On a class of coupled elliptic systems in $\mathbb{R}^N$

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Abstract. It is established the existence and multiplicity of solutions for a class of coupled semilinear elliptic systems in $\mathbb{R}^N$ where none of the potentials are coercive. The main goal is to consider systems where the primitive of the nonlinearity is superquadratic on appropriate directions.

2000 *Mathematics Subject Classification*: 35J50, 35J55.

Key words: Elliptic systems, Variational methods, superquadratic nonlinearity.

1 Introduction

In this paper we present results on the existence and multiplicity of solutions for the system

$$\begin{cases} -\Delta u + a(x)u = F_u(x, u, v), & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = F_v(x, u, v), & x \in \mathbb{R}^N, \end{cases} \quad (P)$$

where $N \geq 3$ and the potentials a and b are continuous functions satisfying

(A₁) there are constants $a_0, b_0 > 0$ such that $a(x) \geq a_0, b(x) \geq b_0$ for all $x \in \mathbb{R}^N$,

(A₂) $\mu(\{x \in \mathbb{R}^N : a(x)b(x) < M\}) < \infty$, for every $M > 0$.

Here μ denotes the Lebesgue measure in $\mathbb{R}^N$. We also suppose that $F \in C^1(\mathbb{R}^N \times \mathbb{R}^2, \mathbb{R})$ satisfies $F(x, 0, 0) \equiv 0$ and the following subcritical coupling condition

The first author was supported by CAPES/Brazil and Pronex. The second author was supported by CNPq/Brazil, Pronex and FINATEC.

(F_1) there are constants $c_1, q_1, q_2 > 0$, $p_1, p_2 > 1$ and functions $\alpha, \beta \in L^\infty(\mathbb{R}^N)$ such that

$$|F_u(x, z)| \leq c_1 |u|^{p_1-1} |v|^{q_1} + \alpha(x) |u| + c_1 |v|, \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$$|F_v(x, z)| \leq c_1 |u|^{q_2} |v|^{p_2-1} + c_1 |u| + \beta(x) |v|, \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

where $z = (u, v)$, $2 \leq p_i + q_i < 2^* \equiv 2N/(N-2)$, $i = 1, 2$ and α and β satisfy

$$\limsup_{|x| \rightarrow \infty} \alpha(x) = \alpha_\infty < a_0 \quad \text{and} \quad \limsup_{|x| \rightarrow \infty} \beta(x) = \beta_\infty < b_0. \quad (1)$$

We observe that there is an extensive bibliography on the study of variational elliptic systems (see [3, 12, 14] and references therein). In two recent articles [4, 6], the problem (P) has been studied under coupled-resonant conditions. The main objective in this work is to complement the results in [4, 6] by considering systems with coupling in $\mathbb{R}^N$ which allow, in particular, a superquadratic growth on the primitive $F(x, z)$ on directions $z = (u, v)$, $u \neq 0, v \neq 0$. In our first two results, we assume the following version of the Ambrosetti-Rabinowitz's [1, 8] superquadratic condition.

(F_2) there are constants $c_2 > 0$, $\theta > 2$, $p, q, \mu, \nu \geq 0$, and functions $h_1, h_2 \in L^1(\mathbb{R}^N)$ such that $2 \leq \mu + \nu, p + q < 2^*$, $\mu + \nu > \frac{N}{2}(p + q - 2)$ (if $p + q > 2$ and $\mu > p$ [$\nu > q$], it is supposed $\nu p > \mu(q - 2)$ [$\mu q > \nu(p - 2)$]),

$$F(x, z) - \frac{1}{\theta} \nabla F(x, z) \cdot z \leq c_2 |u|^p |v|^q + h_1(x), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2,$$

$$\frac{1}{2} \nabla F(x, z) \cdot z - F(x, z) \geq c_2 |u|^\mu |v|^\nu - h_2(x), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

The hypothesis (F_2) is a version for our setting of a condition introduced in [12] (see also [3] for a related condition) to study a class of integro-differential problem on bounded domains of $\mathbb{R}^N$. We note that $c_2 > 0$ and the second relation in (F_2) imply that $F(x, tz)/t^2 \rightarrow \infty$, as $t \rightarrow \infty$, for every $z = (u, v) \in \mathbb{R}^2$, $u \neq 0, v \neq 0$ (see section 3). Considering the coupled linear problem

$$\begin{cases} -\Delta u + a(x)u = \lambda v, & x \in \mathbb{R}^N, \\ -\Delta v + b(x)v = \lambda u, & x \in \mathbb{R}^N, \end{cases} \quad (LP)$$

we denote by λ_1 the first positive eigenvalue for the problem (LP) (we refer the reader to [6] for a study of the eigenvalues of (LP)). Assuming

(F_3) there is $\lambda \in [0, \lambda_1)$ such that

$$F(x, z) - \lambda uv = o(|z|^2), \text{ as } |z| \rightarrow 0, \text{ uniformly in } \mathbb{R}^N,$$

we may state:

Theorem 1.1 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) – (F_3) , then problem (P) possesses a nonzero solution.*

In our next result we verify the existence of infinitely many solutions for (P) under the presence of symmetry. More specifically, we suppose

(F_4) $F(x, z)$ is even with respect to the variable $z \in \mathbb{R}^2$.
Under this condition, we are able to prove

Theorem 1.2 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) , (F_2) and (F_4) , then problem (P) possesses infinitely many pairs of nonzero solutions.*

The existence of a nontrivial solution for the scalar problem in $\mathbb{R}^N$ has been proved by Rabinowitz [9] for subcritical nonlinear terms with superlinear growth and coercive potentials. Afterwards Bartsch and Wang [2] studied the scalar problem under conditions similar to those in [2] [9] and a version of condition (A_2) for that setting. We also observe that this hypothesis has been used in [5] to generalize to $\mathbb{R}^N$ the results on double resonant elliptic problems under a local nonquadraticity condition which had been proved in [7].

Theorems 1.1 and 1.2 may be seen as versions for elliptic systems with coupling of the corresponding results in [9, 2]. We note that the condition (A_2) allows a setting where none of potentials are coercive or satisfy the condition introduced in [2].

In our final two results we establish versions of Theorems 1.1 and 1.2 when c_2 , given by (F_2) , is null. In this case the second relation in (F_2) is not necessary:

$(\widehat{F_2})$ there are $\theta > 2$ and a function $h_1 \in L^1(\mathbb{R}^N)$ such that

$$F(x, z) - \frac{1}{\theta} \nabla F(x, z) \cdot z \leq h_1(x), \quad \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Assuming

(F_5) there is $c_3 \geq 0$ such that

$$\sup_{t \geq 0} \int_{\mathbb{R}^N} [\lambda_1 t^2 u_1 v_1 - F(x, tz_1)] \, dx \leq c_3 < \infty,$$

where $z_1 = (u_1, v_1)$ is the eigenfunction of (LP) associated with λ_1 , we may state:

Theorem 1.3 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) , $(\widehat{F_2})$, (F_3) and (F_5) , then problem (P) possesses a nonzero solution.*

Moreover, supposing F satisfies (F_4) and

(F_6) there exist $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and an open subset Ω of $\mathbb{R}^N$ such that

$$\frac{F(x, ta, tb)}{t^2} \rightarrow \infty, \text{ as } t \rightarrow \infty, \text{ uniformly for } x \in \Omega,$$

we obtain:

Theorem 1.4 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) , $(\widehat{F_2})$, (F_4) and (F_6) , then problem (P) possesses infinitely many pairs of nonzero solutions.*

2 Preliminary results

In order to apply the variational method to study (P) , we follow Rabinowitz [9] and we set $E = E_a \times E_b$ with $E_a = \{u \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla u|^2 + a(x)u^2) dx < \infty\}$ and $E_b = \{v \in W^{1,2}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} (|\nabla v|^2 + b(x)v^2) dx < \infty\}$ endowed with the inner product

$$\langle (u, v), (\phi, \psi) \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla \phi + \nabla v \nabla \psi + a(x)u\phi + b(x)v\psi) dx,$$

and associated norm denoted by $\|\cdot\|$. Standard arguments show that the associated functional $I : E \rightarrow \mathbb{R}$, defined by

$$I(z) = \frac{1}{2} \|z\|^2 - \int_{\mathbb{R}^N} F(x, z) dx, \quad \forall z = (u, v) \in E, \quad (2)$$

is well defined and of class C^1 . Moreover the critical points of I are precisely the weak solutions of the system (P) . The condition (A_1) and the Sobolev Theorem imply that the immersion $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$ is continuous for $2 \leq s \leq 2^*$. However, since we are not supposing that a or b are coercive, i.e. $a(x)$ or $b(x)$ go to $+\infty$ as $|x| \rightarrow 0$, this embedding may not be compact.

Now we will introduce some preliminary results concerning a version of the Palais-Smale condition for the functional I given by (2). Let E be a real Banach space and $I : E \rightarrow \mathbb{R}$ a functional of class C^1 . We recall that a sequence $(z_n) \subset E$ is said to be a *strong Cerami* [(SCe)] sequence [13, 4, 6] if $I(z_n) \rightarrow c$ and $I'(z_n) \rightarrow 0$ as $n \rightarrow \infty$, and $\|z_n\| \|I'(z_n)\|$ is bounded. The functional I satisfies the *strong Cerami condition* [(SCe)] if any strong Cerami sequence $(z_n) \subset E$ possesses a convergent subsequence.

The following result is proved in [6] (see the proof of Theorem 2).

Lemma 2.1 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) , then every bounded (SCe) sequence possesses a convergent subsequence.*

Based on the above result, we may state

Lemma 2.2 *Suppose (A_1) and (A_2) hold. If F satisfies (F_1) and (F_2) or $(\widehat{F_2})$, then the functional I satisfies (SCe).*

Proof. Let $(z_n) = ((u_n, v_n)) \subset E$ be a (SCe) sequence, i.e., (z_n) satisfies (i) $I(z_n) \rightarrow c$ and $I'(z_n) \rightarrow 0$ as $n \rightarrow \infty$, and (ii) $\|z_n\| \|I'(z_n)\| \leq M < \infty$. By

Lemma 2.1, it suffices to show that (z_n) is a bounded sequence. Supposing F satisfies (F_2) , we apply the relations (i) and (ii) to find $M_1 > 0$ such that

$$c_2 \int_{\mathbb{R}^N} |u_n|^\mu |v_n|^\nu dx \leq I(z_n) - \frac{1}{2} \langle I'(z_n), z_n \rangle + \|h_2\|_{L^1} \leq M_1, \quad (3)$$

$$\begin{aligned} \left(\frac{1}{2} - \frac{1}{\theta}\right) \|z_n\|^2 &\leq I(z_n) - \frac{1}{\theta} \langle I'(z_n), z_n \rangle + c_2 \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx + \|h_1\|_{L^1} \\ &\leq c_2 \int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx + M_1. \end{aligned} \quad (4)$$

Considering $\gamma \equiv p + q$ and $\eta \equiv \mu + \nu$, we apply (F_2) to find $r > 1$ such that $rp \geq \mu$, $rq \geq \nu$, $2(r-1) \leq \gamma r - \eta \leq 2^*(r-1)$ and $(\gamma r - \eta)/r < 2$. Since $|u_n(x)|, |v_n(x)| \leq |z_n(x)|$, for every $x \in \mathbb{R}^N$, we may write

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \leq \int_{\mathbb{R}^N} (|u_n|^\mu |v_n|^\nu)^{\frac{1}{r}} |z_n|^{\frac{\gamma r - \eta}{r}} dx.$$

Applying Holder's inequality to the above relation, we get

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \leq \left[\int_{\mathbb{R}^N} |u_n|^\mu |v_n|^\nu dx \right]^{\frac{1}{r}} \left[\int_{\mathbb{R}^N} |z_n|^{\frac{\gamma r - \eta}{r-1}} dx \right]^{\frac{r-1}{r}}.$$

Hence, by the inequality (3), there is $M_2 > 0$ such that

$$\int_{\mathbb{R}^N} |u_n|^p |v_n|^q dx \leq M_2 \|z_n\|_{L^{\frac{\gamma r - \eta}{r-1}}}^{\frac{\gamma r - \eta}{r}}.$$

Since $\frac{\gamma r - \eta}{r-1} \in [2, 2^*]$, from the embedding $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$, $2 \leq s \leq 2^*$, and $\theta > 2$, by the above inequality and (4), we find $M_3, M_4 > 0$ such that

$$\|z_n\|^2 \leq M_3 \|z_n\|_{L^{\frac{\gamma r - \eta}{r-1}}}^{\frac{\gamma r - \eta}{r}} + M_4.$$

Noting that by our choice of r we have $\frac{\gamma r - \eta}{r} < 2$, we may conclude that (z_n) is a bounded sequence. When $(\widehat{F}_2)$ holds, the conclusion that (z_n) is bounded is a direct consequence of (4) with $c_2 = 0$. This concludes the proof of the lemma. $\square$

Next, we state the versions of the abstract results in [1] which will be used in our proofs of Theorems 1.1–1.2.

Theorem 2.3 ([10, 11]) *Let E be a real Banach space. Suppose $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$, (SCe) and*

(I₁) there exists $\rho > 0$ such that $I(z) \geq 0$, for all $z \in \partial B_\rho(0)$.

(I₂) there exist $\beta \in \mathbb{R}$ and $e \in E$ such that $I(te) \leq \beta$ for every $t \geq 0$.

Then I possesses a nonzero critical point.

Theorem 2.4 ([10, 15]) *Let $E = V \oplus W$ be a real Hilbert space with V finite dimensional and $W = V^\perp$. Suppose $I \in C^1(E, \mathbb{R})$ is even and satisfies $I(0) = 0$, (SCe) and*

- (I₃) *there exists $\rho > 0$ such that $I(z) \geq 0$, for all $z \in B_\rho(0) \cap W$.*
- (I₄) *there exists a finite dimensional closed subspace $\hat{V}$ of E and $\beta \in \mathbb{R}$ such that $\hat{V} \supset V$ and $I(z) \leq \beta$, for all $z \in \hat{V}$,*

Then I possesses $\dim \hat{V} - \dim V$ pairs of nontrivial critical points.

Remark 2.5 Theorems 2.3 and 2.4 in [10, 11] and [10, 15] are proved for the Palais-Smale condition [1, 8]. The version for the (SCe) condition is based on a deformation lemma proved in [13].

3 Proofs of theorems 1.1 and 1.3

The proof of Theorem 1.1 will be accomplished by verifying that the functional I , given by (2), satisfies the hypotheses of Theorem 2.3. We start by recalling the variational characterization of the eigenvalue λ_1 :

$$\lambda_1 = \sup\{\mu \geq 0 : \|z\|^2 - 2\mu \int_{\mathbb{R}^N} uv \, dx \geq 0, \forall z \in E\}. \quad (5)$$

Consequently, taking $\lambda \in [0, \lambda_1)$, given by (F_3) , we find $c_3 > 0$ such that

$$\frac{1}{2}\|z\|^2 - \lambda \int_{\mathbb{R}^N} uv \, dx \geq c_3\|z\|^2, \forall z \in E. \quad (6)$$

Now, given $\epsilon > 0$, we use (F_3) to find $\delta > 0$ such that

$$F(x, z) - \lambda uv \leq \epsilon|z|^2, \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, |z| \leq \delta.$$

Furthermore, observing that $F(x, 0, 0) \equiv 0$ and applying (F_1) , we obtain constants $c_4 > 0$ and $2 < \sigma < 2^*$ such that

$$F(x, z) - \lambda uv \leq c_4|z|^\sigma, \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2, |z| > \delta.$$

Combining these two estimates, we get

$$F(x, z) - \lambda uv \leq \epsilon|z|^2 + c_4|z|^\sigma, \forall (x, z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Consequently, by definition (2) and inequality (6),

$$I(z) \geq c_3\|z\|^2 - \epsilon\|z\|_{L^2}^2 - c_4\|z\|_{L^\sigma}^\sigma, \forall z \in E.$$

Using the embedding $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}) \times L^s(\mathbb{R}^N, \mathbb{R})$, $2 \leq s \leq 2^*$, $\sigma > 2$ and choosing $\epsilon > 0$ sufficiently small, we conclude that I satisfies the condition (I_1) .

Now we shall verify the condition (I_2) when $\eta = \mu + \nu > 2$ (the case $\eta = 2$ is proved by a similar argument). Given $(x, z) \in \mathbb{R}^N \times \mathbb{R}^2$, we consider the function $g(t) = F(x, tz)$, for $t \geq 0$, and apply the second relation in (F_2) to get

$$F(x, tz) \geq t^2 F(x, z) + \frac{2c_2 |u|^\mu |v|^\nu}{\eta - 2} (t^\eta - t^2) - |h_2(x)| (t^2 - 1), \quad \forall t \geq 1.$$

Now we fix $\varphi \in C_0^\infty(\mathbb{R}^N)$, $\varphi \not\equiv 0$, and take $e = (\varphi, \varphi)$. Using the above relation, we get

$$\int_{\mathbb{R}^N} F(x, te) \, dx \geq t^2 \int_{\mathbb{R}^N} F(x, e) \, dx + \frac{2c_2 \|\varphi\|_{L^\eta}^\eta}{\eta - 2} (t^\eta - t^2) - (t^2 - 1) \|h_2\|_{L^1},$$

$$\forall t \geq 1.$$

Since $\eta > 2$, we conclude that

$$\frac{1}{t^2} \int_{\mathbb{R}^N} F(x, te) \, dx \rightarrow \infty, \quad \text{as } t \rightarrow \infty. \quad (7)$$

The definition (2) and the limit in (7) show that the functional I satisfies (I_2) whenever the condition (F_2) holds. Since I satisfies $I(0) = 0$, (I_1) , (I_2) and (SCe), by Lemma 2.2, we may apply the Theorem 2.3 to conclude that problem (P) possesses a nonzero solution. This concludes the proof of Theorem 1.1.

For proving the Theorem 1.3, we just need to verify that the functional I satisfies the condition (I_2) . But, taking $e = z_1$ and noting that $\|z_1\|^2 = 2\lambda_1 \int_{\mathbb{R}^N} u_1 v_1 \, dx$, we obtain that (I_2) follows directly from $(\hat{F}_z)$.

4 Proofs of theorems 1.2 and 1.4

We will prove Theorem 1.2 by verifying that the functional I satisfies the hypotheses of Theorem 2.4 with $\dim \hat{V} - \dim V$ as large as we like. Let $\{z_1, z_2, \dots, z_k, \dots\}$ be an orthonormal basis for E . We set $E_k = \text{span}\{z_1, z_2, \dots, z_k\}$ for every $k \in \mathbb{N}$. In order to prove Theorem 1.2 we need the following technical result.

Lemma 4.6 *Suppose (A_1) and (A_2) hold. If $p, q > 0$ and $p + q \geq 2$; then, given $\delta > 0$, there is $k_0 \in \mathbb{N}$ such that*

$$\int_{\mathbb{R}^N} |u|^p |v|^q \, dx \leq \delta \|z\|^{p+q}, \quad \forall z \in E_k^\perp, \quad k \geq k_0.$$

Proof. First, we suppose that $p = q = 1$. Arguing by contradiction, we suppose there exist $c > 0$, and a sequence $z_k = (u_k, v_k) \subset E_k^\perp$ such that, for every $k \in \mathbb{N}$, $\|z_k\| = 1$ and

$$\int_{\mathbb{R}^N} |u_k| |v_k| \, dx \geq c > 0. \quad (8)$$

Without loss of generality, we may suppose that

$$\begin{cases} z_k \rightharpoonup 0, \text{ in } E, \\ u_k, v_k \rightarrow 0, \text{ in } L_{loc}^s(\mathbb{R}^N) \text{ for } 2 \leq s < 2^*. \end{cases} \quad (9)$$

By the condition (A_2) , given $\varepsilon > 0$ we may choose $R > 0$ such that $\mu(C_\varepsilon) < \varepsilon$, where $C_\varepsilon = \{x \in \mathbb{R}^N \setminus B_R(0) : a(x)b(x) \leq \varepsilon^{-2}\}$. Invoking (9) and Holder's inequality, we obtain $M > 0$ such that

$$\int_{C_\varepsilon} |u_k| |v_k| \, dx \leq \mu(C_\varepsilon)^{\frac{2}{N}} \|u_k\|_{L^{2^*}} \|v_k\|_{L^{2^*}} \leq M \mu(C_\varepsilon)^{\frac{2}{N}} \leq M \varepsilon^{\frac{2}{N}}. \quad (10)$$

Now, setting $D_\varepsilon = \mathbb{R}^N \setminus (B_R(0) \cup C_\varepsilon)$, we may use (9) and Holder's inequality one more time to get

$$\int_{D_\varepsilon} |u_k| |v_k| \, dx \leq \varepsilon \int_{D_\varepsilon} \sqrt{a(x)} |u_k| \sqrt{b(x)} |v_k| \, dx \leq \varepsilon. \quad (11)$$

Moreover, by (9), for k sufficiently large,

$$\int_{B_R(0)} |u_k| |v_k| \, dx < \varepsilon. \quad (12)$$

In view of (10)–(12) and the fact that $\varepsilon > 0$ can be chosen arbitrarily small, we obtain a contradiction to (8). This proves the lemma for $p = q = 1$. Now we consider the general case. Supposing $p \geq q$, we claim that there is $c_5 > 0$ such that

$$\int_{\mathbb{R}^N} |u|^p |v|^q \, dx \leq c_5 \|uv\|_{L^{\frac{p+q}{2}}}^q \|z\|^{p-q}, \quad \forall z \in E. \quad (13)$$

The above inequality is trivially true for $p = q$. Hence, we may assume that $p > q$. Applying Holder's inequality, we get

$$\int_{\mathbb{R}^N} |u|^p |v|^q \, dx = \int_{\mathbb{R}^N} (\sqrt{|u||v|})^{2q} |u|^{p-q} \, dx \leq \|uv\|_{L^{\frac{p+q}{2}}}^q \|u\|_{L^{\frac{p}{p-q}}}^{p-q}.$$

Using that $|u(x)| \leq |z(x)|$, for every $x \in \mathbb{R}^N$, and the appropriate embedding, we obtain (13). This proves the claim. Taking $t \in (0, 1]$ such that $p+q = 2t + (1-t)2^*$,

by interpolation, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (|u||v|)^{\frac{p+q}{2}} dx &\leq \left(\int_{\mathbb{R}^N} |uv| dx \right)^t \left(\int_{\mathbb{R}^N} |uv|^{2^*/2} dx \right)^{1-t} \\ &\leq \left(\int_{\mathbb{R}^N} |uv| dx \right)^t \|z\|_{L^{2^*}}^{2^*(1-t)}. \end{aligned}$$

Finally, given $\delta > 0$, using the above inequality and the fact that the lemma holds for $p = q = 1$, we obtain $k_0 \in \mathbb{N}$ such that

$$\|uv\|_{L^{\frac{p+q}{2}}}^q \leq (c_5)^{-1} \delta \|z\|^{2q}, \quad \forall z \in E_k^\perp, \quad k \geq k_0.$$

Combining the above inequality and (13), we conclude the proof of the lemma. $\square$

The next result follows from condition (1) and the fact that the embedding $E \hookrightarrow L^s(U) \times L^s(U)$, $2 \leq s < 2^*$, is compact on bounded open subsets $U \subset \mathbb{R}^N$.

Lemma 4.7 *Suppose (A_1) and (A_2) hold. If α and $\beta \in L^\infty(\mathbb{R}^N)$ satisfy (1), then there is $r > 1$ such that, given $\delta > 0$, we may find $k_1 \in \mathbb{N}$ such that*

$$\begin{aligned} \int_{\mathbb{R}^N} \alpha(x) u^2 dx &\leq \frac{1}{r} \int_{\mathbb{R}^N} a(x) u^2 dx + \delta \|z\|^2, \quad \forall z \in E_k^\perp, k \geq k_1, \\ \int_{\mathbb{R}^N} \beta(x) u^2 dx &\leq \frac{1}{r} \int_{\mathbb{R}^N} b(x) u^2 dx + \delta \|z\|^2, \quad \forall z \in E_k^\perp, k \geq k_1. \end{aligned}$$

Now we may check condition (I_3) . Let $r > 1$ be the constant given by Lemma 4.7. Then, given $\delta > 0$, we may apply (F_1) and Lemmas 4.6–4.7 to get $c_6 > 0$, $2 < \sigma < 2^*$ and $k \in \mathbb{N}$ such that

$$\int_{\mathbb{R}^N} |F(x, z)| dx \leq \frac{1}{2r} \|z\|^2 + \delta \|z\|^\sigma + c_6 \|z\|, \quad \forall z \in E_k^\perp.$$

Consequently,

$$I(z) \geq \left(\frac{r-1}{2r} - \delta \|z\|^{(\sigma-2)} \right) \|z\|^2 - c_6 \|z\|, \quad \forall z \in E_k^\perp.$$

Taking $\rho = \rho(\delta) > 0$ such that $\rho^{(\sigma-2)} = (r-1)/4\delta r$, we have $I(z) \geq (r-1)/(4r)\rho^2 - c_6\rho$, for every $z \in E_k^\perp \cap \partial B_\rho(0)$. Noting that $\rho \rightarrow \infty$ as $\delta \rightarrow 0$, by the above inequality, we may conclude that there is $k \in \mathbb{N}$ such that (I_3) holds for $V = E_k$ and $W = E_k^\perp$.

Now, given $m \in \mathbb{N}$, we take $\{\varphi_1, \dots, \varphi_{m+k}\} \subset C_0^\infty(\mathbb{R}^N)$ such that $\text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) = \emptyset$ for every $i \neq j \in \{1, \dots, m+k\}$. Invoking (7) and considering $\widehat{V} = \text{span}\{(\varphi_1, \varphi_1) \cdots, (\varphi_{m+k}, \varphi_{m+k})\}$, we see easily that the functional I satisfies

(I_4) . Since I satisfies $I(0) = 0$, (I_3) , (I_4) , (SCe) and it is even, by (F_4) , we may apply Theorem 2.4 to conclude that (P) possesses $\dim \widehat{V} - \dim V = m$ pairs of nonzero solutions for every $m \in \mathbb{N}$. This concludes the proof of Theorem 1.2.

For the proof of Theorem 1.4 we need only to check condition (I_4) . We consider Ω given by (F_6) and take $\{\varphi_1, \dots, \varphi_{m+k}\}$ as above with $\text{supp}(\varphi_i) \subset \Omega$, for $1 \leq i \leq k + m$. Then, setting $\widehat{V} = \text{span}\{(a\varphi_1, b\varphi_1) \dots, (a\varphi_{m+k}, b\varphi_{m+k})\}$, we may apply condition (F_6) to check that (I_4) holds.

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Received 21 October 2005; accepted 20 September 2006

Published Online First 21 September 2007

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