39)$$

Using u_n^s as test function in (39), we obtain that

$$\begin{aligned} & \frac{1}{s+1} \int_{\Omega} u_n^{s+1}(x, T) dx + s \left(\frac{p}{p+s-1} \right)^p \int_0^T \int_{\Omega} |\nabla(u_n^{\frac{p+s-1}{p}})|^p |x|^{-p\gamma} dx dt \\ &= \frac{1}{s+1} \int_{\Omega} u_0^{s+1} dx + \int_0^T \int_{\Omega} T_n\left(\frac{1}{|x|^{p(\gamma+1)}}\right) u_n^{\alpha+s} dx dt + c \int_0^T \int_{\Omega} u_n^s(x, t) dx dt. \end{aligned}$$

Using Hölder and Young inequalities we get

$$\begin{aligned} & \frac{1}{s+1} \int_{\Omega} u_n^{s+1}(x, T) dx + s \left(\frac{p}{p+s-1} \right)^p \int_0^T \int_{\Omega} |\nabla(u_n^{\frac{p+s-1}{p}})|^p |x|^{-p\gamma} dx dt \\ & \leq \frac{1}{s+1} \int_{\Omega} u_0^{s+1} dx + \frac{\alpha+2s}{s+1} \int_0^T \int_{\Omega} u_n^{s+1} dx dt + C, \end{aligned}$$

where $C \equiv C(\Omega, N, p, \gamma) > 0$. The conclusion now follows by using Gronwell's lemma and Boccardo-Murat convergence theorem as in the proof of Theorem 3.1. $\square$

Remark 3.5

1. if $f(x, t)$ is a bounded function, then in problem (38) the constant c can be substituted by $f(t, x)$ or by $f(x, t)u$. The same conclusion of the Theorem holds.

2. Assume that $\frac{2N}{N+2} < p < 2$ and let f be such that

$$\int_0^T \left(\int_{\Omega} |f(x, t)|^{\frac{N(p-1)}{p-(2-p)N}} |x|^{\frac{pN}{p-(2-p)N}} dx \right)^{\frac{p-(2-p)N}{N(p-1)}}, \quad (40)$$

for all $T > 0$, then the existence result in Theorem 3.4 holds if we replace the constant c in problem (38) by any function f satisfies the condition (40).

3.2 Existence result in the case $\gamma + 1 \leq 0$

In this section we will not assume any restriction on p . The main result is the following

Theorem 3.6 *Let u_0 be a bounded positive function. Assuming that $\gamma + 1 \leq 0$, then for all $\alpha > p - 1$, problem*

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma} |\nabla u|^{p-2} \nabla u) = \frac{u^\alpha}{|x|^{p(\gamma+1)}}, u \geq 0 \text{ in } \Omega_T, \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega \end{cases} \quad (41)$$

has a nonnegative solution at least for T small.

Proof. Assume that $\alpha > p - 1$ and let y be the solution to the following differential equation

$$y'(t) = C_1 y^\alpha(t), \quad y(0) = C_2$$

where $C_2 \geq \|u_0\|_\infty$ and $C_1 \geq \max\{|x|^{p(\gamma+1)} : x \in \Omega\}$.

If $\alpha < 1$, then $y(t) = \left(y^{1-\alpha}(0) + (1-\alpha)C_1 t\right)^{\frac{1}{1-\alpha}}$ is defined for all $t \geq 0$. If $\alpha > 1$, then $y(t) = \left(y^{1-\alpha}(0) - (\alpha-1)C_1 t\right)^{-\frac{1}{\alpha-1}}$ is defined and positive for $t \leq T^* \equiv \frac{C_2^{1-\alpha}}{(\alpha-1)C_1}$.

Hence we get the existence of $t_0 > 0$ such that $y(t) \geq 0$ for $t \leq t_0$. Moreover, we obtain that $v(x, t) \equiv y(t)$ is a supersolution to problem (41) for $t \leq t_0$. Using an iteration argument we obtain that problem (41) has a local nonnegative positive solution. Notice that in the case where $\alpha < 1$, the solution obtained above is global. $\square$

We refer to [12] to analyze the existence of global solutions, even in a very weak sense, and further related results.

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