$C^{1,\alpha}$ -regularity for electrorheological fluids in two dimensions

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Abstract. We prove the existence of $C_{\text{loc}}^{1,\alpha}$ -solutions to a system of nonlinear partial differential equations describing steady planar motions of electrorheological fluids with Dirichlet boundary conditions for $\inf p > \frac{6}{5}$.

2000 *Mathematics Subject Classification*: 35J60, 76D03, 76A05, 46E30.

Key words: Electrorheological fluids, existence and regularity of solutions, non-standard growth.

1 Introduction

Electrorheological fluids (ERFs), one example of so-called smart fluids, have the special property that their mechanical properties depend in a dramatic way on an applied electric field. The range of applications of ERFs, where the user can influence the properties of the fluids, is enormous and includes vibration absorbers, engine mounts, earthquake-resistant buildings, clutches, actuators, and telemedicine.

In this paper we show the existence of a $C_{\text{loc}}^{1,\alpha}$ solution to the steady electrorheological fluid equations in two dimensions. The governing system for such fluids reads (cf. [RR01], [Růž00])¹

$$\begin{aligned}\operatorname{div}(\mathbf{E} + \mathbf{P}) &= 0, \\ \operatorname{curl} \mathbf{E} &= \mathbf{0},\end{aligned}\tag{1a}$$

¹Here we use the notation $[\nabla \mathbf{v}] \mathbf{w} = (\sum_j \partial_j v_i w_j)_i$.

$$\begin{aligned} -\operatorname{div} \mathbf{S} + [\nabla \mathbf{v}] \mathbf{v} + \nabla \phi &= \mathbf{f} + [\nabla \mathbf{E}] \mathbf{P}, \\ \operatorname{div} \mathbf{v} &= 0, \end{aligned} \quad (1b)$$

where $\mathbf{E}$ is the electric field, $\mathbf{P}$ the polarisation, $\mathbf{v}$ the velocity, $\mathbf{S}$ the extra stress tensor, ϕ the pressure, and $\mathbf{f}$ the mechanical force.

The extra stress $\mathbf{S}$ is a given nonlinear function of the electric field and the symmetric velocity gradient $\mathbf{D}$, i.e. $\mathbf{D}(\mathbf{v}) = \frac{1}{2}(\nabla \mathbf{v} + (\nabla \mathbf{v})^t)$, with non-standard growth $1 < p_\infty \leq p(\cdot) \leq p_0 < \infty$:

$$\begin{aligned} \mathbf{S}(\mathbf{D}, \mathbf{E}) := & \alpha_{21} \left((1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-1}{2}} - 1 \right) \mathbf{E} \otimes \mathbf{E} \\ & + (\alpha_{31} + \alpha_{33} |\mathbf{E}|^2) (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} \mathbf{D} \\ & + \alpha_{51} (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} (\mathbf{D} \mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{D} \mathbf{E}), \end{aligned} \quad (2)$$

where α_{ij} are material constants. In [Růž00] the existence of weak and strong solutions is shown for $p_\infty := \inf p > \frac{3d}{d+2}$ for bounded domains $\Omega \subset \mathbb{R}^d$. In a series of papers Kaplický, Málek, and Stará (see [KMS97], [KMS99], [Kap98]) studied a system very similar to (1b), but with standard growth, i.e. $p \equiv \text{constant}$. They showed that the two-dimensional system possesses a $C_{\text{loc}}^{1,\alpha}$ -solution if $\frac{6}{5} < p < 4$. In the case of the full system (1b) with the extra stress $\mathbf{S}$ given by (2) Acerbi and Mingione [AM02] proved partial regularity in any dimension if $p_\infty > \frac{3d}{d+2}$.

Our proof uses the technique developed in [KMS99]. However, we apply these ideas to modified quantities θ_λ and $V_{\lambda,\rho}$ which allows us to prove the existence of a $C_{\text{loc}}^{1,\alpha}$ -solution in the case of ERFs and to remove the technical upper bound.

Under certain assumptions on α_{ij} one can show that $\mathbf{S}$ has the following properties² (cf. [Ett02] for the requirements for a_{ij} in two and [Růž00] in three dimensions):

$$\begin{aligned} \partial_{kl} S_{ij}(\mathbf{D}, \mathbf{E}) B_{ij} B_{kl} &\geq c (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{B}|^2, \\ \mathbf{S}(\mathbf{D}, \mathbf{E}) : \mathbf{D} &\geq c (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{D}|^2, \\ |\partial_{kl} \mathbf{S}(\mathbf{D}, \mathbf{E})| &\leq C (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}}, \quad k, l = 1, 2, \\ |\partial_n \mathbf{S}(\mathbf{D}, \mathbf{E})| &\leq C (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-1}{2}} (1 + \ln(1 + |\mathbf{D}|^2)), \quad n = 1, 2 \end{aligned} \quad (3)$$

for all $\mathbf{B}, \mathbf{D} \in X := \{\mathbf{D} \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \operatorname{tr} \mathbf{D} = 0\}$ and $\mathbf{E} \in \mathbb{R}^2$, where we used the notation $\partial_{kl} \mathbf{S} = \frac{\partial \mathbf{S}}{\partial D_{kl}}$ and $\partial_n \mathbf{S} = \frac{\partial \mathbf{S}}{\partial E_n}$. Finally we assume that $p \in C^1(\mathbb{R}^+)$ is a given material function satisfying $1 < p_\infty \leq p(\cdot) \leq p_0 < \infty$.

² $\mathbf{A} : \mathbf{B} = \sum_{i,j} A_{ij} B_{ij}$

For $0 \leq \lambda \leq 1$ we define the approximation (note $\mathbf{S}^0 \equiv \mathbf{S}$)

$$\begin{aligned} \mathbf{S}^\lambda(\mathbf{D}, \mathbf{E}) := & \alpha_{21} \left((1 + \lambda|\mathbf{D}|^2)^{\frac{1-p(|\mathbf{E}|^2)}{2}} (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-1}{2}} - 1 \right) \mathbf{E} \otimes \mathbf{E} \\ & + (1 + |\mathbf{D}|^2)^{\frac{p(|\mathbf{E}|^2)-2}{2}} (1 + \lambda|\mathbf{D}|^2)^{\frac{2-p(|\mathbf{E}|^2)}{2}} \\ & \times \left((\alpha_{31} + \alpha_{33}|\mathbf{E}|^2)\mathbf{D} + \alpha_{51}(\mathbf{D}\mathbf{E} \otimes \mathbf{E} + \mathbf{E} \otimes \mathbf{D}\mathbf{E}) \right). \end{aligned}$$

One easily checks that $\mathbf{S}^\lambda(\mathbf{D}, \mathbf{E})$ has the following properties (cf. [Ett02])

$$\begin{aligned} \partial_{kl} S_{ij}^\lambda(\mathbf{D}, \mathbf{E}) B_{ij} B_{kl} & \geq c \left(\frac{1 + |\mathbf{D}|^2}{1 + \lambda|\mathbf{D}|^2} \right)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{B}|^2, \\ \mathbf{S}^\lambda(\mathbf{D}, \mathbf{E}) : \mathbf{D} & \geq c \left(\frac{1 + |\mathbf{D}|^2}{1 + \lambda|\mathbf{D}|^2} \right)^{\frac{p(|\mathbf{E}|^2)-2}{2}} |\mathbf{D}|^2, \\ |\partial_{kl} \mathbf{S}^\lambda(\mathbf{D}, \mathbf{E})| & \leq C \left(\frac{1 + |\mathbf{D}|^2}{1 + \lambda|\mathbf{D}|^2} \right)^{\frac{p(|\mathbf{E}|^2)-2}{2}}, \\ |\partial_n \mathbf{S}^\lambda(\mathbf{D}, \mathbf{E})| & \leq C \left(\frac{1 + |\mathbf{D}|^2}{1 + \lambda|\mathbf{D}|^2} \right)^{\frac{p(|\mathbf{E}|^2)-2}{2}} (1 + |\mathbf{D}|^2)^{\frac{1}{2}} \left(1 + \ln \left(\frac{1 + |\mathbf{D}|^2}{1 + \lambda|\mathbf{D}|^2} \right) \right) \end{aligned} \quad (4)$$

for all $\mathbf{B}, \mathbf{D} \in X$ and $\mathbf{E} \in \mathbb{R}^2$. The constants c, C are independent of λ .

2 The Regularity Theorem and its Proof

In the case $\mathbf{P}(\mathbf{E}) = \chi^E \mathbf{E}$, with constant dielectric susceptibility χ^E , the system (1a) reduces to the well-studied linear Maxwell's equations. Under appropriate assumptions on the boundary data we obtain that $\mathbf{E}$, and consequently $\mathbf{P} = \mathbf{P}(\mathbf{E})$ are smooth functions. Thus we can replace $\mathbf{f} + [\nabla \mathbf{E}] \mathbf{P}$ by $\tilde{\mathbf{f}}$. Omitting the tilde we thus consider the system

$$\begin{aligned} -\operatorname{div} \mathbf{S} + [\nabla \mathbf{v}] \mathbf{v} + \nabla \phi &= \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\Omega, \end{aligned} \quad (5)$$

with $\mathbf{S}$ satisfying (2).

In order to show the existence of a $C_{\text{loc}}^{1,\alpha}(\Omega, \mathbb{R}^2)$ solution we study an approximate system, where $\mathbf{S}$ is replaced by $\mathbf{S}^\lambda$, $0 < \lambda \leq 1$. The weak formulation of this

system reads³

$$\begin{aligned} & \int_{\Omega} \mathbf{S}^{\lambda}(\mathbf{D}(\mathbf{v}^{\lambda}), \mathbf{E}) : \mathbf{D}(\boldsymbol{\varphi}) \, dx + \int_{\Omega} [\nabla \mathbf{v}^{\lambda}] \mathbf{v}^{\lambda} \cdot \boldsymbol{\varphi} \, dx \\ & - \int_{\Omega} \phi \operatorname{div} \boldsymbol{\varphi} \, dx = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\varphi} \, dx \end{aligned} \quad (6)$$

for all $\boldsymbol{\varphi} \in C_0^{\infty}(\Omega, \mathbb{R}^2)$. In [ER02], [Ett02] for $p_{\infty} > \frac{6}{5}$ the following uniform estimates for the associated solution $\mathbf{v}^{\lambda} \in V_{p_{\infty}} := \{\mathbf{u} \in C_0^{\infty}(\Omega, \mathbb{R}^2), \operatorname{div} \mathbf{u} = 0\}^{1, p_{\infty}}$ are proved

$$\int_{\Omega} \left(\frac{1 + |\mathbf{D}(\mathbf{v}^{\lambda})|^2}{1 + \lambda |\mathbf{D}(\mathbf{v}^{\lambda})|^2} \right)^{\frac{p-2}{2}} |\mathbf{D}(\mathbf{v}^{\lambda})|^2 \, dx \leq C(\mathbf{f}, \mathbf{E}), \quad (7)$$

$$\|\nabla^2 \mathbf{v}^{\lambda} \tau\|_{2-\epsilon} \leq C(\mathbf{f}, \mathbf{E}, \nabla \tau, p_0, \epsilon) \quad \forall \epsilon > 0, \quad (8)$$

$$\|\nabla \mathbf{v}^{\lambda} \tau\|_r \leq C(\mathbf{f}, \mathbf{E}, \nabla \tau, p_0, r) \quad \forall r < \infty, \quad (9)$$

$$\int_{\Omega} \left(\frac{1 + |\mathbf{D}(\mathbf{v}^{\lambda})|^2}{1 + \lambda |\mathbf{D}(\mathbf{v}^{\lambda})|^2} \right)^{\frac{p-2}{2}} |\mathbf{D}(\nabla \mathbf{v}^{\lambda})|^2 \tau^2 \, dx \leq C(\mathbf{f}, \mathbf{E}, \nabla \tau, p_0), \quad (10)$$

where τ is a usual cut-off function and $\|\cdot\|_q$ is the usual norm in the Lebesgue space $L^q(\Omega, \mathbb{R}^2)$. Note that in [ER02] the requirement $\mathbf{f} \in L^{p'_{\infty}}(\Omega)$ for $p_{\infty} < 2$ can be weakened to $\mathbf{f} \in L^q(\Omega)$ with $q > 2$ by a simultaneous application of the steps “uniform local estimates in $W^{2, p_{\infty}}$ ” and “uniform local estimates in $W^{2, 2-\epsilon}$ ”. Let us briefly sketch how the restriction $p_{\infty} > \frac{6}{5}$ occurs. Testing equation (6) with $-\Delta \mathbf{v}^{\lambda} \tau^6$ the extra stress $\mathbf{S}^{\lambda}$ essentially yields the term occurring on the left hand side of (10). By embedding we thus have control over $\|\nabla \mathbf{v}^{\lambda} \tau^{\frac{6}{p_{\infty}}}\|_r^{p_{\infty}}$, where $r > 1$ is arbitrary. The convective term produces a term like $\int |\nabla \mathbf{v}^{\lambda}|^2 |\mathbf{v}^{\lambda}| \tau^5 \, dx$, which has to be controlled by $\|\nabla \mathbf{v}^{\lambda} \tau^{\frac{6}{p_{\infty}}}\|_r^{p_{\infty}}$ and (7). This is possible by the Hölder inequality if

$$(2 - p_{\infty}) \frac{1}{p_{\infty}} + \frac{1}{p_{\infty}^*} < 1,$$

which is equivalent to $5p_{\infty} > 6$, because $p_{\infty}^* = \frac{2p_{\infty}}{2-p_{\infty}}$. Note that based on (7) and (9) it is easily shown that there exists a pressure $\phi^{\lambda} \in L^{p_0'}(\Omega)$ satisfying (with abuse of notation of r and τ)

$$\|\phi^{\lambda} \tau\|_r \leq C(\mathbf{f}, \mathbf{E}, \nabla \tau, p_0, r) \quad \forall r < \infty. \quad (11)$$

Using these estimates we are now able to conclude

Theorem 1 *Let $\mathbf{S}$ be given by (2) satisfying (3). Let $\Omega \subset \mathbb{R}^2$ be a smoothly bounded domain, and let $\mathbf{f} \in L^q(\Omega, \mathbb{R}^2)$ with $q > 2$. Then for $p_{\infty} > \frac{6}{5}$ there exists a solution $\mathbf{v} \in V_{p_{\infty}} \cap W_{loc}^{2, q}(\Omega, \mathbb{R}^2) \subset C_{loc}^{1, \alpha}(\Omega, \mathbb{R}^2)$ of (5).*

³ $\mathbf{v} \cdot \mathbf{w} = \sum_i v_i w_i$

Proof. Let $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$ be arbitrary open subsets. At first we show uniform estimates in $W^{2,q_0}(\Omega_1, \mathbb{R}^2)$, where $q_0 > 2$ will be determined later.

We want to use the theory of linear systems with bounded coefficients on suitable small balls $B_\rho := B_\rho(x_0) \subset \Omega_2$ (cited as Theorem 6 in the appendix), where the radius $\rho(x_0) < \frac{1}{2} \text{dist}(\Omega_1, \Omega \setminus \Omega_2)$ will be fixed later. Define therefore

$$A_{ij,\lambda}^{kl}(x) := \partial_{kl} S_{ij}^\lambda(\mathbf{D}(\mathbf{v}^\lambda)(x), \mathbf{E}(x)),$$

$$V_{\lambda,\rho} := V(x_0, \rho, \lambda) := \sup_{B_\rho(x_0)} \left(\frac{1 + |\mathbf{D}(\mathbf{v}^\lambda)|^2}{1 + |\lambda \mathbf{D}(\mathbf{v}^\lambda)|^2} \right)^{\frac{1}{2}} \quad (12)$$

and $p_- := \inf_{B_\rho} p$, $p_+ := \sup_{B_\rho} p$, $\tilde{p}_+ := \max\{p_+, 2\}$ and $\tilde{p}_- := \min\{p_-, 2\}$ (therefore $\tilde{p}_- \leq 2, p_- \leq \tilde{p}_+$). From the properties of $\mathbf{S}^\lambda$ we see that we have for $\mathbf{B} \in X$ and $x \in B_\rho(x_0)$

$$c_1 V_{\lambda,\rho}^{\tilde{p}_- - 2} |\mathbf{B}|^2 \leq A_{ij,\lambda}^{kl}(x) B_{ij} B_{kl} \leq C_2 V_{\lambda,\rho}^{\tilde{p}_+ - 2} |\mathbf{B}|^2. \quad (13)$$

Because of $1 \leq V_{\lambda,\rho} \leq \frac{1}{\lambda}$ we have (cf. [GM82], [KMS99]) that $\mathbf{v}^\lambda \in C^{1,\alpha}(\Omega, \mathbb{R}^2)$ with $\alpha > 0$, where the norms depend on λ .

Let ξ be a cut-off-function. Using $\xi \partial_r \varphi$ as a test-function in (6) one sees after some calculations that $\xi \partial_r \mathbf{v}^\lambda$ and $\xi \partial_r \phi^\lambda$, $r = 1, 2$, solve

$$\int_{\Omega} A_{ij,\lambda}^{kl}(x) D_{kl}(\xi \partial_r \mathbf{v}^\lambda) D_{ij}(\varphi) dx - \int_{\Omega} \xi (\partial_r \phi^\lambda) \text{div } \varphi dx = \langle \mathbf{G}^{r,\lambda}, \varphi \rangle_{1,q} \quad (14)$$

for all $\varphi \in C_0^\infty(\Omega, \mathbb{R}^2)$ with

$$\begin{aligned} \langle \mathbf{G}^{r,\lambda,\xi}, \varphi \rangle_{1,q} := & \int_{\Omega} \left(-\partial_{kl} S_{ij}^\lambda (\partial_r \mathbf{v}^\lambda \otimes \nabla \xi)_{kl}^{\text{sym}} + \partial_n S_{ij}^\lambda \partial_r E_n \xi \right) D_{ij}(\varphi) \\ & - \mathbf{S}^\lambda : (\partial_r \varphi \otimes \nabla \xi)^{\text{sym}} + \mathbf{S}^\lambda : \mathbf{D}(\varphi) \partial_r \xi \\ & - \phi (\partial_r \varphi \cdot \nabla \xi - \text{div } \varphi \partial_r \xi) + (\mathbf{f} - [\nabla \mathbf{v}^\lambda] \mathbf{v}^\lambda) \xi \partial_r \varphi dx, \end{aligned}$$

From now on we take ξ with $\xi \equiv 1$ on $B_{3/4\rho}$, $|\nabla \xi| \leq 5/\rho$ and $\text{supp } \xi \subset B_\rho$ and we suppress the dependence of ξ in $\mathbf{G}^{r,\lambda,\xi}$, i.e. we write $\mathbf{G}^{r,\lambda}$. Note that due to (9) and (11)

$$\|\mathbf{G}^{r,\lambda}\|_{-1,q,B_\rho} \leq C/\rho,$$

where C is independent of λ and ρ . The right hand side of (14) is not in the form required in Theorem 6. Thus we replace the right hand side of (14) for $\varphi \in C_0^\infty(B_\rho, \mathbb{R}^2)$ with the left hand side of the weak formulation of the following auxiliary Stokes problem

$$\begin{aligned} -2 \text{div } \mathbf{D}(\mathbf{g}^{r,\lambda}) + \nabla \tilde{\phi}^\lambda &= \mathbf{G}^{r,\lambda} & \text{in } B_\rho, \\ \text{div } \mathbf{g}^{r,\lambda} &= 0 & \text{in } B_\rho, \\ \mathbf{g}^{r,\lambda} &= \mathbf{0} & \text{on } \partial B_\rho. \end{aligned}$$

For $\mathbf{g}$ with $\operatorname{div} \mathbf{g} = 0$ we have $2 \operatorname{div} \mathbf{D}(\mathbf{g}) = \Delta \mathbf{g}$ and thus according to [Gal98] the unique solution $\mathbf{g}^{r,\lambda} \in W_0^{1,q}(B_\rho)$ fulfils

$$\|\mathbf{g}^{r,\lambda}\|_{1,q,B_\rho} \leq C \|\mathbf{G}^{r,\lambda}\|_{-1,q,B_\rho} \leq C/\rho,$$

where the constant C is independent of λ and ρ . Let $\mathbf{h}^\lambda$ be a solution of

$$\begin{aligned} \operatorname{div} \mathbf{h}^\lambda &= \operatorname{div}(\xi \partial_r \mathbf{v}^\lambda) = \nabla \xi \cdot \partial_r \mathbf{v}^\lambda && \text{in } B_\rho \\ \mathbf{h}^\lambda &= 0 && \text{on } \partial B_\rho, \end{aligned}$$

with $\|\mathbf{h}\|_{1,s} \leq \|\nabla \xi \cdot \partial_r \mathbf{v}^\lambda\|_s \leq C/\rho$. Then $\mathbf{w}^\lambda := \xi \partial_r \mathbf{v}^\lambda + \mathbf{h}^\lambda \in V_2(B_\rho)$ and $\pi^\lambda := \xi \partial_r \phi^\lambda - \tilde{\phi}^\lambda \in L^2(B_\rho)$ solve

$$\begin{aligned} & \int_{\Omega} A_{ij,\lambda}^{kl}(x) D_{kl}(\mathbf{w}^\lambda) D_{ij}(\boldsymbol{\varphi}) dx - \int_{\Omega} \pi^\lambda \operatorname{div} \boldsymbol{\varphi} dx \\ &= \int_{\Omega} A_{ij,\lambda}^{kl}(x) D_{kl}(\mathbf{h}^\lambda) D_{ij}(\boldsymbol{\varphi}) dx + 2 \int_{\Omega} \mathbf{D}(\mathbf{g}^{r,\lambda}) : \mathbf{D}(\boldsymbol{\varphi}) dx. \end{aligned}$$

Now Theorem 6 yields using (13), (9) and $1 \leq V_{\lambda,\rho}$

$$\begin{aligned} \|\partial_r \mathbf{v}^\lambda\|_{1,s,B_{3/4\rho}} &\leq \|\mathbf{w}^\lambda\|_{1,s,B_{3/4\rho}} + \|\mathbf{h}^\lambda\|_{1,s,B_{3/4\rho}} \\ &\leq C/\rho \left(V_{\lambda,\rho}^{2-\tilde{p}-} + 1 \right) \leq C/\rho V_{\lambda,\rho}^{2-\tilde{p}-}, \end{aligned} \quad (15)$$

where $s = s(\lambda)$ is given by

$$s = \min \left\{ 2 \left(1 - \log \left(\frac{1 - \frac{1}{2} V_{\lambda,\rho}^{\tilde{p}- - \tilde{p}+}}{1 - V_{\lambda,\rho}^{\tilde{p}- - \tilde{p}+}} \right) / \log(\gamma_3) \right)^{-1}, q \right\} > 2, \quad \gamma_3 > 1.$$

From the inequality $\log(\frac{1-\frac{1}{2}y}{1-y}) \geq \frac{1}{2}y$ for $y \in [0, 1)$ we conclude that

$$1 - \frac{2}{s} \geq c V_{\lambda,\rho}^{-(\tilde{p}+ - \tilde{p}-)}. \quad (16)$$

Define

$$\theta_\lambda := \left(\frac{1 + |\mathbf{D}(\mathbf{v}^\lambda)|^2}{1 + |\lambda \mathbf{D}(\mathbf{v}^\lambda)|^2} \right)^{\frac{1}{2}}, \quad (17)$$

then for t fixed

$$|\nabla(\theta_\lambda^{t/2})| \leq C \theta_\lambda^{\frac{t-2}{2}} |\mathbf{D}(\nabla \mathbf{v}^\lambda)|.$$

Now because of (cf. (10), (15), $\theta_\lambda \leq 1$ and the definition of $\tilde{p}_-$)

$$\begin{aligned} \|\nabla(\theta_\lambda^{\frac{p_-}{2}})\|_{2,B_\rho} &\leq C, \\ \|\nabla(\theta_\lambda^{\frac{p_-}{2}})\|_{s,B_{3/4\rho}} &\leq C/\rho V_{\lambda,3/4\rho}^{\frac{p_- - \tilde{p}_-}{2}} V_{\lambda,\rho}^{2-\tilde{p}-} \end{aligned}$$

we have by interpolation for $2 < \sigma < s$

$$\|\nabla(\theta_\lambda^{p_-/2})\|_{\sigma, B_{3/4}\rho} \leq C/\rho^a V_{\lambda,\rho}^{a(2+\frac{p_-}{2}-\frac{3\tilde{p}_-}{2})}, \quad (18)$$

where $a \in (0, 1)$ is given by

$$\frac{1}{\sigma} = \frac{1-a}{2} + \frac{a}{s}. \quad (19)$$

The exact dependence of the embedding constant in the Theorem of Morrey/Sobolev (cf. [Zie89] 2.4.1, p. 58) together with Hölder and (9) implies

$$\begin{aligned} V_{\lambda,\rho/2}^{p_-/2} &\leq \|\theta_\lambda^{p_-/2} \tilde{\xi}\|_{\infty, B_{3/4}\rho} \\ &\leq C\rho^{\frac{1}{2}-\frac{1}{\sigma}} \left(\frac{\sigma-1}{\sigma-2}\right)^{\frac{\sigma-1}{\sigma}} \|\nabla(\theta_\lambda^{p_-/2} \tilde{\xi})\|_{\sigma, B_{3/4}\rho} \\ &\leq C \left(\frac{\sigma-1}{\sigma-2}\right)^{\frac{\sigma-1}{\sigma}} \left(\|\nabla(\theta_\lambda^{p_-/2})\|_{\sigma, B_{3/4}\rho} + \frac{1}{\rho} \|\theta_\lambda^{p_-/2}\|_{\sigma, B_\rho}\right) \\ &\leq C \left(\frac{\sigma-1}{\sigma-2}\right)^{\frac{\sigma-1}{\sigma}} \left(\|\nabla(\theta_\lambda^{p_-/2})\|_{\sigma, B_{3/4}\rho} + 1\right), \end{aligned}$$

where $\tilde{\xi}$ is a cut-off-function with $\tilde{\xi} \equiv 1$ on $B_{\rho/2}$, $\nabla \tilde{\xi} \leq 5/\rho$ and $\text{supp } \tilde{\xi} \subset B_{3/4}\rho$. Using (18), $V_{\lambda,\rho} \geq 1$ and (19) we get

$$\begin{aligned} V_{\lambda,\rho/2}^{p_-/2} &\leq C \frac{1}{\rho^a} \left(\frac{\sigma}{\sigma-2}\right)^{\frac{\sigma-1}{\sigma}} V_{\lambda,\rho}^{a(2+\frac{p_-}{2}-\frac{3\tilde{p}_-}{2})} \\ &= C \frac{1}{\rho^a} \left(\frac{1}{a} \frac{s}{s-2}\right)^{\frac{\sigma-1}{\sigma}} V_{\lambda,\rho}^{a(2+\frac{p_-}{2}-\frac{3\tilde{p}_-}{2})} \\ &\leq C \frac{1}{\rho^a} \left(\frac{1}{a}\right)^{\frac{q-1}{q}} V_{\lambda,\rho}^{(\tilde{p}_+ - \tilde{p}_-)(\frac{1}{2} + a\frac{s-2}{2s}) + a(2+\frac{p_-}{2}-\frac{3\tilde{p}_-}{2})}, \end{aligned} \quad (20)$$

where we used in the last line $\frac{\sigma-1}{\sigma} \leq \frac{q-1}{q}$, (16) and $\frac{\sigma-1}{\sigma} = \frac{1}{2} + a\frac{s-2}{2s}$. We now require:

$$\frac{p_-}{2} > \left(\frac{1}{2} + \frac{a(s-2)}{2s}\right) (\tilde{p}_+ - \tilde{p}_-) + a \left(2 + \frac{p_-}{2} - \frac{3\tilde{p}_-}{2}\right). \quad (21)$$

If $a = 0$ then this reads

$$p_- > \tilde{p}_+ - \tilde{p}_- \iff p_- + \tilde{p}_- > \tilde{p}_+,$$

and thus it is always possible to fulfil this requirement if ρ is small enough because p is continuous. Indeed, we choose $\rho_0 = \rho_0(p, \overline{\Omega_2})$ such that for fixed $\beta < 1$ and

fixed small enough⁴ $\epsilon_0 > 0$

$$\beta p_- > \tilde{p}_+ - \tilde{p}_- + \epsilon_0$$

is true on every ball $B_\rho(x_0)$ with centre $x_0 \in \Omega_2$ and radius $\rho \leq \rho_0$. This is possible because p is uniformly continuous on Ω_2 . Define the radius

$$\rho(x) := \min\{\rho_0, 1/2 \operatorname{dist}(x, \Omega \setminus \Omega_2)\}, \quad x \in \Omega_2.$$

So $\rho(x) = \rho_0$ for all $x \in \Omega_1$ and Ω_2 is the union of the balls $B_{\rho(x_0)}(x_0)$, $x_0 \in \Omega_2$, and for $x \in B_{\rho(x_0)}(x_0)$ it holds that $2\rho(x) \geq \rho(x_0)$. We set $a := \frac{\epsilon_0}{6\rho_0}$. Thus one easily checks that the right hand side of (21) is smaller than

$$\frac{1}{2}(\tilde{p}_+ - \tilde{p}_-) + 3ap_0 = \frac{1}{2}(\tilde{p}_+ - \tilde{p}_-) + \frac{\epsilon_0}{2} < \beta \frac{p_-}{2}. \quad (22)$$

Thus (21) is satisfied uniformly with respect to $x_0 \in \Omega_2$ and we conclude from (22) and (20)

$$V_{\lambda, \rho/2}^{p_-/2} \leq C(\epsilon_0, p_0) \frac{1}{\rho(x_0)^a} V_{\lambda, \rho}^{\beta p_-/2}.$$

Multiplying this with $\rho(x_0)^b$ we obtain

$$\begin{aligned} \rho(x_0)^b \theta_\lambda^{p_-/2}(x_0) &\leq \rho(x_0)^b \sup_{x \in B_{\rho(x_0)/2}(x_0)} \theta_\lambda^{p_-/2}(x) \\ &\leq C \left(\sup_{x \in B_{\rho(x_0)}(x_0)} \rho(x_0)^{\frac{b-a}{\beta}} \theta_\lambda^{p_-/2}(x) \right)^\beta. \end{aligned}$$

We demand $\frac{b-a}{\beta} = b$, thus $b = \frac{a}{1-\beta} > 0$. Using $\rho(x_0) \leq 2\rho(x)$ for $x \in B_{\rho(x_0)}(x_0)$ and taking the supremum over all $x_0 \in \Omega_2$ we conclude therefore

$$\sup_{x_0 \in \Omega_2} \rho(x_0)^b \theta_\lambda(x_0) \leq C_b \sup_{x_0 \in \Omega_2} (\rho(x_0)^b \theta_\lambda(x_0))^\beta$$

and further because $\beta < 1$ and $\theta_\lambda \leq \frac{1}{\lambda} < \infty$

$$\sup_{x_0 \in \Omega_2} \rho(x_0)^b \theta_\lambda(x_0) \leq C.$$

Finally we get

$$V_\lambda := \sup_{\Omega_1} \left(\frac{1 + |\mathbf{D}(\mathbf{v}^\lambda)|^2}{1 + |\lambda \mathbf{D}(\mathbf{v}^\lambda)|^2} \right)^{\frac{1}{2}} \leq C,$$

where C is independent of λ . Because the V_λ are uniformly bounded, we get by (16) $s(\lambda) \geq q_0$ for a $q_0 > 2$ and so the $\mathbf{v}^\lambda$ are uniformly bounded in $W^{2, q_0}(\Omega_1)$.

⁴For example $\epsilon_0 < \frac{1}{5}$ and $\max\{\frac{5}{6}, \frac{5p_0-9}{5p_0}\} < \beta < 1$ will do it.

A limit $\mathbf{v} \in W^{2,q_0}(\Omega_1, \mathbb{R}^2) \subset C^{1,\beta}(\Omega_1, \mathbb{R}^2)$ is a solution (see [ER02] or [Ett02] for the limit $\lambda \rightarrow 0$). Using the theory of linear systems with continuous coefficients (for instance [GM82]) the solution $\mathbf{v}$ is in $W^{2,q}(\Omega_1, \mathbb{R}^2)$. So we are finished because Ω_1 was arbitrary. $\square$

Remark 2 By the usual iterating technique one sees that for $\mathbf{f} \in C^\infty$ the solution $\mathbf{v}$ is in $C_{\text{loc}}^\infty(\Omega, \mathbb{R}^2)$.

Remark 3 If $\mathbf{f}$ is small enough, then the solution is unique, cf. [Růž00].

Remark 4 In the theorem we required $\mathbf{S}$ in the form (2), but it would be enough if there exists an approximation $\mathbf{S}^\lambda$ with (4), which is true if $\mathbf{S}$ has a potential and $\mathbf{S}$ fulfils (3).

A Linear Theory

Lemma 5 Let $\mathbf{u} \in V_2$ be the unique solution of

$$\int_{\Omega} \sum_{i,j} D_{ij}(\mathbf{u}(x)) D_{ij}(\boldsymbol{\varphi}(x)) dx = \int_{\Omega} \sum_{i,j} G_{ij}(x) D_{ij}(\boldsymbol{\varphi}(x)) dx \quad \forall \boldsymbol{\varphi} \in V_2. \quad (23)$$

Then for every $q \in [2, 2+r)$ and $G_{ij} \in L^q(\Omega)$, $i, j = 1, 2$, exists a constant $K = K(r) > 1$, such that $\mathbf{u} \in W_0^{1,q}(\Omega)$ and

$$\|\mathbf{D}(\mathbf{u})\|_q \leq K^{1-\frac{2}{q}} \|\mathbf{G}\|_q. \quad (24)$$

Proof. Using that $\text{div } \mathbf{u} = 0$ we get

$$\frac{1}{2} \int_{\Omega} \nabla \mathbf{u}(x) : \nabla \boldsymbol{\varphi}(x) dx = \int_{\Omega} \sum_{i,j} (G_{ij}(x) + G_{ji}(x)) \partial_j \varphi_i(x) dx,$$

and L^p -theory of classical Stokes for $p = 2$ and $p = r + 2$ together with the Korn inequality and Riesz-Thorin interpolation gives the desired result, cf. [KMS99]. $\square$

Theorem 6 Let U be a open bounded domain and $u \in V_2$ and $\pi \in L^2(U)$ with $\int_U \pi dx = 0$ be the solution of

$$\begin{aligned} \int_U A_{ij}^{kl}(x) D_{kl}(\mathbf{u}(x)) D_{ij}(\boldsymbol{\varphi}(x)) dx - \int_U \pi(x) \text{div } \boldsymbol{\varphi}(x) dx \\ = \int_U G_{ij}(x) D_{ij}(\boldsymbol{\varphi}(x)) dx \quad \forall \boldsymbol{\varphi} \in W_0^{1,2}(U), \end{aligned} \quad (25)$$

with $A_{ij}^{kl} \in L^\infty(U)$, $A_{ij}^{kl} = A_{kl}^{ij}$ for $i, j, k, l = 1, 2$,

$$\exists \gamma_1, \gamma_2 > 0 \forall \mathbf{B} \in X \equiv \{\mathbf{D} \in \mathbb{R}_{\text{sym}}^{2 \times 2}, \text{tr } \mathbf{D} = 0\} \text{ and for a.e. } x \in U : \quad (26)$$

$$\gamma_1 |\mathbf{B}|^2 \leq A_{ij}^{kl}(x) B_{ij} B_{kl} \leq \gamma_2 |\mathbf{B}|^2.$$

Let $\epsilon > 0$ be defined by

$$2 + \epsilon = 2 \left(1 - \log \left(\frac{1 - \frac{\gamma_1}{2\gamma_2}}{1 - \frac{\gamma_1}{\gamma_2}} \right) / \log(2K) \right)^{-1}. \quad (27)$$

Let $q \in (2, 2 + \epsilon]$ be arbitrary. $G_{ij} \in L^q(U)$, $i, j = 1, 2$. Then $\mathbf{u} \in W_0^{1,q}(U)$ and

$$\|\mathbf{D}(\mathbf{u})\|_q \leq \frac{2}{\gamma_1} K^{1-\frac{2}{q}} \|G\|_q, \quad (28)$$

where K is the constant from (24).

Proof. A perturbation and fixed point argument, see [KMS99] or [Gia83]. $\square$

Acknowledgements

The research of F. Ettwein was supported by the DFG Graduiertenkolleg “Nichtlineare Differentialgleichungen: Modellierung, Theorie, Numerik, Visualisierung”. The research of L. Diening nad M. Růžička was supported by the DFG Forschergruppe “Nichtlineare Partielle Differentialgleichungen: Theoretische und Numerische Analysis”.

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Received and accepted 29 November 2004

Published Online First 13 August 2007