The stochastic Navier–Stokes equations for heat-conducting, compressible fluids: global existence of weak solutions

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Abstract. We investigate the well posedness of the stochastic Navier–Stokes equations for viscous, compressible, non-isentropic fluids. The global existence of finite-energy weak martingale solutions for large initial data within a bounded domain of $\mathbb{R}^d$ is established under the condition that the adiabatic exponent $\gamma > d/2$. The flow is driven by a stochastic forcing of multiplicative type, white in time and colored in space. This work extends recent results on the isentropic case, the main contribution being to address the issues which arise from coupling with the temperature equation. The notion of solution and corresponding compactness analysis can be viewed as a stochastic counterpart to the work of Feireisl (Dynamics of viscous compressible fluids, vol 26. Oxford University Press, Oxford, 2004).

1. Introduction

1.1. Governing equations

This paper is devoted to the analysis of the initial boundary value problem for the stochastic Navier–Stokes equations for non-isentropic, compressible fluids. This system of stochastic partial differential equations governs the evolution of a viscous, compressible fluid (or gas) subject to random perturbations by noise within a connected, bounded domain D which is an open subset of $\mathbb{R}^d$ with smooth boundary.

The macroscopic state of the fluid is described by a triple (ρ, u, θ) consisting of the scalar, nonnegative density ρ , an $\mathbb{R}^d$ -valued velocity field u , and a scalar, nonnegative temperature θ . The system is posed in the space/time domain $D \times [0, T]$ and is written as follows:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} \mathcal{S} + \rho \sigma_k(\rho, \rho u, \rho \theta, x) \dot{\beta}_k \\ \partial_t(\rho \theta) + \operatorname{div}(\rho \theta u) - \operatorname{div}(\kappa(\theta) \nabla \theta) = \mathcal{S} : \nabla u - \theta p_\theta(\rho) \operatorname{div} u \\ (\rho(0), (\rho u)(0), (\rho \theta)(0)) = (\rho_0, m_0, \rho_0 \theta_0). \end{cases} \quad (1.1)$$

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The pressure is denoted as $P(\rho, \theta)$, the heat conductivity is denoted as $\kappa(\theta)$, and the stress tensor is written as $\mathcal{S}$. In particular, we assume that P and $\mathcal{S}$ are of the form

$$P(\rho, \theta) = p_m(\rho) + \theta p_\theta(\rho). \quad (1.2)$$

$$\mathcal{S}(u) = \mu(\nabla u + \nabla u^t) + \lambda \operatorname{div} u I. \quad (1.3)$$

The velocity field u is taken to satisfy a Dirichlet condition on the boundary ∂D and the temperature θ a Neumann condition. The equation is posed in space dimension $d \geq 3$. Regarding the noise, $\{\beta_k\}_{k=1}^\infty$ denotes a collection of one-dimensional Brownian motions, modulated by noise coefficients $\{\sigma_k\}_{k=1}^\infty$ which may depend on the fluid variables.

In the case that each σ_k vanishes identically, system (1.1) has been well studied in the deterministic literature on compressible fluid mechanics. In fluid dynamic regimes where temperature changes are negligible, the first two equations in (1.1) constitute a closed system of PDE for (ρ, u) , known as the barotropic compressible Navier–Stokes equations. The foundational work on this system is due to P.L. Lions [12], which establishes the existence of global weak solutions in the case that $p_m(\rho) = \rho^\gamma$ for $\gamma > \frac{9}{5}$ (in dimension 3), starting from large initial datum with finite initial energy. The manuscript [12] also discusses compressible models with temperature, but only provides sketches of the proof. A more complete discussion, which requires only $\gamma > \frac{3}{2}$, is due to Feireisl [7]. The work [7] provides the main motivation for our notion of solution and strategy of proof below.

By including the noise, system (1.1) becomes a model for a turbulent compressible flow, with variable density and temperature. The mathematical study of the compressible stochastic Navier–Stokes equations has seen several recent developments. In the barotropic case, the first result is due to [8], which used a deterministic approach in the case where the coefficients σ_k are independent of the fluid variables. In this setting, one can make a convenient change of variables which turns the SPDE into a random PDE. In a more general setting, a stochastic approach is required in order to give a meaning to the noise. Three sets of authors [3, 14, 17] studied independently the case of more general noise coefficients, establishing the existence of global martingale solutions to (1.1) in the barotropic regime. It should be noted that the article [3] of Breit/Hofmanova was the first to appear on ArXiv, while the article [17] of D. Wang/H. Wang was the first to appear in print.

The main focus of the present article is to extend the results of [3, 14, 17] to the more general setting of (1.1), which allows for changes in the temperature of the fluid. We use a combination of the arguments in [7, 14] to treat the stochastic and deterministic sides of the problem, respectively. In particular, we follow the work of E. Feireisl [7] for the notion of solution to the temperature equation. We also adapt various compactness arguments from [7], modified appropriately to the frame of arguments in [14] for building weak martingale solutions. We now turn to a more precise statement of our results.

1.2. Hypotheses

In this section, we layout our assumptions about the various inputs which determine system (1.1). This includes the initial data (ρ_0, m_0, θ_0) , the constitutive laws on the pressure law $P(\rho, \theta)$, and the heat conductivity coefficient $\kappa(\theta)$. In addition, we address the continuity and summability assumptions about the noise coefficients $\{\sigma_k\}_{k=1}^\infty$.

HYPOTHESIS 1.1. *The pressure law takes the form $P(\rho, \theta) = p_m(\rho) + \theta p_\theta(\rho)$, where p_m, p_θ are non-decreasing functions in $C[0, \infty) \cap C^1(0, \infty)$ such that $p_m(0) = p_\theta(0) = 0$. Moreover, we assume p_m is convex and satisfies $p_m(\rho) \sim \rho^\gamma$, while $p_\theta(\rho) \lesssim 1 + \rho^\Gamma$, where $\gamma > \frac{d}{2}$ and $\Gamma = \frac{\gamma}{d}$.*

HYPOTHESIS 1.2. *The heat conductivity coefficient is bounded strictly away from zero, belongs to $C^2[0, \infty)$ and satisfies $\kappa(\theta) \sim \theta^2$.*

HYPOTHESIS 1.3. *The initial data ρ_0, m_0 and θ_0 are deterministic. The initial density $\rho_0 \in L_x^\gamma$ is nonnegative and compatible with the initial momentum in the sense that $m_0 1_{\{\rho_0=0\}}$ vanishes. The initial kinetic energy is finite, meaning that*

$$\int_D \frac{1}{2} \frac{|m_0|^2}{\rho_0} dx < \infty. \quad (1.4)$$

The initial temperature θ_0 belongs to L_x^∞ , and there exists a constant $\underline{\theta}$ such that $\theta_0 \geq \underline{\theta}$ almost everywhere in D .

HYPOTHESIS 1.4. *For each k , the coefficient $\sigma_k : \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+ \times D \rightarrow \mathbb{R}^d$ is bounded and continuous. Moreover, for each $k \geq 1$ there exists a constant C_k such that for each $\rho_1, \rho_2 \in \mathbb{R}_+, m_1, m_2 \in \mathbb{R}^d$, and $\alpha_1, \alpha_2 \in \mathbb{R}_+$*

$$\sup_{x \in D} |\sigma_k(\rho_1, m_1, \alpha_1, x) - \sigma_k(\rho_2, m_2, \alpha_2, x)| \leq C_k (|\rho_1 - \rho_2| + |m_1 - m_2| + |\alpha_1 - \alpha_2|). \quad (1.5)$$

HYPOTHESIS 1.5. *The sequence of coefficients $\{\sigma_k\}_{k=1}^\infty$ satisfy*

$$\sum_{k=1}^{\infty} |\sigma_k|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L_{\rho, m, \alpha}^\infty)} < \infty, \quad (1.6)$$

where we denote for $p \geq 1$

$$|\sigma_k|_{L_x^p(L_{\rho, m, \alpha}^\infty)} = \left| \sup_{\rho, m, \alpha} |\sigma_k(\rho, m, \alpha, \cdot)| \right|_{L_x^p}.$$

Hypotheses 1.1, 1.2, and 1.3 are slightly simplified (for purpose of exposition) versions of the assumptions in [7]. Note, however, that the assumption $\Gamma = \frac{\gamma}{d}$ is not for expositional reasons. This assumption is used crucially to close a priori bounds, analogously to the deterministic theory [7]. The value of considering an unbounded heat conductivity coefficient κ was identified even earlier by P.L. Lions in the monograph [12]. On the other hand, Hypotheses 1.4 and 1.5 are natural analogues of the corresponding assumptions made by the first named author in [14].

1.3. Notion of solution

Next we introduce our notion of weak solution to (1.1). The definition includes both analytic and probabilistic aspects. On the probabilistic side, there is the stochastic basis and the accompanying notion of measurability. This ensures a form of time consistency between the noise and the solution and allows the stochastic integrals to be properly defined. On the analytic side, a natural energy space is identified along with a notion of solution to (1.1) as a function of space and time. The momentum equation is understood in the Ito sense in time and in the sense of distributions (analytically) in space. The continuity equation is understood exactly as in the deterministic setting, $\mathbb{P}$ almost surely in ω . Following [7], the temperature equation is replaced by an inequality in the sense of distributions. To make this precise, we introduce a class of test functions $\mathcal{D}_{temp}$.

DEFINITION 1.6. We write $\varphi \in \mathcal{D}_{temp}$ provided that φ is a nonnegative function in $C^\infty([0, T] \times D)$ such that $\frac{\partial \varphi}{\partial n}(t, x) = 0$ for $(t, x) \in [0, T] \times \partial D$ and $\varphi(T, x) = 0$ for $x \in D$.

DEFINITION 1.7. A triple (ρ, u, θ) is a weak solution to (1.1) provided there exists a stochastic basis

$$(\Omega, \mathcal{F}, (\mathcal{F}^t)_{t=0}^T, \mathbb{P}, \{\beta_k\}_{k \geq 1})$$

such that

1. The quadruple $(\rho, \rho u, \rho \theta, u)$ belongs to $L^2(\Omega \times [0, T]; \mathcal{P}; L^\gamma \times L^{\frac{2\gamma}{\gamma+1}} \times L^q \times [H_0^1]^d)$, where $\mathcal{P}$ is the predictable σ -algebra generated by $(\mathcal{F}_t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$. Moreover, the pair $(\rho, \rho u) : \Omega \times [0, T] \rightarrow [L^\gamma \times L^{\frac{2\gamma}{\gamma+1}}]_w$ is a continuous stochastic process with $\mathbb{P}$ a.s. continuous sample paths.
2. Continuity equation: For all $\phi \in C_c^\infty(D)$ and $t \in [0, T]$, the following equality holds $\mathbb{P}$ a.s.

$$\int_D \rho(t) \phi dx = \int_D \rho_0 \phi dx + \int_0^t \int_D \rho u \cdot \nabla \phi dx ds. \quad (1.7)$$

3. Momentum equation: For all $\phi \in [C_c^\infty(D)]^d$ and $t \in [0, T]$, the following equality holds $\mathbb{P}$ a.s.

$$\begin{aligned} \int_D \rho u(t) \cdot \phi dx &= \int_D m_0 \cdot \phi + \int_0^t \int_D [\rho u \otimes u - \mathcal{S}(u)] : \nabla \phi \\ &\quad + P(\rho, \theta) \operatorname{div} \phi \, dx ds \\ &\quad + \sum_{k=1}^{\infty} \int_0^t \int_D \rho \sigma_k(\rho, \rho u, \rho \theta, x) \cdot \phi dx d\beta_k(s). \end{aligned} \quad (1.8)$$

4. Temperature inequality: For all $\varphi \in \mathcal{D}_{temp}$, the following inequality holds $\mathbb{P}$ a.s.

$$\begin{aligned} & \int_0^T \int_D \rho \theta (\partial_t \varphi + u \cdot \nabla \varphi) + \mathcal{K}(\theta) \Delta \varphi \, dx ds + \int_D \rho_0 \theta_0 \varphi_0 dx \\ & \leq \int_0^T \int_D [\theta p_\theta(\rho) \operatorname{div} u - \mathcal{S}(u) : \nabla u] \varphi dx ds, \end{aligned} \quad (1.9)$$

where $\mathcal{K}(\theta) = \int_1^\theta \kappa(z) dz$.

5. The following energy estimate holds: for all $p \geq 1$,

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[|\sqrt{\rho} u|_{L_t^\infty(L_x^2)}^{2p} + |\rho|_{L_t^\infty(L_x^\gamma)}^{\gamma p} + |\rho \theta|_{L_t^\infty(L_x^1)}^p \right] < \infty. \\ & \mathbb{E}^\mathbb{P} \left[|u|_{L_t^2(H_{0,x}^1)}^{2p} + |\theta|_{L_t^2(H_x^1)}^{2p} + |\nabla \log \theta|_{L_{t,x}^2}^{2p} \right] < \infty. \end{aligned} \quad (1.10)$$

1.4. Main result

Next we present the main result of the article.

THEOREM 1.8. *Let (ρ_0, m_0, θ_0) be an initial density, momentum, temperature triple satisfying Hypothesis 1.3. Suppose the pressure $P(\rho, \theta)$ satisfies Hypothesis 1.1. Let $\{\sigma_k\}_{k \geq 1}$ be a collection of noise coefficients satisfying Hypothesis 1.1 and 1.5. Then there exists a weak solution (ρ, u, θ) to system 1.1 in the sense of Definition 1.7.*

The proof of this theorem relies on a four-level approximating scheme. Three of the levels are inspired by the theory of Feireisl [7] for the treatment of the deterministic Navier–Stokes equations for compressible, non-isentropic fluids. The lowest level uses a time splitting scheme and is similar to the technique used by Berthelin/Vovelle [2] for the 1-d compressible Euler system. Each layer involves a compactness step and an identification procedure. The compactness step involves proving an appropriate tightness result and applying a recent generalization of the Skorohod theorem due to Jakubowski [9] and Vaart/Wellner [16] (Theorem 7.18). The identification procedure involves several ingredients. The first step is to use a martingale method to make a preliminary passage to the limit in the momentum equation, up to a modification in the pressure law. The second is to use this partial stability result to upgrade from strong to weak convergence of the densities. The last step is to use the strong convergence of the density to prove the convergence of the temperature away from vacuum and then use a renormalized limit in the vacuum regions.

1.5. Outline

The outline of this article is as follows. Section 1 presents the fundamental problem introducing the governing equations and the basic assumptions on the system, constitutive relations, noise, boundary, and initial data. In addition, it presents the notion of weak martingale solution to our initial boundary value problem as well as the main result on the global existence of weak solutions.

Section 2 presents the formal a priori bounds. It is worth observing that, unlike the barotropic case, the total energy does not dissipate. Nonetheless, using coloring Hypothesis 1.5, for all $p < \infty$ we obtain $L^p(\Omega)$ bounds on the total energy. Using these bounds together with a suitable variant of the Poincaré inequality, Lemma 2.1, the renormalized form of temperature equation gives $L^p(\Omega; H^1)$ bounds on θ and u .

Section 3 presents the existence for the lowest level of our approximating scheme (the τ layer) to stochastic Navier–Stokes system 1.1. The construction is based on a splitting scheme similar to the one used in [2] for the compressible, stochastic Euler equation. Half of the time, we neglect the evolution of the noise and run the continuity and momentum equation at twice the usual speed. The other half of the time, the noise evolves at $\sqrt{2}$ times the usual speed. However, the splitting of the temperature equation is slightly more subtle. Two of the terms in the equation, $\delta\theta^3$ and $\operatorname{div}(\kappa(\theta)\nabla\theta)$, run at speed 1 for all times, while the remaining terms run at twice the usual speed, but only when the noise is turned off.

Section 4 presents the existence proof for the n layer of the approximating scheme. The compactness step involves two types of uniform bounds on the sequence of approximations $\{(\rho_\tau, u_\tau, \theta_\tau)\}_{\tau>0}$. The first type is analogous to the formal estimates obtained in Sect. 2. The second type of estimates is encoded in a tightness proof. These are necessary to overcome an oscillatory factor $h_{\det}^\tau$ arising in the weak form, which converges (as $\tau \rightarrow 0$) only weakly in time. To combat these oscillations, we need strong convergence in time of any terms in the weak form multiplied by this factor. This difficulty was addressed in detail in the work [14]. The trick is to obtain estimates on the density and velocity which give stronger bounds on these quantities as a function of (t, x) , at the cost of taking a weaker norm as a function of ω . Namely, these estimates are not in terms of $L^p(\Omega)$, but only in measure. The proofs of [14] carry through to this context, so our main focus is to obtain similar improvements on the bounds for the temperature.

Section 5 presents the existence result for the ϵ -layer approximation. As $n \rightarrow \infty$, a subtlety arises in the compactness analysis of the temperature equation which does not seem to have a deterministic counterpart. Namely, in order to apply the Aubin/Lions-type Lemma from Feireisl [7] (Proposition 7.23), we require uniform $L_t^\infty(L_x^1)$ bounds on $\{(\rho_n + \delta)\theta_n\}_{n=1}^\infty$ which hold pointwise in ω . However, due to the presence of stochastic integrals in the total energy balance, we only have uniform $L^p(\Omega; L_t^\infty(L_x^1))$ bounds for $p < \infty$. This requires an additional argument in the Skorohod compactness step.

In Sect. 6, we send $\epsilon \rightarrow 0$ to obtain an existence proof for the δ -layer. In comparison with $\tau \rightarrow 0$ and $n \rightarrow \infty$, the main difficulty in this section is the complexity of the proof of strong convergence of the density. This is essentially already proved in [14], but the pressure law now depends on both density and temperature, so we sketch a proof for completeness.

The proof of main result Theorem 1.8 is presented in Sect. 7. As $\delta \rightarrow 0$, two regularizations in the temperature equation degenerate. In particular, we no longer

have a free $L^p(\Omega; L^3_{t,x})$ bound on the sequence $\{\theta_\delta\}_{\delta>0}$. Instead, this has to be proved directly from the weak form at the δ layer. Additional difficulties lie in the strong convergence of the densities, due to the fact that the limiting density may not lie in $L^2_{t,x}$.

1.6. Notation

Here, and in what follows, we use the notation:

Functional spaces: (i) The shorthand notation $L^q_t(L^p_x)$, $L^q_t(W^{k,p}_x)$, $W^{k,q}_t(L^p_x)$ is used to denote the spaces $L^q([0, T]; L^p(D))$, $L^q([0, T]; W^{k,p}(D))$, $W^{k,q}([0, T]; L^p(D))$, respectively, where each space is understood to be endowed with its strong topology. Also, we use M_x to denote the finite, signed radon measures on D . The abbreviation $L^\infty_t(M_x)$ denotes $L^\infty([0, T]; M_x)$. We will often use the same notation to denote scalar functions in $L^q_t(L^p_x)$ and vector-valued functions (with d components) in $[L^q_t(L^p_x)]^d$, but the meaning will always be clear from the context. To emphasize when one of the spaces above is endowed with its weak topology, we write $[L^q_t(L^p_x)]_w$, $[L^q_t(W^{k,p}_x)]_w$. Also, the abbreviation $C_t([L^p_x]_w)$ denotes the topological space of weakly continuous functions $f : [0, T] \rightarrow L^p(D)$. The space $W^{k,p}_{0,x}$ is the closure of the smooth compactly supported functions, $C^\infty_c(D)$, with respect to the $W^{k,p}(D)$ norm. Moreover, we denote $W^{1,2}_{0,x}$ as $H^1_{0,x}$.

Probability space: (ii) Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a Banach space E , let $L^p(\Omega; E)$ be the collection of equivalence classes of $\mathcal{F}$ measurable mappings $X : \Omega \rightarrow E$ such that the p^{th} moment of the E norm is finite. Again, we write $L^p_w(\Omega; E)$ when emphasizing that the space is endowed with its weak topology. To define the sigma algebra generated by various random variables, we use a restriction operator $r_t : C([0, T]; E) \rightarrow C([0, t]; E)$ which realizes a mapping $f : [0, T] \rightarrow E$ as a mapping $r_t f : [0, t] \rightarrow E$. The same notation is used for the restriction of an equivalence class $f \in L^p([0, T]; E)$ to $r_t f \in L^p([0, t]; E)$.

Operators and operations: (iii) We denote $\mathcal{A} = \nabla \Delta^{-1}$, understood to be well defined on compactly supported distributions in $\mathbb{R}^d$. Given two $d \times d$ matrices A, B , $A : B$ denotes a Frobenius matrix product. The notation $A \lesssim B$ denotes inequality up to an insignificant constant. The notion of insignificance will be clear from the context.

2. Formal energy estimates

In this section, we derive the formal a priori bounds for SPDE (1.1). In Sect. 2.1, we derive the evolution of the total energy, which is the sum of internal and kinetic contributions. The evolution of the internal energy is obtained through the renormalized form of the continuity equation. Namely, given $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}$, multiplying the continuity equation by $\beta'(\rho)$ yields:

$$\partial_t(\beta(\rho)) + \operatorname{div}(\beta(\rho)u) + [\rho\beta'(\rho) - \beta(\rho)] \operatorname{div} u = 0. \quad (2.1)$$

On the other hand, the evolution of the kinetic energy is obtained from the momentum equation and the Ito formula. Since the stochastic forcing is non-conservative, the total energy is not conserved. In fact, it undergoes random fluctuations produced by a stochastic integral. Moreover, since the noise is understood in the Ito sense, a nonnegative correction term appears. As the stochastic integral is an unbounded stochastic process, it is not possible to obtain $L^\infty(\Omega)$ bounds for the total energy. However, as in [14], coloring Hypothesis 1.5 leads to bounds in $L^p(\Omega)$ for every $p \geq 1$.

In subsections 2.2 and 2.3, we obtain further bounds on the temperature and the velocity. The basic tool is the following: for $H : \mathbb{R}_+ \rightarrow \mathbb{R}$, multiplying the temperature equation by $H'(\theta)$ and combining with the continuity equation yields:

$$\begin{aligned} \partial_t(\rho H(\theta)) + \operatorname{div}(\rho u H(\theta) - H'(\theta)\kappa(\theta)\nabla\theta) + H''(\theta)\kappa(\theta)|\nabla\theta|^2 \\ = H'(\theta)[S : \nabla u - \theta p_\theta(\rho) \operatorname{div} u]. \end{aligned} \quad (2.2)$$

Integrating over D , using the boundary conditions, and rearranging gives:

$$\begin{aligned} \int_D (H'(\theta)S : \nabla u - H''(\theta)\kappa(\theta)|\nabla\theta|^2) dx \\ = \frac{d}{dt} \int_D \rho H(\theta) dx + \int_D \theta H'(\theta) p_\theta(\rho) \operatorname{div} u dx. \end{aligned} \quad (2.3)$$

For concave H , the main difficulty in using this identity to control temperature and velocity gradients is the last term on the RHS of (2.3). Using a trick of Lions from [12], in Sect. 2.2 we introduce an entropy to remove this term and obtain a preliminary bound on θ . In Sect. 2.3, we revisit identity (2.3) to obtain an additional temperature estimate and a bound for the velocity. Finally, to transition from gradient estimates to H^1 bounds, we will need the following variant of the Poincaré inequality:

LEMMA 2.1. *For all $M > 0$ and $\beta \geq 1$, there exists a positive constant $C_{M,\beta}$ such that for all nonnegative $f, g : D \rightarrow \mathbb{R}$ with $|g|_{L^1(D)} \geq M$, the following inequality holds:*

$$|f^\beta|_{L^2(D)} \leq C_M (|\nabla(f^\beta)|_{L^2(D)} + |g|^{\frac{\gamma\beta}{\gamma-1}}_{L^\gamma} [|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2(D)}]). \quad (2.4)$$

The proof of Lemma 2.1 uses the method of Mellet/Vasseur [13], but tracking a bit more carefully the dependence of the estimate on $|g|_{L^\gamma}$. This is important for our purposes since this quantity will be a random process.

2.1. Estimates for the total energy

Start by defining

$$P_m(\rho) = \int_1^\rho \frac{p_m(z)}{z^2} dz. \quad (2.5)$$

Recall that p_m was defined in Hypothesis 1.1. Set $\beta(\rho) = \rho P_m(\rho)$ in renormalized form (2.1) and use the identity $\rho[(\rho P_m(\rho))' - P_m(\rho)] = p_m(\rho)$. Combining with the temperature equation, we find the evolution of the internal energy:

$$\begin{aligned} & \partial_t(\rho(P_m(\rho) + \theta)) + \operatorname{div}(\rho u(P_m(\rho) + \theta) - \kappa(\theta)\nabla\theta) \\ &= \mathcal{S} : \nabla u - P(\rho, \theta) \operatorname{div} u. \end{aligned} \quad (2.6)$$

Using Ito's formula together with the momentum equation in (1.1) yields the identity:

$$\partial_t \left(\frac{1}{2} \rho |u|^2 \right) + \operatorname{div} \left(\frac{1}{2} \rho u |u|^2 \right) = (\operatorname{div} \mathcal{S} - \nabla P) \cdot u + \rho |\sigma_k|^2 + \rho \sigma_k \cdot u \dot{\beta}_k. \quad (2.7)$$

Combining (2.6) and (2.7), integrating over $D \times [0, t]$, and using the boundary conditions give:

$$\begin{aligned} & \int_D \left(\rho \theta(t) + \frac{1}{2} \rho |u|^2(t) + \rho P_m(\rho)(t) \right) dx \\ &= \int_D \left(\rho_0 \theta_0 + \frac{1}{2} \frac{|m_0|^2}{\rho_0} + \rho_0 P_m(\rho_0) \right) dx \\ &+ \sum_{k=1}^{\infty} \int_0^t \int_D \rho u \cdot \sigma_k(\rho, \rho u, \rho \theta, x) dx d\beta_k(s) \\ &+ \frac{1}{2} \sum_{k=1}^{\infty} \int_0^t \int_D \rho |\sigma_k(\rho, \rho u, \rho \theta, x)|^2 dx ds. \end{aligned} \quad (2.8)$$

Applying the Burkholder/Davis/Gundy inequality followed by Hölder and Hypothesis 1.5 yields for all $p > 1$

$$\begin{aligned} & \mathbb{E} \left[\sup_{t \in [0, T]} \left| \sum_{k=1}^{\infty} \int_0^t \int_D \rho u \cdot \sigma_k dx d\beta_k(s) \right|^p \right] + \mathbb{E} \left[\left| \sum_{k=1}^{\infty} \int_0^T \int_D \rho |\sigma_k|^2 dx ds \right|^p \right] \\ & \lesssim \mathbb{E} \left[\left| \sum_{k=1}^{\infty} \int_0^T \left(\int_D \rho u \cdot \sigma_k dx \right)^2 ds \right|^{p/2} \right] + \mathbb{E} \left[\left| \sum_{k=1}^{\infty} \int_0^T \int_D \rho |\sigma_k|^2 dx ds \right|^p \right] \\ & \lesssim \left(\sum_{k=1}^{\infty} |\sigma_k|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L_{\rho, m, \alpha}^{\infty})} \right)^{\frac{p}{2}} \mathbb{E} \left[\left| \int_0^T |\rho u(s)|^2_{L_x^{\frac{2\gamma}{\gamma+1}}} ds \right|^{p/2} \right] \\ & + \left(\sum_{k=1}^{\infty} |\sigma_k|^2_{L_x^{\frac{2\gamma}{\gamma-1}}(L_{\rho, m, \alpha}^{\infty})} \right)^p \mathbb{E} \left[\left| \int_0^T |\rho(s)|_{L_x^{\gamma}} ds \right|^p \right] \\ & \lesssim \mathbb{E} \left[|\rho|_{L_t^{\infty}(L_x^{\gamma})}^{\frac{p}{2}} |\sqrt{\rho} u|_{L_t^{\infty}(L_x^2)}^p \right] + \mathbb{E} \left[|\rho|_{L_t^{\infty}(L_x^{\gamma})}^p \right]. \end{aligned}$$

Maximizing over $[0, T]$ and taking $L^p(\Omega)$ norms of both sides of the total energy identity yields:

$$\mathbb{E} \left[|\rho \theta|_{L_t^{\infty}(L_x^1)}^p + |\sqrt{\rho} u|_{L_t^{\infty}(L_x^2)}^{2p} + |\rho P_m(\rho)|_{L_t^{\infty}(L_x^1)}^p \right]$$

$$\begin{aligned} &\lesssim \mathbb{E} \left[|\rho_0 \theta_0|_{L_x^1}^p + \left| \frac{m_0}{\sqrt{\rho_0}} \right|_{L_x^2}^{2p} + |\rho_0 P_m(\rho_0)|_{L_x^1}^p \right] \\ &\quad + \mathbb{E} \left[|\rho|_{L_t^\infty(L_x^\gamma)}^{\frac{p}{2}} |\sqrt{\rho} u|_{L_t^\infty(L_x^2)}^p \right] + \mathbb{E} \left[|\rho|_{L_t^\infty(L_x^\gamma)}^p \right] + 1. \end{aligned}$$

In view of Hypothesis 1.1, $\rho P_m(\rho) \sim \rho^\gamma$. By Young's inequality we deduce the following a priori bound:

$$\mathbb{E} \left[|\rho \theta|_{L_t^\infty(L_x^1)}^p + |\sqrt{\rho} u|_{L_t^\infty(L_x^2)}^{2p} + |\rho|_{L_t^\infty(L_x^\gamma)}^{\gamma p} \right] \lesssim C. \quad (2.9)$$

The constant C depends only on p, d, D, T and the norms of the initial data discussed in Hypothesis 1.3.

2.2. Further bounds on the temperature

Begin by letting

$$P_\theta(\rho) = \int_1^\rho \frac{p_\theta(z)}{z^2} dz. \quad (2.10)$$

Define the entropy $s(\rho, \theta) = \log(\theta) - P_\theta(\rho)$. Use the temperature renormalization $H(\theta) = \log(\theta)$ in (2.3) and the density renormalization $\beta(\rho) = \rho P_\theta(\rho)$ in (2.1). Taking the difference of these two equations yields the $\mathbb{P}$ a.s. inequality:

$$\int_D \theta^{-2} \kappa(\theta) |\nabla \theta|^2 dx \leq \frac{d}{dt} \int_D \rho s dx. \quad (2.11)$$

The integral over $[0, T]$ of the RHS of inequality (2.11) can be controlled as follows:

$$\begin{aligned} &\int_D \rho s(T) dx - \int_D \rho_0 s_0 \leq \int_D \rho \log(\theta)(T) - \int_D \rho_0 s_0 \\ &\leq \int_{\{\theta(T) \geq 1\}} \rho \log(\theta)(T) dx + \int_D \rho_0 P_\theta(\rho_0) dx - \int_{\theta_0 < 1} \rho_0 \log(\theta_0) dx \\ &\lesssim |\rho \theta|_{L_t^\infty(L_x^1)} + \int_D \rho_0^\Gamma dx + |\log(\underline{\theta} \wedge 1)| |\rho_0|_{L_x^1}. \end{aligned}$$

In the last inequality, we used Hypothesis 1.3 to bound θ_0 from below and Hypothesis 1.1 to deduce $P_\theta(\rho_0) \sim \rho_0^{\Gamma-1}$. Finally, since $\Gamma \geq \frac{\gamma}{d}$, we can integrate (2.11) over $[0, T]$, take $L^p(\Omega)$ norms on both sides, and appeal to (2.9) to deduce:

$$\mathbb{E} |\theta^{-1} \kappa^{\frac{1}{2}}(\theta) \nabla \theta|_{L_{t,x}^2}^{2p} \leq C. \quad (2.12)$$

Hypothesis 1.2 on the heat conductivity coefficient now implies

$$\mathbb{E} [|\nabla \theta|_{L_{t,x}^2}^{2p} + |\nabla \log(\theta)|_{L_{t,x}^2}^{2p}] \leq C. \quad (2.13)$$

Finally, we seek an estimate on θ in L_x^2 . For each $(\omega, t) \in \Omega \times [0, T]$, apply Lemma 2.1 (with $\beta = 1$) to $f(x) = \theta(t, x, \omega)$ and $g(x) = \rho(t, x, \omega)$. Conservation of mass together with Hypothesis 1.3 implies that the stochastic process

$(\omega, t) \rightarrow |\rho(t, \omega)|_{L^1(D)}$ is deterministic and stationary through time. Hence, integrating over $[0, T]$ yields the following $\mathbb{P}$ a.s. inequality:

$$\begin{aligned} |\theta|_{L^2_{t,x}} &\leq C \left[|\nabla \theta|_{L^2_{t,x}} + |\rho|^{\frac{\gamma}{\gamma-1}}_{L^\gamma_t(L^1_x)} (|\rho \theta|_{L^2_t(L^1_x)} + |\nabla \theta|_{L^2_{t,x}}) \right] \\ &\leq C \left[1 + |\nabla \theta|^2_{L^2_{t,x}} + |\rho \theta|^2_{L^\infty_t(L^1_x)} + |\rho|^{\frac{2\gamma}{\gamma-1}}_{L^\infty_t(L^1_x)} \right]. \end{aligned}$$

Taking $L^p(\Omega)$ norms on both sides and using bounds (2.9) for the total energy together with (2.13), we deduce for all $p \geq 1$:

$$\mathbb{E} \left[|\theta|^{2p}_{L^2_t(H^1_x)} \right] \lesssim C. \quad (2.14)$$

2.3. Velocity estimates and an improved temperature bound

Use (2.3) with $H(\theta) = \theta$ and $H_\sigma(\theta) = \theta^{1-\sigma}$, then integrate over $[0, T]$ to find the $\mathbb{P}$ a.s. inequality

$$\begin{aligned} &\int_0^T \int_D (|\nabla u|^2 + \theta^{-(\sigma+1)} \kappa(\theta) |\nabla \theta|^2) dx \\ &\lesssim |\rho \theta^{1-\sigma}|_{L^\infty_t(L^1_x)} + |\rho \theta|_{L^\infty_t(L^1_x)} + |\theta[\theta^{-\sigma} + 1] p_\theta(\rho) \operatorname{div} u|_{L^1_{t,x}} \\ &\lesssim |\rho|_{L^\infty_t(L^\gamma_x)} + |\rho \theta|_{L^\infty_t(L^1_x)} \\ &\quad + \left(|\theta|_{L^2_t(L^{\frac{2d}{d-2}}_x)} + |\theta|^{1-\sigma}_{L^2_t(L^{\frac{2d}{d-2}}_x)} \right) |p_\theta(\rho)|_{L^\infty_t(L^d_x)} |\operatorname{div} u|_{L^2_{t,x}}. \end{aligned}$$

Use Hypotheses (1.2) and (1.1) together with Young's inequality to obtain the $\mathbb{P}$ a.s. inequality:

$$|\nabla u|^2_{L^2_{t,x}} + |\nabla(\theta^{\frac{3-\sigma}{2}})|^2_{L^2_{t,x}} \leq C \left(1 + |\rho|^\gamma_{L^\infty_t(L^\gamma_x)} + |\rho \theta|_{L^\infty_t(L^1_x)} + |\theta|^{\frac{1}{2}-\frac{1}{d}}_{L^2_t(H^1_x)} \right).$$

Taking $L^p(\Omega)$ norms on both sides yields:

$$\mathbb{E} \left[|\nabla u|^{2p}_{L^2_{t,x}} + |\nabla(\theta^{\frac{3-\sigma}{2}})|^{2p}_{L^2_{t,x}} \right] \leq C.$$

Applying Lemma 2.1 with $\beta = (3 - \sigma)/2$ and arguing as in the last section, we find that:

$$\mathbb{E} \left[|\nabla u|^{2p}_{L^2_{t,x}} + |\theta^{\frac{3-\sigma}{2}}|^{2p}_{L^2_t(H^1_x)} \right] \leq C.$$

3. τ layer existence

In this section, we build the first layer of our approximating scheme, the τ layer. Each of the parameters n, ϵ , and δ is present in the notion of solution, Definition

3.3 below, but they are frozen in this section, so we only indicate dependence of the approximating sequence on τ , the time splitting parameter. We partition the time interval $[0, T]$ into $\frac{T}{\tau}$ time intervals of length τ , where $\frac{T}{\tau}$ is assumed to be an even integer. Denoting $t_j = j\tau$, we define the functions $h_{\det}^\tau$ and h_{st}^τ via

$$h_{\det}^\tau(s) = \sum_{j=0}^{\frac{T}{2\tau}-1} 1_{(t_{2j}, t_{2j+1}]}(s) = 1 - h_{\text{st}}^\tau(s). \quad (3.1)$$

The main result of this section is the following:

THEOREM 3.1. *For each $\tau > 0$, there exists a τ layer approximation $(\rho_\tau, u_\tau, \theta_\tau)$ to 1.1 (in the sense of Definition 3.3 below).*

Now we give the precise definition of a τ layer approximation. There are three elements of our approximating scheme we must introduce: a finite dimensional space where the velocity evolves, a regularization of the multiplicative structure of the noise, and an artificial pressure.

To truncate the high modes of the velocity field, we introduce of collection $\{\Pi_n\}_{n=1}^\infty$ of linear operators satisfying the following:

HYPOTHESIS 3.2. *For each $n \geq 1$, Π_n is a bounded linear operator from $L^1(D; \mathbb{R}^d)$ to $C^3(D; \mathbb{R}^d) \cap C_c(D; \mathbb{R}^d)$ with a finite dimensional range. For all $1 < p < \infty$, $s = 0, 1, 2, 3$, and $u \in W^{s,p}(D; \mathbb{R}^d)$,*

$$\lim_{n \rightarrow \infty} |\Pi_n u - u|_{W^{s,p}(D; \mathbb{R}^d)} = 0. \quad (3.2)$$

The collection $\{\Pi_n\}_{n=1}^\infty$ can be constructed using a wavelet expansion. For more details on wavelet expansions in domains, see [15]. These operators are accompanied by a collection of finite dimensional spaces $\{X_n\}_{n=1}^\infty$ defined by $X_n = \Pi_n(L^2(D; \mathbb{R}^d))$.

In view of these remarks, let $C^+(D)$ be the cone of positive functions in $C(D)$ and η_δ a standard mollifier space/time mollifier. Define the operator $\sigma_{k,\tau,n,\delta} : C^+(D) \times L^1(D; \mathbb{R}^d) \times L^1(D) \rightarrow X_n$ by

$$\sigma_{k,\tau,n,\delta}(\rho, m, \alpha) = \Pi_n \circ \sigma_k(\rho * \eta_\delta(\cdot), \left[\left(\rho \wedge \frac{1}{\tau} \right) \frac{m}{\rho} \right] * \eta_\delta(\cdot), \alpha * \eta_\delta(\cdot), \cdot),$$

where (ρ, m, θ) are understood to be extended by zero outside of $[0, T] \times D$ to give a meaning to the convolution. For the remainder of this section, we will use the abbreviation $\sigma_{k,\tau}$.

Finally, the original pressure in the momentum equation will be replaced by an “artificial” one of the form $p_m(\rho) + \delta\rho^\beta + \theta p_\theta(\rho)$ for a sufficiently large power β . For technical reasons which will be clear later in the article, we require that

$$\beta > \max(d, 2\gamma, 4). \quad (3.3)$$

DEFINITION 3.3. A triple $(\rho_\tau, u_\tau, \theta_\tau)$ is defined to be a τ layer approximation to (1.1) provided there exists a stochastic basis $(\Omega_\tau, \mathcal{F}_\tau, (\mathcal{F}_\tau^t)_{t=0}^T, \mathbb{P}_\tau, \{\beta_\tau^k\}_{k=1}^n)$ such that:

1. For all $t \in [0, T]$, $\mathcal{F}_\tau^t = \sigma(\{\beta_\tau^k(s)\}_{k=1}^n : s \leq t)$.
2. The quadruple $(\rho_\tau, \rho_\tau u_\tau, \rho_\tau \theta_\tau, u_\tau)$ belongs in $L^2(\Omega \times [0, T]; \mathcal{P}; L^\beta \times L^\beta \times L^q \times X_n)$, where $\mathcal{P}$ is the predictable σ -algebra generated by $(\mathcal{F}_\tau^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.
3. For all $\phi \in C^\infty(D)$ and $t \in [0, T]$, the following equality holds $\mathbb{P}_\tau$ a.s.

$$\int_D \rho_\tau(t) \phi dx = \int_D \rho_{0,\delta} \phi dx + \int_0^t \int_D 2h_{\det}^\tau(s) [\rho_\tau u_\tau \cdot \nabla \phi + \epsilon \rho_\tau \Delta \phi] dx ds. \quad (3.4)$$

4. For all $\phi \in X_n$ and $t \in [0, T]$, the following equality holds $\mathbb{P}_\tau$ a.s.

$$\begin{aligned} \int_D \rho_\tau u_\tau(t) \cdot \phi dx &= \int_D m_{0,\delta} \cdot \phi dx + \int_0^t \int_D 2h_{\det}^\tau [\rho_\tau u_\tau \otimes u_\tau - \mathcal{S}(u_\tau)] : \nabla \phi dx ds \\ &+ \int_0^t \int_D 2h_{\det}^\tau [(P(\rho_\tau, \theta_\tau) + \delta \rho_\tau^\beta) \operatorname{div} \phi - \epsilon \nabla u_\tau \nabla \rho_\tau \cdot \phi] dx ds \\ &+ \sum_{k=1}^n \int_0^t \int_D \sqrt{2} h_{st}^\tau \rho_\tau \sigma_{k,\tau}(\rho_\tau, \rho_\tau u_\tau, \rho_\tau \theta_\tau) \cdot \phi dx d\beta_\tau^k(s). \end{aligned} \quad (3.5)$$

5. For all $\varphi \in C^\infty(D)$ with $\frac{\partial \varphi}{\partial n} |_{\partial D} = 0$, the following equation holds $\mathbb{P}_\tau$ almost surely:

$$\begin{aligned} &\int_D (\rho_\tau(t) + \delta) \theta_\tau(t) \varphi dx - \int_0^t \int_D (\delta \theta_\tau^3 \varphi + K(\theta_\tau) \Delta \varphi) dx ds \\ &= \int_D (\rho_{0,\delta} + \delta) \theta_{0,\delta} \varphi dx + \int_0^t \int_D 2h_{\det}^\tau \rho_\tau \theta_\tau u_\tau \cdot \nabla \varphi dx ds \\ &+ \int_0^t \int_D 2h_{\det}^\tau [(1 - \delta) \mathcal{S}(u_\tau) : \nabla u_\tau - \theta_\tau p_\theta(\rho_\tau) \operatorname{div} u_\tau] \varphi dx ds, \end{aligned} \quad (3.6)$$

where $\mathcal{K}(\theta) = \int_1^\theta \kappa(z) dz$.

In the definition above, we have replaced the initial data (ρ_0, m_0, θ_0) by a triple $(\rho_{0,\delta}, m_{0,\delta}, \theta_{0,\delta})$ satisfying:

HYPOTHESIS 3.4. For each $\delta > 0$, $\rho_{0,\delta}$ and $\theta_{0,\delta}$ belong to $C^\infty(D)$ and obey the bounds

$$\delta \leq \rho_{0,\delta} \leq \delta^{-\frac{1}{2\beta}} \quad \underline{\theta} \leq \theta_{0,\delta} \leq |\theta_0|_{L^\infty}. \quad (3.7)$$

Both are assumed to satisfy a Neumann boundary condition. The sequence $\{(\rho_{0,\delta}, \theta_{0,\delta})\}_{\delta>0}$ converges strongly to (ρ_0, θ_0) in $L_x^\gamma \times L_x^1$. Finally, we define $m_{0,\delta} = m_0 1_{\{\rho_{0,\delta} \geq \rho_0\}}$ and assume $|\{\rho_{0,\delta} < \rho_0\}| \rightarrow 0$.

3.1. Machinery from the deterministic theory

For each $\rho \in C^+(D)$, define a multiplication-type operator $\mathcal{M}[\rho] : X_n \rightarrow X_n^*$ as follows: for $u, \eta \in X_n$,

$$\langle \mathcal{M}[\rho]u, \eta \rangle = \int_D \rho(x)u(x) \cdot \eta(x)dx.$$

LEMMA 3.5. *For each $\rho \in C^+(D)$, $\mathcal{M}[\rho] : X_n \rightarrow X_n^*$ is an invertible(linear) mapping.*

Proof. This is a very short exercise left to the reader, see Section 7 in Chapter 7 of [7] for more discussion. $\square$

Let us also introduce the mapping $\mathcal{N} : C(D) \times X_n \times L^2(D) \rightarrow X_n^*$ by the relation

$$\langle \mathcal{N}[\rho, u, \theta], \eta \rangle = \int_D [\rho u \otimes u - \mathcal{S}(u) + (P(\rho, \theta) + \delta \rho^\beta)I] : \nabla \eta - \epsilon \nabla u \nabla \rho \cdot \eta \, dx.$$

PROPOSITION 3.6. *Let $s < t$ be initial and final times and suppose initial data $\rho_{in} \in C^+(D)$, $u_{in} \in C^1(D; \mathbb{R}^d)$, and $\theta_{in} \in C^+(D)$ are given. Then there exists a unique pair*

$$(\rho, u, \theta) \in C\left((s, t]; C^2(D) \cap C^+(D)\right) \times C\left((s, t]; X_n\right) \times C\left((s, t]; L^2(D)\right),$$

solving the system

$$\begin{cases} \partial_t \rho = 2\epsilon \Delta \rho - 2 \operatorname{div}(\rho u) & \text{in } D \times (s, t] \\ u(S) = \mathcal{M}^{-1}[\rho(S)] \circ \left(m_{in}^* + \int_s^S 2\mathcal{N}[u(r), \rho(r), \theta(r)]dr\right) & \text{in } D \times [s, t] \\ \partial_t((\rho + \delta)\theta) + \delta \theta^3 - \operatorname{div}(\kappa(\theta)\nabla \theta) + 2 \operatorname{div}(\rho \theta u) = 2(1 - \delta)\mathcal{S} : \nabla u - 2\theta p_\theta(\rho) \operatorname{div} u & \text{in } D \times (s, t] \\ \frac{\partial \rho}{\partial n} = \frac{\partial \theta}{\partial n} = 0 & \text{in } \partial D \times (s, t] \\ (\rho(s), \theta(s)) = (\rho_{in}, \theta_{in}) & \text{in } D \end{cases} \quad (3.8)$$

where $m_{in}^* \in X_n^*$ is defined for $\eta \in X_n$ via the relation

$$\langle m_{in}^*, \eta \rangle = \int_D \rho_{in} u_{in} \cdot \eta.$$

If $u_{in} \in X_n$, then $u(s) = u_{in}$. The temperature equation is understood to hold almost everywhere in $[s, t] \times D$. Moreover, the solution map

$$\begin{aligned} \mathcal{S} : C^+(D) \times X_n \times L^2(D) &\rightarrow C\left((s, t]; C^+(D)\right) \\ &\times C\left((s, t]; X_n\right) \times C\left((s, t]; L^2(D)\right) \end{aligned}$$

is continuous.

Proof. The proof of this result is given in Section 7 of Chapter 7 of [7]. $\square$

We also require the following classical result on nonlinear parabolic equations.

PROPOSITION 3.7. *Let $\rho \in C^\infty(D)$ be nonnegative and $\delta > 0$. For all times $s < t$ and $\theta_{in} \in L^2(D)$, there exists a unique classical solution to*

$$\begin{cases} \partial_t((\rho + \delta)\theta) + \delta\theta^3 - \operatorname{div}(\kappa(\theta)\nabla\theta) = 0 & \text{in } D \times (s, t] \\ \theta(s) = \theta_{in}. \end{cases} \quad (3.9)$$

Proof. Since ρ is time-independent and smooth, we may appeal to Theorem 8.1 in Chapter 5 of [11]. $\square$

3.2. A classical SPDE result

Let $s < t$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}^r)_{r=s}^t, \{\beta_k\}_{k=1}^n)$ be a stochastic basis such the filtration $(\mathcal{F}^r)_{r=s}^t$ is generated by the collection $\{\beta_k\}_{k=1}^n$. Let $\mathcal{P}$ be the predictable σ -algebra generated by $(\mathcal{F}^r)_{r=s}^t$. Suppose $\rho \in L^2(\Omega; \mathcal{F}_s; C_x^+)$ and $\theta \in L^2(\Omega \times [s, t]; \mathcal{P}; C^+(D))$ are given. A stochastic process $u \in L^2(\Omega \times [s, t]; \mathcal{P}; X_n)$ is said to be a solution to the SPDE

$$\begin{cases} \partial_t u = \sum_{k=1}^n \sigma_{k,\tau}(\rho, \rho u, \rho\theta) \dot{\beta}_k(t) & \text{in } (s, t] \times D \\ u(s) = u_{in} & \text{in } D \end{cases} \quad (3.10)$$

provided that for all $S \in [s, t]$ the following equality (in X_n) holds $\mathbb{P}$ a.s.

$$u(S) = u_{in} + \sum_{k=1}^n \int_s^S \sigma_{k,\tau}(\rho, \rho u(r), \rho\theta(r)) d\beta_k(r). \quad (3.11)$$

PROPOSITION 3.8. *There exists a unique solution $u \in L^2(\Omega; C([s, T]; X_n))$ to (3.10) in the sense of (3.11).*

Proof. The τ layer regularization of the noise coefficient ensures that the proposition can be established in the classical way using the Cauchy–Lipschitz theorem. For more details on this style of argument, see Chapter 7 of [5]. $\square$

3.3. Proof of Theorem 3.1

We are now prepared to establish an existence theorem for the lowest level of our scheme.

Proof. Let $\tau > 0$ be given. We will define the solution inductively. Namely, suppose that $(\rho_\tau, u_\tau, \theta_\tau)$ have been constructed to satisfy continuity equation (3.4), momentum equation (3.5), and temperature equation (3.6) on the time interval $[0, t_{2j}]$. To extend the solution to the interval $(t_{2j}, t_{2j+1}]$, apply Proposition 3.6 to find a unique triple (ρ, u, θ) satisfying:

$$\begin{cases}
\partial_t \rho + 2 \operatorname{div}(\rho u) - 2\epsilon \Delta \rho = 0 & \text{in } D \times (t_{2j}, t_{2j+1}] \\
u(t) = \mathcal{M}^{-1}[\rho(t)] \circ \left(\rho_\tau u_\tau(t_{2j})^* + \int_{t_{2j}}^t 2\mathcal{N}[u(s), \rho(s), \theta(s)] ds \right) & \text{in } D \times (t_{2j}, t_{2j+1}] \\
\partial_t((\rho + \delta)\theta) + 2 \operatorname{div}(\rho \theta u) + \delta \theta^3 - \Delta \mathcal{K}(\theta) = 2(1 - \delta)\mathcal{S} : \nabla u - 2\theta p_\theta(\rho) \operatorname{div} u & \text{in } D \times (t_{2j}, t_{2j+1}] \\
\frac{\partial \rho}{\partial n} = \frac{\partial \theta}{\partial n} = 0 & \text{in } \partial D \times (t_{2j}, t_{2j+1}] \\
(\rho(t_{2j}), u(t_{2j}), \theta(t_{2j})) = (\rho_\tau(t_{2j}), u_\tau(t_{2j}), \theta_\tau(t_{2j})) & \text{in } D
\end{cases} \quad (3.12)$$

To extend the solution to the interval $(t_{2j+1}, t_{2j+2}]$ we appeal first to Proposition 3.7 to solve for the temperature and then to Propositions 3.8 to solve for the velocity. Observe that the evolution of the temperature does not involve the velocity, allowing us to decouple the two equations. In this manner, we find a unique triple (ρ, u, θ) satisfying

$$\begin{cases}
\partial_t \rho = 0 & \text{in } D \times (t_{2j+1}, t_{2j+2}] \\
\partial_t u = \sqrt{2} \sum_{k=1}^n \sigma_{k,\tau}(\rho, \rho u, \rho \theta) \dot{\beta}_k & \text{in } D \times (t_{2j+1}, t_{2j+2}] \\
\partial_t((\rho + \delta)\theta) + \delta \theta^3 = \operatorname{div}(\kappa(\theta) \nabla \theta) & \text{in } D \times (t_{2j+1}, t_{2j+2}] \\
(\rho(t_{2j+1}), u(t_{2j+1}), \theta(t_{2j+1})) = (\rho_\tau(t_{2j+1}), u_\tau(t_{2j+1}), \theta_\tau(t_{2j+1})) & \text{in } D.
\end{cases} \quad (3.13)$$

Using the Ito formula and the inductive hypothesis, one may check that (3.4)–(3.6) continue to hold for $t \in [t_{2j}, t_{2j+2}]$. The desired measurability, part 2 of Definition 3.3, follows from the continuity of the solution map to the deterministic problem (guaranteed by Proposition 3.6), together with the fact that we obtain a stochastically strong solution during each time interval where the stochastic forcing evolves. $\square$

4. n layer existence

In this section, we apply Theorem 3.1 to build the next layer of the approximating scheme, the n layer. Our goal is to establish the following:

THEOREM 4.1. *For each $n \geq 1$, there exists an n layer approximation (in the sense of Definition 4.2 below) (ρ_n, u_n, θ_n) of (1.1) relative to a stochastic basis $(\Omega_n, \mathcal{F}_n, (\mathcal{F}_n^t)_{t=0}^T, \mathbb{P}_n, \{\beta_n^k\}_{k=1}^n)$.*

Let us introduce the n layer regularization of the multiplicative noise structure. Define an operator $\sigma_{k,n,\delta} : L^1(D) \times L^1(D; \mathbb{R}^d) \times L^1(D) \rightarrow X_n$ via the relation

$$\sigma_{k,n,\delta}(\rho, m, \alpha) = \Pi_n \circ \sigma_k(\rho * \eta_\delta(\cdot), m * \eta_\delta(\cdot), \alpha * \eta_\delta(\cdot), \cdot). \quad (4.1)$$

For the remainder of this section, we will use the abbreviation $\sigma_{k,n}$.

DEFINITION 4.2. A triple (ρ_n, u_n, θ_n) is defined to be an n layer approximation to (1.1) provided there exists a stochastic basis $(\Omega_n, \mathcal{F}_n, (\mathcal{F}_n^t)_{t=0}^T, \mathbb{P}_n, \{\beta_n^k\}_{k=1}^n)$ such that

1. The quadruple $(\rho_n, \rho_n u_n, \theta_n, u_n)$ belongs in $L^2(\Omega \times [0, T]; \mathcal{P}; L^\beta \times L^\beta \times L^q \times X_n)$, where $\mathcal{P}$ is the predictable σ -algebra generated by $(\mathcal{F}_n^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

2. For all $\phi \in C^\infty(D)$ and all times $t \in [0, T]$ the following equality holds $\mathbb{P}_n$ a.s.

$$\int_D \rho_n(t) \phi dx = \int_D \rho_{0,\delta} \phi + \int_0^t \int_D [\rho_n u_n \cdot \nabla \phi + \epsilon \rho_n \Delta \phi] dx ds. \quad (4.2)$$

3. For all $\phi \in X_n$ and all times $t \in [0, T]$ the following equality holds $\mathbb{P}_n$ a.s.

$$\begin{aligned} \int_D \rho_n u_n(t) \cdot \phi dx &= \int_D m_{0,\delta} \cdot \phi dx + \int_0^t \int_D [\rho_n u_n \otimes u_n - \mathcal{S}(u_n)] : \nabla \phi dx ds \\ &+ \int_0^t \int_D [(P(\rho_n, \theta_n) + \delta \rho_n^\beta) \operatorname{div} \phi - \epsilon \nabla u_n \nabla \rho_n \cdot \phi] dx ds \\ &+ \sum_{k=1}^n \int_0^t \int_D \rho_n \sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n) \cdot \phi dx d\beta_n^k(s). \end{aligned} \quad (4.3)$$

4. For all $\varphi \in C^\infty(D)$ with $\frac{\partial \varphi}{\partial n} \big|_{\partial D} = 0$, the following equation holds $\mathbb{P}_n$ almost surely:

$$\begin{aligned} \int_D (\rho_n(t) + \delta) \theta_n(t) \varphi dx &= \int_D (\rho_{0,\delta} + \delta) \theta_{0,\delta} \varphi dx \\ &+ \int_0^t \int_D [\rho_n \theta_n u_n - \kappa(\theta_n) \nabla \theta_n] \cdot \nabla \varphi dx ds \\ &+ \int_0^t \int_D [(1 - \delta) \mathcal{S}(u_n) : \nabla u_n - \theta_n p_\theta(\rho_n) \operatorname{div} u_n - \delta \theta_n^3] \varphi dx ds. \end{aligned} \quad (4.4)$$

For each n fixed we apply Theorem 3.1 to obtain a sequence of τ layer approximations $\{(\rho_{\tau,n}, u_{\tau,n}, \theta_{\tau,n})\}_{\tau>0}$. In Sect. 4.1, we prove a compactness result for this sequence and extract a candidate n layer approximation (ρ_n, u_n, θ_n) built on a convenient choice of probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. In Sect. 4.2, we use the compactness result to verify (ρ_n, u_n, θ_n) is an n layer approximation in the sense of Definition 4.2.

4.1. $\tau \rightarrow 0$ Compactness step

A key tool in this section is a τ layer analogue of renormalized temperature equation (2.1). For the convenience of the reader, we will now explain briefly how to derive this in the current context. For simplicity of notation, we drop dependence on τ for the moment. The following computations can be justified using the further regularity properties proved in Lemma 7.4 of [7]. Begin by observing that the continuity and temperature equations can be written in the compact form:

$$\begin{cases} \partial_t \rho = 2h_{\det}^\tau [\operatorname{div}(\rho u) - \epsilon \Delta \rho] \\ \partial_t ((\rho + \delta)\theta) + \delta \theta^3 + \operatorname{div}(2h_{\det}^\tau \rho u \theta - \kappa(\theta) \nabla \theta) \\ = 2h_{\det}^\tau [(1 - \delta) \mathcal{S}(u) : \nabla u - \theta p_\theta(\rho) \operatorname{div} u]. \end{cases} \quad (4.5)$$

Letting $H : \mathbb{R}_+ \rightarrow \mathbb{R}$ and multiplying by $H'(\theta)$, we use the parabolic equation to deduce:

$$\begin{aligned}
& [\partial_t((\rho + \delta)\theta) + \operatorname{div}(2h_{\det}^\tau \rho u \theta - \kappa(\theta) \nabla \theta)] H'(\theta) \\
&= \partial_t((\rho + \delta)H(\theta)) + \operatorname{div}(2h_{\det}^\tau \rho u H(\theta) - \kappa(\theta) H'(\theta) \nabla \theta) \\
&\quad + 2\epsilon h_{\det}^\tau \Delta \rho [\theta H'(\theta) - H(\theta)] + \kappa(\theta) H''(\theta) |\nabla \theta|^2.
\end{aligned}$$

Hence, we obtain the following renormalized form:

$$\begin{aligned}
& \partial_t((\rho + \delta)H(\theta)) + \kappa(\theta) H''(\theta) |\nabla \theta|^2 + \delta \theta^3 H'(\theta) \\
& \quad + \operatorname{div}(2h_{\det}^\tau \rho u H(\theta) - \kappa(\theta) H'(\theta) \nabla \theta) \\
&= 2h_{\det}^\tau [(1 - \delta)S(u) : \nabla u - \theta p_\theta(\rho) \operatorname{div} u] H'(\theta) \\
& \quad - 2\epsilon h_{\det}^\tau \Delta \rho [\theta H'(\theta) - H(\theta)].
\end{aligned} \tag{4.6}$$

The next lemma obtains estimates of the type derived in Sect. 2.

LEMMA 4.3. *For all $p \geq 1$, there exists $C(p, n, \epsilon, \delta)$ such that:*

$$\begin{aligned}
& \sup_{\tau > 0} \mathbb{E}^\mathbb{P} \left[|\sqrt{\rho_\tau} u_\tau|_{L_t^\infty(L_x^2)}^{2p} + |\rho_\tau|_{L_t^\infty(L_x^\beta)}^{\beta p} + |(\rho_\tau + \delta)\theta_\tau|_{L_t^\infty(L_x^1)}^p \right] \leq C. \\
& \sup_{\tau > 0} \mathbb{E}^\mathbb{P} \left[|h_{\det}^\tau u_\tau|_{L_t^2(H_{0,x}^1)}^{2p} + |\nabla(h_{\det}^\tau \rho_\tau^{\frac{\beta}{2}})|_{L_{t,x}^2}^{2p} + |\theta_\tau|_{L_{t,x}^3}^{3p} \right] \leq C. \\
& \sup_{\tau > 0} \mathbb{E}^\mathbb{P} \left[|\theta_\tau|_{L_t^2(H_x^1)}^{2p} + |\nabla \log(\theta_\tau)|_{L_{t,x}^2}^{2p} \right] \leq C.
\end{aligned} \tag{4.7}$$

Proof. The first two lines of (4.7) may be obtained all at once. Indeed, we can apply the Ito formula to find the evolution of the total energy. For all $t \in [0, T]$:

$$\begin{aligned}
& \int_D \frac{1}{2} \rho_\tau |u_\tau|^2(t) + \rho_\tau P_m(\rho_\tau(t)) + \frac{\delta}{\beta - 1} \rho_\tau^\beta(t) + (\rho_\tau + \delta) \theta_\tau(t) dx \\
& \quad + \int_0^t \int_D \delta [S(u_\tau) : \nabla u_\tau + \theta_\tau^3] dx ds \\
& \quad + \int_0^t \int_D 2\epsilon h_{\det}^\tau \left(\frac{p'_m(\rho_\tau)}{\rho_\tau} + \delta \beta \rho_\tau^{\beta-2} \right) |\nabla \rho_\tau|^2 dx ds \\
&= \sum_{k=1}^n \int_0^t \int_D \sqrt{2} h_{\det}^\tau \rho_\tau u_\tau \cdot \sigma_{k,\tau}(\rho_\tau, \rho_\tau u_\tau, \rho_\tau \theta_\tau) dx d\beta_\tau^k(s) \\
& \quad + \sum_{k=1}^n \int_0^t \int_D h_{\det}^\tau \rho_\tau |\sigma_{k,\tau}(\rho_\tau, \rho_\tau u_\tau, \rho_\tau \theta_\tau)|^2 dx dt + E_n(0).
\end{aligned} \tag{4.8}$$

Moreover, the sequence $\{E_n(0)\}_{n=1}^\infty$ satisfies the uniform bounds

$$\sup_n E_n(0) \leq E_\delta(0) = \frac{1}{2} \int_D \left[\frac{|m_{0,\delta}|^2}{\rho_{0,\delta}} + \rho_{0,\delta} P_m(\rho_{0,\delta}) + \rho_{0,\delta} \theta_{0,\delta} \right] dx.$$

Using the same approach as in Sect. 2, the $L^p(\Omega; L_t^\infty)$ norms of the RHS of (4.8) can be estimated in terms of the $L^p(\Omega; L_t^\infty)$ norms of the LHS of (4.8). Indeed, the only

additional fact needed is the L_x^p boundedness of the operators Π_n . This follows from Hypothesis 3.2 and the Banach/Steinhaus theorem.

To obtain the remaining estimates, use (4.6) with $H(\theta) = \log(\theta)$ and integrate over $[0, T] \times D$ to find the following $\mathbb{P}$ a.s. inequality:

$$\begin{aligned} & \int_0^T \int_D \theta^{-2} \kappa(\theta_\tau) |\nabla \theta_\tau|^2 dx ds \\ & \leq 2 \int_0^T \int_D p_\theta(\rho_\tau) \operatorname{div}(h_{\det}^\tau u_\tau) + \epsilon \Delta(h_{\det}^\tau \rho)(\log(\theta_\tau) - 1) dx ds \\ & \quad + \delta \int_0^T \int_D \theta_\tau^2 dx ds + \int_D \rho \log(\theta_\tau)(T) dx - \int_D \rho_{0,\delta} \log(\theta_{0,\delta}) dx. \end{aligned}$$

Integrating by parts, we observe that:

$$\begin{aligned} & \int_0^T \int_D \epsilon \Delta(h_{\det}^\tau \rho)(\log(\theta_\tau) - 1) dx ds = - \int_0^T \int_D \epsilon \nabla(h_{\det}^\tau \rho_\tau) \cdot \nabla(\log(\theta_\tau)) dx ds \\ & \leq \gamma' |\nabla \log(\theta_\tau)|_{L_{t,x}^2}^2 + C_{\gamma'} |\epsilon \nabla(h_{\det}^\tau \rho_\tau)|_{L_{t,x}^2}^2. \end{aligned}$$

Hence, for any $\gamma' > 0$, there exists a $C_{\gamma'}$ such that

$$\begin{aligned} & \int_0^T \int_D \theta^{-2} \kappa(\theta_\tau) |\nabla \theta_\tau|^2 dx ds \leq \gamma' |\nabla \log(\theta)|_{L_{t,x}^2}^2 + |p_\theta(\rho)|_{L_t^2(L_x^{\frac{2d}{d+2}})} |h_{\det}^\tau u|_{L_t^2(H_x^1)} \\ & \quad + C_\gamma (|\epsilon \nabla(h_{\det}^\tau \rho)|_{L_{t,x}^2}^2 + |\rho \theta|_{L_t^\infty(L_x^1)} + \delta |\theta|_{L_{t,x}^3}^3 + 1). \end{aligned}$$

Observe that $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$ and $L_x^\gamma \hookrightarrow L_x^{\frac{2\gamma}{d+2}}$. Thus, we obtain:

$$\sup_{\tau > 0} \mathbb{E}(|\nabla \theta_\tau|_{L_{t,x}^2}^{2p} + |\nabla \log(\theta_\tau)|_{L_{t,x}^2}^{2p}) \leq C(p, n, \epsilon, \delta). \quad (4.9)$$

Finally, using modified Poincaré inequality (2.1) as in Sect. 2 yields the last line in (4.7). $\square$

The next proposition is the main compactness step, yielding a candidate n layer approximation and a new sequence of τ layer approximations with improved compactness properties.

PROPOSITION 4.4. *There exists a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, a collection of independent Brownian motions $\{\beta_n^k\}_{k=1}^n$, a limit point (ρ_n, u_n, θ_n) , and a sequence of measurable maps $\{\tilde{T}_\tau\}_{\tau > 0}$ such that:*

1. For all $\tau > 0$, $\tilde{T}_\tau : (\Omega_n, \mathcal{F}_n, \mathbb{P}_n) \rightarrow (\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{P} = (\tilde{T}_\tau)_\# \mathbb{P}_n$.
2. The new sequence $\{(\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau)\}_{\tau > 0}$ defined by $(\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau) = (\rho_\tau, u_\tau, \theta_\tau) \circ \tilde{T}_\tau$ constitutes a τ layer approximation to (1.1) relative to the stochastic basis $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n, (\tilde{\mathcal{F}}_\tau^t)_{t=0}^T, \tilde{W}_\tau)$, where $\tilde{W}_\tau := W \circ \tilde{T}_\tau$ and $(\tilde{\mathcal{F}}_\tau^t)_{t=0}^T$ is the filtration generated by W_τ . Moreover, the initial data are recovered in the sense that $\tilde{\rho}_n(0) = \rho_{0,\delta}$, $\tilde{u}_n(0) = \mathcal{M}^{-1}[\rho_{0,\delta}]m_{0,\delta}$, and $\tilde{\theta}_n(0) = \theta_{0,\delta}$.

3. The uniform bounds in Lemma 4.3 hold with $\tilde{\rho}_\tau, \tilde{\theta}_\tau, \tilde{u}_\tau$ replacing $\rho_\tau, \theta_\tau, u_\tau$ and $\mathbb{P}_n$ replacing $\mathbb{P}$.
4. As $\tau \rightarrow 0$, the following convergences hold pointwise Ω_n :

$$\tilde{\rho}_\tau \rightarrow \rho_n \quad \text{in} \quad C_t(L_x^\beta) \cap L_t^\beta(W_x^{1,\beta}) \quad (4.10)$$

$$\tilde{u}_\tau \rightarrow u_n \quad \text{in} \quad C_t(X_n) \quad (4.11)$$

$$(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \rightarrow (\rho_n + \delta)\theta_n \quad \text{in} \quad C_t([L_x^2]_w) \quad (4.12)$$

$$\tilde{\theta}_\tau \rightarrow \theta_n \quad \text{in} \quad [L_t^2(H_x^1) \cap L_{t,x}^4]_w \quad (4.13)$$

$$\tilde{W}_\tau \rightarrow W_n \quad \text{in} \quad [C_t]^n, \quad (4.14)$$

where $W = \{\beta^k\}_{k=1}^n$ and $W_n = \{\beta_n^k\}_{k=1}^n$.

The proof of Proposition 4.4 begins with a tightness lemma. For each $\tau > 0$, let

$$Y_\tau = (\rho_\tau, u_\tau, (\rho_\tau + \delta)\theta_\tau, \theta_\tau, W),$$

where $W = \{\beta_k\}_{k=1}^n$. Observe that Y_τ induces a measure on the topological space

$$E = C_t(L_x^\beta) \cap L_t^\beta(W_x^{1,\beta}) \times C_t(X_n) \times C_t([L_x^2]_w) \times [L_t^2(H_x^1) \cap L_{t,x}^4]_w \times [C_t]^n.$$

LEMMA 4.5. *The sequence of laws $\{\mathbb{P} \circ Y_\tau^{-1}\}_{\tau>0}$ are tight on E .*

Proof. The tightness of $\{\mathbb{P} \circ (\rho_\tau, u_\tau, W)^{-1}\}_{\tau>0}$ on $C_t(L_x^\beta) \cap L_t^\beta(W_x^{1,\beta}) \times C_t(X_n) \times [C_t]^n$ can be established using the bootstrapping arguments in [14]. In the course of these arguments, the following useful fact is proved:

$$\lim_{M \rightarrow \infty} \sup_{\tau>0} \mathbb{P}(|u_\tau|_{C_t(X_n)} + |\rho_\tau|_{L_{t,x}^\infty} \geq M) = 0. \quad (4.15)$$

To prove the tightness of $\{\mathbb{P} \circ ((\rho_\tau + \delta)\theta_\tau, \theta_\tau)^{-1}\}_{\tau>0}$ on $C_t([L_x^2]_w) \times [L_t^2(H_x^1) \cap L_{t,x}^4]_w$, we require some further estimates. Use renormalized form (4.6) with $H(\theta) = \frac{1}{2}\theta^2$ and integrate over D to find:

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int_D (\rho_\tau + \delta) \theta_\tau^2 dx + \int_D (\kappa(\theta_\tau) |\nabla \theta_\tau|^2 + \delta \theta_\tau^4) dx \\ &= 2h_{\det}^\tau \int_D \left[(1 - \delta) S(u_\tau) : \nabla u_\tau \theta_\tau - \theta_\tau^2 p_\theta(\rho_\tau) \operatorname{div} u_\tau - \frac{1}{2} \epsilon \Delta \rho_\tau \theta_\tau^2 \right] dx. \end{aligned}$$

Integrating by parts, we find that for any small $\gamma' > 0$,

$$\begin{aligned} \int_D \Delta \rho_\tau \theta_\tau^2 dx &= - \int_D \nabla \rho_\tau \cdot \nabla \theta_\tau \theta_\tau dx \\ &\leq \gamma' \int_D \theta_\tau^2 |\nabla \theta_\tau|^2 dx + \int_D |\nabla \rho|^2. \end{aligned}$$

Using that $\kappa(\theta) \sim \theta^2$, the first term above can be absorbed into the LHS of the estimate after integrating over $[0, T]$. Hence, we obtain the following $\mathbb{P}$ a.s. inequality:

$$\begin{aligned} |\sqrt{\rho_\tau + \delta} \theta_\tau|_{L_t^\infty(L_x^2)}^2 + \delta |\theta_\tau|_{L_{t,x}^4}^4 &\lesssim 1 + |u_\tau|_{C_t(X_n)}^2 |\theta_\tau|_{L_{t,x}^1} \\ &\quad + |u_\tau|_{C_t(X_n)} |\rho_\tau|_{L_{t,x}^\infty}^{\frac{\gamma}{d}} |\theta_\tau|_{L_{t,x}^2}^2 + |\nabla \rho_\tau|_{L_{t,x}^2}^2. \end{aligned}$$

Using (4.15) and the $L^p(\Omega; L_t^2(H_x^1))$ bounds on θ_τ, ρ_τ , we find that:

$$\lim_{M \rightarrow \infty} \sup_{\tau > 0} \mathbb{P}(|\theta_\tau|_{L_{t,x}^4}^4 + |\sqrt{\rho_\tau + \delta} \theta_\tau|_{L_t^\infty(L_x^2)}^2 \geq M) = 0. \quad (4.16)$$

Using once more the $L^p(\Omega; L_t^2(H_x^1))$ bounds on θ_τ , we can apply (4.16) and Banach–Alaoglu to deduce the tightness of $\{\mathbb{P} \circ \theta_\tau^{-1}\}_{\tau > 0}$ on $[L_t^2(H_x^1) \cap L_{t,x}^4]_w$.

Our final task is to prove the tightness of $\{\mathbb{P} \circ ((\rho_\tau + \delta)u_\tau)^{-1}\}_{\tau > 0}$ on $C_t([L_x^2]_w)$. Toward this end, recall that for any $p > 1$ and $M > 0$, the following set is compact in $C_t([L_x^2]_w)$:

$$\{f \in L_t^\infty(L_x^2) \mid |f|_{L_t^\infty(L_x^2)} + |\partial_t f|_{L_t^p(W_x^{-2,p})} \leq M\}. \quad (4.17)$$

We will show that for $p = \frac{4}{3}$, the induced measures above become uniformly concentrated on such sets, up to small probability. Start by noting the inequality:

$$|(\rho_\tau + \delta)\theta_\tau|_{L_t^\infty(L_x^2)} \leq |\sqrt{\rho_\tau + \delta}|_{L_{t,x}^\infty} |\sqrt{\rho_\tau + \delta} \theta_\tau|_{L_t^\infty(L_x^2)}.$$

Hence, (4.15) and (4.16) imply:

$$\lim_{M \rightarrow \infty} \sup_{\tau > 0} \mathbb{P}(|(\rho_\tau + \delta)\theta_\tau|_{L_t^\infty(L_x^2)} \geq M) = 0. \quad (4.18)$$

Next, write the temperature equation as follows:

$$\begin{aligned} \partial_t((\rho_\tau + \delta)\theta_\tau) &= -2h_{\det}^\tau \operatorname{div}(\rho_\tau u_\tau \theta) - \delta \theta_\tau^3 + \Delta \mathcal{K}(\theta_\tau) \\ &\quad + 2h_{\det}^\tau [(1 - \delta)\mathcal{S}(u) : \nabla u_\tau - \theta p_\theta(\rho_\tau) \operatorname{div} u_\tau]. \end{aligned}$$

Estimating each term on the RHS gives:

$$\begin{aligned} |h_{\det}^\tau \operatorname{div}(\rho_\tau \theta_\tau u_\tau)|_{L_t^2(W_x^{-1, \frac{2d}{d-2}})} &\leq C |\rho_\tau \theta_\tau u_\tau|_{L_t^2(L_x^{\frac{2d}{d-2}})} \leq C |\rho_\tau u_\tau|_{L_{t,x}^\infty} |\theta_\tau|_{L_t^2(L_x^{\frac{2d}{d-2}})} \\ |-\delta \theta_\tau^3 + 2(1 - \delta)h_{\det}^\tau \mathcal{S}(u_\tau) : \nabla u_\tau|_{L_{t,x}^{\frac{4}{3}}} &\leq C (|\theta_\tau|_{L_{t,x}^4}^3 + |u_\tau|_{C_t(X_n)}^2) \\ |h_{\det}^\tau \theta_\tau p_\theta(\rho_\tau) \operatorname{div} u_\tau|_{L_t^2(L_x^{\frac{2d}{d-2}})} &\leq C |\theta_\tau|_{L_t^2(L_x^{\frac{2d}{d-2}})} |\rho_\tau|_{L_{t,x}^\infty}^{\gamma/d} |u_\tau|_{C_t(X_n)} \\ |\Delta \mathcal{K}(\theta_\tau)|_{L_t^{\frac{4}{3}}(W_x^{-2, \frac{4}{3}})} &\leq C |\theta_\tau|_{L_{t,x}^4}^3. \end{aligned}$$

Applying (4.15) and Lemma 4.3, we find that

$$\lim_{M \rightarrow \infty} \sup_{\tau > 0} \mathbb{P}(|\partial_t((\rho_\tau + \delta)\theta_\tau)|_{L_t^{\frac{4}{3}}(W_x^{-2, \frac{4}{3}})} \geq M) = 0. \quad (4.19)$$

□

Next we apply the tightness result above together with a version of Skorohod Theorem 7.18 to complete our compactness step.

Proof of Proposition 4.4. Note that (E, τ) is a Jakubowski space. Hence, in view of Lemma 4.5, we may apply Jakubowski/Skorohod Theorem 7.18 to the sequence $\{Y_\tau\}_{\tau>0}$ in order to obtain a probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$, a sequence of measurable maps $\{\tilde{T}_\tau\}_{\tau>0}$, and a limit point Y such that $Y_\tau \circ \tilde{T}_\tau$ converges pointwise to Y . In components, we write $Y = (\rho_n, u_n, T_n, \theta_n, W_n)$. Noting that $(\tilde{\rho}_\tau + \delta)\tilde{u}_\tau \rightarrow (\rho_n + \delta)u_n$ in $D'_{t,x}$ pointwise in Ω_n , we identify $T_n = (\rho_n + \delta)\theta_n$. This yields parts 1 and 4 of Proposition 4.4. The uniform energy bounds in Part 3 now follow from the fact that $\tilde{T}_\tau$ pushes forward $\mathbb{P}_n$ to $\mathbb{P}$ together with the estimates obtained already in Lemma 4.3. Finally, using the explicit relationship between $\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau$ and $\rho_\tau, u_\tau, \theta_\tau$, we are able to check that the equation is preserved, that is, Part 2 holds. Similarly, Part 3 can be deduced from the bounds in Lemma 4.3. For more details on this last point, see [1]. $\square$

4.2. $\tau \rightarrow 0$ identification step

LEMMA 4.6. *For all $\omega \in \Omega_n$, $\tilde{\theta}_\tau(\omega) \rightarrow \theta_n(\omega)$ in $L^3_{t,x}$.*

Proof. Let us fix $\omega \in \Omega_n$ and mostly omit dependence of $\tilde{\theta}_\tau, \theta_n$ on this parameter within the context of this proof. By Part 4 of Proposition 4.4, $\tilde{\theta}_\tau \rightarrow \theta_n$ weakly in $L^4_{t,x}$. Hence, it suffices to check that $\tilde{\theta}_\tau^2 \rightarrow \theta_n^2$ weakly in $L^1_{t,x}$.

Toward this end, we will use Part 4 of Proposition 4.4 several more times. First observe that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \rightarrow (\rho_n + \delta)\theta_n$ in $C_t([L^2_x]_w)$. Moreover, applying a standard compactness upgrade, we deduce that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \rightarrow (\rho_n + \delta)\theta_n$ strongly in $L^2_t(H_x^{-1})$. Since $\tilde{\theta}_\tau \rightarrow \theta_n$ weakly in $L^2_t(H_x^{-1})$, we obtain $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau^2 \rightarrow (\rho_n + \delta)\theta_n^2$ in $D'_{t,x}$. Moreover, there exists a $\hat{q} > 1$ and $C(\omega)$ such that

$$\sup_{\tau>0} |(\tilde{\rho}_\tau(\omega) + \delta)\tilde{\theta}_\tau(\omega)^2|_{L^{\hat{q}}_{t,x}} \leq C(\omega).$$

This implies that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau^2 \rightarrow (\rho_n + \delta)\theta_n^2$ in $L^q_{t,x}$ for a $q > 1$. Finally, since $(\tilde{\rho}_\tau + \delta)^{-1} \rightarrow (\rho_n + \delta)^{-1}$ in $L^{q'}_{t,x}$, we deduce that $\tilde{\theta}_\tau^2 \rightarrow \theta_n^2$ weakly in $L^1_{t,x}$. $\square$

Next we define a filtration $(\mathcal{F}_t^n)_{t=0}^T$ via $\mathcal{F}_t^n = \sigma(r_t X_n)$ where $X_n = (\rho_n, \rho_n u_n, \rho_n u_n, W_n, \rho_n \theta_n)$ and $r_t : E_T \rightarrow E_t$, where

$$E_s = C([0, s]; L^\beta) \cap L^\beta([0, s]; W^{1,\beta}) \times C([0, s]; L^\beta \times [L^2]_w \times X_n \times \mathbb{R}^n) \\ \times L^2([0, s]; L^q),$$

$$\text{and } \frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}.$$

LEMMA 4.7. *The triple (ρ_n, u_n, θ_n) satisfies parabolic equation (4.2) and projected momentum equation, (4.3) of Definition 4.2.*

Proof. By Part 2, $\tilde{\rho}_\tau, \tilde{u}_\tau$ satisfy (on Ω_n) the same parabolic equation as ρ_τ, u_τ . Using pointwise convergences (5.10) and (4.11), together with the observation that $h_{\det}^\tau \rightarrow \frac{1}{2}$ weakly in $L^p([0, T])$ for all $p \geq 1$, we may pass to the limit in the weak form and deduce that ρ_n, u_n satisfy parabolic equation (4.2).

Now let $\phi \in X_n$ and define the continuous, $(\mathcal{F}_t^n)_{t=0}^T$ adapted process $(M_\phi^n(t))_{t=0}^T$ via

$$\begin{aligned} M_\phi^n(t) &= \int_D \rho_n u_n(t) \cdot \phi dx - \int_0^t \int_D [\rho_n u_n \otimes u_n \\ &\quad - \mathcal{S}(u_n) - (P(\rho_n, \theta_n) + \delta \rho_n^\beta) I] : \nabla \phi dx ds \\ &\quad - \epsilon \int_0^t \int_D \nabla u_n \nabla \rho_n dx ds. \end{aligned}$$

Similarly, we define the process $(\tilde{M}_\tau^\phi(t))_{t=0}^T$ in terms of $\tilde{\rho}_\tau, \tilde{u}_\tau, \tilde{\theta}_\tau$ and an additional oscillating factor $2h_{\det}^\tau$. Let us check that

$$\sup_{\tau > 0} \mathbb{E}^{\mathbb{P}_n} \left[\sup_{t \in [0, T]} |\tilde{M}_\tau^\phi(t)|^4 \right] \leq C(n, \epsilon, \delta). \quad (4.20)$$

Indeed, estimating each term we find that

$$\begin{aligned} \sup_{\tau > 0} \mathbb{E}^{\mathbb{P}_n} \left[\sup_{t \in [0, T]} |\tilde{M}_\tau^\phi(t)|^4 \right] &\lesssim (\mathbb{E}^{\mathbb{P}_n} |\sqrt{\tilde{\rho}_\tau} \tilde{u}_\tau|_{L_t^\infty(L_x^2)}^4)^{\frac{1}{2}} (\mathbb{E}^{\mathbb{P}_n} |\tilde{\rho}_\tau|_{L_t^\infty(L_x^\beta)}^2)^{\frac{1}{2}} \\ &(\mathbb{E}^{\mathbb{P}_n} (|\sqrt{\tilde{\rho}_\tau} \tilde{u}_\tau|_{L_t^\infty(L_x^2)}^{12})^{\frac{1}{3}} (\mathbb{E}^{\mathbb{P}_n} |\tilde{\rho}_\tau|_{L_t^\infty(L_x^\beta)}^6)^{\frac{1}{3}} (\mathbb{E}^{\mathbb{P}_n} |h_{\det}^\tau \tilde{u}_\tau|_{L_t^2(H_x^1)}^{12})^{\frac{1}{3}} \\ &+ \mathbb{E}^{\mathbb{P}_n} |h_{\det}^\tau u_\tau|_{L_t^2(H_x^2)}^8 + \mathbb{E}^{\mathbb{P}_n} [|\tilde{\rho}_\tau|_{L_t^\infty(L_x^{4\beta})}^{4\beta} + |\tilde{\rho}_\tau|_{L_t^\infty(L_x^{4\gamma})}^{4\gamma}] \\ &+ (\mathbb{E}^{\mathbb{P}_n} |\tilde{\theta}_\tau|_{L_t^2(H_x^1)}^8)^{\frac{1}{2}} \mathbb{E}^{\mathbb{P}_n} [|\tilde{\rho}_\tau|_{L_t^\infty(L_x^\gamma)}^{\frac{8\gamma}{d}}]^{\frac{1}{2}} + \mathbb{E}^{\mathbb{P}_n} (|h_{\det}^\tau \tilde{u}_\tau|_{L_t^2(H_x^1)}^8)^{\frac{1}{2}} \mathbb{E}^{\mathbb{P}_n} (|h_{\det}^\tau \nabla \tilde{\rho}_\tau|_{L_{t,x}^2}^8)^{\frac{1}{2}}. \end{aligned} \quad (4.21)$$

Using the pointwise convergences in Part 4 of Proposition 4.4 we find that $\tilde{M}_\tau^\phi(t) \rightarrow M_n^\phi(t)$ for all $\omega \in \Omega_n$ and $t \in [0, T]$. A similar analysis of the quadratic variation and cross-variation terms allows us to appeal to Lemma 7.21 and identify

$$M_n^\phi(t) = \sum_{k=1}^n \int_0^t \int_D \rho_n \sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n) \cdot \phi dx d\beta_k(s). \quad (4.22)$$

□

LEMMA 4.8. *The triple (ρ_n, u_n, θ_n) satisfies modified temperature equation (4.2) of Definition 4.2.*

Proof. By part 2 of Proposition 4.4, for all $\varphi \in C^\infty(D)$ with $\frac{\partial \varphi}{\partial n}|_{\partial D} = 0$, the following equality holds $\mathbb{P}_n$ almost surely:

$$\int_D (\tilde{\rho}_\tau(t) + \delta) \tilde{\theta}_\tau(t) \varphi dx - \int_0^t \int_D (\delta \tilde{\theta}_\tau^3 + K(\tilde{\theta}_\tau) \Delta \varphi) dx ds$$

$$\begin{aligned}
&= \int_D (\rho_{0,\delta} + \delta) \theta_{0,\delta} \varphi dx + \int_0^t \int_D 2h_{\det}^\tau \tilde{\rho}_\tau \tilde{\theta}_\tau \tilde{u}_\tau \cdot \nabla \varphi dx ds \\
&\quad + \int_0^t \int_D 2h_{\det}^\tau [(1 - \delta) \mathcal{S}(\tilde{u}_\tau) : \nabla \tilde{u}_\tau - \theta_\tau p_\theta(\rho_\tau) \operatorname{div} u_\tau] \varphi dx ds.
\end{aligned}$$

We will pass to the limit pointwise in $\omega \in \Omega_n$, proceeding term by term in the equality above from left to right. For the first term, use that $(\tilde{\rho}_\tau + \delta)\tilde{\theta}_\tau \rightarrow (\rho_n + \delta)\theta_n$ in $C_t([L_x^2]_w)$. For the next two terms, use that $\tilde{\theta}_\tau \rightarrow \theta_n$ in $L_{t,x}^4$. This is sufficient since $\kappa(\theta) \sim \theta^2$ implies $\mathcal{K}(\theta) \sim \theta^3$. The regularized data do not depend on n and can be left alone.

For the next three terms, recall that $h_{\det}^\tau \rightarrow \frac{1}{2}$ weakly in L_t^p for any $p \in [1, \infty)$. For the flux term, use Lemma 4.6 together with Proposition 4.4 to deduce that $\tilde{\rho}_\tau \tilde{u}_\tau \tilde{\theta}_\tau \rightarrow \rho_n u_n \theta_n$ in $L_t^3(L_x^p)$ where $\frac{1}{p} = \frac{1}{\beta} + \frac{1}{3}$. Note that $p > 1$ since $\beta > 4$, so we find that $2h_{\det}^\tau \tilde{\rho}_\tau \tilde{u}_\tau \tilde{\theta}_\tau \rightarrow \rho_n u_n \theta_n$ in $D'_{t,x}$. For the next term, the $C_t(X_n)$ convergence of the velocity clearly implies $2(1 - \delta)h_{\det}^\tau \mathcal{S}(\tilde{u}_\tau) : \nabla \tilde{u}_\tau \rightarrow \mathcal{S}(u_n) : \nabla u_n$ in $D'_{t,x}$. For the last term, note that $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$ implies $p_\theta(\tilde{\rho}_\tau) \rightarrow p_\theta(\rho_n)$ in $C_t(L_x^q)$ for any $q < \frac{\beta d}{\gamma}$. Thus, $\tilde{\theta}_\tau p_\theta(\tilde{\rho}_\tau) \operatorname{div} \tilde{u}_\tau \rightarrow \theta_n p_\theta(\rho_n) \operatorname{div} u_n$ in $L_t^4(L_x^r)$ provided $\frac{1}{r} > \frac{1}{4} + \frac{\gamma}{\beta d}$. Since $\beta > \gamma$ and $d \geq 3$, we can ensure $r > 1$. Hence, we find that $2h_{\det}^\tau \tilde{\theta}_\tau p_\theta(\tilde{\rho}_\tau) \operatorname{div} \tilde{u}_\tau \rightarrow \theta_n p_\theta(\rho_n) \operatorname{div} u_n$ in $L_t^4(L_x^{r-})$ in $D'_{t,x}$. This completes the proof. $\square$

5. ϵ layer existence

This section is devoted to the ϵ layer existence theory; sending $n \rightarrow \infty$ our goal is to prove:

THEOREM 5.1. *For every $\epsilon > 0$, there exists an ϵ layer approximation (in the sense of Definition 5.2 below) $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ to (1.1), relative to a stochastic basis $(\Omega_\epsilon, \mathcal{F}_\epsilon, (\mathcal{F}_\epsilon^t)_{t=0}^T, \mathbb{P}_\epsilon, \{\beta_\epsilon^k\}_{k=1}^\infty)$.*

For all $p \geq 1$, there exists C_p independent of ϵ, δ such that

$$\begin{aligned}
&\sup_{\epsilon > 0} \mathbb{E}^{\mathbb{P}_\epsilon} \left[|\sqrt{\rho_\epsilon} u_\epsilon|_{L_t^\infty(L_x^2)}^{2p} + |\rho_\epsilon|_{L_t^\infty(L_x^\gamma)}^{\gamma p} + |\delta^{\frac{1}{\beta}} \rho_\epsilon|_{L_t^\infty(L_x^\beta)}^{\beta p} + |(\rho_\epsilon + \delta) \theta_\epsilon|_{L_t^\infty(L_x^1)}^p \right] \leq C_p. \\
&\sup_{\epsilon > 0} \mathbb{E}^{\mathbb{P}_\epsilon} \left[|\delta^{\frac{1}{2}} u_\epsilon|_{L_t^2(H_{0,x}^1)}^{2p} + |\epsilon^{\frac{1}{2}} \nabla(\rho_\epsilon^{\frac{\gamma}{2}} + \delta^{\frac{1}{2}} \rho_\epsilon^{\frac{\beta}{2}})|_{L_t^2(L_x^2)}^{2p} + |\delta^{\frac{1}{3}} \theta_\epsilon|_{L_{t,x}^3}^{3p} \right] \leq C_p. \quad (5.1)
\end{aligned}$$

Moreover, there exists $C'(p, \delta)$ such that

$$\sup_{n \geq 1} \mathbb{E}^{\mathbb{P}_n} \left[|\theta_n|_{L_t^2(H_x^1)}^{2p} \right] \leq C'(p, \delta). \quad (5.2)$$

For each $k \geq 1$ and $\delta > 0$, define an operator $\sigma_{k,\delta} : L^1(D) \times L^1(D; \mathbb{R}^d) \times L^1(D) \rightarrow C(D)$ via

$$\sigma_{k,\delta}(\rho, m, \alpha) = \sigma_k(\rho * \eta_\delta(\cdot), m * \eta_\delta(\cdot), \alpha * \eta_\delta(\cdot)).$$

Following Feiresil [7], we define a set $\mathcal{R}$ of admissible renormalizations of the temperature equation. Namely, $\mathcal{R}$ consists of non-increasing real-valued functions $h \in C^2[0, \infty)$ which satisfy $h(0) = 1$, $\lim_{z \rightarrow \infty} h(z) = 0$ and $h''(z)h(z) \geq 2(h'(z))^2$ for all $z \geq 0$.

DEFINITION 5.2. A triple $(\rho_\epsilon, u_\epsilon, \theta_\epsilon)$ is an ϵ layer approximation to (1.1) provided there exists a stochastic basis $(\Omega_\epsilon, \mathcal{F}_\epsilon, (\mathcal{F}_\epsilon^t)_{t=0}^T, \mathbb{P}_\epsilon, \{\beta_\epsilon^k\}_{k \geq 1})$ such that

1. The quadruple $(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon \theta_\epsilon, u_\epsilon)$ belongs in $L^2(\Omega \times [0, T]; \mathcal{P}; L^\beta \times L^{\frac{2\beta}{\beta+1}} \times L^q \times [H_0^1]^d)$, where $\mathcal{P}$ is the predictable σ -algebra generated by $(\mathcal{F}_\epsilon^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.
2. For all $\phi \in C^\infty(D)$ and all $t \in [0, T]$, the following equality holds $\mathbb{P}_\epsilon$ a.s.

$$\int_D \rho_\epsilon(t) \phi dx = \int_D \rho_{0,\delta} + \int_0^t \int_D [\rho_\epsilon u_\epsilon \cdot \nabla \phi + \epsilon \rho_\epsilon \Delta \phi] dx ds. \quad (5.3)$$

3. For all $\phi \in [C_c^\infty(D)]^d$ and all $t \in [0, T]$, the following equality holds $\mathbb{P}_\epsilon$ a.s.

$$\begin{aligned} \int_D \rho_\epsilon u_\epsilon(t) \cdot \phi dx &= \int_D m_{0,\delta} \cdot \phi + \int_0^t \int_D [\rho_\epsilon u_\epsilon \otimes u_\epsilon - \mathcal{S}(u_\epsilon)] : \nabla \phi dx ds \\ &+ \int_0^t \int_D [(P(\rho_\epsilon, \theta_\epsilon) + \delta \rho_\epsilon^\beta) \operatorname{div} \phi - \epsilon \nabla u_\epsilon \nabla \rho_\epsilon \cdot \phi] dx ds \\ &+ \sum_{k=1}^\infty \int_0^t \int_D \rho_\epsilon \sigma_{k,\delta}(\rho_\epsilon, \rho_\epsilon u_\epsilon, \rho_\epsilon \theta_\epsilon) \cdot \phi dx d\beta_\epsilon^k(s). \end{aligned} \quad (5.4)$$

4. For all nonnegative $\varphi \in C^\infty([0, T] \times D)$ with $\frac{\partial \varphi}{\partial n}|_{\partial D} = 0$ and all $h \in \mathcal{R}$, the inequality below holds $\mathbb{P}_\epsilon$ a.s.

$$\begin{aligned} &\int_0^T \int_D (\rho_\epsilon + \delta) H(\theta_\epsilon) \partial_t \varphi + \rho_\epsilon H(\theta_\epsilon) u_\epsilon \cdot \nabla \varphi + \mathcal{K}_h(\theta_\epsilon) \Delta \varphi - \delta \theta_\epsilon^3 H'(\theta_\epsilon) \varphi dx dt \\ &\leq \int_0^T \int_D h(\theta_\epsilon) [\theta_\epsilon p_\theta(\rho_\epsilon) \operatorname{div} u_\epsilon - \mathcal{S}(u_\epsilon) : \nabla u_\epsilon] \varphi dx dt \\ &\quad + \int_0^T \int_D \nabla \rho_\epsilon \cdot \nabla [(H(\theta_\epsilon) - \theta_\epsilon h(\theta_\epsilon)) \varphi] dx ds \\ &\quad + \int_0^T \int_D h'(\theta_\epsilon) \kappa(\theta_\epsilon) |\nabla \theta_\epsilon|^2 dx dt - \int_D \rho_\epsilon(0+) H(\theta_\epsilon) \varphi(0) dx, \end{aligned} \quad (5.5)$$

where $H(\theta) = \int_0^\theta h(z) dz$ and $\mathcal{K}_h(\theta) = \int_0^\theta \kappa(z) h(z) dz$.

5.1. $n \rightarrow \infty$ compactness step

Let us begin with the following uniform energy bounds:

LEMMA 5.3. *For all $p \geq 1$, there exists C_p independent of n, ϵ, δ such that*

$$\begin{aligned} \sup_{n \geq 1} \mathbb{E}^{\mathbb{P}_n} \left[|\sqrt{\rho_n} u_n|_{L_t^\infty(L_x^2)}^{2p} + |\rho_n|_{L_t^\infty(L_x^\gamma)}^{\gamma p} + |\delta^{\frac{1}{\beta}} \rho_n|_{L_t^\infty(L_x^\beta)}^{\beta p} + |(\rho_n + \delta) \theta_n|_{L_t^\infty(L_x^1)}^p \right] &\leq C_p. \\ \sup_{n \geq 1} \mathbb{E}^{\mathbb{P}_n} \left[|\delta^{\frac{1}{2}} u_n|_{L_t^2(H_{0,x}^1)}^{2p} + |\epsilon^{\frac{1}{2}} \nabla \left(\rho_n^{\frac{\gamma}{2}} + \delta^{\frac{1}{2}} \rho_n^{\frac{\beta}{2}} \right)|_{L_t^2(L_x^2)}^{2p} + |\delta^{\frac{1}{3}} \theta_n|_{L_{t,x}^3}^{3p} \right] &\leq C_p. \end{aligned} \quad (5.6)$$

Moreover, there exists $C(p, \delta)$ such that

$$\sup_{n \geq 1} \mathbb{E}^{\mathbb{P}_n} \left[|\theta_n|_{L_t^2(H_x^1)}^{2p} \right] \leq C(p, \delta). \quad (5.7)$$

Proof. An application of Ito's formula yields for all $t \in [0, T]$:

$$\begin{aligned} &\int_D \frac{1}{2} \rho_n |u_n|^2(t) + \rho_n P_m(\rho_n(t)) + \frac{\delta}{\beta - 1} \rho_n^\beta(t) + (\rho_n + \delta) \theta_n(t) dx \\ &\quad + \int_0^t \int_D \delta [S(u_n) : \nabla u_n + \theta_n^3] dx ds \\ &\quad + \int_0^t \int_D \epsilon \left(\frac{P'_m(\rho_n)}{\rho_n} + \delta \beta \rho_n^{\beta-2} \right) |\nabla \rho_n|^2 dx ds \\ &= \sum_{k=1}^n \int_0^t \int_D \rho_n u_n \cdot \sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n) dx d\beta_n^k(s) \\ &\quad + \sum_{k=1}^n \int_0^t \int_D \rho_n |\sigma_{k,n}(\rho_n, \rho_n u_n, \rho_n \theta_n)|^2 dx dt + E_n(0). \end{aligned}$$

By Hypothesis 3.2 and the Banach/Steinhaus theorem, the operators $\{\Pi_n\}_{n=1}^\infty$ satisfy

$$\sup_{n \geq 1} |\Pi_n|_{\mathcal{L}(L^{\frac{2\gamma}{\gamma+1}}, L^{\frac{2\gamma}{\gamma+1}})} < \infty. \quad (5.8)$$

Hence, the first part of the lemma is obtained in the same way as the formal estimates Sect. 2.

The second part of the lemma can be proved with the same technique as in Lemma 4.3. The δ dependence of the constant arises from the $L^p(\Omega; L_{t,x}^2)$ bound for $\operatorname{div} u_n$. $\square$

Next we establish the following compactness result:

PROPOSITION 5.4. *There exists a probability space $(\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon)$, a collection of independent Brownian motions $\{\beta_\epsilon^k\}_{k=1}^\infty$, limit points $(\rho_\epsilon, u_\epsilon, \theta_\epsilon, \sqrt{\rho_\epsilon} u_\epsilon)$, and a sequence of measurable maps $\{\tilde{T}_n\}_{n=1}^\infty$ such that*

1. *For each $n \geq 1$, $\tilde{T}_n : (\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon) \rightarrow (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ and $(\tilde{T}_n)_\# \mathbb{P}_\epsilon = \mathbb{P}_n$.*
2. *The new sequence $\{(\tilde{\rho}_n, \tilde{u}_n, \tilde{\theta}_n)\}_{n=1}^\infty$ defined by $(\tilde{\rho}_n, \tilde{u}_n, \tilde{\theta}_n) = (\rho_n, u_n, \theta_n) \circ \tilde{T}_n$ constitutes an n layer approximation relative to the stochastic basis $(\tilde{\Omega}_\epsilon, \tilde{\mathcal{F}}_\epsilon, \tilde{\mathbb{P}}_\epsilon, (\tilde{\mathcal{F}}_n^t)_{t=0}^T, \tilde{W}_n)$, where $\tilde{W}_n := W_n \circ \tilde{T}_n$ and $\tilde{\mathcal{F}}_n^t = \tilde{T}_n^{-1} \circ \mathcal{F}_n^t$.*

3. The uniform bounds in Lemma 5.3 hold with ρ_n, θ_n, u_n replaced by $\tilde{\rho}_n, \tilde{\theta}_n, \tilde{u}_n$ and $\mathbb{P}_n$ replaced by $\mathbb{P}_\epsilon$. Moreover, $\mathbb{P}_\epsilon$ almost surely,

$$\sup_{n \in \mathbb{N}} \|(\tilde{\rho}_n + \delta)\tilde{\theta}_n\|_{L_t^\infty(L_x^1)} < \infty. \quad (5.9)$$

4. The following convergences hold pointwise on Ω_ϵ

$$\tilde{\rho}_n \rightarrow \rho_\epsilon \quad \text{in} \quad C_t([L_x^\beta]_w) \cap L_t^2(L_x^2) \quad (5.10)$$

$$\tilde{u}_n \rightarrow u_\epsilon \quad \text{in} \quad [L_t^2(H_{0,x}^1)]_w \quad (5.11)$$

$$\tilde{\rho}_n \tilde{u}_n \rightarrow \rho_\epsilon u_\epsilon \quad \text{in} \quad C_t([L_x^{\frac{2\beta}{\beta+1}}]_w) \quad (5.12)$$

$$\tilde{\theta}_n \rightarrow \theta_\epsilon \quad \text{in} \quad [L_t^2(H_x^1) \cap L_{t,x}^3]_w \quad (5.13)$$

$$(\tilde{\rho}_n + \delta)\tilde{\theta}_n \rightarrow (\rho_\epsilon + \delta)\theta_\epsilon \quad \text{in} \quad [L_t^\infty(M_x)]_{w-\star} \quad (5.14)$$

$$\tilde{W}_n \rightarrow W_\epsilon \quad \text{in} \quad [C_t]^\infty, \quad (5.15)$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{d} + \frac{\gamma}{\beta d}$.

5. The following additional convergences hold

$$\sqrt{\tilde{\rho}_n} \tilde{u}_n \rightarrow \sqrt{\rho_\epsilon} u_\epsilon \quad \text{in} \quad L_{w^*}^p(\Omega_\epsilon; L_t^\infty(L_x^2)) \quad (5.16)$$

$$\tilde{u}_n \rightarrow u_\epsilon \quad \text{in} \quad L_w^p(\Omega_\epsilon; L_t^2(H_{0,x}^1)) \quad (5.17)$$

$$\tilde{\rho}_n \rightarrow \rho_\epsilon \quad \text{in} \quad L_{w^*}^p(\Omega_\epsilon; L_t^\infty(L_x^\beta)) \cap L_w^p(\Omega_\epsilon; L_t^2(W_x^{1,2})) \quad (5.18)$$

for all $q \geq 1$.

Now we proceed to a tightness lemma. Define the sequence of random variables $\{Y_n\}_{n \geq 1}$, where

$$Y_n = \left(\rho_n, u_n, \Pi_n(\rho_n u_n), \theta_n, (\rho_n + \delta)\theta_n, \{\beta_n^k\}_{k=1}^n \right)$$

Our convention is that given a topological vector space G , a finite sequence $\{x_k\}_{k=1}^n$ is viewed as an element of G^∞ where $x_j = 0$ for $j \geq n$. For each $n \geq 1$, Y_n induces a measure on the topological space E , where

$$\begin{aligned} E = & C_t([L_x^\beta]_w) \cap L_{t,x}^2 \times [L_t^2(H_{0,x}^1)]_w \times C_t([L_x^{\frac{2\beta}{\beta+1}}]_w) \\ & \times [L_t^2(H_x^1) \cap L_{t,x}^3]_w \times [L_t^1(C_0(D))]'_* \times [C_t]^\infty. \end{aligned}$$

LEMMA 5.5. The sequence of induced measures $\{\mathbb{P}_n \circ Y_n^{-1}\}_{n \geq 1}$ are tight on E .

Proof. The tightness of $\{\mathbb{P}_n \circ (\rho_n, u_n, \Pi_n(\rho_n u_n), W_n)^{-1}\}_{n=1}^\infty$ parallels the treatment in [14]. Indeed, the only additional fact needed is the $\mathbb{P}_n$ a.s. inequality

$$\begin{aligned} \int_s^t \int_D \theta_n p_\theta(\rho_n) \operatorname{div} \varphi dx ds &\leq C |\varphi|_{W_x^{1,\infty}} \int_s^t |\theta(r)|_{L^{\frac{2d}{d-2}}} |\rho(r)^{\frac{\gamma}{d}}|_{L^{\frac{2d}{d+2}}} dr \\ &\leq C(t-s)^{\frac{1}{2}} |\rho_n|_{L_t^\infty(L_x^\gamma)}^{\frac{\gamma}{d}} |\theta_n|_{L_t^2(H_x^1)}. \end{aligned}$$

Next we note that for all $M \geq 0$, the ball of radius M in $L_t^\infty(L_x^1)$ is compact with respect to the weak- $\star$ topology on in $L_t^\infty(M_x)$. Hence, by Chebyshev and the uniform bounds from Lemma 5.3, we obtain the tightness of $\{\mathbb{P}_n \circ ((\rho_n + \delta)\theta_n)^{-1}\}_{n=1}^\infty$ in $[L_t^1(C_0(D))]'_*$. Similarly, using Banach–Alaoglu and the uniform bounds in Lemma 5.3, we obtain the tightness of $\{\mathbb{P}_n \circ \theta_n^{-1}\}_{n=1}^\infty$ on $[L_t^2(H_x^1) \cap L_{t,x}^3]_w$. $\square$

Now we can complete the proof of our compactness step.

Proof of Proposition 5.4. Since E is a Jakubowski space, Lemma 5.5 allows us to apply Theorem 7.18. The remainder of the proof follows essentially the same arguments as in [14]. The one point worthy of a remark regards the proof of (5.9) and (5.14). Indeed, Lemma 5.5 yields directly the pointwise convergence $(\tilde{\rho}_n + \delta)\tilde{\theta}_n \rightarrow (\rho_\epsilon + \delta)\theta_\epsilon$ in $[L_t^1(C_0(D))]'_*$. However, due to the explicit representation $\tilde{\rho}_n(\tilde{\theta}_n + \delta) = \rho_n(\theta_n + \delta) \circ \tilde{T}_n$, we obtain also pointwise uniform bounds (5.9). These two facts together are now sufficient to deduce pointwise convergence (5.14). $\square$

5.2. $n \rightarrow \infty$ identification step

Next we define a filtration $(\mathcal{F}_t^\epsilon)_{t=0}^T$ via $\mathcal{F}_t^\epsilon = \sigma(r_t X_\epsilon)$ where $X_\epsilon = (\rho_\epsilon, \rho_\epsilon u_\epsilon, W_\epsilon, u_\epsilon, \rho_\epsilon \theta_\epsilon)$ and $r_t : E_T \rightarrow E_t$

$$\begin{aligned} E_s &= C([0, s]; [L^\beta]_w) \cap L^2([0, s]; H^1(D)) \times C([0, s]; [L^{\frac{2\beta}{\beta+1}}]_w \times \mathbb{R}^\infty) \\ &\quad \times L^2([0, s]; H_0^1) \times L^2([0, s]; L^q(D)), \end{aligned}$$

where $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

LEMMA 5.6. *The pair $(\rho_\epsilon, u_\epsilon)$ satisfies parabolic equation, (5.3) of Definition 5.2. Moreover, we have the following convergence upgrade: for all $p \geq 1$,*

$$\lim_{n \rightarrow \infty} \mathbb{E}^{\tilde{\mathbb{P}}_\epsilon} [|\tilde{\rho}_n - \rho_\epsilon|_{L_t^2(W_x^{1,2})}^p] = 0 \quad (5.19)$$

Proof. The proof uses Proposition 5.4 with the same arguments as in [14]. $\square$

LEMMA 5.7. *For $\mathbb{P}_\epsilon$ almost all $\omega \in \Omega_\epsilon$, $\tilde{\theta}_n(\omega) \rightarrow \theta_\epsilon(\omega)$ in $L_{t,x}^q$ for each $q < 3$. Moreover, $\rho_\epsilon, u_\epsilon$, and θ_ϵ satisfy renormalized temperature inequality (6.39).*

Proof. The basic strategy of the proof is the same as in Lemma 4.6. Namely, by Proposition 5.4, for all $\omega \in \Omega_\epsilon$, $\tilde{\theta}_n(\omega) \rightarrow \theta_\epsilon(\omega)$ weakly in $L_{t,x}^3$. Hence, to prove the Lemma, it suffices to prove $\tilde{\theta}_n^2(\omega) \rightarrow \theta_\epsilon^2(\omega)$ in $D'_{t,x}$ for arbitrary $\omega \in \Omega_\epsilon$.

As in Lemma 4.6, we will begin by proving that $(\tilde{\rho}_n(\omega) + \delta)\theta_n(\omega) \rightarrow (\rho_\epsilon(\omega) + \delta)\theta_\epsilon(\omega)$ strongly in $L_t^2(H_x^{-1})$. However, this is not as simple as in Lemma 4.6 because

we no longer know that $(\tilde{\rho}_n(\omega) + \delta)\theta_n(\omega) \rightarrow (\rho_\epsilon(\omega) + \delta)\theta_\epsilon(\omega)$ in $C_t([L_x^2]_w)$. Instead our approach will be to verify Hypotheses of Proposition 7.23.

Fix $\omega \in \Omega_\epsilon$. In the language of Proposition 7.23, let $f_n = (\tilde{\rho}_n(\omega) + \delta)\tilde{\theta}_n(\omega)$, $f = (\rho_\epsilon(\omega) + \delta)\theta_\epsilon(\omega)$ and

$$g_n = -\operatorname{div}(\tilde{\rho}_n(\omega)\tilde{u}_n(\omega)\tilde{\theta}_n(\omega)) + \Delta\mathcal{K}(\tilde{\theta}_n(\omega)) - \delta\tilde{\theta}_n(\omega)^3 \\ + (1 - \delta)S(\tilde{u}_n(\omega)) : \nabla\tilde{u}_n(\omega) - \tilde{\theta}_n(\omega)p_\theta(\tilde{\rho}_n(\omega))\operatorname{div}\tilde{u}_n(\omega).$$

The temperature equation implies $\partial_t f_n \leq g_n$ in $D'_{t,x}$. By (5.9), the sequence $\{f_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^\infty(L_x^1)$, pointwise in ω . It remains to verify that $\{f_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^2(L_x^{\frac{2d}{d+2}})$ and $\{g_n\}_{n=1}^\infty$ is uniformly bounded in $L_t^1(W_x^{-k,p})$ for some $p > 1$. For this purpose, we use the pointwise convergences in Proposition 5.4 to select a constant $C(\omega)$ independent of $n \geq 1$ such that

$$\sup_{n \geq 1} [|\tilde{\rho}_n(\omega)|_{L_t^\infty(L_x^\gamma)} + |\tilde{u}_n(\omega)|_{L_t^2(H_x^1)}] \leq C(\omega). \\ \sup_{n \geq 1} [|\tilde{\theta}_n(\omega)|_{L_t^2(H_x^1)} + |\tilde{\theta}_n(\omega)|_{L_{t,x}^3}] \leq C(\omega).$$

Thus, we find that $f_n = (\tilde{\rho}_n + \delta)\tilde{\theta}_n \in L_t^2(L_x^r)$ for $\frac{1}{r} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$, uniformly in $n \geq 1$. Moreover, $\frac{1}{r} < \frac{1}{2} + \frac{1}{d}$ since $\gamma > \frac{d}{2}$.

Next we will estimate each term in the definition of g_n . For simplicity of notation, we drop dependence on ω . Let $\frac{1}{p} = 1 + \frac{1}{\gamma} - \frac{2}{d}$, then

$$|\operatorname{div}(\tilde{\rho}_n\tilde{\theta}_n\tilde{u}_n)|_{L_t^1(W_x^{-1,q})} \leq |\tilde{\rho}_n\tilde{\theta}_n\tilde{u}_n|_{L_t^1(L_x^q)} \\ \leq |\tilde{\rho}_n|_{L_t^\infty(L_x^\gamma)}|\tilde{\theta}_n|_{L_t^2(H_x^1)}|\tilde{u}_n|_{L_t^2(H_x^1)}.$$

Since $\mathcal{K}(\theta) \sim \theta^3$, we find that

$$|\Delta\mathcal{K}(\tilde{\theta}_n)|_{L_t^1(W_x^{-2,1})} + \delta|\tilde{\theta}_n^3|_{L_{t,x}^1} + 2(1 - \delta)|S(\tilde{u}_n) : \nabla\tilde{u}_n|_{L_{t,x}^1} \\ \leq C[|\tilde{\theta}_n|_{L_{t,x}^3}^3 + |\tilde{u}_n|_{L_t^2(H_x^1)}^2].$$

Finally, since $p_\theta(\rho) \sim \rho^{\frac{\gamma}{d}}$,

$$|\tilde{\theta}_n p_\theta(\tilde{\rho}_n)\operatorname{div}\tilde{u}_n|_{L_{t,x}^1} \leq C|\tilde{\theta}_n|_{L_t^2(L_x^{\frac{2d}{d-2}})}|\tilde{\rho}_n|^{\gamma/d}_{L_t^\infty(L_x^\gamma)}|\tilde{u}_n|_{L_t^2(H_x^1)}.$$

Choose a $p > 1$ such that $L_{t,x}^1 + L_t^1(W_x^{-2,1}) + L_t^1(W_x^{-1,q}) \hookrightarrow L_t^1(W_x^{-3,p})$.

With these observations at hand, we may apply Proposition 7.23 and deduce $(\tilde{\rho}_n + \delta)\tilde{\theta}_n \rightarrow (\rho_\epsilon + \delta)\theta_\epsilon$ in $L_t^2(H_x^{-1})$. Moreover, by Proposition 5.4, $\tilde{\theta}_n \rightarrow \theta_\epsilon$ in $L_t^2(H_x^1)$. Hence, $(\tilde{\rho}_n + \delta)\tilde{\theta}_n^2 \rightarrow (\rho_\epsilon + \delta)\theta_\epsilon^2$ in $D'_{t,x}$. This completes the proof of the strong convergence of the temperature.

Finally, to pass the limit in the temperature equation, work pointwise in Ω_ϵ and follow exactly the arguments in [7], page 174. $\square$

LEMMA 5.8. *The pair $(\rho_\epsilon, u_\epsilon)$ satisfies energy corrected momentum equation 5.4 from Definition 5.2.*

Proof. In view of Proposition 5.4 and Lemmas 5.6/5.7, we may proceed as in [14] to prove the lemma. Indeed, there are just two differences with the corresponding proposition in [14]: the first is the form of the pressure, and the second is the dependence of $\sigma_{k,n}$ on $\rho_n \theta_n$. To treat the first point, the only additional fact required is that $P(\tilde{\rho}_n, \tilde{\theta}_n) + \delta \tilde{\rho}_n^\beta \rightarrow P(\rho_\epsilon, \theta_\epsilon) + \delta \rho_\epsilon^\beta$ strongly in $L^1_{t,x}$, $\mathbb{P}_\epsilon$ almost surely. This follows from Lemmas 5.6, 5.7, and the pointwise(in ω) uniform control of $\tilde{\theta}_n p_\theta(\tilde{\rho}_n)$ in $L^2_t(L^r_x)$ for $\frac{1}{r} > \frac{1}{2} - \frac{1}{d} + \frac{d}{\gamma}$. Moreover, the fourth moments of the pressure contribution to the weak form can be estimated as in (4.21), from $\tau \rightarrow 0$ identification of the momentum martingale. To treat the second point use the $\mathbb{P}_\epsilon$ almost sure $L^2_t(L^q_x)$ convergence of $\{\tilde{\rho}_n \tilde{\theta}_n\}_{n=1}^\infty$ to $\rho_\epsilon \theta_\epsilon$ together with the compactness of the mollification operator. $\square$

5.3. Conclusion of the proof

Proof of Theorem 5.1. For each $\epsilon > 0$, we obtain an ϵ layer approximation $(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon)$ using our compactness step, Proposition 5.4 together with Lemmas 5.6 and 5.8. To obtain the uniform bounds, use the weak and weak- $\star$ $L^p(\Omega)$ convergences in Proposition 5.4 as in [14] to treat all the terms which do not involve the temperature. To treat the uniform bounds for the temperature use the lower semicontinuity of the $L^p(\Omega; L^2_t(H^1_x) \cap L^3_{t,x})$ with respect to weak convergence. Finally, note that by another lower-semicontinuity argument,

$$\mathbb{E}^{\mathbb{P}_\epsilon} |(\rho_\epsilon + \delta)\theta_\epsilon|_{L^\infty(M_x)}^p \leq C_p. \quad (5.20)$$

Since the total variation norm and the L^1_x norm agree for absolutely continuous measures, we find that for any $q > 1$, we have the continuous embedding $L^q_{t,x} \cap L^\infty_t(M_x) \hookrightarrow L^\infty_t(L^q_x)$. Since $(\rho_\epsilon(\omega) + \delta) \in L^\infty_t(L^\gamma_x)$ with $\gamma > \frac{3}{2}$ and $\theta_\epsilon(\omega) \in L^3_{t,x}$ for all $\omega \in \Omega_\epsilon$, we deduce that

$$\mathbb{E}^{\mathbb{P}_\epsilon} |(\rho_\epsilon + \delta)\theta_\epsilon|_{L^\infty(L^1_x)}^p \leq C_p. \quad (5.21)$$

$\square$

6. δ layer existence

This section is devoted to the δ layer existence theory; sending $\epsilon \rightarrow 0$ our goal is to prove:

THEOREM 6.1. *For every $\delta > 0$, there exists a δ layer approximation (in the sense of Definition 6.2 below) $\rho_\delta, u_\delta, \theta_\delta$ to (1.1), relative to a stochastic basis $(\Omega_\delta, \mathcal{F}_\delta, (\mathcal{F}_\delta^t)_{t=0}^T, \mathbb{P}_\delta, \{\beta_\delta^k\}_{k=1}^\infty)$. Moreover, for all $p \geq 1$, there exists a constant $C_p > 0$ independent of δ such that*

$$\sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} \left[|\sqrt{\rho_\delta} u_\delta|_{L_t^\infty(L_x^2)}^{2p} + |\rho_\delta|_{L_t^\infty(L_x^\gamma)}^{\gamma p} + |\delta^{\frac{1}{\beta}} \rho_\delta|_{L_t^\infty(L_x^\beta)}^{\beta p} + |(\rho_\delta + \delta) \theta_\delta|_{L_t^\infty(L_x^1)}^p \right] \leq C_p \cdot$$

$$\sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} \left[|\delta^{\frac{1}{2}} u_\delta|_{L_t^2(H_{0,x}^1)}^{2p} + |\delta^{\frac{1}{3}} \theta_\delta|_{L_{t,x}^3}^{3p} \right] \leq C_p. \quad (6.1)$$

Recall that $\mathcal{R}$ is the collection of admissible renormalizations defined in the previous section.

DEFINITION 6.2. A triple $(\rho_\delta, u_\delta, \theta_\delta)$ is a δ layer approximation to (1.1) provided there exists a stochastic basis $(\Omega_\delta, \mathcal{F}_\delta, (\mathcal{F}_\delta^t)_{t=0}^T, \mathbb{P}_\delta, \{\beta_\delta^k\}_{k \geq 1})$ such that

1. The quadruple $(\rho_\delta, \rho_\delta u_\delta, \rho_\delta \theta_\delta, u_\delta)$ belongs in $L^2(\Omega \times [0, T]; \mathcal{P}; L^\beta \times L^{\frac{2\beta}{\beta+1}} \times L^1 \times [H_0^1]^d)$, where $\mathcal{P}$ is the predictable σ -algebra generated by $(\mathcal{F}_\delta^t)_{t=0}^T$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.
2. For all $\phi \in C^\infty(D)$ and all $t \in [0, T]$, the following equality holds $\mathbb{P}_\delta$ a.s.

$$\int_D \rho_\delta(t) \phi dx = \int_D \rho_{0,\delta} + \int_0^t \int_D \rho_\delta u_\delta \cdot \nabla \phi dx ds. \quad (6.2)$$

3. For all $\phi \in [C_c^\infty(D)]^d$ and all $t \in [0, T]$, the following equality holds $\mathbb{P}_\delta$ a.s.

$$\begin{aligned} \int_D \rho_\delta u_\delta(t) \cdot \phi dx &= \int_D m_{0,\delta} \cdot \phi + \int_0^t \int_D [\rho_\delta u_\delta \otimes u_\delta - \mathcal{S}(u_\delta)] : \nabla \phi dx ds \\ &+ \int_0^t \int_D [(P(\rho_\delta, \theta_\delta) + \delta \rho_\delta^\beta) \operatorname{div} \phi] dx ds \\ &+ \sum_{k=1}^\infty \int_0^t \int_D \rho_\delta \sigma_{k,\delta}(\rho_\delta, \rho_\delta u_\delta, \rho_\delta \theta_\delta) \cdot \phi dx d\beta_\delta^k(s). \end{aligned} \quad (6.3)$$

4. For all nonnegative $\varphi \in \mathcal{D}_{temp}$ and all $h \in \mathcal{R}$, the inequality below holds $\mathbb{P}_\delta$ a.s.

$$\begin{aligned} &\int_0^T \int_D (\rho_\delta + \delta) H(\theta_\delta) \partial_t \varphi + \rho_\delta H(\theta_\delta) u_\delta \cdot \nabla \varphi + \mathcal{K}_h(\theta_\delta) \Delta \varphi - \delta \theta_\delta^3 H'(\theta_\delta) \varphi dx dt \\ &\leq \int_0^T \int_D h(\theta_\delta) [\theta_\delta p_\theta(\rho_\delta) \operatorname{div} u_\delta - \mathcal{S}(u_\delta) : \nabla u_\delta] \varphi dx dt \\ &+ \int_0^T \int_D h'(\theta_\delta) \kappa(\theta_\delta) |\nabla \theta_\delta|^2 dx dt - \int_D \rho_\delta^0 H(\theta_\delta) \varphi(0) dx, \end{aligned} \quad (6.4)$$

where $H(\theta) = \int_0^\theta h(z) dz$ and $\mathcal{K}_h(\theta) = \int_0^\theta \kappa(z) h(z) dz$.

6.1. $\epsilon \rightarrow 0$ compactness step

The main goal of this subsection is to prove the following compactness result:

PROPOSITION 6.3. *There exists a probability space $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta)$, a collection of independent Brownian motions $\{\beta_\delta^k\}_{k=1}^\infty$, limit points $(\rho_\delta, u_\delta, \theta_\delta, \sqrt{\rho_\delta} u_\delta, p_m(\rho_\delta) + \delta \rho_\delta^\beta, \overline{\theta_\delta p_\theta(\rho_\delta)})$, and a sequence of measurable maps $\{\tilde{T}_\epsilon\}_{\epsilon > 0}$ such that*

1. For each $\epsilon > 0$, $\tilde{T}_\epsilon : (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta) \rightarrow (\Omega_\epsilon, \mathcal{F}_\epsilon, \mathbb{P}_\epsilon) \rightarrow$ and $(\tilde{T}_\epsilon)_\# \mathbb{P}_\delta = \mathbb{P}_\epsilon$.
2. The new sequence $\{(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon)\}_{\epsilon>0}$ defined by $(\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon) = (\rho_\epsilon, u_\epsilon, \theta_\epsilon) \circ \tilde{T}_\epsilon$ constitutes an ϵ layer approximation relative to the stochastic basis $(\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta, (\tilde{\mathcal{F}}_\epsilon^t)_{t=0}^T, \tilde{W}_\epsilon)$, where $\tilde{W}_\epsilon := W_\epsilon \circ \tilde{T}_\epsilon$ and $\tilde{\mathcal{F}}_\epsilon^t = \tilde{T}_\epsilon^{-1} \circ \mathcal{F}_\epsilon^t$.
3. The uniform bounds in 5.1 hold with $\rho_\epsilon, u_\epsilon, \theta_\epsilon$ replaced by $\tilde{\rho}_\epsilon, \tilde{u}_\epsilon, \tilde{\theta}_\epsilon$ and $\mathbb{P}_\epsilon$ replaced by $\mathbb{P}_\delta$. Moreover, it holds that

$$\sup_{n \in \mathbb{N}} \|(\tilde{\rho}_\epsilon + \delta)\tilde{\theta}_\epsilon\|_{L_t^\infty(L_x^1)} < \infty, \quad (6.5)$$

pointwise in Ω_δ .

4. The following convergences hold pointwise on Ω_δ :

$$\tilde{\rho}_\epsilon \rightarrow \rho_\delta \quad \text{in } C_t([L_x^\beta]_w) \quad (6.6)$$

$$\tilde{u}_\epsilon \rightarrow u_\delta \quad \text{in } [L_t^2(H_{0,x}^1)]_w \quad (6.7)$$

$$\tilde{\rho}_\epsilon \tilde{u}_\epsilon \rightarrow \rho_\delta u_\delta \quad \text{in } C_t([L_x^{\frac{2\beta}{\beta+1}}]_w) \quad (6.8)$$

$$p_m(\tilde{\rho}_\epsilon) + \delta \overline{\tilde{\rho}_\epsilon^\beta} \rightarrow \overline{p_m(\rho_\delta)} + \delta \overline{\rho_\delta^\beta} \quad \text{in } [L_{t,x}^{1+\beta^{-1}}]_w \quad (6.9)$$

$$\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) \rightarrow \overline{\theta_\delta p_\theta(\rho_\delta)} \quad \text{in } [L_t^2(L_x^q)]_w \quad (6.10)$$

$$\tilde{\theta}_\epsilon \rightarrow \theta_\delta \quad \text{in } [L_t^2(H_x^1) \cap L_{t,x}^3]_w \quad (6.11)$$

$$(\tilde{\rho}_\epsilon + \delta)\tilde{\theta}_\epsilon \rightarrow (\rho_\delta + \delta)\theta_\delta \quad \text{in } [L_t^\infty(M_x)]_{w*} \quad (6.12)$$

$$\tilde{W}_\epsilon \rightarrow W_\delta \quad \text{in } [C_t]^\infty, \quad (6.13)$$

where $\frac{1}{q} = \frac{1}{2} - \frac{1}{d} + \frac{\gamma}{\beta d}$.

5. The following additional convergences hold

$$\sqrt{\tilde{\rho}_\epsilon} \tilde{u}_\epsilon \rightarrow \sqrt{\rho_\delta} u_\delta \quad \text{in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^2)) \quad (6.14)$$

$$\tilde{u}_\epsilon \rightarrow u_\delta \quad \text{in } L_w^p(\Omega_\delta; L_t^2(H_{0,x}^1)) \quad (6.15)$$

$$\tilde{\rho}_\epsilon \rightarrow \rho_\delta \quad \text{in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^\beta)) \cap L_w^p(\Omega_\delta; L_t^2(W_x^{1,2})) \quad (6.16)$$

$$\tilde{\rho}_\epsilon \operatorname{div} \tilde{u}_\epsilon \rightarrow \overline{\rho_\delta \operatorname{div} u_\delta} \quad \text{in } L_w^p(\Omega_\delta; L_t^2(L_x^{\frac{2\beta}{\beta+2}})) \quad (6.17)$$

$$\tilde{\rho}_\epsilon \log \tilde{\rho}_\epsilon \rightarrow \overline{\rho_\delta \log \rho_\delta} \quad \text{in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^2)). \quad (6.18)$$

To prove the tightness, we need the following integrability gains:

PROPOSITION 6.4. *For every $p \geq 1$, there exists $C_p > 0$ such that*

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta} \left[\left| \int_0^T \int_D \rho_\delta P(\rho_\delta, \theta_\delta) + \delta \rho_\delta^{\beta+1} dx dt \right|^p \right] \leq C_p. \quad (6.19)$$

Proof. The only difference with the proof in [14] is the pressure law, and this has been accounted for in the statement of the proposition. $\square$

To finish the proof of 6.3, follow the approach in [14], proving an appropriate tightness result and then appealing to Jakubowski/Skorohod Theorem 7.18.

6.2. Preliminary identification step

Next we define a filtration $(\mathcal{F}_t^\delta)_{t=0}^T$ via $\mathcal{F}_t^\epsilon = \sigma(r_t X_\delta)$ where $X_\delta = (\rho_\delta, \rho_\delta u_\delta, W_\delta, u_\delta, \rho_\delta \theta_\delta)$ and $r_t : E_T \rightarrow E_t$

$$E_s = C([0, s]; [L^\beta]_w) \times C([0, s]; [L^{\frac{2\beta}{\beta+1}}]_w \times \mathbb{R}^\infty) \\ \times L^2([0, s]; H_0^1) \times L^2([0, s]; L^q(D)),$$

where $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

LEMMA 6.5. *The pair (ρ_δ, u_δ) satisfies continuity equation, 6.2 of Definition 6.2.*

Proof. The result follows using similar line of argument as in the earlier section. $\square$

LEMMA 6.6. *The pair $(\tilde{\rho}_\delta, \tilde{u}_\delta)$ satisfies momentum equation (6.3) from Definition 6.2, with a modified pressure law $\overline{p_m(\rho_\delta)} + \overline{\theta_\delta p_\theta(\rho_\delta)} + \delta \rho_\delta^\beta$.*

Proof. In view of Proposition 6.3 and Lemmas 5.6/5.7, we may proceed following similar line of argument as in [14] to prove the lemma. Since our work treats the non-isentropic case, and in light of the form of the pressure $P(\tilde{\rho}_\epsilon, \tilde{\theta}_\epsilon)$ the convergence of the pressure term

$$P(\tilde{\rho}_\epsilon, \tilde{\theta}_\epsilon) + \delta \tilde{\rho}_\epsilon^\beta \rightarrow P(\rho_\delta, \theta_\delta) + \delta \rho_\delta^\beta$$

strongly in $L_{t,x}^1, \mathbb{P}_\epsilon$ almost surely will be of use. The proof of the lemma follows by combining Lemma 5.6, Lemma 5.7, and the pointwise(in ω) uniform control of $\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon)$ in $L_t^2(L_x^r)$ for $\frac{1}{r} > \frac{1}{2} - \frac{1}{d} + \frac{d}{\gamma}$. Moreover, the fourth moments of the pressure contribution to the weak form can be estimated as in the earlier layer. $\square$

6.3. Strong convergence of the density

Now to proceed to the proof of the strong convergence of the density. The first step is the following weak continuity result:

LEMMA 6.7. *Let $K \subset\subset D$ be arbitrary, then the weak continuity of the effective viscous pressure holds on average, that is:*

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K ((2\mu + \lambda) \operatorname{div} \tilde{u}_\epsilon - P_m(\tilde{\rho}_\epsilon) - \tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) - \delta \tilde{\rho}_\epsilon^\beta) \tilde{\rho}_\epsilon dx dt \right] \\ = \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K ((2\mu + \lambda) \operatorname{div} u_\delta - \overline{P_m(\rho_\delta)} - \overline{\theta_\delta p_\theta(\rho_\delta)} - \delta \overline{\rho_\delta^\beta}) \rho_\delta dx dt \right].$$

Proof. Consider the operator $\mathcal{A} = \nabla \Delta^{-1}$ and let η be a bump function supported in D . Define the following two random test functions $\tilde{\varphi}_\epsilon = \eta \mathcal{A}[\eta \tilde{\rho}_\epsilon]$ and $\varphi_\delta = \eta \mathcal{A}[\eta \rho_\delta]$ with $\tilde{\varphi}_\epsilon(0) = \varphi_\delta(0)$. Using these test functions in the weak formulation of the momentum equation and following the line of argument in [14], applying Ito rule as needed and taking expectation in the resulting relation (as a way to eliminate the stochastic integrals) we arrive after some analysis at the following relations:

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \left(\eta^2 \left[(2\mu + \lambda) \operatorname{div} \tilde{u}_\epsilon - P_m(\tilde{\rho}_\epsilon) - \tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) - \delta \tilde{\rho}_\epsilon^\beta \right] \right) \tilde{\rho}_\epsilon dx dt \right] \\ = I^0 + I_1^{A,\epsilon} + I_2^{A,\epsilon} + I_1^{C,\epsilon} + I_2^{C,\epsilon} + I_3^{C,\epsilon} + I_1^P + I_2^P. \\ \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \left(\eta^2 \left[(2\mu + \lambda) \operatorname{div} u_\delta - P_m(\rho_\delta) - \theta_\delta p_\theta(\rho_\delta) - \delta \rho_\delta^\beta \right] \right) \rho_\delta dx dt \right] \\ = I^0 + I_1^C + I_2^C + I_3^C + I_1^P + I_2^P. \end{aligned}$$

More precisely,

$$I^0 = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_D \eta m_{0,\delta} \cdot \mathcal{A}[\eta \rho_{0,\delta}] dx \right] \quad (6.20)$$

The terms $I_1^{A,\epsilon}$, $I_2^{A,\epsilon}$ account for the presence of artificial viscosity, are given as follows, and tend to 0 as $\epsilon \rightarrow 0$.

$$I_1^{A,\epsilon} = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \epsilon \eta \tilde{\rho}_\epsilon \tilde{u}_\epsilon \cdot \mathcal{A}[\operatorname{div}(\eta \tilde{\rho}_\epsilon) - \nabla \eta \cdot \nabla \tilde{\rho}_\epsilon] dx ds \right] \quad (6.21)$$

$$I_2^{A,\epsilon} = -\mathbb{E}^{p dh} \left[\int_0^T \int_K \epsilon \nabla \tilde{u}_\epsilon \nabla \tilde{\rho}_\epsilon \cdot \varphi_\epsilon dx ds \right] \quad (6.22)$$

The terms $I_1^{C,\epsilon}$, $I_2^{C,\epsilon}$, $I_3^{C,\epsilon}$ account for the localization of the estimate.

$$\begin{aligned} I_1^{C,\epsilon} = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K [\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon - 2\mu \nabla \tilde{u}_\epsilon] : \nabla \eta \otimes \mathcal{A}[\eta \tilde{\rho}_\epsilon] dx ds \right] \\ + \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K [-\lambda \operatorname{div} \tilde{u}_\epsilon + P_m(\tilde{\rho}_\epsilon) + \tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) + \delta \tilde{\rho}_\epsilon^\beta] \mathbb{I} : \nabla \eta \otimes \mathcal{A}[\eta \tilde{\rho}_\epsilon] dx ds \right] \end{aligned} \quad (6.23)$$

$$I_2^{C,\epsilon} = \int_0^T \int_K \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \tilde{\rho}_\epsilon \tilde{u}_\epsilon \cdot \mathcal{A}[\nabla \eta \cdot \tilde{\rho}_\epsilon \tilde{u}_\epsilon] dx ds \right] \quad (6.24)$$

$$I_3^{C,\epsilon} = \int_0^T \int_K \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \tilde{u}_\epsilon \cdot [\mathcal{A}(\eta \tilde{\rho}_\epsilon) \nabla - \nabla \eta \tilde{\rho}_\epsilon] dx ds \right]. \quad (6.25)$$

The terms $I_1^{P,\epsilon}$, $I_2^{P,\epsilon}$ are given as follows

$$I_1^{P,\epsilon} = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \eta [\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon : \nabla \mathcal{A}[\eta \tilde{\rho}_\epsilon] - \tilde{\rho}_\epsilon \tilde{u}_\epsilon \cdot \mathcal{A} \circ \operatorname{div}(\eta \tilde{\rho}_\epsilon \tilde{u}_\epsilon)] dx ds \right] \quad (6.26)$$

$$I_2^{P,\epsilon} = -\mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_K \tilde{\rho}_\epsilon \tilde{u}_\epsilon(T) \cdot \varphi_\epsilon(T) dx \right]. \quad (6.27)$$

In the limit as $\epsilon \rightarrow 0$ we obtain

$$\begin{aligned} I_1^C &= \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K [\rho_\delta u_\delta \otimes u_\delta - 2\mu \nabla u_\delta] : \nabla \eta \otimes \mathcal{A}[\eta \rho_\delta] dx ds \right] \\ &\quad + \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K [-\lambda \operatorname{div} u_\delta + P_m(\rho_\delta) + \theta_\delta p_\theta(\rho_\delta) + \delta \rho_\delta^\beta] \mathbb{I} : \right. \\ &\quad \left. \times \nabla \eta \otimes \mathcal{A}[\eta \rho_\delta] dx ds \right] \end{aligned} \quad (6.28)$$

$$I_2^C = \int_0^T \int_K \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \rho_\delta u_\delta \cdot \mathcal{A}[\nabla \eta \cdot \rho_\delta u_\delta] dx ds \right] \quad (6.29)$$

$$I_3^C = \int_0^T \int_K \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K u_\delta \cdot [\mathcal{A}(\eta \rho_\delta) \nabla - \nabla \eta \rho_\delta] dx ds \right]. \quad (6.30)$$

The terms I_1^P, I_2^P are given as follows

$$I_1^P = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_K \eta [\rho_\delta u_\delta \otimes u_\delta : \nabla \mathcal{A}[\eta \rho_\delta] - \rho_\delta u_\delta \cdot \mathcal{A} \circ \operatorname{div}(\eta \rho_\delta u_\delta)] dx ds \right] \quad (6.31)$$

$$I_2^P = -\mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_K \rho_\delta u_\delta(T) \cdot \varphi_\delta(T) dx \right]. \quad (6.32)$$

The result is obtained noting that

- For all $q \geq 1$ the operator $\mathcal{A} : L_x^\beta \rightarrow L_x^q$ is compact. Therefore, $\mathcal{A}[\eta \tilde{\rho}_\epsilon] \rightarrow \mathcal{A}[\eta \rho_\delta]$ strongly in $L_{t,x}^q$.
- Following similar line of argument as in the earlier layer, we get $\tilde{\rho}_\epsilon \tilde{u}_\epsilon \otimes \tilde{u}_\epsilon \rightarrow \rho_\delta u_\delta \otimes u_\delta$ in $L_t^2(L^{\frac{\beta d}{\beta(d-1)+d}})$.
- Combining these two observations with (7.17), (7.18) and (7.20) yield the result.

□

Making the appropriate adaptations to the arguments in [14] and using the monotonicity of p_θ as in [7], one can apply Lemma 6.7 to deduce the following strong convergence result for the density. We present here an outline of the proof for completeness.

LEMMA 6.8. *The sequence of densities $\{\tilde{\rho}_\epsilon\}_{\epsilon>0}$ converge strongly to ρ_δ in the sense that for all $p \geq 1$ and $r < \beta + 1$*

$$\lim_{\epsilon \rightarrow 0} |\tilde{\rho}_\epsilon - \rho_\delta|_{L^p(\tilde{\Omega}_\delta; L_{t,x}^r)} = 0. \quad (6.33)$$

Proof. Our goal is to establish that

$$\overline{\rho_\delta \log \rho_\delta} = \rho_\delta \log(\rho_\delta) \text{ a.e. in } \Omega_\delta \times [0, T] \times D.$$

Following similar line of argument as in [14] we arrive at

$$\begin{aligned}
 & \int_0^T \int_D \psi_t \mathbb{E}^{\mathbb{P}_\delta} [\rho_\delta \log \rho_\delta - \overline{\rho_\delta \log \rho_\delta}] dx ds \\
 & \leq \liminf_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K \psi \rho_\delta (\overline{P(\rho_\delta, \theta_\delta)} + \delta \overline{\rho_\delta^\beta}) \right. \\
 & \quad \left. - \psi \rho_\epsilon (P(\rho_\epsilon, \theta_\epsilon) + \delta \rho_\epsilon^\beta) dx ds \right] + \overline{R_K}(\psi). \\
 & = \liminf_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K \psi \left(\rho_\delta \overline{P(\rho_\delta, \theta_\delta)} - \rho_\epsilon P(\rho_\epsilon, \theta_\epsilon) \right) \right. \\
 & \quad \left. + \psi \delta \left(\rho_\delta \overline{\rho_\delta^\beta} - \rho_\epsilon^{\beta+1} \right) dx ds \right] + \overline{R_K}(\psi),
 \end{aligned}$$

with

$$\overline{R_K}(\psi) = \mathbb{E}^{\tilde{\mathbb{P}}_\delta} \left[\int_0^T \int_{D \setminus K} \psi [\rho_\delta \operatorname{div} u_\delta - \overline{\rho_\delta \operatorname{div} u_\delta}] dx dr \right].$$

Now,

$$\overline{P(\rho_\delta, \theta_\delta)} = \overline{p_m(\rho_\delta)} + \overline{\theta_\delta p_\theta(\rho_\delta)},$$

and

$$\overline{\theta_\epsilon p_\theta(\rho_\epsilon)} = \theta_\delta \overline{p_\theta(\rho_\delta)}, \quad \overline{\theta_\epsilon p_\theta(\rho_\epsilon) \rho_\epsilon} = \theta_\delta \overline{p_\theta(\rho_\epsilon) \rho_\epsilon}. \quad (6.34)$$

In particular, as p_θ is a non-decreasing function of the density we get

$$\overline{\theta_\epsilon p_\theta(\rho_\epsilon) \rho_\epsilon} = \theta_\delta \overline{p_\theta(\rho_\epsilon)} \rho_\epsilon \geq \theta_\delta \overline{p_\theta(\rho_\epsilon) \rho_\epsilon} = \overline{\theta_\epsilon p_\theta(\rho_\epsilon) \rho_\delta}. \quad (6.35)$$

In addition, as the function $z \rightarrow \delta z^\beta$ is increasing, we get

$$\liminf_{\epsilon \rightarrow 0} \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_K \left(\rho_\epsilon^{\beta+1} - \rho_\delta \overline{\rho_\delta^\beta} \right) dx ds \right] \geq 0. \quad (6.36)$$

Taking into consideration (6.34), (6.35), (6.36) and noting that for any Lebesgue point $s \in [0, T]$ of the function $s \rightarrow \mathbb{E}^{\mathbb{P}} [\rho_\delta \log \rho_\delta - \overline{\rho_\delta \log \rho_\delta}] dx (s)$, we may choose a sequence of test functions that approximate $1_{[0,s]}(t)$, so that their time derivatives approximate the negative of a dirac mass centered at the point s , we conclude that the following estimate holds almost everywhere in time

$$\begin{aligned}
 & \mathbb{E}^{\mathbb{P}_\delta} \left[\int_D [\overline{\rho_\delta \log \rho_\delta} - \rho_\delta \log \rho_\delta] dx \right] (s) \\
 & \leq \mathbb{E}^{\mathbb{P}_\delta} \left[\int_0^T \int_{D \setminus K} \psi [\rho_\delta \operatorname{div} u_\delta - \overline{\rho_\delta \operatorname{div} u_\delta}] dx dt \right].
 \end{aligned} \quad (6.37)$$

Choosing K close enough to D and taking into consideration the convexity of the function $\rho \log \rho$ we conclude that

$$\overline{\rho_\delta \log \rho_\delta} = \rho_\delta \log \rho_\delta \quad \text{a.e. in } \tilde{\Omega}_\delta \times D.$$

□

LEMMA 6.9. For $\mathbb{P}_\epsilon$ almost all $\omega \in \Omega_\delta$ and $q < 3$, $\tilde{\theta}_\epsilon(\omega) \rightarrow \theta_\delta(\omega)$ in $L^q_{t,x}$. Moreover, ρ_δ , u_δ , and θ_δ satisfy renormalized temperature inequality, (6.4)

Proof. The proof of the strong convergence claim follows the argument in Lemma 5.7. The only additional detail is to explain how to pass from the renormalized form to an inequality in $\mathcal{D}'_{t,x}$ directly on $\partial_t[(\tilde{\rho}_\epsilon + \delta)\tilde{\theta}_\epsilon]$. Toward this end, for each $m \in \mathbb{N}$, introduce the renormalization $h_m(z) = (1 + z)^{-\frac{1}{m}}$. It is straightforward to verify $h_m \in \mathcal{R}$. Using the recovery maps, we may transfer the renormalized temperature inequality to the new probability space to find for each $\epsilon > 0$ and $m \in \mathbb{N}$, the $\mathbb{P}_\delta$ a.s. inequality

$$\begin{aligned} & \int_0^T \int_D (\tilde{\rho}_\epsilon + \delta) H_m(\tilde{\theta}_\epsilon) \partial_t \varphi + \tilde{\rho}_\epsilon H_m(\tilde{\theta}_\epsilon) \tilde{u}_\epsilon \cdot \nabla \varphi + \mathcal{K}_m(\tilde{\theta}_\epsilon) \Delta \varphi - \delta \tilde{\theta}_\epsilon^3 h_m(\tilde{\theta}_\epsilon) \varphi \, dx \, dt \\ & \leq \int_0^T \int_D h_m(\tilde{\theta}_\epsilon) [\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) \operatorname{div} \tilde{u}_\epsilon - \mathcal{S}(\tilde{u}_\epsilon) : \nabla \tilde{u}_\epsilon] \varphi \, dx \, dt \\ & \quad + \int_0^T \int_D \nabla \tilde{\rho}_\epsilon \cdot \nabla [(H_m(\tilde{\theta}_\epsilon) - \tilde{\theta}_\epsilon h_m(\tilde{\theta}_\epsilon)) \varphi] \, dx \, ds \\ & \quad + \int_0^T \int_D h'_m(\tilde{\theta}_\epsilon) \kappa(\tilde{\theta}_\epsilon) |\nabla \tilde{\theta}_\epsilon|^2 \, dx \, dt - \int_D \tilde{\rho}_\epsilon(0+) H_m(\theta_\epsilon) \varphi(0) \, dx, \end{aligned} \quad (6.38)$$

where $H_m(\theta) = \int_0^\theta h_m(z) \, dz$ and $\mathcal{K}_m(\theta) = \int_0^\theta \kappa(z) h_m(z) \, dz$. Therefore, sending $m \rightarrow \infty$ yields the inequality

$$\begin{aligned} & \int_0^T \int_D (\tilde{\rho}_\epsilon + \delta) \tilde{\theta}_\epsilon \partial_t \varphi + \tilde{\rho}_\epsilon \tilde{\theta}_\epsilon \tilde{u}_\epsilon \cdot \nabla \varphi + \mathcal{K}(\tilde{\theta}_\epsilon) \Delta \varphi - \delta \tilde{\theta}_\epsilon^3 \varphi \, dx \, dt \\ & \leq \int_0^T \int_D [\tilde{\theta}_\epsilon p_\theta(\tilde{\rho}_\epsilon) \operatorname{div} \tilde{u}_\epsilon - \mathcal{S}(\tilde{u}_\epsilon) : \nabla \tilde{u}_\epsilon] \varphi \, dx \, dt \\ & \quad - \int_D \tilde{\rho}_\epsilon(0+) \theta_\epsilon \varphi(0) \, dx. \end{aligned} \quad (6.39)$$

Finally, passing to the limit in the renormalized temperature equation follows the same arguments as in the proof of Lemma 5.7. The only additional detail is to check that the term arising from the vanishing viscosity regularization tends to zero as $\epsilon \rightarrow 0$. Given an $H \in \mathcal{R}$, this term contributes

$$\int_0^T \int_D \epsilon \nabla \tilde{\rho}_\epsilon \cdot \nabla (\beta(\tilde{\theta}_\epsilon) \varphi) \, dx \, ds, \quad (6.40)$$

where $\beta(\theta) = \theta H'(\theta) - H(\theta)$. As we mentioned in the proof of Lemma 5.7, β' is globally bounded on $\mathbb{R}$. This yields the inequality

$$\mathbb{E}^{\mathbb{P}_\delta} [\epsilon |\nabla \tilde{\rho}_\epsilon \cdot \nabla (\beta(\tilde{\theta}_\epsilon) \varphi)|_{L^1_{t,x}}] \leq \epsilon^{\frac{1}{2}} [\mathbb{E}^{\mathbb{P}_\delta} |\nabla (\epsilon^{\frac{1}{2}} \tilde{\rho}_\epsilon)|_{L^2_{t,x}}^2]^{\frac{1}{2}} [\mathbb{E}^{\mathbb{P}_\delta} |\tilde{\theta}_\epsilon|_{L^2_t(H^1_x)}^2]^{\frac{1}{2}}. \quad (6.41)$$

In view of the uniform bounds guaranteed by Proposition 6.3, this term tends to zero. $\square$

6.4. Conclusion of the proof

Proof of Theorem 6.1. For each $\delta > 0$ we apply Proposition 6.3 to construct a candidate δ layer approximation $(\rho_\delta, u_\delta, \theta_\delta)$. Combining Lemmas 6.5 and 6.6 with the strong convergence of the density we are able to identify

$$\overline{P(\rho_\delta, \theta_\delta)} + \delta \overline{\rho_\delta^\beta} = P(\rho_\delta, \theta_\delta) + \delta \rho_\delta^\beta$$

and complete the identification procedure. The uniform bounds can be argued as in the proof of Theorem 5.1. $\square$

7. Proof of the main result: $\delta \rightarrow 0$

7.1. Some further estimates

As $\delta \rightarrow 0$, the bounds based on the total energy are stable. However, the bounds obtained from the renormalized form of the temperature equation degenerate and need to be re-derived using the δ layer temperature inequality. In particular, the following estimates can be proved.

LEMMA 7.1. *For all $\sigma \in (0, 1)$ and $p \in [1, \infty)$, there exists a constant $C_{\sigma,p} > 0$ such that*

$$\begin{aligned} \sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta|_{L_t^2(H_x^1)}^{2p} + |\nabla \log(\theta_\delta)|_{L_{t,x}^2}^{2p} + |u_\delta|_{L_t^2(H_x^1)}^{2p}] &\leq C_p. \\ \sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta^{(3-\sigma)/2}|_{L_t^2(H_x^1)}] &\leq C_p. \end{aligned}$$

Proof. The proof of this lemma is essentially the same as in the formal estimates section, so we will only sketch the proof. The main difference is that we must use δ layer temperature inequality (6.4), which a priori holds only in a weak, renormalized form.

Begin by using the renormalization H_m defined by:

$$H_m(\theta) = \int_1^\theta \frac{1}{1+mz} dz. \quad (7.1)$$

Derive the analogue of (2.3) and multiply by a factor of $m \in \mathbb{N}$ to obtain

$$\mathbb{E}^{\mathbb{P}_\delta} \left(\int_0^T \int_D \frac{\kappa(\theta_\delta) |\nabla \theta_\delta|^2}{(\frac{1}{m} + \theta_\delta)^2} dx ds \right)^p \lesssim 1 + \mathbb{E} |p_\theta(\rho_\delta) \operatorname{div} u_\delta|_{L_{t,x}^1}^p + \mathbb{E} |\delta^{\frac{1}{2}} \theta_\delta|_{L_{t,x}^2}^{2p}. \quad (7.2)$$

Applying Hölder's inequality yields

$$\mathbb{E} |\delta^{\frac{1}{2}} \theta_\delta|_{L_{t,x}^2}^{2p} \leq \mathbb{E} |\delta^{\frac{1}{3}} \theta_\delta|_{L_{t,x}^3}^{2p} \lesssim 1.$$

Using the renormalized continuity equation and sending $m \rightarrow \infty$ yields

$$\mathbb{E}|\nabla\theta_\delta|_{L^2_{t,x}}^{2p} \leq C_p. \quad (7.3)$$

Using the modified Poincaré inequality from the formal estimates section gives:

$$\mathbb{E}|\theta_\delta|_{L^2_t(H^1_x)}^{2p} \leq C_p. \quad (7.4)$$

The remaining estimates are obtained in an analogous way using the renormalizations $\tilde{H}_m$ and $\hat{H}_m$ defined by

$$\begin{aligned} \tilde{H}_m(\theta) &= \int_1^\theta \frac{dz}{(1+z)^{\frac{1}{m}}}, \\ \hat{H}_m(\theta) &= \int_1^\theta \frac{dz}{\frac{1}{m}+z}. \end{aligned}$$

A limited amount of additional details is provided in [7]. $\square$

Observe that the bounds above do not control the $L^p(\Omega; L^3_{t,x})$ norm of the $\{\theta_\delta\}_{\delta>0}$, which is essential for passing to the limit in the temperature inequality. Instead, this estimate must be derived directly from the weak form of the δ layer renormalized temperature inequality. For this purpose, we adapt the approach of Feirisel [7] to the stochastic case. Namely, we begin by proving estimates away from vacuum (Lemma 7.2); then, we bootstrap these bounds and use a random test function approach to get estimates in the low density regions (Proposition 7.3)).

LEMMA 7.2. *For all $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that for all $v < 1$*

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta 1_{\{\rho_\delta \geq v\}}|_{L^3_{t,x}}^p] \leq v^{-p} C_p. \quad (7.5)$$

Proof. For each $\sigma \in (0, 1)$ let $r(\sigma) = \frac{(3-\sigma)d}{d-2}$. By interpolation, there exists $\alpha(\sigma) \in (0, 1)$ and $q(\sigma) > 1$ such that for all $f \in L^\infty_t(L^1_x) \cap L^{3-\sigma}_t(L^{r(\sigma)}_x)$, the following interpolation inequality holds:

$$|f|_{L^{q(\sigma)}_{t,x}} \leq |f|_{L^\infty_t(L^1_x)}^{\alpha(\sigma)} |f|_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}^{1-\alpha(\sigma)}.$$

Choosing σ sufficiently small, we can guarantee $q(\sigma) > 3$, leading to the following chain of inequalities:

$$\begin{aligned} \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta 1_{\{\rho_\delta \geq v\}}|_{L^p_{t,x}}^p] &\leq \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta 1_{\{\rho_\delta \geq v\}}|_{L^\infty_t(L^1_x)}^{\alpha p} |\theta_\delta 1_{\{\rho_\delta \geq v\}}|_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}^{(1-\alpha)p}] \\ &\leq v^{-\alpha p} \mathbb{E}^{\mathbb{P}_\delta} [|\rho_\delta \theta_\delta|_{L^\infty_t(L^1_x)}^{\alpha p} |\theta_\delta|_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}^{(1-\alpha)p}] \\ &\leq v^{-\alpha p} \mathbb{E}^{\mathbb{P}_\delta} [|\rho_\delta \theta_\delta|_{L^\infty_t(L^1_x)}^p]^\alpha \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta|_{L^{3-\sigma}_t(L^{r(\sigma)}_x)}^p]^{1-\alpha}. \end{aligned}$$

By the uniform $L^p(\Omega_\delta; H^1)$ bounds in $\{\theta_\delta^{\frac{3-\sigma}{2}}\}_{\delta>0}$ from Lemma 7.1 and the Sobolev embedding theorem, this concludes the proof. $\square$

Now we improve on our initial estimate by adapting a method from [7] to obtain estimates on the temperature near the vacuum.

PROPOSITION 7.3. *For all $p \in [1, \infty)$ there exists a constant $C_p > 0$ such that*

$$\sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} [|\theta_\delta|_{L^3_{t,x}}^p] \leq C_p. \quad (7.6)$$

Proof. For each level $\nu > 0$, we introduce a sequence of random variables $\{X_\delta^\nu\}_{\delta > 0}$, where $X_\delta^\nu : \Omega_\delta \rightarrow \mathbb{R}$ is defined by:

$$X_\delta^\nu(\omega) = \inf_{t \in [0, T]} \lambda\{x \in D \mid \rho_\delta(t, x, \omega) \geq \nu\}.$$

Here, λ denotes the Lebesgue measure on $\mathbb{R}^d$. We will begin by proving there exists a $\nu_0 > 0$ such that for all $\nu \leq \nu_0$, $X_\delta^\nu > 0$ a.s. with respect to $\mathbb{P}_\delta$. Moreover, for all $p \geq 1$, there exists a constant $C_{p,\nu} > 0$ such that the following uniform bound holds:

$$\sup_{\delta > 0} \mathbb{E}^{\mathbb{P}_\delta} [X_\delta^\nu]^{-p} \leq C_{p,\nu}. \quad (7.7)$$

To establish the claim above, begin by applying Holder in space, then minimizing in time to deduce the following $\mathbb{P}_\delta$ a.s. inequality:

$$\inf_{t \in [0, T]} |\rho_\delta(t) 1_{\{\rho_\delta(t) \geq \nu\}}|_{L_x^1} \leq [X_\delta^\nu]^{1-\frac{1}{\gamma}} |\rho_\delta|_{L_t^\infty(L_x^\gamma)}. \quad (7.8)$$

On the other hand, conservation of mass implies that $\mathbb{P}_\delta$ almost surely,

$$\inf_{t \in [0, T]} |\rho_\delta(t) 1_{\{\rho_\delta(t) \geq \nu\}}|_{L_x^1} \geq |\rho_\delta^0|_{L_x^1} - \nu \lambda(D). \quad (7.9)$$

Combining (7.8) and (7.9) yields the following lower bound for X_δ^ν :

$$X_\delta^\nu \geq |\rho_\delta|_{L_t^\infty(L_x^\gamma)}^{-\frac{\gamma}{\gamma-1}} [|\rho_\delta^0|_{L_x^1} - \nu \lambda(D)]^{\frac{\gamma}{\gamma-1}}.$$

By the strong convergence of the initial densities, Hypothesis 3.4, we can choose a small enough ν_0 to ensure $X_\delta^\nu > 0$ a.s. with respect to $\mathbb{P}_\delta$. Hence, inverting the lower bound above and applying the uniform $L^p(\Omega_\delta; L_t^\infty(L_x^\gamma))$ bounds on $\{\rho_\delta\}_{\delta > 0}$ yields (7.7).

The next step of the proof is to construct a suitable random test function. For each $\nu < \nu_0$, let $B_\nu : \mathbb{R}_+ \rightarrow [-1, 0]$ be a smooth function such that $B_\nu(z) = 0$ for $z \leq \nu$ and $B_\nu(z) = -1$ for $z \geq 2\nu$. For each $\delta > 0$, construct $\eta_\nu^\delta : [0, T] \times D \times \Omega_\delta \rightarrow \mathbb{R}$ such that for each $(t, \omega) \in [0, T] \times \Omega_\delta$, the function $x \rightarrow \eta_\nu^\delta(t, x, \omega)$ solves the following Neumann problem:

$$\begin{cases} \Delta \eta = B_\nu(\rho_\delta(t, \omega)) - \lambda^{-1}(D) \int_D B_\nu(\rho_\delta(t, \omega, x)) dx & \text{in } D \\ \frac{\partial \eta}{\partial n} = 0 & \text{on } \partial D \\ \int_D \eta dx = 0 \end{cases} \quad (7.10)$$

Let $\psi : [0, T] \rightarrow [0, 1]$ be smooth and compactly supported. For each $\delta > 0$ and $\nu > \nu_0$, we may now define our random test function $\varphi_\nu^\delta : [0, T] \times D \times \Omega_\delta \rightarrow \mathbb{R}$ by setting

$$\varphi_\nu^\delta(t, x, \omega) = \psi(t) [\eta_\nu^\delta(t, x, \omega) - \inf_{(t,x) \in [0,T] \times D} \eta_\nu^\delta(t, x, \omega)]. \quad (7.11)$$

Applying directly the argument in [7], we find that for each $\omega \in \Omega_\delta$, the function $(t, x) \rightarrow \varphi_\nu^\delta(t, x, \omega)$ is a valid test function in (6.4) for the renormalized form satisfied by $(\rho_\delta, \theta_\delta, u_\delta)$ in the state $\omega \in \Omega_\delta$. Taking a sequence of renormalizations $\{H_k\}_{k=1}^\infty$ which converge to the identity, then taking $L^p(\Omega_\delta)$ norms we find that:

$$\mathbb{E}^{\mathbb{P}^\delta} \left| \int_0^T \int_D \psi \theta_\delta^3 1_{\{\rho_\delta \geq \nu\}} |^{p/2} \right| \lesssim [\mathbb{E}^{\mathbb{P}^\delta} |X_\delta^\nu|^{-p}]^{\frac{1}{2}} \left(\sum_{k=1}^6 I_k^p \right)^{\frac{1}{2}}. \quad (7.12)$$

The terms I_1^p to I_6^p are given by:

$$\begin{aligned} I_1^p &= \mathbb{E}^{\mathbb{P}^\delta} |\theta_\delta 1_{\{\rho_\delta \geq \nu\}}|_{L_{t,x}^3}^{3p} \\ I_2^p &= \mathbb{E}^{\mathbb{P}^\delta} |\rho_\delta \theta_\delta u_\delta \nabla \varphi_\nu^\delta|_{L_{t,x}^1}^p \\ I_3^p &= \mathbb{E}^{\mathbb{P}^\delta} |\delta \theta_\delta^3 \varphi_\nu^\delta|_{L_{t,x}^1}^p \\ I_4^p &= \mathbb{E}^{\mathbb{P}^\delta} |\theta_\delta p_\theta(\rho_\delta) \operatorname{div} u_\delta \varphi_\nu^\delta|_{L_{t,x}^1}^p \\ I_5^p &= \mathbb{E}^{\mathbb{P}^\delta} |(\rho_{0,\delta} + \delta) \theta_{0,\delta} \varphi_\nu^\delta|_{L_{t,x}^1}^p \\ I_6^p &= \mathbb{E}^{\mathbb{P}^\delta} |(\rho_\delta + \delta) \theta_\delta \partial_t \varphi_\nu^\delta|_{L_{t,x}^1}^p \end{aligned}$$

Begin by applying Lemma 7.2 to control I_1^p . The remaining terms can now be estimated as follows:

$$\begin{aligned} I_2^p &\leq \mathbb{E}^{\mathbb{P}^\delta} [|\rho_\delta|_{L_t^\infty(L_x^\gamma)}^{4p}]^{\frac{1}{4}} \mathbb{E}^{\mathbb{P}^\delta} [|\theta_\delta|_{L_t^2(L_x^{\frac{2d}{d-2}})}^{4p}]^{\frac{1}{4}} \mathbb{E}^{\mathbb{P}^\delta} [|\rho_\delta|_{L_t^2(H_x^1)}^{4p}]^{\frac{1}{4}} \mathbb{E}^{\mathbb{P}^\delta} [|\nabla_x \eta_\nu^\delta|_{L_{t,x}^\infty}^{4p}]^{\frac{1}{4}} \\ I_3^p + I_5^p &\leq (1 + \mathbb{E}^{\mathbb{P}^\delta} |\delta^{\frac{1}{3}} \theta_\delta|_{L_{t,x}^3}^{3p}) |\varphi_\nu^\delta|_{L^\infty([0,T] \times D \times \Omega_\delta)}^p \\ I_4^p &\leq [\mathbb{E}^{\mathbb{P}^\delta} |\theta_\delta|_{L_t^2(L_x^{\frac{2d}{d-2}})}^{3p}]^{\frac{1}{3}} [\mathbb{E}^{\mathbb{P}^\delta} |\rho_\delta|_{L_t^\infty(L_x^\gamma)}^{\frac{3p\gamma}{d}}]^{\frac{1}{3}} [\mathbb{E}^{\mathbb{P}^\delta} |u_\delta|_{L_t^2(H_x^1)}^{3p}]^{\frac{1}{3}} |\varphi_\nu^\delta|_{L^\infty([0,T] \times D \times \Omega_\delta)}^p \end{aligned}$$

Combining classical results on the Neumann problem with the uniform bounds in Lemma 7.1, each of these terms $I_1^p - I_5^p$ is controlled uniformly in $\delta > 0$. Finally, use the renormalized form of the continuity equation as in [7] to estimate I_6^p . $\square$

In addition, the following integrability gains can be proved for the $\{\rho_\delta\}_{\delta>0}$. The approach is entirely analogous to [14], so we omit the proof.

PROPOSITION 7.4. *For all $p \geq 1$ and $\kappa < \min(\frac{2\gamma}{d} - 1, \frac{\gamma}{2})$, there exists a constant $C_{p,\kappa}$ such that*

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}^\delta} \left[\left| \int_0^T \int_D \rho_\delta^\kappa P(\rho_\delta, \theta_\delta) dx dt \right|^p \right] \leq C_{p,\kappa}. \quad (7.13)$$

7.2. $\delta \rightarrow 0$ compactness step

In preparation for the compactness analysis of the temperature equation, we define a sequence of temperature renormalizations as in [7].

$$\mathcal{K}_m(\theta) = \int_0^\theta k(z)h_m(z) \, dz, \quad h_m(z) = \frac{1}{(1+z)^{\frac{1}{m}}}.$$

We also introduce a sequence of cutoffs T_k defined by

$$T_k(z) = kT\left(\frac{z}{k}\right), \quad z \in \mathbb{R}, \quad k \in \mathbb{N},$$

with T being a smooth concave function on $\mathbb{R}$ such that

$$T(z) = \begin{cases} z & \text{for } z \leq 1 \\ 2 & \text{for } z \geq 3. \end{cases}$$

Finally, we introduce a sequence of $\{L_k\}_{k=1}^\infty$ by setting. Introduce the function L_k as

$$L_k(z) = \begin{cases} z \ln z, & \text{for } 0 \leq z < k, \\ z \ln k + z \int_k^z \frac{T_k(s)}{s^2} \, ds, & \text{for } z \geq k, \end{cases}$$

The main compactness result for the $\delta \rightarrow 0$ step is the following:

PROPOSITION 7.5. *There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a collection of independent Brownian motions $\{\beta_k\}_{k=1}^\infty$, a sequence of measurable maps $\{T_\delta\}_{\delta>0}$, and limit points*

$$\begin{aligned} & \left(\rho, u, \bar{\theta}, \sqrt{\rho}u, \overline{p_m(\rho)}, \overline{\theta p_\theta(\rho)} \right) \\ & \left(\{\overline{L_k(\rho)}\}_{k=1}^\infty, \{\overline{T_k(\rho) \operatorname{div} u}\}_{k=1}^\infty, \{\overline{(\rho T'_k(\rho) - T_k(\rho)) \operatorname{div} u}\}_{k=1}^\infty, \right. \\ & \left. \{\overline{T_k(\rho) \theta p_\theta(\rho)}\}_{k=1}^\infty, \{\overline{\mathcal{K}_m(\theta)}\}_{m=1}^\infty \right) \\ & T_\delta : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\Omega_\delta, \mathcal{F}_\delta, \mathbb{P}_\delta), \end{aligned}$$

such that the following hold:

1. The measure $\mathbb{P}_\delta$ satisfies $(T_\delta)_\# \mathbb{P} = \mathbb{P}_\delta$.
2. The new sequence $\{(\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta)\}_{\delta>0}$ defined by $(\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta) = (\rho_\delta, u_\delta, \theta_\delta) \circ T_\delta$ constitutes a δ layer approximation relative to the stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\tilde{\mathcal{F}}_\delta^t)_{t=0}^T, \tilde{W}_\delta)$, where $\tilde{W}_\delta := W_\delta \circ T_\delta$ and $\tilde{\mathcal{F}}_\delta^t = T_\delta^{-1} \circ \mathcal{F}_\delta^t$.
3. The uniform bounds in Lemma 7.1 and Proposition 7.3 hold with $\rho_\delta, u_\delta, \theta_\delta$ replaced by $\tilde{\rho}_\delta, \tilde{u}_\delta, \tilde{\theta}_\delta$ and $\mathbb{P}_\delta$ replaced by $\mathbb{P}$. Moreover, it holds that

$$\sup_{\delta>0} \|(\tilde{\rho}_\delta + \delta)\tilde{\theta}_\delta\|_{L_t^\infty(L_x^1)} < \infty \quad (7.14)$$

pointwise in Ω .

4. The following convergences hold pointwise on Ω :

$$\tilde{\rho}_\delta \rightarrow \rho \text{ in } C_t([L_x^\gamma]_w) \quad (7.15)$$

$$\tilde{u}_\delta \rightarrow u \text{ in } [L_t^2(H_{0,x}^1)]_w \quad (7.16)$$

$$\tilde{\rho}_\delta \tilde{u}_\delta \rightarrow \rho u \text{ in } C_t([L_x^{\frac{2\gamma}{\gamma+1}}]_w) \quad (7.17)$$

$$p_m(\tilde{\rho}_\delta) \rightarrow \overline{p_m(\rho)} \text{ in } [L_{t,x}^{q_1}]_w \quad (7.18)$$

$$\tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta) \rightarrow \overline{\theta p_\theta(\rho)} \text{ in } [L_{t,x}^{q_2}]_w \cap L_t^2(H_x^{-1}) \quad (7.19)$$

$$T_k(\tilde{\theta}_\delta) \tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta) \rightarrow \overline{\theta p_\theta(\rho) T_k(\rho)} \text{ in } [L_{t,x}^{q_2}]_w \cap L_t^2(H_x^{-1}) \quad (7.20)$$

$$\tilde{\theta}_\delta \rightarrow \theta \text{ in } [L_t^2(H_x^1) \cap L_{t,x}^3]_w \quad (7.21)$$

$$\tilde{\rho}_\delta \tilde{\theta}_\delta \rightarrow \rho \theta \text{ in } [L_t^\infty(M_x)]_{w*} \quad (7.22)$$

$$\tilde{W}_\delta \rightarrow W \text{ in } [C_t]^\infty \quad (7.23)$$

$$T_k(\tilde{\rho}_\delta) \rightarrow \overline{T_k(\rho)} \text{ in } [C_t]([L_x^{\frac{4\gamma d}{2\gamma-d}}]_w) \quad (7.24)$$

$$L_k(\tilde{\rho}_\delta) \rightarrow \overline{L_k(\rho)} \text{ in } [C_t]([L_x^q]_w) \quad (7.25)$$

$$(\tilde{\rho}_\delta T_k'(\tilde{\rho}_\delta) - T_k(\tilde{\rho}_\delta)) \operatorname{div} \tilde{u}_\delta \rightarrow \overline{(\rho T_k'(\rho) - T_k(\rho)) \operatorname{div} u} \text{ in } L_t^\infty(L_x^2) \quad (7.26)$$

$$\mathcal{K}_m(\tilde{\theta}_\delta) \rightarrow \overline{\mathcal{K}_m(\theta)} \text{ in } [L_{t,x}^1]_w \quad (7.27)$$

5. The following additional convergences hold

$$\sqrt{\tilde{\rho}_\delta} \tilde{u}_\delta \rightarrow \sqrt{\overline{\rho u}} \text{ in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^2)) \quad (7.28)$$

$$\tilde{\rho}_\delta \rightarrow \rho \text{ in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^\gamma)) \cap L_w^p(\Omega_\delta; L_t^2(W_x^{1,2})) \quad (7.29)$$

$$T_k(\tilde{\rho}_\delta) \operatorname{div} \tilde{u}_\delta \rightarrow \overline{T_k(\rho) \operatorname{div} u} \text{ in } L_w^p(\Omega_\delta; L_t^2(L_x^{\frac{2\gamma}{\gamma+2}})) \quad (7.30)$$

$$\tilde{\rho}_\delta \log \tilde{\rho}_\delta \rightarrow \overline{\rho \log \rho} \text{ in } L_{w*}^p(\Omega_\delta; L_t^\infty(L_x^2)) \quad (7.31)$$

The proof follows a similar line of argument as in [14] and the previous layers of this article. The only key difference is to obtain a tightness result for the renormalizations of the temperature equation. Toward this end, we define for each $\delta > 0$ the $[L_{t,x}^1]^\infty$ -valued random variable $Z_\delta = \{\mathcal{K}_m(\theta_\delta)\}_{m=1}^\infty$. Using the $L^p(\Omega; L_{t,x}^3)$ estimates from Proposition 7.3, we will establish the following:

LEMMA 7.6. *The sequence of induced measures $\{\mathbb{P}_\delta \circ Z_\delta^{-1}\}_{\delta>0}$ are tight on $[(L_{t,x}^1)_w]^\infty$.*

Proof. By Proposition 7.3, there exists a constant C such that

$$\sup_{\delta>0} \mathbb{E}^{\mathbb{P}_\delta} |\theta_\delta|_{L_{t,x}^3}^3 \leq C. \quad (7.32)$$

Moreover, by Hypothesis 1.2, there exists another constant D such that for all $m \geq 1$ and $\theta \geq 0$,

$$\mathcal{K}_m(\theta) \leq D\theta^{3-\frac{1}{m}}. \quad (7.33)$$

Fix an $\epsilon > 0$. For each $m \geq 1$ define the set $E_m^\epsilon \subset L_{t,x}^1$ by

$$E_m^\epsilon = \{f \in L_{t,x}^1 \mid |f|_{L_{t,x}^{\frac{3m}{3m-1}}} \leq \epsilon^{-1} 2^m\}.$$

Since every sequence in E_m^ϵ is uniformly bounded and uniformly integrable in $L_{t,x}^1$, we may conclude that E_m^ϵ is a compact set in $(L_{t,x}^1)_w$. In addition, define

$$E^\epsilon = \prod_{m=1}^{\infty} E_m^\epsilon.$$

By Tychonoff's theorem, E^ϵ is a compact set in $[(L_{t,x}^1)_w]^\infty$. Applying Chebyshev, we find that

$$\mathbb{P}_\delta(Z_\delta \notin E_\epsilon) \leq \sum_{m=1}^{\infty} \mathbb{P}_\delta(\mathcal{K}_m(\theta_\delta) \notin E_m^\epsilon) \leq \epsilon \sum_{m=1}^{\infty} 2^{-m} \mathbb{E}^{\mathbb{P}_\delta} |\mathcal{K}_m(\theta_\delta)|_{L_{t,x}^{\frac{3m}{3m-1}}}.$$

Hence, applying inequalities (7.32) and (7.33) yields

$$\mathbb{P}_\delta(Z_\delta \notin E_\epsilon) \leq D\epsilon \sum_{m=1}^{\infty} 2^{-m} \mathbb{E}^{\mathbb{P}_\delta} |\theta_\delta|_{L_{t,x}^3}^{3-\frac{1}{m}} \leq CD\epsilon.$$

Since $\epsilon > 0$ was arbitrary and C, D were fixed in advance, this completes the proof. $\square$

7.3. $\delta \rightarrow 0$ preliminary limit passage

Next we define a filtration $(\mathcal{F}_t)_{t=0}^T$ via $\mathcal{F}_t = \sigma(r_t X)$ where $X = (\rho, \rho u, W, u, \rho \bar{\theta})$ and $r_t : E_T \rightarrow E_t$

$$\begin{aligned} E_s &= C([0, s]; [L^\gamma]_w) \times C([0, s]; [L^{\frac{2\gamma}{\gamma+1}}]_w \times \mathbb{R}^\infty) \\ &\quad \times L^2([0, s]; H_0^1) \times L^2([0, s]; L^q(D)), \end{aligned}$$

where $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$.

LEMMA 7.7. *The pair (ρ, u) satisfies continuity equation 1.7 of Definition 1.7.*

Proof. In view of the strong convergence of the initial density (the initial data are assumed to be deterministic) this immediately follows from the pointwise convergences in Proposition 7.5. $\square$

At this point, we cannot make the same preliminary passage to the limit in the momentum equation as in the $\epsilon \rightarrow 0$ step. Recall that this preliminary passage to the limit was crucial for establishing the averaged weak continuity of the effective viscous pressure. Instead, one proves a partial result in this direction (Lemma 7.8), which essentially amounts to a preliminary passage to the limit in each of the terms besides the stochastic integrals. Indeed, as $\delta \rightarrow 0$, passing to the limit in the stochastic

integrals is even more difficult than passing to the limit in the pressure. This lemma turns out to be enough to prove the strong convergence of the density and temperature. We are then able to return to the task of passing to the limit in the stochastic integrals at the very end of the proof.

LEMMA 7.8. *For all $\phi \in C_c^\infty(D)$, the process $(M_t(\phi))_{t=0}^T$ defined by*

$$M_t(\phi) = \int_D \rho u(t) \cdot \phi dx - \int_D m_0 \cdot \phi - \int_0^t \int_D [\rho u \otimes u - \mathcal{S}(u)] : \nabla \phi \\ + [\overline{p_m(\rho)} + \overline{\theta p_m(\rho)}] \operatorname{div} \phi dx ds.$$

is a continuous, $(\mathcal{F}^t)_{t=0}^T$ martingale satisfying for all $p \geq 1$

$$\mathbb{E} \left[\sup_{t \in [0, T]} |M_t(\phi)|^p \right] < \infty. \quad (7.34)$$

Proof. The proof uses our compactness result Proposition 7.5 following the arguments in [14]. $\square$

LEMMA 7.9. *Let $K \subset\subset D$ be arbitrary, then the weak continuity of the effective viscous pressure holds on average, that is:*

$$\lim_{\delta \rightarrow 0} \mathbb{E}^\mathbb{P} \left[\int_0^T \int_K ((2\mu + \lambda) \operatorname{div} \tilde{u}_\delta - p_m(\tilde{\rho}_\delta) - \tilde{\theta}_\delta p_\theta(\tilde{\rho}_\delta)) T_k(\tilde{\rho}_\delta) dx dt \right] \\ = \mathbb{E}^\mathbb{P} \left[\int_0^T \int_K ((2\mu + \lambda) \operatorname{div} u - \overline{p_m(\rho)} - \overline{\theta p_\theta(\rho)}) \overline{T_k(\rho)} dx dt \right].$$

Proof. The proof is established by proving an averaged Ito product rule based on the two random test functions. For more details, see the argument in [14]. The key trick is that:

$$\mathbb{E}^\mathbb{P} \int_0^T \int_D \eta \tilde{u}_\delta \{T_k(\tilde{\rho}_\delta) \mathcal{R}_{i,j}[\tilde{\rho}_\delta \tilde{u}_\delta] - \tilde{\rho}_\delta \tilde{u}_\delta \mathcal{R}_{i,j}[T_k(\tilde{\rho}_\delta)]\} dx dt \\ \rightarrow \mathbb{E}^\mathbb{P} \int_0^T \int_D \eta u \{\overline{T_k(\rho)} \mathcal{R}_{i,j}[\rho u] - \rho u \mathcal{R}_{i,j}[\overline{T_k(\rho)}]\} dx dt.$$

The other key point is that

$$\delta \tilde{\rho}_\delta^\beta \rightarrow 0 \quad \text{in } L^1((0, T) \times D \times \Omega),$$

as a consequence of the integrability gains on the density. $\square$

7.4. Strong convergence of the density

Since γ may be close to $\frac{d}{2}$, the limiting density ρ may not belong to $L_t^\infty(L_x^2)$ for all $\omega \in \Omega$. Hence, it is not a priori clear that the continuity equation for ρ may be

renormalized (which is crucial for the proof of strong convergence). In [7], this issue is addressed via a thorough analysis of the so-called oscillations defect measure:

$$\operatorname{osc}_p[\tilde{\rho}_\delta \rightarrow \rho](\mathcal{O}) = \sup_{k \geq 1} \left(\limsup_{\delta \rightarrow 0} \iint_{\mathcal{O}} |T_k(\tilde{\rho}_\delta) - T_k(\rho)|^p dx ds \right). \quad (7.35)$$

In our framework, the quantity above is random. If we could show that for some $p > 2$ and all $\mathcal{O} \subset [0, T] \times D$, $\operatorname{osc}_p[\rho_\delta \rightarrow \rho](\mathcal{O}) < \infty$ $\mathbb{P}$ almost surely, we could appeal directly to the results in [7] and deduce that ρ is a renormalized solution of the transport equation $\mathbb{P}$ almost surely. This would be the case, for instance, if we could show that

$$\mathbb{E} \mathbb{P} [\operatorname{osc}_p[\rho_\delta \rightarrow \rho](\mathcal{O})] < \infty.$$

However, it seems that this would require proving the weak continuity of the effective viscous pressure, Lemma 7.9, in the $\mathbb{P}$ almost sure sense, rather than in expectation only. It is our point of view that this is most likely not even true based on the information at this stage in the proof. The issue is the contribution of the stochastic integrals, which seems to lead only to a convergence in probability law, not $\mathbb{P}$ almost sure convergence. Effectively, the way to cure this problem is to reverse the order of operations. Namely, in the notion of oscillations defect measure, one should take an expectation prior to passing the limit supremum in δ . This remark is made precise through the following lemma. The proof follows closely the method in [7].

LEMMA 7.10. *Let $K \subset\subset D$ be given. There exists a positive constant C_K such that*

$$\sup_{k \geq 1} \limsup_{\delta \rightarrow 0} \mathbb{E} \mathbb{P} \int_0^T \int_K |T_k(\tilde{\rho}_\delta) - T_k(\rho)|^{\gamma+1} dx dt \leq C_K. \quad (7.36)$$

Proof. In view of our convexity and growth Hypothesis 1.1 on P_m , it follows that for all $y, z \geq 0$,

$$|T_k(z) - T_k(y)|^{\gamma+1} \lesssim (p_m(z) - p_m(y))[T_k(z) - T_k(y)].$$

Combining this with observation with Lemma 7.9 and the monotonicity of p_θ yields:

$$\begin{aligned} & \limsup_{\delta \rightarrow 0+} \mathbb{E} \mathbb{P} \int_0^T \int_K |T_k(\tilde{\rho}_\delta) - T_k(\rho)|^{\gamma+1} dx dt \\ & \leq \lim_{\delta \rightarrow 0+} \mathbb{E} \mathbb{P} \int_0^T \int_K (p_m(\tilde{\rho}_\delta) - p_m(\rho))[T_k(\tilde{\rho}_\delta) - T_k(\rho)] dx dt \\ & \leq \lim_{\delta \rightarrow 0+} \mathbb{E} \mathbb{P} \int_0^T \int_K (p_m(\tilde{\rho}_\delta) - p_m(\rho))[T_k(\tilde{\rho}_\delta) - T_k(\rho)] dx dt \\ & \quad + \mathbb{E} \mathbb{P} \int_0^T \int_K (\overline{p_m(\rho)} - p_m(\rho))[T_k(\tilde{\rho}_\delta) - T_k(\rho)] dx dt \\ & = \lim_{\delta \rightarrow 0+} \mathbb{E}^p \int_0^T \int_K p_m(\tilde{\rho}_\delta) T_k(\tilde{\rho}_\delta) - \overline{p_m(\rho) T_k(\rho)} dx dt \end{aligned}$$

$$\leq \left| \limsup_{\delta \rightarrow 0} \mathbb{E}^{\mathbb{P}} \int_K \left[\operatorname{div} \tilde{u}_\delta T_k(\tilde{\rho}_\delta) - \operatorname{div} \tilde{u}_\delta \overline{T_k(\rho)} \right] dx dt \right|,$$

Finally, using the uniform estimates for $\{\tilde{u}_\delta\}_{\delta>0}$ in $L^2(\Omega; L_t^2(H_x^1))$ guaranteed by Proposition 7.5, it follows that:

$$\begin{aligned} & \limsup_{\delta \rightarrow 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K \left[\operatorname{div} u_\delta T_k(\rho_\delta) - \operatorname{div} u_\delta \overline{T_k(\rho)} \right] dx dt \\ &= \limsup_{\delta \rightarrow 0+} \mathbb{E}^{\mathbb{P}} \int_0^T \int_K \left[T_k(\tilde{\rho}_\delta) - T_k(\rho) + T_k(\rho) - \overline{T_k(\rho)} \right] \operatorname{div} \tilde{u}_\delta dx dt \\ &\leq 2 \limsup_{\delta \rightarrow 0} \mathbb{E}^{\mathbb{P}} \left(\|\operatorname{div} \tilde{u}_\delta\|_{L^2((0,T) \times D)} \|T_k(\tilde{\rho}_\delta) - T_k(\rho)\|_{L^2((0,T) \times K)} \right) \\ &\leq \tilde{\epsilon} \limsup_{\delta \rightarrow 0} \mathbb{E}^{\mathbb{P}} \left(\|T_k(\tilde{\rho}_\delta) - T_k(\rho)\|_{L^{\gamma+1}((0,T) \times K)}^{\gamma+1} \right) + C\tilde{\epsilon}, \end{aligned}$$

where $\tilde{\epsilon}$ may be chosen arbitrarily small. Combining these inequalities and bringing the last term back to the LHS of the estimate gives the claim. $\square$

Using Lemma 7.10, we can now check that ρ is a renormalized solution of the continuity equation driven by u , $\mathbb{P}$ almost surely.

LEMMA 7.11. *Extend ρ, u by zero outside D . Let $\beta \in C^1(\mathbb{R}_+)$ and suppose that β' is compactly supported. Then we have the $\mathbb{P}$ almost sure identity:*

$$\partial_t \beta(\rho) + \operatorname{div}(\beta(\rho)u) + [\beta'(\rho)\rho - \beta(\rho)] \operatorname{div} u = 0, \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3). \quad (7.37)$$

Proof. Start by renormalizing the continuity equation for $\tilde{\rho}_\delta$ with T_k to obtain the $\mathbb{P}$ almost sure identity:

$$\partial_t T_k(\tilde{\rho}_\delta) + \operatorname{div}(T_k(\tilde{\rho}_\delta)\tilde{u}_\delta) = -[T'_k(\tilde{\rho}_\delta)\tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] \operatorname{div} \tilde{u}_\delta \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3). \quad (7.38)$$

Send $\delta \rightarrow 0$ and apply $\mathbb{P}$ a.s. convergences in Proposition 7.5 to pass the limit in each term above. Noting that the equation satisfied by $\overline{T_k(\rho)}$ can be renormalized (due to the unlimited integrability), we find that $\mathbb{P}$ almost surely,

$$\begin{aligned} & \partial_t \beta(\overline{T_k(\rho)}) + \operatorname{div}(\beta(\overline{T_k(\rho)})u) + \left[\beta'(\overline{T_k(\rho)})(\overline{T_k(\rho)}) - \beta(\overline{T_k(\rho)}) \right] \operatorname{div} u \\ &= -\beta'(\overline{T_k(\rho)})[\overline{T'_k(\rho)\rho} - \overline{T_k(\rho)}] \operatorname{div} u \text{ in } \mathcal{D}'((0, T) \times \mathbb{R}^3). \end{aligned} \quad (7.39)$$

By assumption, we may choose $M > 0$ such that β' is supported on $[0, M]$. Passing $k \rightarrow \infty$ on both sides of the identity above and using the $\mathbb{P}$ almost sure $L^1_{t,x}$ strong convergence of $\overline{T_k(\rho)}$ toward ρ , we see that the proof of the Lemma will be complete as soon as we establish:

$$\lim_{k \rightarrow \infty} \|\overline{[T'_k(\rho)\rho - T_k(\rho)] \operatorname{div} u}\|_{1_{\{\overline{T_k(\rho)} \geq M\}}} = 0, \quad (7.40)$$

where $K \subset\subset D$ is arbitrary and fixed in advance. Let us now prove convergence (7.40). Using a lower-semicontinuity argument together with the uniform bound for $\{\tilde{u}_\delta\}_\delta$ in $L^2(\Omega \times [0, T] \times D)$, we see that it suffices to prove:

$$\lim_{k \rightarrow \infty} \liminf_{\delta \rightarrow 0} \| [T'_k(\tilde{\rho}_\delta) \tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^2(\Omega \times [0, T] \times K)} = 0. \quad (7.41)$$

Observe that as $k \rightarrow \infty$, we have $T'_k(\tilde{\rho}_\delta) \tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta) \rightarrow 0$ in $L^1(\Omega \times [0, T] \times D)$, uniformly in δ . Interpolating, we see that it is enough to have the following uniform bound:

$$\sup_{k \geq 1} \liminf_{\delta \rightarrow 0} \| [T'_k(\tilde{\rho}_\delta) \tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^{\gamma+1}(\Omega \times [0, T] \times K)} < \infty. \quad (7.42)$$

Using the fact that $T'_k(z)z \leq T_k(z)$, adding and subtracting $T_k(\rho)$, then appealing to lower-semicontinuity arguments once more yield:

$$\begin{aligned} & \sup_{k \geq 1} \liminf_{\delta \rightarrow 0} \| [T'_k(\tilde{\rho}_\delta) \tilde{\rho}_\delta - T_k(\tilde{\rho}_\delta)] 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^{\gamma+1}(\Omega \times [0, T] \times K)} \\ & \lesssim \sup_{k \geq 1} \liminf_{\delta \rightarrow 0} \| [T_k(\tilde{\rho}_\delta)] 1_{\{\overline{T_k(\rho)} \geq M\}} \|_{L^{\gamma+1}(\Omega \times [0, T] \times K)} \\ & \lesssim 1 + \sup_{k \geq 1} \liminf_{\delta \rightarrow 0} \| [T_k(\tilde{\rho}_\delta) - T_k(\rho)] \|_{L^{\gamma+1}(\Omega \times [0, T] \times K)}. \end{aligned}$$

The proof is now complete in view of Lemma 7.10. $\square$

Using Lemma 7.10 together with our renormalization Lemma 7.11, we may now obtain the following strong convergence of the density.

LEMMA 7.12. *The sequence of densities $\{\rho_\delta\}_{\delta>0}$ converge strongly to ρ in the sense that for all $p \geq 1$ and $r < \gamma + \kappa$*

$$\lim_{\delta \rightarrow 0} |\rho_\delta - \rho|_{L^p(\tilde{\Omega}_\delta; L^r_{t,x})} = 0. \quad (7.43)$$

Proof. Since ρ is a renormalized solution, the proof now follows along the lines of the arguments in [7, 14]. The role of the higher integrability ρ in the case $\gamma > 2$ is now replaced by the use of Lemma 7.10. The treatment of the temperature part of the pressure does not differ substantially from the arguments given at the $\epsilon \rightarrow 0$ construction of the δ layer. $\square$

7.5. Strong convergence of the temperature

With the strong convergence of the density at hand, we may now deduce the following strong convergence of the temperature. Recall that $\bar{\theta}$ is the limit extracted in Proposition 7.5.

LEMMA 7.13. *For all $q < 3$, $\omega \in \Omega$, we have $\tilde{\theta}_\delta(\omega) 1_{\{\rho(\omega)>0\}} \rightarrow \bar{\theta}(\omega) 1_{\{\rho(\omega)>0\}}$ in $L^q_{t,x}$.*

Proof. Following [7], it suffices to verify that $\sqrt{\rho}(\omega)\tilde{\theta}_\delta(\omega) \rightarrow \sqrt{\rho}(\omega)\theta(\omega)$ in $L^2_{t,x}$, then use the $L^3_{t,x}$ weak convergence guaranteed by Proposition 7.5.

Following the same strategy as in Lemma 6.9, one verifies that $(\tilde{\rho}_\delta(\omega) + \delta)\tilde{\theta}_\delta^2(\omega) \rightarrow \rho(\omega)\theta^2(\omega)$ in $\mathcal{D}'_{t,x}$. Using the uniform estimates, the convergence actually holds in $L^r_{t,x}$ for some $r > 1$. Finally, use the strong convergence, uniform bounds, and the decomposition $\rho\tilde{\theta}_\delta^2 = (\rho - \tilde{\rho}_\delta)\tilde{\theta}_\delta^2 + \rho\tilde{\theta}_\delta^2$ to deduce $\rho(\omega)\tilde{\theta}_\delta(\omega)^2 \rightarrow \rho(\omega)\theta(\omega)^2$ in $L^1_{t,x}$. This completes the proof. $\square$

7.6. Defining the limiting temperature: renormalized limits

By Prop 7.5, we know that for all $m \geq 1$ and $\omega \in \Omega$, $\mathcal{K}_m(\tilde{\theta}_\delta)(\omega) \rightarrow \overline{\mathcal{K}_m(\tilde{\theta})}(\omega)$ weakly in $L^1_{t,x}$. Since weak convergence is order preserving, in view of the definition of $\mathcal{K}_m$, we have that for each $\omega \in \Omega$, the sequence $\{\overline{\mathcal{K}_m(\tilde{\theta})}(\omega)\}_{m=1}^\infty$ is monotone. Hence, by the monotone convergence theorem, there exists $\overline{\mathcal{K}}(\omega)$ such that for all $\omega \in \Omega$, we have the convergence $\overline{\mathcal{K}_m}(\omega) \rightarrow \overline{\mathcal{K}}(\omega)$ strongly in $L^1_{t,x}$.

We are now in a position to define the limiting temperature θ via the relation $\theta(\omega) = \mathcal{K}^{-1} \circ \overline{\mathcal{K}}(\omega)$. Let us observe that, away from the vacuum, this definition is consistent with our initial limit $\bar{\theta}$. Indeed, Lemma 7.13 implies that $1_{\rho>0}\mathcal{K}_m(\tilde{\theta}_\delta)$ converges to $1_{\rho>0}\mathcal{K}_m(\bar{\theta})$ strongly in $L^1_{t,x}$, pointwise in Ω . Therefore, $\mathcal{K}_m(\bar{\theta})1_{\rho>0} = 1_{\rho>0}\overline{\mathcal{K}_m(\bar{\theta})}$ and passing $m \rightarrow \infty$ gives $\mathcal{K}(\bar{\theta})1_{\rho>0} = \overline{\mathcal{K}}1_{\rho>0}$. Since $\mathcal{K}$ is invertible, it follows that $\bar{\theta}1_{\rho>0} = \theta1_{\rho>0}$. In particular, it follows that the definition of $(\mathcal{F}_t)_{t=0}^T$ is unchanged after replacing the role of $\bar{\theta}$ by θ , so this will still define our limiting filtration.

To summarize, we have the following Corollary of Lemma 7.13.

COROLLARY 7.14. *For all $q < 3$, $\omega \in \Omega$, we have the following strong convergence away from vacuum:*

$$\tilde{\theta}_\delta(\omega)1_{\{\rho(\omega)>0\}} \rightarrow \theta(\omega)1_{\{\rho(\omega)>0\}} \quad \text{in } L^q_{t,x}.$$

7.7. Conclusion of the proof

Proof of Theorem 1.8.

LEMMA 7.15. *The quantities ρ, θ, u satisfy the temperature inequality.*

Proof. Use the δ layer renormalized temperature inequality satisfied by $\tilde{\rho}_\delta, \tilde{\theta}_\delta, \tilde{u}_\delta$ with the renormalization H_m defined above. Sending $m \rightarrow \infty$ and using Corollary 7.14 together with the arguments from Lemma 6.9 yields:

$$\begin{aligned} \int_0^T \int_\Omega \rho \theta \partial_t \varphi + \rho \theta \mathbf{u} \cdot \nabla \varphi + \overline{\mathcal{K}(\theta)} \Delta \varphi \, dx \, dt &\leq - \int_0^T \int_\Omega \mathbb{S} : \nabla \mathbf{u} \varphi \, dx \, dt \\ &+ \int_0^T \int_\Omega \varphi (\theta p_\theta(\rho) \operatorname{div} \mathbf{u}) \, dx \, dt - \int_\Omega \rho_0 \theta_0 \varphi(0) \, dx, \end{aligned} \quad (7.44)$$

By the definition of θ , it follows that $\mathcal{K}(\theta)(\omega) = \overline{\mathcal{K}}(\omega)$, pointwise in Ω . This completes the lemma. $\square$

LEMMA 7.16. *The pair (ρ, u) satisfies momentum equation 1.8 of Definition 1.7.*

Proof. For each $\phi \in C_c^\infty(D)$ we introduce the continuous $(\mathcal{F}^t)_{t=0}^T$ adapted process $(M_t(\phi))_{t=0}^T$ defined by

$$\begin{aligned} M_t(\phi) &= \int_D \rho u(t) \cdot \phi dx - \int_D m_0 \cdot \phi dx \\ &\quad - \int_0^t \int_D [\rho u \otimes u - \mathcal{S}(u)] : \nabla \phi + [p_m(\rho) + \theta p_\theta(\rho)] \operatorname{div} \phi dx ds. \end{aligned}$$

Recall that since $\bar{\theta}$ and θ agree away from vacuum, $(\mathcal{F}^t)_{t=0}^T$ remains the same after replacing $\bar{\theta}$ by θ . Hence, we may combine our preliminary martingale Lemma 7.8 with strong convergence upgrades Corollary 7.14 and Lemma 7.12 to conclude that $(M_t)_{t=0}^T$ is a martingale. Moreover, our strong convergence results yield: $\tilde{\rho}_\delta \tilde{\theta}_\delta * \eta_\delta \rightarrow \rho \theta$ strongly in $L^p(\Omega; L_t^2(L_x^q))$ for all $p \in [1, \infty)$ and $\frac{1}{q} = \frac{1}{\gamma} + \frac{1}{2} - \frac{1}{d}$. Using Lipschitz continuity Hypothesis 1.4 together with the arguments from [14], we may identify

$$M_t(\phi) = \sum_{k=1}^{\infty} \int_0^t \int_D \rho \sigma_k(\rho, \rho u, \rho \theta, x) dx d\beta_k(s).$$

This yields the momentum equation. □

The proof of our main result is now complete. □

Appendix A

A.1. Random variables on topological spaces and the Skorohod theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(E, \tau, \mathcal{B}_\tau)$ be a topological space endowed with its Borel sigma algebra. A mapping $X : \Omega \rightarrow (E, \tau)$ is called an “ E -valued random variable” provided it is a measurable mapping between these spaces. Every E -valued random variable induces a probability measure on $(E, \tau, \mathcal{B}_\tau)$ by pushforward, which we denote $\mathbb{P} \circ X^{-1}$. A sequence of probability measures $\{\mathbb{P}_n\}_{n=1}^\infty$ on $\mathcal{B}_\tau$ is said to be “tight” provided that for each $\xi > 0$ there exists a τ compact set K_ξ such that $\mathbb{P}_n(K_\xi) \geq 1 - \xi$ for all $n \geq 1$.

A collection $\{X_t\}_{t=0}^T$ is an E -valued stochastic process provided that for each t , X_t is an E -valued random variable. An E -valued stochastic process is progressively measurable with respect to the filtration $\{\mathcal{F}^t\}_{t=0}^T$ provided that for each $t \leq T$,

$$X|_{[0,t]} : \Omega \times [0, t] \rightarrow (E, \tau, \mathcal{B}_\tau)$$

is measurable with respect to the product sigma algebra $\mathcal{F}_t \times \mathcal{B}([0, t])$.

DEFINITION 7.17. A topological space (E, τ) is called a Jakubowski space provided there exists a countable sequence $\{G_k\}_{k=1}^\infty : E \rightarrow \mathbb{R}$ of τ continuous functionals which separate points in E .

Our main interest in such spaces is the following fundamental result:

THEOREM 7.18. *Let (E, τ) be a Jakubowski space. Suppose that $\{\hat{X}_k\}_{k \geq 1}$ is a sequence of E -valued random variables on a sequence of probability spaces $\{(\hat{\Omega}_k, \hat{\mathcal{F}}_k, \hat{\mathbb{P}}_k)\}_{k \geq 1}$ such that $\{\hat{\mathbb{P}}_k \circ \hat{X}_k^{-1}\}_{k=1}^{\infty}$ is tight.*

Then there exists a new probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with an E -valued random variable X and a sequence of “recovery” maps $\{\hat{T}_k\}_{k=1}^{\infty}$

$$\hat{T}_k : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow (\hat{\Omega}_k, \hat{\mathcal{F}}_k, \hat{\mathbb{P}}_k)$$

with the following two properties:

1. *For each k , the measure $\hat{\mathbb{P}}_k$ may be recovered from $\mathbb{P}$ by pushing forward $\hat{T}_k$.*
2. *The new sequence $\{X_k\}_{k \geq 1} := \{\hat{X}_k \circ \hat{T}_k\}_{k \geq 1}$ converges pointwise to X (with respect to the topology τ).*

Proof. This result is a combination of the versions of the Skorohod theorem proved in [9, 16]. It can be proved by modifying the proof in [9] in a very slight way. Namely, at the point in the proof where the classical Skorohod theorem for metric spaces is applied, one may apply the Skorohod theorem in [16] to obtain the recovery maps. $\square$

LEMMA 7.19. *The space $L^1(D)$ is a Jakubowski space.*

Proof. It suffices to show that there is a constable family $\{e_i\}$ of functions in $L^\infty(D)$ such that for any $f \in L^1(\mathbb{R}^d)$, the linear-functional

$$L_i(f) := \int_D e_i f$$

vanishes for all i if and only if $f = 0$. Clearly if $f = 0$, $L_i(f) = 0$ regardless of the choice of $\{e_i\}$. In view of the Sobolev embedding $W^{1,d+1}(D) \hookrightarrow C_0(D) \subseteq L^\infty(D)$ and the separability of $W^{1,d+1}(D)$, we choose $\{e_i\}$ any countable dense subset of $W^{1,d+1}(D)$. Suppose that $L_i(f) = 0$ for all i . Clearly the map $e_i \rightarrow L_i(f)$ is continuous in $W^{1,d+1}(D)$ since

$$|L_i(f)| \leq \|f\|_{L^1} \|e_i\|_{L^\infty} \leq C \|f\|_{L^1} \|e_i\|_{W^{1,d+1}}.$$

Therefore by the density of $\{e_i\}$ in $W^{1,d+1}(\mathbb{R}^d)$ we conclude that

$$\int_{\mathbb{R}^d} \varphi f = 0 \quad \forall \varphi \in W^{1,d+1}(D).$$

It follows that $f = 0$. Note that we can always normalize define $K_i = L_i / \|L_i\|$ so that K_i takes values in $[-1, 1]$. $\square$

A.2. Series of one-dimensional stochastic integrals

By a stochastic basis, we mean a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ together with a filtration $\{\mathcal{F}^t\}_{t=0}^T$ and a collection $\{\beta_k\}_{k=1}^\infty$ of $\{\mathcal{F}^t\}_{t=0}^T$ one-dimensional Brownian motions.

PROPOSITION 7.20. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}^t\}_{t=0}^T, \{\beta_k\}_{k=1}^\infty)$ be a stochastic basis endowed with a collection of $\{\mathcal{F}^t\}_{t=0}^T$ progressively measurable processes $\{f_k\}_{k=1}^\infty : \Omega \times [0, T] \rightarrow \mathbb{R}$, such that*

$$\sum_{k=1}^{\infty} \int_0^T \mathbb{E}^{\mathbb{P}} \left[f_k^2(s) \right] ds < \infty.$$

Then we may construct an $\{\mathcal{F}^t\}_{t=0}^T$ martingale $\{M_t\}_{t=0}^T$ with $\mathbb{P}$ a.s. continuous paths of the form

$$M_t = \sum_{k=1}^{\infty} \int_0^t f_k(s) d\beta_k(s).$$

The series above converges uniformly in time in probability, and the quadratic variation process is given by

$$\langle M \rangle_2^t(\omega) = \sum_{k=1}^{\infty} \int_0^t f_k^2(s, \omega) ds.$$

Proof. This is a consequence of the Kolmogorov three series theorem and the construction of the one-dimensional stochastic integral. See Krylov [10] for more discussion. $\square$

The next lemma, taken from [4], provides a procedure for identifying a continuous, adapted process as a series of one-dimensional stochastic integrals.

LEMMA 7.21. *Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}^t\}_{t=0}^T, \{\beta_k\}_{k=1}^\infty)$ be a stochastic basis endowed with a continuous $\{\mathcal{F}^t\}_{t=0}^T$ martingale $\{M_t\}_{t=0}^T$. Moreover, suppose the following are also $\{\mathcal{F}^t\}_{t=0}^T$ martingales*

1. $(\omega, t) \rightarrow M_t^2(\omega) - \sum_{k=1}^{\infty} \int_0^t f_k^2(\omega, s) ds$
2. $(\omega, t) \rightarrow M_t(\omega) \beta_t^k(\omega) - \int_0^t f_k(s) ds$ (for each $k \geq 1$)

then the process $\{M_t\}_{t=0}^T$ may be identified as

$$M_t = \sum_{k=1}^{\infty} \int_0^t f_k(s) d\beta_k(s).$$

Appendix B

B.1. Weak convergence upgrades

LEMMA 7.22. *Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^\infty$ in $L^p(\Omega, \mathcal{F}, \mu)$ converge weakly to $f \in L^p(\Omega, \mathcal{F}, \mu)$. Moreover, assume there is a convex function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\{\varphi(f_n)\}_{n=1}^\infty$ converges weakly to $\varphi(f)$ in $L^1(\Omega, \mathcal{F}, \mu)$. Denote by $\mathcal{C}$ the subset of $\mathbb{R}$ where φ is strictly convex.*

Then there is a full μ measure set Ω' such that $\{f_n(\omega)\}_{n=1}^\infty$ converges pointwise to $f(\omega)$ for all $\omega \in \mathcal{C} \cap \Omega'$.

The following proposition is proved in [7].

PROPOSITION 7.23. *Let $\{f_n\}_{n=1}^\infty$ be a sequence converging to f in $D'_{t,x}$. Suppose there exists $\{g_n\}_{n=1}^\infty$ such that $\partial_t f_n \geq g_n$ in $D'_{t,x}$ for all $n \geq 1$.*

If $f_n \in L^2_t(L^q_x) \cap L^\infty_t(L^1_x)$ with $q < \frac{2d}{d+2}$ and $g_n \in L^1_t(W_x^{-m,r})$ with $m, r > 1$, uniformly in n , then $f_n \rightarrow f$ in $L^2_t(H_x^{-1})$.

B.2. Some tools from the deterministic compressible theory

The following result is a consequence of the Div Curl lemma. Denote $\mathcal{R}_{ij} = \partial_{ij} \Delta^{-1}$, understood to be well defined on compactly supported distributions.

LEMMA 7.24. *Let D be a smooth, bounded domain and η a smooth cutoff. Let B be a Banach space. Suppose $\{f_n\}_{n=1}^\infty$ converges to f in $C_t([L^p_x]_w)$ and $\{g_n\}_{n=1}^\infty$ converges to g in $C_t([L^q_x]_w)$. Also, assume the embedding $L^r_x \hookrightarrow B$ is compact, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} < 1$.*

Then the following convergence holds:

$$\eta(f_n \mathcal{R}_{ij}[\eta g_n] - g_n \mathcal{R}_{i,j}[\eta f_n]) \rightarrow \eta(f \mathcal{R}_{ij}[\eta g] - g \mathcal{R}_{i,j}[\eta f])$$

weakly in $L^m_t(B)$ for all $1 \leq m < \infty$.

Proof. Combine the corresponding result in Feireisl [6] (using the compact injection operator from L^r_x to B). $\square$

B.3. A weighted Poincaré inequality

The following variant of the Poincaré inequality can be established with the Rellich lemma via the classical argument by contradiction (for instance).

LEMMA 7.25. *For all $\beta \geq 1$ there exists a positive constant C_β such that for all nonnegative $f : D \rightarrow \mathbb{R}$, the following inequality holds:*

$$|f|_{L^2(D)} \leq C \left[|\nabla f|_{L^2(D)} + |f|^{\frac{1}{\beta}}|_{L^1(D)} \right]. \quad (\text{B.1})$$

We will employ Lemma 7.25 to prove the following:

LEMMA 7.26. *For all $M > 0$ and $\beta \geq 1$, there exists a positive constant $C_{M,\beta}$ such that for all nonnegative $f, g : D \rightarrow \mathbb{R}$ with $|g|_{L^1(D)} \geq M$, the following inequality holds:*

$$|f^\beta|_{L^2(D)} \leq C_M \left(|\nabla(f^\beta)|_{L^2(D)} + |g|^{\frac{\gamma\beta}{\gamma-1}}|_{L^\gamma(D)} [|fg|^\beta_{L^1(D)} + |\nabla f|^\beta_{L^2(D)}] \right). \quad (7.2)$$

Proof. We will apply Lemma 7.25 to f^β . Hence, it suffices to estimate $|f|_{L^1(D)}$. Toward this end, introduce a good set G as follows:

$$G = \{x \in D \mid g(x) > \frac{1}{2|D|} \int_D g dx\}.$$

First we check the following lower bound:

$$|G| \geq \left(\frac{|g|_{L_x^1}}{2|g|_{L^\gamma}} \right)^{\frac{\gamma}{\gamma-1}}. \quad (7.3)$$

This follows from the following decomposition

$$\int_D g dx = \int_G g dx + \int_{D \setminus G} g dx \leq |G|^{\frac{\gamma-1}{\gamma}} |g|_{L_x^\gamma} + \frac{1}{2} \frac{|D \setminus G|}{|G|} \int_D g dx. \quad (7.4)$$

Absorbing the last term into the LHS, then raising both sides to the power $\frac{\gamma}{\gamma-1}$ gives the claim. Next we write

$$\begin{aligned} \int_D f dx &= |G|^{-1} (|D| \int_G f dx + [|G| \int_D f dx - |D| \int_G f dx]) \\ &= |G|^{-1} (|D| \int_G f dx + \iint_{G \times D} |f(y) - f(x)| dx dy) \\ &\lesssim |G|^{-1} (2|D| |g|_{L^1}^{-1} \int_G g f dx + |\nabla f|_{L^2}) \end{aligned}$$

Raising to the power β we find:

$$|f|_{L^1(D)}^\beta \leq |G|^{-\beta} 2^{\beta-1} [2^\beta |D|^\beta |g|_{L^1}^{-\beta} |fg|_{L^1(D)}^\beta + |\nabla f|_{L^2}^\beta]. \quad (7.5)$$

Using our lower bound for $|G|$, we find:

$$\begin{aligned} |f|_{L^1(D)}^\beta &\leq \left(\frac{|g|_{L_x^\gamma}}{|g|_{L^1}} \right)^{\frac{\beta\gamma}{\gamma-1}} 2^{\beta-1} [2^\beta |D|^\beta |g|_{L^1}^{-\beta} |fg|_{L^1(D)}^\beta + |\nabla f|_{L^2}^\beta] \\ &\leq C_{\beta,M} |g|_{L^\gamma}^{\frac{\gamma\beta}{\gamma-1}} [|fg|_{L^1(D)}^\beta + |\nabla f|_{L^2(D)}^\beta]. \end{aligned}$$

Combining this estimate with the lemma gives the claim. $\square$

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REFERENCES

- [1] Lubomír Bañas, Zdzisław Brzeźniak, Mikhail Neklyudov, and Andreas Prohl. A convergent finite-element-based discretization of the stochastic landau–lifshitz–gilbert equation. *IMA Journal of Numerical Analysis*, page drt020, 2013.
- [2] Florent Berthelin and Julien Vovelle. Stochastic isentropic euler equations. *arXiv preprint arXiv:1310.8093*, 2013.

- [3] Dominic Breit and Martina Hofmanova. Stochastic navier stokes equations for compressible fluids. *Indiana University of Mathematics Journal*, 65:1183–1250, 2016.
- [4] Zdzisław Brzeźniak and Martin Ondreját. Stochastic geometric wave equations with values in compact Riemannian homogeneous spaces. *Ann. Probab.*, 41(3B):1938–1977, 2013.
- [5] Giuseppe Da Prato and Jerzy Zabczyk. *Stochastic equations in infinite dimensions*. Cambridge university press, 2014.
- [6] Eduard Feireisl. On compactness of solutions to the compressible isentropic navier-stokes equations when the density is not square integrable. *Commentationes Mathematicae Universitatis Carolinae*, 42(1):83–98, 2001.
- [7] Eduard Feireisl. *Dynamics of viscous compressible fluids*, volume 26. Oxford University Press Oxford, 2004.
- [8] Eduard Feireisl, Bohdan Masłowski, and Antonín Novotný. Compressible fluid flows driven by stochastic forcing. *Journal of Differential Equations*, 254(3):1342–1358, 2013.
- [9] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen.*, 42(1):209–216, 1997.
- [10] N. V. Krylov. On the Itô–Wentzell formula for distribution-valued processes and related topics. *Probab. Theory Related Fields*, 150(1-2):295–319, 2011.
- [11] Olga Aleksandrovna Ladyzhenskaja, Vsevolod Alekseevich Solonnikov, and Nina N Ural'tseva. *Linear and quasi-linear equations of parabolic type*, volume 23. American Mathematical Soc., 1988.
- [12] Pierre-Louis Lions. Mathematical topics in fluid mechanics. volume 2: Compressible models. 1998.
- [13] Antoine Mellet and Alexis Vasseur. A bound from below for the temperature in compressible navier–stokes equations. *Monatshefte für Mathematik*, 157(2):143–161, 2009.
- [14] Scott Smith. Random perturbations of viscous compressible fluids: Global existence of weak solutions. *SIAM Journal of Mathematical Analysis*, to appear.
- [15] Hans Triebel. *Theory of function spaces. III*, volume 100 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 2006.
- [16] Aad W. van der Vaart and Jon A. Wellner. *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York, 1996. With applications to statistics.
- [17] Dehua Wang, Huaqiao Wang, et al. Global existence of martingale solutions to the three-dimensional stochastic compressible navier-stokes equations. *Differential and Integral Equations*, 28(11/12):1105–1154, 2015.

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