A non trivial solution for the non linear differential equation $\delta d\xi = f'(\langle \xi, \xi \rangle)\xi$

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1 Introduction

The non linear elliptic problem

$$-\Delta u = f'(u)$$

has been largely studied by many authors in the last forty years and a lot of variational tools have been developed (see for example [1]).

Recently in [2] the following equation has been studied

$$\nabla \times \nabla \times A = f'(A), \quad A : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \quad (1)$$

where $\nabla \times$ denotes the curl operator and f' is the gradient of a convex function $f : \mathbb{R}^3 \rightarrow \mathbb{R}$.

(1) represents the semilinear Maxwell equations (SME) introduced in [2] in the magnetostatic case.

The solutions of (1) are the stationary points of the functional

$$J(A) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \times A|^2 dx - \int_{\mathbb{R}^3} f(A) dx. \quad (2)$$

The difficulty in solving (1) lies in the strongly degenerate nature of the functional (2). In fact it is not bounded from above and from below, and there is

an infinite dimensional subspace where $\int_{\mathbb{R}^3} f(A)$ is not controlled by the norm $\|\nabla \times A\|_2$.

In [2] a solution has been found splitting A in the sum of a conservative field and a zero divergence field, and using some technical min-max devices. The aim of this work is to extend this theory to the case in which the vector field A is substituted by a k -form ξ , in a compact Riemannian n -manifold M .

The equation (1) becomes

$$\delta d\xi = f'(\langle \xi, \xi \rangle) \xi \quad (3)$$

where d is the exterior differential, δ is its adjoint with respect to the inner product

$$(\eta, \xi)_2 = \int_M \langle \eta, \xi \rangle \omega = \int_M *(\eta \wedge * \xi) \omega$$

$*$ is the Hodge operator and ω is a volume n -form.

We assume that f has a subcritical growth, say

$$f(t) = (2/p)t^{p/2}, \quad t > 0, \quad 2 < p < 2^* = \frac{2n}{n-2}, \quad n \geq 3. \quad (4)$$

We are looking for weak solutions of (3), namely for k -differential forms ξ belonging to the Sobolev space $H_k^1(M)$ (see section 2 for definition) such that

$$\int_M \langle d\xi, d\eta \rangle \omega = \int_M f'(\langle \xi, \xi \rangle) \langle \xi, \eta \rangle \omega, \quad \forall \eta \in H_k^1(M). \quad (5)$$

The main result is

Theorem 1 *If f satisfies assumption (4), then (3) has a non trivial weak solution $\xi \in H_k^1(M)$.*

The paper is organized as follows:

In section 2, we introduce the functional framework and some preliminary results

In section 3 we prove Theorem 1 by adapting to our situation some variational tools developed in [2]

Finally, in section 4, we consider the particular case of a compact, connected three-manifold, where generic k -forms are substituted by vectorial fields.

2 The functional framework

The functional associated with (3) is

$$J(\xi) = \frac{1}{2} \int_M \langle d\xi, d\xi \rangle \omega - \frac{1}{2} \int_M f(\langle \xi, \xi \rangle) \omega. \quad (6)$$

Now we use Hodge decomposition (see [3]) to split ξ in the sum $d\alpha + \beta$, where $\delta\beta = 0$. Replacing in (6) and taking (4) into account we get

$$J(\alpha, \beta) = \frac{1}{2} \int_M \langle d\beta, d\beta \rangle \omega - \frac{1}{p} \int_M (\langle d\alpha + \beta, d\alpha + \beta \rangle)^{p/2} \omega. \quad (7)$$

Actually, this functional makes still sense if α and β are less regular than differential forms.

For any $q > 1$ and $k \in \mathbb{N}$, let $H_k^{1,q}(M)$, $H_k^1(M)$ and $L_k^q(M)$ be defined as the closure of the spaces $\Lambda^k(M)$ of the k -forms on M with respect to the norms $\|\cdot\|_{1,q}$, $\|\cdot\|$ and $\|\cdot\|_q$

$$\begin{aligned} H_k^{1,q}(M) &:= \overline{\Lambda^k(M)}^{\|\cdot\|_{1,q}}, \\ H_k^1(M) &:= \overline{\Lambda^k(M)}^{\|\cdot\|}, \\ L_k^q(M) &:= \overline{\Lambda^k(M)}^{\|\cdot\|_q} \end{aligned}$$

where, for every $\xi \in \Lambda^k(M)$,

$$\begin{aligned} \|\xi\|_{1,q}^q &:= \int_M \langle d\xi, d\xi \rangle^{q/2} \omega + \int_M \langle \delta\xi, \delta\xi \rangle^{q/2} \omega + \int_M \langle \xi, \xi \rangle^{q/2} \omega \\ \|\xi\|^2 &:= \int_M \langle d\xi, d\xi \rangle \omega + \int_M \langle \delta\xi, \delta\xi \rangle \omega + \int_M \langle \xi, \xi \rangle \omega \\ \|\xi\|_q^q &:= \int_M \langle \xi, \xi \rangle^{q/2} \omega. \end{aligned}$$

By the Sobolev immersion $H_k^1(M) \hookrightarrow L_k^q(M)$ for $1 \leq q \leq \frac{2n}{n-2}$, we have $J(\alpha, \beta) < +\infty$ for $\alpha \in H_{k-1}^{1,p}(M)$ and $\beta \in H_k^1(M)$.

For $\alpha \in H_{k-1}^{1,p}(M)$, $\beta \in H_k^1(M)$, $\xi \in L_k^p(M)$ we set

$$\begin{aligned} F(\alpha, \beta) &:= \int_M \langle d\alpha + \beta, d\alpha + \beta \rangle^{p/2} \omega, \\ F(\xi) &:= \int_M \langle \xi, \xi \rangle^{p/2} \omega \\ F_\beta &:= F(\cdot, \beta). \end{aligned}$$

Moreover

$$\begin{aligned} \mathcal{W} &:= \{\eta \in H_k^1(M) \mid \delta\eta = 0\} \\ \mathcal{C} &:= \{\eta \in H_{k-1}^{1,p}(M) \mid d\eta = 0\}. \end{aligned}$$

In order to solve (3) we shall fix $\beta \in \mathcal{W}$ and minimize F_β . However we see that F_β is not coercive on the space $H_{k-1}^{1,p}(M)$, for any $\beta \in \mathcal{W}$.

In fact, if $\alpha \in \Lambda^{k-1}(M)$ is any non trivial closed form, we define the indexed set

$$(\alpha_t)_{t>0} := (t\alpha)_{t>0}$$

and we have that

$$\begin{aligned} \lim_{t \rightarrow +\infty} F_\beta(\alpha_t) &= F_\beta(0) \\ \lim_{t \rightarrow +\infty} \|\alpha_t\|_{1,p} &= +\infty. \end{aligned}$$

Now consider the orthogonal complement to $\mathcal{C}$

$$\mathcal{V} := \mathcal{C}^\perp. \quad (8)$$

Theorem 2 *For every $\beta \in H_k^1(M)$, F_β is coercive on $\mathcal{V}$.*

In order to attain such a result, we need some spectral properties of the Laplace-Beltrami operator on k -forms.

Since $H_k^1(M)$ is an Hilbert space by the scalar product

$$(\xi, \eta)_{H_k^1(M)} = \int_M (\langle d\xi, d\eta \rangle + \langle \delta\xi, \delta\eta \rangle + \langle \xi, \eta \rangle) \omega,$$

we can consider the canonical Riesz isomorphism L between $H_k^1(M)$ and its dual.

We set

$$\begin{aligned} D(-\Delta) &:= L^{-1}(L_k^2(M)) \\ \tilde{L} &:= L|_{D(-\Delta)} \\ -\Delta &:= \tilde{L} - I, \end{aligned}$$

where I is the identity map.

Observe that $\Lambda^k(M) \subset D(-\Delta)$ and

$$-\Delta|_{\Lambda^k(M)} = d\delta + \delta d$$

i.e. $-\Delta$ is an extension of the Laplace-Beltrami operator.

Moreover

$$(-\Delta\xi, \eta)_2 = \int_M \langle d\xi, d\eta \rangle \omega + \int_M \langle \delta\xi, \delta\eta \rangle \omega, \quad \forall \xi, \eta \in D(-\Delta). \quad (9)$$

Lemma 3 *$\tilde{L}$ is a self-adjoint, invertible operator, and its inverse is compact.*

Proof. By definition, $\tilde{L} : D(-\Delta) \rightarrow L_k^2(M)$ is invertible and symmetric together with its inverse.

Actually, they are self-adjoint, since $\tilde{L}^{-1}$ is defined in the whole space $L_k^2(M)$.

Finally $\tilde{L}^{-1}$ is compact since the embedding $H_k^1(M) \hookrightarrow L_k^2(M)$ is compact. $\square$

Remark 4 Since $\tilde{L}^{-1}$ is compact and self-adjoint, the spectrum of $\tilde{L}$ is discrete and divergent and each eigenvalue is positive and has finite multiplicity.

Moreover there exists an orthonormal countable basis of $L_k^2(M)$ constituted by eigenvectors of $\tilde{L}$.

Lemma 5 1 λ is an eigenvalue of $-\Delta$ iff and $\mu = \lambda + 1$ is an eigenvalue of $\tilde{L}$;

2 ξ is an eigenvector of $-\Delta$ iff ξ is an eigenvector of $\tilde{L}$.

Proof. Let λ be an eigenvalue of $-\Delta$ and $\xi \neq 0$ be the corresponding eigenvector. Then

$$-\Delta \xi = \lambda \xi \Leftrightarrow (\tilde{L} - I)\xi = \lambda \xi \Leftrightarrow \tilde{L}\xi = \lambda \xi + \xi = (\lambda + 1)\xi. \quad (10)$$

□

Lemma 6 Let $\xi \in H_k^1(M)$ and ξ^0 be its orthogonal projection on $\ker(-\Delta)$. The norm

$$\|\xi\|_\sim^2 := \int_M (\langle d\xi, d\xi \rangle + \langle \delta\xi, \delta\xi \rangle + \langle \xi^0, \xi^0 \rangle) \omega \quad (11)$$

is equivalent to the usual norm on $H_k^1(M)$.

Proof. Of course $\|\cdot\|_\sim \leq \|\cdot\|$ since

$$\int_M \langle \xi^0, \xi^0 \rangle \omega \leq \int_M \langle \xi, \xi \rangle \omega, \quad \forall \xi \in H_k^1(M).$$

By Remark 4 and Lemma 5, the spectrum of $-\Delta$ is nonnegative, discrete and divergent, and there exists an orthonormal countable basis of $L_k^2(M)$, composed by eigenvectors of $-\Delta$.

Let

$$\lambda_1 \leq \lambda_2 \leq \dots \quad (12)$$

be the sequence of eigenvalues different from 0 of $-\Delta$, repeated according to their multiplicity, and

$$f_1, f_2, \dots \quad (13)$$

be the corresponding eigenvectors of $-\Delta$.

The eigenvectors $f_1, f_2, \dots$ constitute a basis of $(\ker(-\Delta))^\perp$, so we take a basis $h_1, h_2, \dots, h_N$ of $\ker(-\Delta)$ to complete that of $L_k^2(M)$.

Now let $\xi \in H_k^1(M)$, then

$$\xi = \sum_{i=1}^{+\infty} \xi_i f_i + \sum_{i=1}^N \xi_i^0 h_i = \sum_{i=1}^{+\infty} \xi_i f_i + \xi^0, \quad (14)$$

where ξ_i and ξ_i^0 are the Fourier coefficients of ξ .

Since $H_k^1(M) = D(-\Delta^{\frac{1}{2}})$, we have that

$$\begin{aligned}\|\xi\|^2 &= \sum_{i=1}^{+\infty} \lambda_i \xi_i^2 + \sum_{i=1}^{+\infty} \xi_i^2 + \sum_{i=1}^N (\xi_i^0)^2 \\ &\leq \sum_{i=1}^{+\infty} (\lambda_i + 1) \xi_i^2 + (\xi^0, \xi^0)_2 \leq M \|\xi\|_\sim^2\end{aligned}$$

where $M = (\lambda_1 + 1)/\lambda_1$. $\square$

Now we can prove Theorem 2:

Proof (of Theorem 2). First we prove that there exists $c_1 > 0$ s.t.

$$\|\alpha\| \leq c_1 \left(\int_M \langle d\alpha, d\alpha \rangle \omega \right)^{\frac{1}{2}}, \quad \forall \alpha \in \mathcal{V}. \quad (15)$$

By Lemma 6, we have just to prove that, if $\alpha \in \mathcal{V}$ and α^0 is its projection on $\ker(-\Delta)$,

$$\delta\alpha = 0 \quad (16)$$

and

$$\alpha^0 = 0. \quad (17)$$

(16) is true since (8) implies that

$$(\delta\alpha, \eta)_2 = (\alpha, d\eta)_2 = 0, \quad \forall \eta \in \Lambda^{k-2}(M).$$

Moreover

$$(-\Delta\alpha^0, \alpha^0)_2 = 0 \quad (18)$$

then, using (9), (18) implies that $\alpha^0 \in \mathcal{C}$. But $\alpha_0 \in \mathcal{C}^\perp$ too, so (17) holds.

Now, take any $\alpha \in \mathcal{V}$ and $\beta \in H_k^1(M)$. By (15) and since the inclusions $H_{k-1}^1(M) \subset L_{k-1}^p(M)$ and $L_k^p(M) \subset L_k^2(M)$ are continuous, certainly

$$\|\alpha\|_p \leq k_1 \|\alpha\| \leq k_2 \|d\alpha\|_2 \leq k_3 \|d\alpha\|_p, \quad (19)$$

so, by (16) and (19), we have

$$\|\alpha\|_{1,p}^p = \|d\alpha\|_p^p + \|\alpha\|_p^p \leq k_4 \|d\alpha\|_p^p \simeq F_\beta(\alpha). \quad (20)$$

Hence the conclusion clearly follows. $\square$

Corollary 6.1 *For every $\beta \in H_k^1(M)$ there exists a unique minimum for the functional F_β on $\mathcal{V}$.*

Proof. By the previous theorem $F_\beta \simeq \|\cdot\|_{1,p}^p$ on $\mathcal{V}$, so it is coercive and weakly lower semicontinuous. Hence F_β has a minimizer on $\mathcal{V}$ by Weierstrass theorem, and this minimizer is unique by the strict convexity of F_β . $\square$

3 Proof of Theorem 1

Set $\Phi : H_k^1(M) \rightarrow \mathcal{V}$ such that for every $\beta \in H_k^1(M)$, F_β achieves at $\Phi(\beta)$ its minimizer on $\mathcal{V}$.

Moreover, let

$$G(\beta) := F_\beta(\Phi(\beta)) \quad (21)$$

for $\beta \in H_k^1(M)$.

Now consider the functional

$$I(\beta) = J(\Phi(\beta), \beta) = \frac{1}{2} \int_M \langle d\beta, d\beta \rangle \omega - \frac{1}{p} G(\beta). \quad (22)$$

for $\beta \in H_k^1(M)$. If I were C^1 , it could be easily seen that its critical points are weak solutions of (3), by following analogous arguments as those used in [4]. However we don't know so much about regularity of I , so we proceed as follows.

First we prove the following

Lemma 7 *G is weakly continuous.*

Proof. Let

$$\beta_n \rightharpoonup \beta \text{ in } H_k^1(M). \quad (23)$$

Since $H_k^1(M) \hookrightarrow L_k^p(M)$ compactly, from (23) we deduce

$$\beta_n \rightarrow \beta \text{ in } L_k^p(M), \quad (24)$$

therefore $(F(\beta_n))_n$ is bounded.

Since for every $n \in \mathbb{N}$: $0 \leq G(\beta_n) \leq F(\beta_n)$,

$$G(\beta_n) \text{ is bounded} \quad (25)$$

so there exists $c \in \mathbb{R}$ such that, up to a subsequence,

$$G(\beta_n) \rightarrow c. \quad (26)$$

We want to show that

$$G(\beta) = c. \quad (27)$$

From (24) and (25) it follows that $(d\Phi(\beta_n))_n$ is bounded in $L_k^p(M)$, and then, by (19),

$$(\Phi(\beta_n))_n \text{ is bounded in } \mathcal{V},$$

so, up to subsequences,

$$\Phi(\beta_n) \rightharpoonup \bar{\alpha} \text{ in } \mathcal{V}. \quad (28)$$

Since F is weakly lower semicontinuous, by (24) and (28) we have that

$$\begin{aligned} c &= \lim_n G(\beta_n) = \lim_n F(\beta_n + d\Phi(\beta_n)) \geq \\ &\geq F(\beta + d\bar{\alpha}) \geq G(\beta), \end{aligned} \quad (29)$$

the last inequality follows from (21) and the definition of the map Φ .

To prove $G(\beta) \geq c$, let's consider the following function

$$g_q : \xi \in H_k^1(M) \mapsto \langle \xi, \xi \rangle_q^{p/2} \in \mathbb{R}, \quad \forall q \in M. \quad (30)$$

Obviously $g_q \in C^2(H_k^1(M), \mathbb{R})$, for all $q \in M$. Using Taylor formula, we get

$$\begin{aligned} g_q(\beta + d\Phi(\beta)) &= g_q(\beta_n + d\Phi(\beta_n) + \beta + d\Phi(\beta) - \beta_n - d\Phi(\beta_n)) \\ &= g_q(\beta_n + d\Phi(\beta_n)) \\ &\quad + \langle g'_q(\beta_n + d\Phi(\beta_n)), \beta + d\Phi(\beta) - \beta_n - d\Phi(\beta_n) \rangle \\ &\quad + \frac{1}{2} g''_q(\vartheta_n) [\beta + d\Phi(\beta) - \beta_n - d\Phi(\beta_n)]^2. \end{aligned} \quad (31)$$

where $\vartheta_n \in H_k^1(M)$ is a suitable convex combination of $\beta_n + d\Phi(\beta_n)$ and $\beta + d\Phi(\beta)$.

Since g_q is convex from (31) we get

$$\begin{aligned} g_q(\beta + d\Phi(\beta)) &\geq g_q(\beta_n + d\Phi(\beta_n)) \\ &\quad + \langle g'_q(\beta_n + d\Phi(\beta_n)), \beta + d\Phi(\beta) - \beta_n - d\Phi(\beta_n) \rangle. \end{aligned} \quad (32)$$

Since (32) holds in every point, we can omit q . By (30) and (32) we have that

$$\begin{aligned} G(\beta) &\geq \int_M g(\beta_n + d\Phi(\beta_n)) \omega \\ &\quad + \int_M \langle g'(\beta_n + d\Phi(\beta_n)), \beta + d\Phi(\beta) - \beta_n - d\Phi(\beta_n) \rangle \omega \end{aligned} \quad (33)$$

Now observe that, by Holder inequality,

$$\begin{aligned} \int_M \langle g'(\beta), \gamma \rangle \omega &= \frac{p}{2} \int_M \langle g(\beta)^{\frac{p-2}{p}} \beta, \gamma \rangle \omega \\ &\leq \frac{p}{2} \left(\int_M \langle g(\beta)^{\frac{p-2}{p}} \beta, g(\beta)^{\frac{p-2}{p}} \beta \rangle^{\frac{p}{2(p-1)}} \omega \right)^{\frac{2(p-1)}{p}} \|\gamma\|_p \end{aligned} \quad (34)$$

$$= \frac{p}{2} \|\beta\|_p \|\gamma\|_p \quad (35)$$

for β and $\gamma \in L_k^p(M)$. From (34) we deduce that, if $\beta \in L_k^p(M)$, the linear map

$$\gamma \in L_k^p(M) \mapsto \int_M \langle g'(\beta), \gamma \rangle \omega \in \mathbb{R} \quad (36)$$

is continuous and

$$\|g'(\beta)\| \leq \|\beta\|_p. \quad (37)$$

So from (25) and (37) we deduce that

$$\|g'(\beta_n + d\Phi(\beta_n))\| \text{ is uniformly bounded} \quad (38)$$

and by (24) and (38) we can conclude that

$$\int_M \langle g'(\beta_n + d\Phi(\beta_n)), (\beta - \beta_n) \rangle \omega \longrightarrow 0. \quad (39)$$

Since $\Phi(\beta_n)$ is a critical point for F_{β_n} ,

$$\int_M \langle g'(\beta_n + d\Phi(\beta_n)), d\Phi(\beta) - d\Phi(\beta_n) \rangle \omega = 0 \quad (40)$$

and this implies, together with (33) and (39), that

$$G(\beta) \geq c. \quad (41)$$

Finally (29) and (41) imply (27). $\square$

Remark 8 If $\beta \in \mathcal{W}$ is different from 0, then $\beta + d\Phi(\beta) \neq 0$. In fact, arguing by contradiction, if $\beta + d\Phi(\beta) = 0$, then $\delta d\Phi(\beta) = -\delta\beta = 0$. This should imply that

$$0 = \int_M \langle \delta d\Phi(\beta), \Phi(\beta) \rangle \omega = \int_M \langle d\Phi(\beta), d\Phi(\beta) \rangle \omega$$

from which $\beta = d\Phi(\beta) = 0$.

Now we introduce a suitable cut function in the functional I , in order to get a new functional bounded from below.

Let $\beta_0 \in \mathcal{W} - \{0\}$ and $s : \mathbb{R} \rightarrow \mathbb{R}$ be a C^1 map such that

1. $s(t) \geq 0$ and $s(t) = 0$ when $t \leq 0$,
2. $s'(t) > 0$ when $t > 0$,
3. $s(G(\beta_0)) \geq 1 + \frac{1}{2} \int_M \langle d\beta_0, d\beta_0 \rangle \omega$,
4. s is bounded

and set

$$I_s(\beta) := \frac{1}{2} \int_M \langle d\beta, d\beta \rangle \omega - s(G(\beta)). \quad (42)$$

The following Lemma holds

Lemma 9 I_s takes a negative minimum.

Proof. I_s is coercive and weakly lower semicontinuous by definition of s and Lemma 7. So there exists a $\bar{\beta} \in \mathcal{W}$ such that

$$I_s(\bar{\beta}) \leq I_s(\beta), \quad (43)$$

for every $\beta \in \mathcal{W}$.

By 3. we have

$$I_s(\bar{\beta}) \leq I_s(\beta_0) \leq -1.$$

□

Remark 10 A trivial (but important) consequence of (43) is that $\bar{\beta} \neq 0$.

Lemma 11 Suppose $\beta \in \mathcal{W}$ and that there exists $\mu \neq 0$ such that

$$\int_M \langle d\beta, d\gamma \rangle \omega - \mu^2 \int_M f' \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \langle \beta + d\Phi(\beta), \gamma \rangle \omega = 0 \quad (44)$$

for every $\gamma \in \mathcal{W}$.

Then $\beta_\lambda = \lambda\beta$, $\lambda = \mu^{\frac{2}{p-2}}$ is a solution of

$$\int_M \langle d\beta, d\gamma \rangle \omega - \int_M f' \langle \beta_\lambda + d\Phi(\beta_\lambda), \beta_\lambda + d\Phi(\beta_\lambda) \rangle \langle \beta_\lambda + d\Phi(\beta_\lambda), \gamma \rangle \omega = 0, \quad (45)$$

for every $\gamma \in \mathcal{W}$.

Proof. Since f is strictly convex, we have that for every $\beta \in \mathcal{W}$, the equation

$$dF_\beta(\alpha) = 0, \quad \alpha \in \mathcal{V} \quad (46)$$

has the unique solution $\Phi(\beta)$.

In a more explicit way, (46) means that, for every $\beta \in \mathcal{W}$, $\Phi(\beta)$ is the unique form in $\mathcal{V}$ such that for any $\vartheta \in \mathcal{V}$

$$\int_M f' \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \langle \beta + d\Phi(\beta), d\vartheta \rangle \omega = 0 \quad (47)$$

If we set $(\Phi(\beta))_\lambda = \lambda\Phi(\beta)$ and take any $\vartheta \in \mathcal{V}$, from (47) we have

$$\begin{aligned} & \int_M f'(\langle \beta_\lambda + d(\Phi(\beta))_\lambda, \beta_\lambda + d(\Phi(\beta))_\lambda \rangle) \langle \beta_\lambda + d(\Phi(\beta))_\lambda, d\vartheta \rangle \omega \\ &= \int_M \lambda^{p-1} f'(\langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle) \langle \beta + d\Phi(\beta), d\vartheta \rangle \omega = 0 \end{aligned} \quad (48)$$

so, by uniqueness, $\Phi(\beta_\lambda) = (\Phi(\beta))_\lambda$.

Now a direct calculation shows that

$$\begin{aligned}
 & \int_M \langle d\beta_\lambda, d\gamma \rangle \omega - \int_M f' \langle \beta_\lambda + d\Phi(\beta_\lambda), \beta_\lambda + d\Phi(\beta_\lambda) \rangle \langle \beta_\lambda + d\Phi(\beta_\lambda), \gamma \rangle \omega \\
 &= \lambda \int_M \langle d\beta, d\gamma \rangle \omega - \lambda^{p-1} \int_M f' \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \langle \beta + d\Phi(\beta), \gamma \rangle \omega \\
 &= \lambda \left[\int_M \langle d\beta, d\gamma \rangle \omega - \mu^2 \int_M f' \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \langle \beta + d\Phi(\beta), \gamma \rangle \omega \right] \\
 &= 0,
 \end{aligned} \tag{49}$$

for every $\gamma \in \mathcal{W}$. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. We shall prove that if β is a minimizer for I_s in $\mathcal{W}$ then, for a suitable $\lambda > 0$, $\xi = \beta_\lambda + d\Phi(\beta_\lambda)$ is a nontrivial weak solution for (3).

Since β is a minimizer for I_s in $\mathcal{W}$, for every $\varepsilon > 0$ and $\gamma \in \mathcal{W}$ we have that

$$\begin{aligned}
 0 &\leq I_s(\beta + \varepsilon\gamma) - I_s(\beta) \\
 &= \int_M \langle d\beta, \varepsilon d\gamma \rangle \omega + \frac{1}{2} \varepsilon^2 \int_M \langle d\gamma, d\gamma \rangle \omega - s(G(\beta + \varepsilon\gamma)) + s(G(\beta)) \\
 &= \varepsilon \int_M \langle d\beta, d\gamma \rangle \omega + \frac{1}{2} \varepsilon^2 \int_M \langle d\gamma, d\gamma \rangle \omega - s'(\vartheta_\varepsilon)(G(\beta + \varepsilon\gamma) - G(\beta)) \\
 &= \varepsilon \int_M \langle d\beta, d\gamma \rangle \omega + \frac{1}{2} \varepsilon^2 \int_M \langle d\gamma, d\gamma \rangle \omega \\
 &\quad - s'(\vartheta_\varepsilon) \left(\int_M g(\beta + \varepsilon\gamma + d\Phi(\beta + \varepsilon\gamma)) \omega - \int_M g(\beta + d\Phi(\beta)) \omega \right) \\
 &= \varepsilon \int_M \langle d\beta, d\gamma \rangle \omega + \frac{1}{2} \varepsilon^2 \int_M \langle d\gamma, d\gamma \rangle \omega \\
 &\quad - s'(\vartheta_\varepsilon) \int_M \langle g'(\beta + d\Phi(\beta)), \varepsilon\gamma + d\Phi(\beta + \varepsilon\gamma) - d\Phi(\beta) \rangle \omega \\
 &\quad - \frac{1}{2} s'(\vartheta_\varepsilon) \int_M g''(v_\varepsilon) [\varepsilon\gamma + d\Phi(\beta + \varepsilon\gamma) - d\Phi(\beta)]^2 \omega
 \end{aligned} \tag{50}$$

where ϑ_ε is a convex combination of $G(\beta)$ and $G(\beta + \varepsilon\gamma)$ and v_ε is a convex combination of $\beta + d\Phi(\beta)$ and $\beta + \varepsilon\gamma + d\Phi(\beta + \varepsilon\gamma)$.

Since $\Phi(\beta)$ is a minimizer for F_β ,

$$\int_M \langle g'(\beta + d\Phi(\beta)), d\Phi(\beta + \varepsilon\gamma) - d\Phi(\beta) \rangle \omega = 0,$$

so, since g_q is strictly convex and s' is positive, (50) implies

$$0 \leq \varepsilon \int_M \langle d\beta, d\gamma \rangle \omega + \frac{1}{2} \varepsilon^2 \int_M \langle d\gamma, d\gamma \rangle \omega - s'(\vartheta_\varepsilon) \int_M \langle g'(\beta + d\Phi(\beta)), \varepsilon\gamma \rangle \omega. \tag{51}$$

Now, if we divide (51) by ε and let $\varepsilon \rightarrow 0$, we get

$$\int_M \langle d\beta, d\gamma \rangle \omega - s'(\vartheta) \int_M \langle g'(\beta + d\Phi(\beta)), \gamma \rangle \omega \geq 0, \quad \forall \gamma \in \mathcal{W} \quad (52)$$

where $\vartheta = \lim_{\varepsilon \rightarrow 0} \vartheta_\varepsilon = F(\beta + d\Phi(\beta))$ by continuity of G .

From (52) we deduce that

$$\int_M \langle d\beta, d\gamma \rangle \omega - s'(\vartheta) \int_M f' \langle \beta + d\Phi(\beta), \beta + d\Phi(\beta) \rangle \langle \beta + d\Phi(\beta), \gamma \rangle \omega = 0 \quad (53)$$

holds for every $\gamma \in \mathcal{W}$.

Note that $s'(\vartheta) > 0$, because $\vartheta \neq 0$ by Remark 8 and Remark 10. By Lemma 11, if we put $\lambda = s'(\vartheta)^{\frac{1}{p-2}}$ then

$$\int_M \langle d\beta_\lambda, d\gamma \rangle \omega - \int_M f' \langle \beta_\lambda + d\Phi(\beta_\lambda), \beta_\lambda + d\Phi(\beta_\lambda) \rangle \langle \beta_\lambda + d\Phi(\beta_\lambda), \gamma \rangle \omega = 0 \quad (54)$$

for every $\gamma \in \mathcal{W}$.

Now it remains to prove that $\xi = \beta_\lambda + d\Phi(\beta_\lambda)$ is a weak solution for the problem (3) i.e.

$$\int_M \langle d\xi, d\eta \rangle \omega - \int_M f' \langle \xi, \xi \rangle \langle \xi, \eta \rangle \omega = 0, \quad (55)$$

for every $\eta \in \Lambda^k(M)$.

By Hodge decomposition

$$\eta = d\eta_1 + \eta_2$$

where $\eta_1 \in \mathcal{V}$ and $\eta_2 \in \mathcal{W}$.

Clearly

$$\int_M f' \langle \xi, \xi \rangle \langle \xi, d\eta_1 \rangle \omega = \langle dF_{\beta_\lambda}(\Phi(\beta_\lambda)), \eta_1 \rangle = 0, \quad (56)$$

so, since $d\xi = d\beta_\lambda$ and $d\eta = d\eta_2$, (55) is a trivial consequence of (54) and (56). $\square$

4 A particular case

In this section, we just want to show how the problem (3) is formulated when we assume that (M, g) is a compact, connected Riemannian three-manifold.

The equation we consider now is

$$\nabla \times \nabla \times A = f'(A) \text{ on } M, \quad (57)$$

where $\nabla \times$ is the curl operator, $f(A) = \frac{1}{p} g(A, A)^{\frac{p}{2}}$, $2 < p < 2^*$ and $A \in \chi(M)$ (i.e. A is a vector field on M).

The Hodge decomposition produces the following splitting on A

$$A = u + \nabla v \quad (58)$$

where $u \in \chi(M)$ is a vector field such that $\nabla \cdot u = 0$ and $v : M \rightarrow \mathbb{R}$ is a smooth map.

The functional associated with (57) is

$$J(u, v) = 1/2 \int_M |\nabla u|^2 \omega - 1/p \int_M |u + \nabla v|^p \omega \quad (59)$$

where $|\cdot|$ replaces $g(\cdot, \cdot)^{\frac{1}{2}}$, and ω is again a volume form.

$J(u, v)$ is well defined if $u \in H_1^1(M)$ and $v \in H^{1,p}(M)$ where

$$\begin{aligned} H^{1,p}(M) &:= \overline{C^\infty(M)}^{\|\cdot\|_{1,p}} \\ H_1^1(M) &:= \overline{\chi(M)}^{\|\cdot\|} \end{aligned}$$

and for any $h \in C^\infty(M)$ and $w \in \chi(M)$

$$\begin{aligned} \|h\|_{1,p}^p &:= \int_M |\nabla h|^p \omega + \int_M |h|^p \omega \\ \|w\|^2 &:= \int_M |\nabla w|^2 \omega + \int_M |w|^2 \omega. \end{aligned}$$

Now, arguing as in section 2, we fix $u \in H^1(M)$ and minimize the functional $F_u(v) = \int_M |u + \nabla v|^p \omega$ with respect to v . As in section 2, we can conclude that the right space for v is

$$\mathcal{V} := \mathcal{C}^\perp,$$

where

$$\mathcal{C} = \{\eta \in H^{1,p}(M) \mid d\eta = 0\}.$$

Since M is connected, the space of closed 0-forms is made up of constant maps. Then

$$\mathcal{V} = \left\{ v \in H^{1,p}(M) \mid \int_M v \omega = 0 \right\}. \quad (60)$$

The particular nature of the space $\mathcal{V}$, allows to prove Theorem 2 using the following classical result (see for example in [5])

Lemma 12 (Poincaré inequality). *There exists $c(M, p) > 0$ such that for any $h \in \mathcal{V}$:*

$$\|h\|_p \leq c \|\nabla h\|_p. \quad (61)$$

Observe that (61) is nothing but (19) when $k = 0$, so, a posteriori, we can look at (19) as a generalized Poincaré inequality for k -forms.

Finally, Theorem 1 says that equation (57) has a weak solution.

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