Stable tameness of automorphisms of $F\langle x, y, z \rangle$ fixing z

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Abstract It is proved that every z -automorphism (z -coordinates, respectively) of the free associative algebra $F\langle x, y, z \rangle$ over an arbitrary field F is stably tame.

Keywords Automorphisms · Coordinates · Polynomial algebras · Free associative algebras · Stably tameness · Lifting problem

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Secondary 13F20 · 13W20 · 14R10 · 16W20 · 16Z05

1 Introduction and main results

An F -automorphism of a free associative algebra $F\langle x_1, \dots, x_n \rangle$ (a polynomial algebra $F[x_1, \dots, x_n]$) is elementary if it fixes all variables except one. An F -automorphism is tame if it is the product of elementary automorphisms. An F -automorphisms $(f_1, \dots, f_n)$ of $F\langle x_1, \dots, x_n \rangle (F[x_1, \dots, x_n])$ is *stably tame* if

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there exists a nonnegative integer m , the automorphism $(f_1, \dots, f_n, x_{n+1}, \dots, x_{n+m})$ of $F\langle x_1, \dots, x_{n+m} \rangle$ ($F[x_1, \dots, x_{n+m}]$) is tame.

From now on all automorphisms are F -automorphisms.

Whether every automorphism of a free associative algebra (polynomial algebra) is stably tame is a long-standing and interesting open question.

In [5], stable tameness of some special kind of automorphisms of polynomial and free associative algebras was obtained. In [6], it is proved that every fixing z automorphism of the polynomial algebra $F[x, y, z]$ over a field F of characteristic zero is stably tame, among other things. It is the first big step for attacking the stably tameness problem.

In this paper, based on our recent result of the lifting problem, we prove the following

Theorem 1.1 *Every fixing z automorphism of the free associative algebra $F\langle x, y, z \rangle$ over an arbitrary field F is stably tame and becomes tame after adding one variable.*

A polynomial $f \in F\langle x_1, \dots, x_n \rangle$ is a coordinate if $(f, f_2, \dots, f_n)$ is an automorphism for some $f_2, \dots, f_n \in F\langle x_1, \dots, x_n \rangle$. The coordinate f is stably tame if $(f, f_2, \dots, f_n)$ is stably tame. A coordinate f is a x_n -coordinate if there exists an automorphism $(f, f_2, \dots, f_{n-1}, x_n)$.

As a direct consequence of Theorem 1.1, we obtain

Theorem 1.2 *Every z -coordinate of $F\langle x, y, z \rangle$ is stably tame.*

2 Proofs

To prove Theorem 1.1, we only need to prove the following

Theorem 2.1 *For every automorphism (f, g) in $\text{Aut}_{F[z]} F\langle x, y, z \rangle$, (f, g, t) is a tame automorphism in $\text{Aut}_{F[z]} F\langle x, y, z, t \rangle$.*

To prove Theorem 2.1, we need some preliminaries.

Theorem 2.2 *An automorphism (f, g) in $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ can be canonically decomposed as product of the following type of automorphisms: (i) Linear automorphisms in $\text{Aut}_{F[z]} F\langle x, y, z \rangle$; (ii) Automorphisms that can be obtained by an elementary automorphism in*

$$\text{Aut}_{F[z]} F\langle x, y, z \rangle,$$

conjugated by a linear automorphism in

$$\text{Aut}_{F(z)} F(z) *_F F\langle x, y \rangle$$

Proof It is Theorem 3.4 in [1]. □

Lemma 2.3 *Suppose a polynomial $f \in F\langle x, y, z \rangle$ is neither a left multiple nor a right multiple of any nontrivial polynomial in $F[z] - F$. Then, $(F(z) * F[f]) \cap (F\langle x, y, z \rangle) = F(z) * F[f]$.*

Proof Suppose $R \in F(z) * F[f] \setminus F\langle f, z \rangle = (F(z) * F[f]) - F$. We need to prove that $R \in F(z) * F\langle x, y \rangle \setminus F\langle x, y, z \rangle$, that is, $R \notin F\langle x, y, z \rangle$.

As $R \in F(z) * F[f]$, it has the form

$$R = \sum_i \alpha_i f M_i,$$

where M_i are linear independent over F , $\alpha_i \in F(z)$.

It is easy to see that if $\alpha_k \in F(z) \setminus F[z]$ for some k , then $R \notin F\langle x, y, z \rangle$. Hence, $\alpha_k \in F[z]$ for all k . Then, R has the form $R = \sum_{k=0}^n \gamma_k z^k f N_k$. The set $\{z^k\}_{k=0}^\infty$ can be completed to a basis $\{e_i\}$ of $F(z)$ as a vector space over F , and every element x in the coproduct $F(z) * F[f]$ can be expressed as $x = e_i f h_i$. Hence, all $N_k \in F(z) * F[f]$. On the other hand, if some of $N_k \notin F\langle x, y, z \rangle$, then $f N_k \notin F\langle x, y, z \rangle$, because f is not right divisible by any polynomial from $F[z] - F$. Hence, $(N_k \in F(z) * F[f]) \cap (F\langle x, y, z \rangle) = F(z) * F[f]$ for all k and $\deg(N_k) < \deg(R)$. We conclude by induction. $\square$

Let z_l denote the left multiplication operator on z , z_r the right multiplication operator. An automorphism ψ in $\text{Aut}_{F[z]} F\langle x, y, z \rangle$ linear in both x and y has the following form: $\psi : x \rightarrow a_{11}x + a_{12}y, y \rightarrow a_{21}x + a_{22}y$ where $a_{ij} \in F[z_l, z_r]$. It should be pointed out, the study of such automorphisms is equivalent to the study of invertible 2×2 matrices over the polynomial ring of two commuting variables z_l and z_r over a field.

Lemma 2.4 *Let $\psi : x \rightarrow a_{11}x + a_{12}y, y \rightarrow a_{21}x + a_{22}y$ be a linear automorphism of $F(z_l, z_r)\langle x, y \rangle$ where $a_{ij} \in F(z_l, z_r)$ and let $\varphi : x \rightarrow x, y \rightarrow Q(x)$ be an elementary z -automorphism of $F\langle x, y, z \rangle$. Suppose $\phi = \psi \circ \varphi \circ \psi^{-1} \in \text{Aut}_{F[z]} F\langle x, y, z \rangle$. Then, it has the following form*

$$\phi : x \rightarrow x + bh(ax + by), y \rightarrow y - ah(ax + by); \quad a, b \in F[z_l, z_r],$$

where $h(t) \in F[z_l, z_r][t]$ is a one-variable polynomial over $F[z_l, z_r]$.

Proof Let $\alpha \in F[z_l, z_r]$ be the least common multiple of the denominators of $a_{21}, a_{22}, a = a_{21}\alpha, b = a_{22}\alpha$. Then

$$\phi : x \rightarrow x + bQ((ax + by)/\alpha), y \rightarrow y - aQ((ax + by)/\alpha);$$

where $a, b \in F[z_l, z_r]$. Note that the coefficients of $bQ((ax + by)/\alpha)$ and $aQ((ax + by)/\alpha)$ must be in $F[z_l, z_r]$, hence by Lemma 2.3, the coefficients of $Q((ax + by)/\alpha)$ must be in $F[z_l, z_r]$ as well. Therefore, $Q((ax + by)/\alpha)$ must have the form $h(ax + by)$ for some $h(t) \in F[z_l, z_r][t]$. $\square$

Lemma 2.5 *Let ψ be a z -automorphism of $F\langle x, y, z \rangle$ in the form*

$$x \rightarrow x + bh(ax + by), y \rightarrow y - ah(ax + by); \quad a, b \in F[z_l, z_r]$$

for some polynomial $h(t) \in F[z_l, z_r][t]$. Then, it is stably tame and becomes tame after adding one variable.

Proof Based on the method of Smith [5],

$$(x + bh(ax + by), y - ah(ax + by), t) = (x, y, t - h(ax + by)) \\ \times (x - bt, y + at, t)(x, y, z, t + h(ax + by))(x + bt, y - at, t).$$

Remark 2.6 The general Anick type automorphisms (see [3,4])

$$(x + zg(xz - zy, z), y + g(xz - zy, z)z),$$

in $\text{Aut}_{F[z]} F\langle x, y, z \rangle$, with an arbitrary polynomial $g(t, s) \in F\langle t, s \rangle$, are obviously covered by Lemma 2.5.

Proof of Theorem 1.1 First, an automorphism of Type (i) in Theorem 2.1 is stably tame and becomes tame after adding one variable, according to [2].

Now, suppose we have an automorphism of Type (ii) in Theorem 2.1, we are done by Lemma 2.5. $\square$

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