Obstacle problem for semilinear parabolic equations with measure data

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Abstract. We consider the obstacle problem with two irregular barriers for the Cauchy–Dirichlet problem for semilinear parabolic equations with measure data. We prove the existence and uniqueness of renormalized solutions of the problem as well as results on approximation of the solutions by the penalization method. In the proofs, we use probabilistic methods of the theory of Markov processes and the theory of backward stochastic differential equations.

1. Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be an open bounded set and let μ be a bounded soft measure on $D_T := (0, T] \times D$ (we call a Radon measure μ soft if it does not charge sets of zero parabolic capacity). Suppose we are also given $f : D_T \times \mathbb{R} \rightarrow \mathbb{R}$, $\varphi : D \rightarrow \mathbb{R}$ and two functions $h_1, h_2 : D_T \rightarrow \bar{\mathbb{R}}$ such that $h_1 \leq h_2$. In the present paper, we investigate the obstacle problem

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u \leq -f_u - \mu \text{ on } \{u \geq h_1\}, \\ \frac{\partial u}{\partial t} + A_t u \geq -f_u - \mu \text{ on } \{u \leq h_2\}, \\ h_1 \leq u \leq h_2 \text{ on } D_T, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases} \quad (1.1)$$

Here, $f_u(t, x) = f(t, x, u(t, x))$, $(t, x) \in D_T$, and

$$A_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij}(t, x) \frac{\partial}{\partial x_i} \right), \quad (1.2)$$

where $a : D_T \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that for some $\Lambda \geq 1$,

$$\Lambda^{-1} |y|^2 \leq \sum_{i,j=1}^d a_{ij}(t, x) y_i y_j \leq \Lambda |y|^2, \quad y \in \mathbb{R}^d \quad (1.3)$$

for a.e. $(t, x) \in D_T$.

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Problem (1.1) with regular barriers and L^2 data is quite well investigated (see, e.g., the classical monograph [2, Section 3.2]). There are only few papers devoted to problem (1.1) with one irregular time-dependent barrier, and to our knowledge, there is no paper on problem (1.1) with two irregular barriers and general soft measure on the right-hand side. In [20, 26], the linear problem with one barrier and L^2 data is considered. A semilinear problem (1.1) with L^2 data and f satisfying a Lipschitz and a linear growth condition is considered in [11] in the case of one barrier and in [13] in the case of two barriers. Note, however, that unlike the present paper, in [11, 13], the function f may depend on the solution of (1.1) as well as its gradient.

The main goal of the paper is to prove the existence and uniqueness of solutions of (1.1) in the case where the barriers h_1, h_2 are merely measurable and satisfy some kind of separation condition, $\varphi \in L^1(D)$, f satisfies a monotonicity condition in u and some mild growth condition considered earlier in the theory of PDEs with measure data (see, e.g., [4]). We are also interested in the approximation of solutions of (1.1) by the penalization method. As in [11, 13], to study these problems, we adopt a stochastic approach based on the theory of backward stochastic differential equations.

The problem of existence and uniqueness of solutions of (1.1) with measure data is delicate even for regular barriers. If the barriers are irregular, then the additional serious problem is to define solutions of (1.1) in a way that ensures their uniqueness. The classical approach via variational inequalities is not suitable even in the case of one barrier and L^2 data, because even in that case, the solution of the variational inequality with irregular barrier is in general not unique.

To overcome the difficulty with uniqueness of solutions, one could try to adapt to the parabolic case the method of minimal solutions, which was applied successfully in [17] to one-sided elliptic obstacle problems with general Radon measure on the right-hand side. Unfortunately, the concept of minimal solutions is not directly applicable to the two-sided obstacle problem.

To address the nonuniqueness problem, one can also try to adopt the approach from the paper [26] devoted to linear parabolic one-sided obstacle problems with L^2 data and define a solution of (1.1) as a pair (u, ν) consisting of a measurable function $u : D_T \rightarrow \mathbb{R}$ and a soft measure ν on D_T such that

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u = -f_u - \mu - \nu & \text{in } D, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)|_{\partial D} = 0, & t \in (0, T) \end{cases} \quad (1.4)$$

is satisfied in some weak sense, $h_1 \leq u \leq h_2$ a.e. and ν satisfies some minimality condition. In case h_1, h_2 and u are regular, the natural minimality condition says that

$$\int_{D_T} (u - h_1) d\nu^+ = \int_{D_T} (h_2 - u) d\nu^- = 0, \quad (1.5)$$

where ν^+, ν^- denote the positive and the negative part of the Jordan decomposition of ν , i.e., the reaction measure acts only when u touches the barriers. If h_1, h_2 are merely measurable, the problem is to make sense of (1.5), because in general, ν is not

absolutely continuous with respect to the Lebesgue measure. A further complication arises from the fact that in general u is not quasi-continuous. It is, however, quasi-l.s.c., so determined q.e. Therefore, one can hope that when the barriers are quasi-l.s.c. or quasi-u.s.c., and hence determined q.e., then the minimality condition holds (the integrals in (1.5) are then well defined since v is soft). Unfortunately, even in that case, the integrals in (1.5) may be strictly positive (a simple example is to be found in [11]).

Our definition of a solution of (1.1) is based on a nonlinear Feynman–Kac formula. It may be viewed as a probabilistic extension of the definition described above, because in the linear case with L^2 data, our probabilistic solution (u, ν) coincides with the solution considered in [26]. Let us also note that in the case of one-sided problem, the first component u of the probabilistic solution is a minimal solution in the sense that

$$u = \text{quasi-essinf}\{v \geq h_1, m_1\text{-a.e.} : v \text{ is a supersolution of problem (1.7)}\}, \quad (1.6)$$

where m_1 is the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$ and

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u = -f_u - \mu \text{ in } D, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases} \quad (1.7)$$

Thus, our probabilistic approach leads to a parabolic analogue of a solution considered in [17]. In the case of two-sided problem, it leads to some variant of (1.6).

Let $\mathbb{X} = (X, P_{s,x})$ be a Markov family associated with the operator A_t , and for a soft measure γ on D_T , let A^γ denote the additive functional of $\mathbb{X}$ associated with γ in the Revuz sense (see Sect. 2). Set

$$\xi^s = \inf\{t \geq s : X_t \notin D\}. \quad (1.8)$$

By a solution of (1.1), we mean a pair (u, ν) consisting of a function u on D_T and a bounded soft measure ν on D_T such that

$$u(s, x) = E_{s,x} \left(\mathbf{1}_{\{\xi^s > T\}} \varphi(X_T) + \int_s^{\xi^s \wedge T} f_u(t, X_t) dt + \int_s^{\xi^s \wedge T} d(A_{s,t}^\mu + A_{s,t}^\nu) \right) \quad (1.9)$$

for q.e. $(s, x) \in D_T$ and ν satisfies a minimality condition introduced in [11]. The minimality condition says that for any measurable functions h_1^*, h_2^* on D_T having the property that $h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2$ a.e. and the processes $[s, T] \ni t \mapsto h_i^*(t, X_t)$, $i = 1, 2$, are càdlàg under $P_{s,x}$ for q.e. $(s, x) \in D_T$, the following equalities

$$\begin{aligned} & \int_s^{\xi^s \wedge T} (u_-(t, X_t) - h_{1-}^*(t, X_t)) dA_{s,t}^{\nu^+} \\ &= \int_s^{\xi^s \wedge T} (h_{2-}^*(t, X_t) - u_-(t, X_t)) dA_{s,t}^{\nu^-} = 0 \end{aligned} \quad (1.10)$$

hold $P_{s,x}$ -a.s. for q.e. $(s, x) \in D_T$ [In (1.10), $g_-(t, X_t) = \lim_{s \leq t, s \rightarrow t} g(s, X_s)$ for $g := u, h_1^*, h_2^*$].

Our main result says that under natural mild assumptions on the data, there exists a unique solution (u, ν) of (1.1). Moreover, from the nonlinear Feynman–Kac formula (1.9), we deduce that u is a renormalized (and an entropy as well) solution of problem (1.4). If A^μ is continuous and h_1, h_2 are quasi-continuous, condition (1.10) has purely analytic interpretation. We show that under this additional assumption, u is quasi-continuous and (1.10) reduces to condition (1.5). In the case of irregular barriers, an analytical formulation of the minimality condition is possible in the linear case with L^2 data. In that case (1.10) may be expressed by using the notion of precise version of function introduced in [27].

The reason for adopting here a probabilistic definition of a solution of (1.1) not only pertains to the difficulties with the analytic formulation of the minimality condition for the reaction measure ν . One major advantage of the probabilistic definition is that it fits well to a general scheme of proving existence of solutions of equations with measure data, which was successfully adopted in the paper [15] devoted to semilinear elliptic equations with operators associated with general symmetric regular Dirichlet forms. The scheme comprises two essentially different parts. In the present context, the first part consists in using stochastic methods to show the existence of a pair (u, ν) satisfying (1.9), (1.10) and such that the process $t \mapsto u(t, X_t)$ has some integrability properties. As a matter of fact, this part follows rather easily from results on doubly reflected BSDEs proved recently in [12]. The second part consists in using the nonlinear Feynman–Kac formula (1.9) to prove additional regularity properties of u and to show that u is a renormalized solution of (1.4).

Results from [12] are also used to show that the solution of (1.1) can be approximated by the penalization method. For instance, we show that under the same assumptions under which there exists a unique solution (u, ν) of (1.1), if u_n is a renormalized solution of the problem

$$\begin{cases} \frac{\partial u_n}{\partial t} + A_t u_n = -f_{u_n} - \mu - n(u_n - h_1)^- + n(u_n - h_2)^-, \\ u_n(T, \cdot) = \varphi, \quad u_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T), \end{cases}$$

then $u_n \rightarrow u$ q.e. on D_T and $\nabla u_n \rightarrow \nabla u$ a.e. From [12] and the main result of the present paper, it also follows that q.e. on D_T ,

$$u = \text{quasi-essinf}\{v \geq h_1, m_1\text{-a.e.: } v \text{ is a supersolution of (1.7) with } \mu \text{ replaced by } \mu - \nu^-\}. \quad (1.11)$$

If $h_2 \equiv +\infty$, then $\nu^- = 0$, and so (1.11) reduces to (1.6). Finally, let us mention that from [12] and our main result, it follows that u can also be characterized as a solution of the following stopping time problem (sometimes called Dynkin game): for any h_1^*, h_2^* as in condition (1.10),

$$\begin{aligned}
u(s, x) = & \sup_{\sigma \in T^s} \inf_{\delta \in T^s} E_{s,x} \left(\int_s^{\sigma \wedge \delta \wedge T} f_u(t, X_t) dt + \int_s^{\sigma \wedge \delta \wedge T} dA_{s,t}^\mu \right. \\
& + h_1^*(\delta, X_\delta) \mathbf{1}_{\{\delta \leq \sigma < T\}} \mathbf{1}_{\{\delta < \xi^s \wedge T\}} + h_2^*(\sigma, X_\sigma) \mathbf{1}_{\{\sigma < \delta\}} \mathbf{1}_{\{\sigma < \xi^s \wedge T\}} \\
& \left. + \varphi(X_T) \mathbf{1}_{\{\sigma = \delta = T\}} \right)
\end{aligned}$$

for q.e. $(s, x) \in D_T$, where T^s denotes the set of all stopping times with values in $[s, T]$ with respect to the completion of the filtration generated by X .

In the present paper, we are mostly interested in the investigation of renormalized (or entropy) solutions of (1.1). But it is worth mentioning that as a by-product of our proofs, we obtain new results on stochastic representation of solutions of (1.1) and the Cauchy–Dirichlet problem (1.7), which can be regarded as problem (1.1) with $h_1 \equiv -\infty$, $h_2 \equiv +\infty$. Some of these results seem to be new even in the case of problem (1.7) with L^2 data. To our knowledge in all the existing results on stochastic representation of solutions to that problem, some regularity of the boundary of the domain is assumed. In the present paper, we do not require any regularity of D . Let us also note that in the recent paper [25], the existence and uniqueness of renormalized solutions of the Cauchy–Dirichlet problem with more general than (1.2) divergence form operator (for instance p -Laplace operator) and f not depending on x and satisfying the so-called sign condition is proved. In our paper, we consider equations with A given by (1.2), but we allow f to depend on x .

2. Preliminaries

In this section, we have compiled some basic facts on diffusions associated with the operator A_t defined by (1.2) and their additive functionals associated with soft measures on $\mathbb{R}_+ \times \mathbb{R}^d$. Here, and in the next sections, we assume that a satisfies (1.3). By putting $a_{ij} = \delta_j^i$ outside D_T , we can and will assume that a is defined and satisfy (1.3) in all $\mathbb{R}_+ \times \mathbb{R}^d$.

2.1. Time-inhomogeneous diffusions and additive functionals

Let $\Omega = C(\mathbb{R}_+; \mathbb{R}^d)$ be the space of continuous $\mathbb{R}^d$ -valued functions on $\mathbb{R}_+ = [0, +\infty)$, X be the canonical process on Ω , $\mathcal{F}_t^s = \sigma(X_u, u \in [s, t])$ and for given $T > 0$ let $\bar{\mathcal{F}}_t^s = \sigma(X_u, u \in [T + s - t, T])$. We define $\mathcal{G}_T^s$ as the completion of $\mathcal{F}_T^s$ with respect to the family $\mathcal{P} = \{P_{s,\mu} : \mu \text{ is a probability measure on } \mathcal{B}(\mathbb{R}^d)\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx)$, and then, we define $\mathcal{G}_t^s$ ($\bar{\mathcal{G}}_t^s$) as the completion of $\mathcal{F}_t^s$ ($\bar{\mathcal{F}}_t^s$) in $\mathcal{G}_T^s$ with respect to $\mathcal{P}$.

Let p denote the fundamental solution for the operator A_t and let $\mathbb{X} = \{(X, P_{s,x}) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ be a time-inhomogeneous Markov process for which p is the transition density function, i.e.,

$$P_{s,x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_\Gamma p(s, x, t, y) dy, \quad t > s$$

for any $\Gamma \in \mathcal{B}(\mathbb{R}^d)$. The process $\mathbb{X}$ admits the so-called strict Fukushima decomposition, i.e., for every $T > 0$,

$$X_t = X_s + A_{s,t} + M_{s,t}, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \quad (2.1)$$

for every $(s, x) \in [0, T] \times \mathbb{R}^d$, where $\{A_{s,t}, 0 \leq s \leq t \leq T\}$ is a two parameter continuous additive functional (CAF) of $\mathbb{X}$ (with respect to the filtration $\{\mathcal{G}_t^s\}$) of zero energy and $\{M_{s,t}, 0 \leq s \leq t \leq T\}$ is a two parameter martingale additive functional (MAF) of $\mathbb{X}$ (with respect to $\{\mathcal{G}_t^s\}$) of finite energy (see [29]). In particular, $M_{s,\cdot}$ is a $(\{\mathcal{G}_t^s\}, P_{s,x})$ -martingale. Moreover, for every $(s, x) \in [0, T] \times \mathbb{R}^d$,

$$\langle M_{s,\cdot}^i, M_{s,\cdot}^j \rangle_t = \int_s^t a_{ij}(\theta, X_\theta) d\theta, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \quad (2.2)$$

(see [29]). It follows that under $P_{s,x}$, the process $B_{s,\cdot}$ defined as

$$B_{s,t} = \int_s^t \sigma^{-1}(\theta, X_\theta) dM_{s,\theta}, \quad t \in [s, T],$$

where $\sigma \cdot \sigma^* = a$ and σ^{-1} is the inverse matrix of σ , is a Brownian motion on $[s, T]$. It is also known (see [14]) that $\mathbb{X}$ admits the so-called Lyons–Zheng decomposition, i.e., for any $T > 0$,

$$X_t - X_u = \frac{1}{2}M_{u,t} + \frac{1}{2} \left(N_{T+s-t}^{s,x} - N_{s,T+s-u}^{s,x} \right) - \alpha_{u,t}^{s,x}, \quad s \leq u \leq t \leq T$$

for every $(s, x) \in [0, T] \times \mathbb{R}^d$, where $N_{s,\cdot}^{s,x}$ is an $(\{\bar{G}_t^s\}, P_{s,x})$ -martingale and

$$\alpha_{u,t}^{s,x} = \sum_{j=1}^d \int_u^t \frac{1}{2} a_{ij}(\theta, X_\theta) p^{-1} \frac{\partial p}{\partial y_j}(s, x, \theta, X_\theta) d\theta.$$

For a measurable $\mathbb{R}^d$ -valued function $\mathbf{f}$ and $s \leq r \leq t \leq T$, we put

$$\begin{aligned} \int_r^t \mathbf{f}(\theta, X_\theta) d^* X_\theta &:= - \int_r^t \mathbf{f}(\theta, X_\theta) \left(dM_{s,\theta} + d\alpha_{s,\theta}^{s,x} \right) \\ &\quad - \int_{T+s-t}^{T+s-r} \mathbf{f}(T+s-\theta, X_{T+s-\theta}) dN_{s,\theta}^{s,x}. \end{aligned}$$

The usefulness of the backward integral defined above comes from the fact that for regular $\mathbf{f}$,

$$\int_u^t \operatorname{div} \mathbf{f}(\theta, X_\theta) d\theta = \int_u^t a^{-1} \mathbf{f}(\theta, X_\theta) d^* X_\theta, \quad s \leq u \leq t \leq T, \quad P_{s,x}\text{-a.s.},$$

where a^{-1} is the inverse matrix of a (see, e.g., [30]).

Let us recall that for every $\Phi \in L^2(0, T; H^{-1}(\mathbb{R}^d))$, there exist $f \in L^2([0, T] \times \mathbb{R}^d)$ and $G = (G^1, \dots, G^d)$ with $G^i \in L^2([0, T] \times \mathbb{R}^d)$, $i = 1, \dots, d$, such that

$$\Phi = f + \operatorname{div}(G),$$

i.e., for every $\eta \in L^2(0, T; H^1(\mathbb{R}^d))$,

$$\Phi(\eta) = \int_{[0, T] \times \mathbb{R}^d} \eta f \, dm_1 - \int_{[0, T] \times \mathbb{R}^d} G \cdot \nabla \eta \, dm_1,$$

where m_1 denote the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$.

The following lemma will be needed in the next section to prove regularity of parabolic potentials. For the meaning of the notion of “quasi-every” used below, see Sect. 2.3.

LEMMA 2.1. *Assume that $\Phi_1, \Phi_2 \in L^2(0, T; H^{-1}(\mathbb{R}^d))$, $D \subset \mathbb{R}^d$ is an open bounded set and $\Phi_1 = \Phi_2$ in $L^2(0, T; H^{-1}(D))$. If $\Phi_1 = f_1 + \operatorname{div}(G_1)$, $\Phi_2 = f_2 + \operatorname{div}(G_2)$, then for quasi-every $(s, x) \in D_T$,*

$$\begin{aligned} & \int_s^{t \wedge \xi^s} f_1(\theta, X_\theta) \, d\theta + \int_s^{t \wedge \xi^s} a^{-1} G_1(\theta, X_\theta) \, d^* X_\theta \\ &= \int_s^{t \wedge \xi^s} f_2(\theta, X_\theta) \, d\theta + \int_s^{t \wedge \xi^s} a^{-1} G_2(\theta, X_\theta) \, d^* X_\theta, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \end{aligned}$$

Proof. Let $D_\varepsilon = \{x \in D : \operatorname{dist}(x, D^c) > \varepsilon\}$ and let $\Phi_i^\varepsilon = f_i^\varepsilon + \operatorname{div}(G_i^\varepsilon)$, $i = 1, 2$, where $f_i^\varepsilon, G_i^\varepsilon$ are standard regularizations of f_i, G_i , respectively. It is an elementary check that

$$\Phi_1^\varepsilon = \Phi_2^\varepsilon \quad \text{on } L^2\left(0, T; H^{-1}(D_\varepsilon)\right).$$

Since $\Phi_1^\varepsilon, \Phi_2^\varepsilon \in L^2([0, T] \times \mathbb{R}^d)$, it follows from the above equality that

$$\Phi_1^\varepsilon = \Phi_2^\varepsilon, \quad m_1\text{-a.e. on } [0, T] \times D_\varepsilon.$$

In [14], it is proved that

$$\int_s^\cdot f_i^\varepsilon(\theta, X_\theta) \, d\theta + \int_s^\cdot a^{-1} G_i^\varepsilon(\theta, X_\theta) \, d^* X_\theta \rightarrow \int_s^\cdot f_i(\theta, X_\theta) \, d\theta + \int_s^\cdot a^{-1} G_i(\theta, X_\theta) \, d^* X_\theta$$

uniformly on $[s, T]$ in probability $P_{s,x}$ for q.e. $(s, x) \in [s, T] \times \mathbb{R}^d$. Let $\xi^{s,\varepsilon} = \inf\{t \geq s, X_t \notin D_\varepsilon\}$. Then,

$$\begin{aligned} & \int_0^{t \wedge \xi^{s,\varepsilon}} f_1^\varepsilon(\theta, X_\theta) \, d\theta + \int_0^{t \wedge \xi^{s,\varepsilon}} a^{-1} G_1^\varepsilon(\theta, X_\theta) \, d^* X_\theta = \int_0^{t \wedge \xi^{s,\varepsilon}} \Phi_1^\varepsilon(\theta, X_\theta) \, d\theta \\ &= \int_0^{t \wedge \xi^{s,\varepsilon}} \Phi_2^\varepsilon(\theta, X_\theta) \, d\theta = \int_0^{t \wedge \xi^{s,\varepsilon}} f_2^\varepsilon(\theta, X_\theta) \, d\theta + \int_0^{t \wedge \xi^{s,\varepsilon}} a^{-1} G_2^\varepsilon(\theta, X_\theta) \, d^* X_\theta, \end{aligned}$$

from which the desired result follows, because $\xi^{s,\varepsilon} \nearrow \xi^s$, $P_{s,x}$ -a.s. as $\varepsilon \searrow 0$. $\square$

2.2. Time-homogeneous diffusions

In the next sections, it will be advantageous to consider a certain time-homogeneous diffusion determined by $\mathbb{X}$. The standard construction of it is as follows. We set

$$\Omega' = \mathbb{R}_+ \times \Omega, \quad P'_{s,x}(B) = P_{s,x}(\{\omega \in \Omega : (s, \omega) \in B\}) \quad (2.3)$$

and consider the process $\mathbf{X}$ on Ω' defined as

$$\mathbf{X}_t(s, \omega) = (s + t, X_{s+t}(\omega)), \quad t \geq 0. \quad (2.4)$$

Let $\mathcal{F}'_t = \sigma(\mathbf{X}_u, u \leq t)$, $\mathcal{F}'_\infty = \sigma(\mathbf{X}_u, u < +\infty)$ and let $\mathcal{G}'_\infty$ denote the completion of $\mathcal{F}'_\infty$ with respect to the family $\mathcal{P}' = \{P'_\mu : \mu \text{ is a probability measure on } \mathbb{R}_+ \times \mathbb{R}^d\}$, where $P'_\mu(\cdot) = \int_{\mathbb{R}_+ \times \mathbb{R}^d} P'_{s,x}(\cdot) \mu(ds dx)$. Finally, let $\mathcal{G}'_t$ denote the completion of $\mathcal{F}'_t$ in $\mathcal{G}'_\infty$ with respect to $\mathcal{P}'$. Then, $\mathbb{X}' = \{(\mathbf{X}_t, P'_{s,x}); (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a time-homogeneous Markov process with respect to the filtration $\{\mathcal{G}'_t\}$ with the transition density

$$P'(t, (s, x), \Gamma) = P(s, x, s + t, \Gamma_{s+t}), \quad (2.5)$$

where $\Gamma_{s+t} = \{x \in \mathbb{R}^d : (s + t, x) \in \Gamma\}$.

For $h, t \geq 0$, we define $\theta_h : \Omega' \rightarrow \Omega'$ and $\tau(t) : \Omega' \rightarrow \mathbb{R}_+$ by putting

$$\theta_h \omega' = (s + h, \omega), \quad \tau(t)(\omega') = s + t = \tau(0)(\omega') + t$$

for $\omega' = (s, \omega)$. Then,

$$\mathbf{X}_t(\theta_h \omega') = \mathbf{X}_{t+h}(\omega'), \quad \tau(t) \circ \theta_h = \tau(t) + h, \quad h, t \geq 0.$$

Since the decomposition (2.1) is unique, we may assume that the two parameter AFs A, M of (2.1) are defined for all $0 \leq s \leq t$ and that for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ the decomposition (2.1) holds true for all $t \geq s$. Set

$$A_t(\omega') = A_{s,s+t}(\omega), \quad M_t(\omega') = M_{s,s+t}(\omega), \quad t \geq 0 \quad (2.6)$$

for $\omega' = (s, \omega)$. Since $P_{s,x}(A_{s,t} = A_{s,u} + A_{u,t}, 0 \leq s \leq u \leq t) = 1$ for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$ and M has the same property, it follows from (2.6) and the fact that the energy of A equals zero and the energy of M is finite that $A = \{A_t, t \geq 0\}$ is a CAF of $\mathbb{X}'$ of zero energy and $M = \{M_t, t \geq 0\}$ is a MAF of $\mathbb{X}'$ of finite energy. In particular, M is a $(\{\mathcal{G}'_t\}, P'_{s,x})$ -martingale for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Moreover, from (2.1) and (2.4), it follows that for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$X_{\tau(t)} = X_{\tau(0)} + A_t + M_t, \quad t \geq 0 \quad P'_{s,x}\text{-a.s.}$$

(By convention, if ξ is a random variable defined on Ω , we set $\xi(\omega') = \xi(\omega)$ for any $\omega' = (s, \omega) \in \Omega'$. Thus, in particular, $X_{\tau(t)}(\omega') = X_{\tau(t)(\omega')}(\omega)$).

Set

$$B_t = \int_0^t \sigma^{-1}(\mathbf{X}_\theta) dM_\theta, \quad t \geq 0. \quad (2.7)$$

By (2.2) and (2.6),

$$\langle M^i, M^j \rangle_t = \langle M_{s,\cdot}^i, M_{s,\cdot}^j \rangle_{s+t} = \int_s^{s+t} a_{ij}(\theta, X_\theta) d\theta, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}$$

for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. Therefore, B is a Brownian motion under $P'_{s,x}$ for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$. In fact, by [16, Theorem 12], it is an $\{\mathcal{G}'_t\}$ -Brownian motion.

2.3. Capacity and soft measures

Let $\mathcal{W}$ be the space of $u \in L^2(\mathbb{R}_+; H_0^1(\mathbb{R}^d))$ such that $\frac{\partial u}{\partial t} \in L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}^d))$ endowed with the usual norm $\|u\|_{\mathcal{W}} = \|u\|_{L^2(\mathbb{R}_+; H_0^1(\mathbb{R}^d))} + \|\frac{\partial u}{\partial t}\|_{L^2(\mathbb{R}_+; H^{-1}(\mathbb{R}^d))}$. We define the parabolic capacity of an open set $U \subset \mathbb{R}_+ \times \mathbb{R}^d$ as

$$\text{cap}(U) = \inf \left\{ \|u\|_{\mathcal{W}} : u \in \mathcal{W}, u \geq \mathbf{1}_U \text{ a.e. in } \mathbb{R}_+ \times \mathbb{R}^d \right\}.$$

The parabolic capacity of a Borel set $B \subset \mathbb{R}_+ \times \mathbb{R}^d$ is defined as

$$\text{cap}(B) = \inf \left\{ \text{cap}(U) : U \text{ is an open subset of } \mathbb{R}_+ \times \mathbb{R}^d, B \subset U \right\}.$$

It is known that for a Borel $B \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\text{cap}(B) = 0$ iff

$$\int_{\mathbb{R}_+} P_{s,m}(\exists t > s : (t, X_t) \in B) ds = 0,$$

where $P_{s,m}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) dx$ and m denotes the Lebesgue measure on $\mathbb{R}^d$ (see the argument following Eq. (5.2) in [22]).

We will say that some property is satisfied quasi-everywhere (q.e. for short) if it is satisfied except for some Borel subset of $\mathbb{R}_+ \times \mathbb{R}^d$ of capacity zero.

Let μ be a signed Borel measure on $\mathbb{R}_+ \times \mathbb{R}^d$, and let $|\mu|$ denote the total variation of μ . We say that μ is smooth if $|\mu|(B) = 0$ for every Borel set $B \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that $\text{cap}(B) = 0$, and there exists an ascending sequence $\{F_n\}$ of closed subsets of $\mathbb{R}_+ \times \mathbb{R}^d$ such that for every compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$, $\lim_{n \rightarrow \infty} \text{cap}(K \setminus F_n) \rightarrow 0$, $|\mu|(F_n) < \infty$, $n \geq 1$, and $\mu(\mathbb{R}_+ \times \mathbb{R}^d \setminus \bigcup F_n) = 0$. We say that a smooth measure μ is soft if μ is a Radon measure. We say that μ is a soft measure on $V \subset \mathbb{R}_+ \times \mathbb{R}^d$ if it is soft and $\mu(\mathbb{R}_+ \times \mathbb{R}^d \setminus V) = 0$.

In what follows by S , we will denote the set of all positive smooth measures on $\mathbb{R}_+ \times \mathbb{R}^d$ and by S_0 the subset of S consisting of all measures of finite energy integral (see, e.g., [21, Section 5] for the definition). By $\mathcal{M}_b(D_T)$, we denote the set of all signed Borel measures on $\mathbb{R}_+ \times \mathbb{R}^d$ having bounded variation and such that $\mu(\mathbb{R}_+ \times \mathbb{R}^d \setminus D_T) = 0$. By $\mathcal{M}_b^+(D_T)$, we denote the subset of $\mathcal{M}_b(D_T)$ consisting of all positive measures. We write $\|\mu\|_{TV}$ for the total variation norm of $\mu \in \mathcal{M}_b(D_T)$. By $\mathcal{M}_0(D_T)$, we denote the set of all soft measures on D_T , by $\mathcal{M}_{0,b}(D_T)$, the set of all bounded soft measures on D_T , and by $\mathcal{M}_{0,b}^+(D_T)$, the subset of $\mathcal{M}_{0,b}(D_T)$ consisting of all positive measures.

Let us note that in the literature soft measures are usually defined on $(0, T) \times D$. In the present paper, we consider soft measures on $(0, T] \times D$ because in general the obstacle reaction measure is concentrated on that set.

2.4. Additive functionals and soft measures

Let $E_{s,x}$ (resp. $E'_{s,x}$) denote the expectation with respect to $P_{s,x}$ (resp. $P'_{s,x}$). Let us recall that a positive additive functional A of $\mathbb{X}'$ and a positive soft measure μ on $\mathbb{R}_+ \times \mathbb{R}^d$ are in the Revuz correspondence if

$$\int_{\mathbb{R}_+ \times \mathbb{R}^d} f \, d\mu = \lim_{\alpha \rightarrow \infty} \alpha \int_{\mathbb{R}_+ \times \mathbb{R}^d} (E'_{s,x} \int_0^\infty e^{-\alpha t} f(\mathbf{X}_t) \, dA_t) \, dm_1(s, x) \quad (2.8)$$

for every $f \in \mathcal{B}^+(\mathbb{R}_+ \times \mathbb{R}^d)$. If $\langle \mu, 1 \rangle < +\infty$, then A is called integrable. By Theorems 6.4.7 and 6.4.9 in [23], for every $\mu \in S_0$, there exists a unique positive natural additive functional (NAF) A of $\mathbb{X}'$ in the Revuz correspondence with μ (see also (5.9), (5.10) in [21]). Since each measure $\mu \in S$ may be approximated by measures in S_0 (see [21, Theorem 5.6]), repeating step by step the proof of [23, Theorem 4.1.16] (see also [21, Theorem 5.6]), one can show that for every $\mu \in S$ there exists a unique positive NAF A of $\mathbb{X}'$ in the Revuz correspondence with μ . In what follows, we will denote it by A^μ . For $\mu \in \mathcal{M}_0(D_T)$, we put $A^\mu = A^{\mu^+} - A^{\mu^-}$. Let us also note that if μ belongs to the subset of S consisting of smooth measures with respect to the capacity determined by the coercive part of the time-dependent form associated with $\mathbb{X}'$ (see [31] for details), then by [31, Theorem 2.2], the AF A^μ is continuous. In fact, in this case, an explicit representation of A^μ is known (see Theorem 2.1 and Corollary 4.4 in [14]).

Let

$$\zeta = \inf\{t \geq 0, \mathbf{X}_t \notin \mathbb{R}_+ \times D\}, \quad \zeta_\tau = \zeta \wedge (T - \tau(0))$$

and let p'_D denote the transition density of the process $\mathbb{X}'$ killed at first exit from $\mathbb{R}_+ \times D$, i.e.,

$$p'_D(t, (s, x), z) = p'(t, (s, x), z) - E'_{s,x} [\zeta < t, p'(t - \zeta, \mathbf{X}_\zeta, z)],$$

where p' is the transition density of $\mathbb{X}'$. It is known that A^μ corresponds to μ iff for quasi-every $(s, x) \in D_T$,

$$E'_{s,x} \int_0^{\zeta_\tau} f(\mathbf{X}_t) \, dA_t^\mu = \int \int_{[0, T-s] \times D} f(z) p'_D(t, (s, x), z) \, d\mu(z) \, dt \quad (2.9)$$

for every $f \in \mathcal{B}^+(D_T)$ (see [19]).

In [11], a correspondence between two parameter additive functionals of $\mathbb{X}$ and soft measures on D_T is considered. An AF $\{A_{s,t}^\mu, 0 \leq s \leq t \leq T\}$ of $\mathbb{X}$ corresponds to a bounded soft measure μ on D_T in the sense of [11] if for q.e. $(s, x) \in D_T$,

$$E_{s,x} \int_s^{\xi^s \wedge T} f(t, X_t) \, dA_{s,t}^\mu = \int_{[s, T] \times D} f(t, y) p_D(s, x, t, y) \, d\mu(t, y) \quad (2.10)$$

for every $f \in \mathcal{B}^+(D_T)$, where ξ^s is defined by (1.8) and p_D is the transition density of the process $\mathbb{X}$ killed at the first exit from D , i.e.,

$$p_D(s, x, t, y) = p(s, x, t, y) - E_{s,x} [\xi^s < t, p(t - \xi^s, X_{\xi^s}, y)].$$

Let us define $A = \{A_t, t \geq 0\}$ as

$$A_t(\omega') = A_{s, s+t}^\mu(\omega), \quad \omega' = (s, \omega), \quad t \geq 0. \quad (2.11)$$

Using (2.3)–(2.5) and the fact that $\zeta_\tau(\omega') = (\xi^s(\omega) - s) \wedge (T - s)$, one can show that (2.10) is satisfied iff (2.9) is satisfied with A^μ replaced by A . Therefore, a two parameter AF $\{A_{s,t}^\mu\}$ of $\mathbb{X}$ corresponds to μ in the sense of [11] iff the one-parameter AF A of $\mathbb{X}'$ defined by (2.11) corresponds to μ in the sense of (2.8).

LEMMA 2.2. Assume that $\mu \in \mathcal{M}_{0,b}^+(D_T)$. Then for q.e. $(s, x) \in D_T$,

$$E'_{s,x} \int_0^{\zeta_\tau} dA_t^\mu < +\infty.$$

Proof. Follows from [11, Theorem 6.2]. $\square$

We say that a measurable function $f : D_T \rightarrow \mathbb{R}$ is quasi-integrable if the function $\mathbb{R}_+ \ni t \mapsto f(\mathbf{X}_t)$ belongs to $L_{loc}^1(\mathbb{R}_+)$ $P_{s,x}$ -a.s. for q.e. $(s, x) \in D_T$, where we put $f(s, x) = 0$ for $(s, x) \in D_T^c$. The set of all quasi-integrable functions on D_T will be denoted by $qL^1(D_T)$.

COROLLARY 2.3. If $f \in L^1(D_T)$, then $f \in qL^1(D_T)$.

REMARK 2.4. Analysis similar to that in [15, Remark 4.4] shows that if for every $\varepsilon > 0$, there exists an open set $G_\varepsilon \subset D_T$ such that $\text{cap}(G_\varepsilon) < \varepsilon$ and $f|_{D_T \setminus G_\varepsilon} \in L^1(D_T \setminus G_\varepsilon)$ then $f \in qL^1(D_T)$. Since D_T is bounded, the reverse implication is also true. Indeed, assume that $f \in qL^1(D_T)$, $f \geq 0$. Then, extending f on $\mathbb{R}_+ \times \mathbb{R}^d$ by putting zero outside D_T , we get that $A_t = \int_0^t f(\mathbf{X}_r) dr$ is finite for every $t \geq 0$, which implies that it is a positive continuous AF of $\mathbb{X}'$. Directly from the definition of the Revuz correspondence, it follows that $f \cdot m_1$ corresponds to the AF A . This implies in particular that $f \cdot m_1$ is a smooth measure. Therefore, for every compact $K \subset \mathbb{R}_+ \times \mathbb{R}^d$ and for every $\varepsilon > 0$, there exists an open set $U_\varepsilon \subset \mathbb{R}_+ \times \mathbb{R}^d$ such that $\text{cap}(U_\varepsilon) \leq \varepsilon$ and $f \in L^1(K \setminus U_\varepsilon)$. Taking $K = \overline{D_T}$, we get the desired result.

LEMMA 2.5. Let μ be a positive smooth measure on D_T and let $\nu \in \mathcal{M}_{0,b}^+(D_T)$. If

$$E'_{s,x} \int_0^{\zeta_\tau} dA_r^\mu \leq E'_{s,x} \int_0^{\zeta_\tau} dA_r^\nu$$

for q.e. $(s, x) \in D_T$, then $\|\mu\|_{TV} \leq \|\nu\|_{TV}$.

Proof. By the assumptions, for q.e. $(s, x) \in D_T$,

$$\int_0^T \int_D p_D(s, x, r, y) d\mu(r, y) \leq \int_0^T \int_D p_D(s, x, r, y) d\nu(r, y). \quad (2.12)$$

Let $D^k = \{x \in D; \text{dist}(x, \partial D) > 1/k\}$, $D_T^k = [1/k, T] \times D^k$ and $h_k = \mathbf{1}_{\{D_T^k\}}$. By [20], for every k , there exists a minimal solution $e_k \in L^2(0, T; H^1(D))$, in the variational sense, of the obstacle problem

$$\begin{cases} \frac{\partial e_k}{\partial t} - A_t e_k \geq 0, & \frac{\partial e_k}{\partial t} - A_t e_k = 0 \text{ on } \{e_k > h_k\}, \\ e_k \geq h_k, \\ e_k(0, \cdot) = 0, & e_k(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases}$$

In particular, $\beta_k = \frac{\partial e_k}{\partial t} - A_t e_k$ belongs to $\mathcal{M}_b^+(D_T) \cap \mathcal{W}'(D_T)$, where $\mathcal{W}'(D_T)$ is the dual space of $\mathcal{W}(D_T)$. In fact, $\beta_k \in \mathcal{M}_{0,b}^+(D_T) \cap \mathcal{W}'(D_T)$ because it is known that $\mathcal{M}_b^+(D_T) \cap \mathcal{W}'(D_T) = \mathcal{M}_{0,b}^+(D_T) \cap \mathcal{W}'(D_T)$ (see, e.g., [8]). By [1], for q.e. $(s, x) \in D_T$,

$$e_k(s, x) = \int_0^T \int_D p_D(r, y, s, x) d\beta_k(r, y). \quad (2.13)$$

From (2.12), (2.13) and the fact that $0 \leq e_k \leq 1$ q.e. on D_T , it follows that

$$\int_{D_T} e_k(s, x) d\mu(s, x) \leq \int_{D_T} e_k(s, x) d\nu(s, x) \leq \|v\|_{TV}. \quad (2.14)$$

The fact that $0 \leq e_k \leq 1$ q.e. follows from the fact that e_k is the minimal potential majorizing h_k (see [2, Theorem 3.2.28]). Indeed, it is an elementary check that $e_k \wedge 1$ is a potential majorizing h_k . Therefore, $e_k \leq e_k \wedge 1$, which implies that $e_k \leq 1$. Of course, $e_k \geq 0$ since $e_k \geq h_k$. From this, we get in particular that $e_k(s, x) = 1$ for $(s, x) \in D_T^k$. Therefore, $e_k(s, x) \nearrow 1$ q.e. on D_T . Consequently, letting $k \rightarrow \infty$ in (2.14) and using Fatou's lemma, we get the desired result. $\square$

3. Backward stochastic differential equations

Results of this section together with some results on BSDEs proved in [12] form the basis for our main theorems on the obstacle problem (1.1).

3.1. General BSDEs

Suppose we are given a filtered probability space $(\Omega, \mathcal{G}, \{\mathcal{G}_t\}_{t \geq 0}, P)$. We will need the following spaces of processes.

$\mathcal{V}$ (resp. $\mathcal{V}^+$) is the space of all progressively measurable càdlàg processes V of finite variation (resp. increasing processes) such that $V_0 = 0$. $\mathcal{V}^1$ is the space of all processes $V \in \mathcal{V}$ such that $E|V|_T < +\infty$, where $|V|_T$ denotes the total variation of V on $[0, T]$.

$\mathcal{D}^q$, $q > 0$, is the space all progressively measurable càdlàg processes η such that $E \sup_{0 \leq t \leq T} |\eta_t|^q < +\infty$ for every $T > 0$.

M^q , $q > 0$, is the space of all progressively measurable processes η such that $E(\int_0^T |\eta_t|^2 dt)^{q/2} < +\infty$ for every $T > 0$.

Let us also recall that a càdlàg adapted process η is said to be of Doob's class (D) if the collection $\{\eta_\tau : \tau \text{ is a finite } \{\mathcal{G}_t\}_{t \geq 0}\text{-stopping time}\}$ is uniformly integrable.

Let B be a $\{\mathcal{G}_t\}$ -Brownian motion, σ be a bounded stopping time, ξ be a $\mathcal{G}_\sigma$ -measurable random variable, $A \in \mathcal{V}$ and let $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a function such that $f(\cdot, y, z)$ is progressively measurable for $y \in \mathbb{R}$, $z \in \mathbb{R}^d$ (for brevity, in our notation, we omit the dependence of f on $\omega \in \Omega$). Let us recall that a pair (Y, Z) of $\mathbb{R} \times \mathbb{R}^d$ -valued processes is called a solution of BSDE($\xi, \sigma, f + dA$) if

- (a) Y, Z are $\{\mathcal{G}_t\}$ -progressively measurable, Y is càdlàg, $P(\int_0^\sigma |Z_t|^2 dt < +\infty) = 1$,
- (b) $t \mapsto f(t, Y_t, Z_t) \in L^1(0, \sigma)$, P -a.s.,
- (c) $Y_t = \xi + \int_t^\sigma f(s, Y_s, Z_s) ds + \int_t^\sigma dA_s - \int_t^\sigma Z_s dB_s$, $0 \leq t \leq \sigma$, P -a.s.

REMARK 3.1. *The processes Y, Z in the above definition may be considered as processes on $\mathbb{R}_+$. Indeed, if we set $Z_t = 0$ for $t \geq \sigma$ and $Y_t = Y_\sigma$ for $t \geq \sigma$, then the equation in condition (c) is satisfied for every $t \geq 0$ if we adopt the convention that $\int_a^b = 0$ for $a \geq b$.*

Let us consider the following assumptions.

- (A1) $f(\cdot, y, z)$ is $\{\mathcal{G}_t\}$ -progressively measurable for every $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ and $f(t, \cdot, z)$ is continuous for every $(t, z) \in \mathbb{R} \times \mathbb{R}^d$, P -a.s.,
- (A2) There is $L > 0$ such that $|f(t, y, z) - f(t, y, z')| \leq L|z - z'|$ for a.e. $t \geq 0$ and every $y \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, P -a.s.,
- (A3) There is $\kappa \in \mathbb{R}$ such that $(f(t, y, z) - f(t, y', z))(y - y') \leq \kappa|y - y'|^2$ for a.e. $t \geq 0$ and every $y, y' \in \mathbb{R}$, $z \in \mathbb{R}^d$, P -a.s.,
- (A4) $E|\xi| + E \int_0^\sigma |f(t, 0, 0)| dt + E \int_0^\sigma d|A|_t < +\infty$,
- (A5) For every $r > 0$, $t \mapsto \sup_{|y| \leq r} |f(t, y, 0)| \in L^1(0, \sigma)$, P -a.s.,
- (AZ) There exist $\alpha \in (0, 1)$, $\gamma \geq 0$ and a nonnegative progressively measurable process g such that $E \int_0^\sigma |g_t| dt < +\infty$ and P -a.s.,

$$|f(t, y, z) - f(t, y, 0)| \leq \gamma(g_t + |y| + |z|)^\alpha$$

for a.e. $t \geq 0$ and every $y \in \mathbb{R}$, $z \in \mathbb{R}^d$.

It is known (see [12, Theorem 3.11]) that under (A1)–(A5), (AZ) there exists a unique solution (Y, Z) of BSDE($\xi, \sigma, f + dA$) such that Y is of class (D), $Y \in \mathcal{D}^q$ and $Z \in M^q$ for any $q \in (0, 1)$.

3.2. Markov-type BSDEs

Let $T > 0$, $f : D_T \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function, $\varphi \in L^1(D)$. By putting $\varphi(t, x) = \varphi(x)$ for $(t, x) \in D_T$, we will regard φ as defined on D_T . Let μ be a soft measure on D_T and B be the Brownian motion defined by (2.7).

Let $(s, x) \in D_T$. We say that a pair $(Y^{s,x}, Z^{s,x})$ consisting of an $\mathbb{R}$ -valued process $Y^{s,x}$ and an $\mathbb{R}^d$ -valued process $Z^{s,x}$ is a solution of BSDE $_{s,x}(\varphi, D, f + d\mu)$ if

$Y^{s,x}, Z^{s,x}$ are $\{\mathcal{G}_t'\}$ -progressively measurable, $Y^{s,x}$ is càdlàg, $t \mapsto f(\mathbf{X}_t, Y_t^{s,x}, Z_t^{s,x}) \in L^1(0, \zeta_\tau)$, $P'_{s,x}$ -a.s., $P'_{s,x}(\int_0^{\zeta_\tau} |Z_r^{s,x}|^2 dr < +\infty) = 1$ and

$$\begin{aligned} Y_t^{s,x} &= \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T - \tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, Y_r^{s,x}, Z_r^{s,x}) dr \\ &\quad + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} Z_r^{s,x} dB_r, \quad t \in [0, \zeta_\tau] \end{aligned} \quad (3.1)$$

holds true $P'_{s,x}$ -a.s. In other words, $(Y^{s,x}, Z^{s,x})$ is a solution of (3.1) if it is a solution of BSDE $(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T - \tau(0)}), \zeta_\tau, f(\mathbf{X}, \cdot, \cdot) + dA^\mu)$ on the filtered probability space $(\Omega', \mathcal{G}'_\infty, \{\mathcal{G}_t'\}, P'_{s,x})$ with the Brownian motion B .

PROPOSITION 3.2. *Let $(s, x) \in D_T$ and let $(Y^{s,x}, Z^{s,x}) = (Y, Z)$ be a solution of (3.1). Assume that (A1), (A2), (AZ) are satisfied, $Z \in M^q$ for some $q > \alpha$, there exist $0 < h \leq T - s$ and $\Lambda \in \mathcal{G}'_\infty$ such that $P'_{s,x}(\Lambda) = 1$, $P'_{s,x}(\theta_h^{-1}(\Lambda)) = 1$ and (3.1) is satisfied for every $\omega \in \Lambda$. Then,*

$$Y_{t-h} \circ \theta_h = Y_t, \quad t \geq h, \quad P'_{s,x}\text{-a.s.}$$

and

$$Z_{-h} \circ \theta_h = Z, \quad dt \otimes P'_{s,x}\text{-a.s. on } [h, \zeta_\tau] \times \Omega'.$$

Proof. Put $\kappa_h = (\zeta \circ \theta_h) \wedge (T - \tau(0) \circ \theta_h)$. Then,

$$\begin{aligned} Y_t \circ \theta_h &= \mathbf{1}_{\{\zeta \circ \theta_h > T - \tau(0) \circ \theta_h\}} \varphi(\mathbf{X}_{T - \tau(0) \circ \theta_h}) + \int_t^{\kappa_h} f(\mathbf{X}_r \circ \theta_h, Y_r \circ \theta_h, Z_r \circ \theta_h) dr \\ &\quad + \int_t^{\kappa_h} d(A_r^\mu \circ \theta_h) - \left(\int_t^{\zeta_\tau} Z_r dB_r \right) \circ \theta_h, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.} \end{aligned}$$

Fix $t \geq h$. Since $\zeta \circ \theta_h = (\zeta - h)^+$ and $\tau(0) \circ \theta_h = \tau(0) + h$, we have $\kappa_h = (\zeta - h) \wedge (T - \tau(0) - h)$ on the set $\{\zeta \geq h\}$. Consequently, $h + \kappa_h = \zeta_\tau$ on the set $\{\zeta \geq h\}$. Therefore, since $\mathbf{X}_r \circ \theta_h = \mathbf{X}_{r+h}$ and A^μ is additive, we have

$$\int_t^{\kappa_h} f(\mathbf{X}_r \circ \theta_h, Y_r \circ \theta_h, Z_r \circ \theta_h) dr = \int_{t+h}^{\zeta_\tau} f(\mathbf{X}_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) dr$$

and

$$\int_t^{\kappa_h} d(A_r^\mu \circ \theta_h) = \int_{t+h}^{\zeta_\tau} dA_r^\mu$$

$P'_{s,x}$ -a.s. on $\{\zeta \geq h\}$. Let $t = t_0 < t_1 < \dots < t_n = \zeta_\tau$ and let $t'_i = t_i + h, i = 0, \dots, n$. Since B is additive, $(Z_{t_i}(B_{t_{i+1}} - B_{t_i})) \circ \theta_h = Z_{t'_i - h} \circ \theta_h (B_{t'_{i+1}} - B_{t'_i})$, from which it follows that

$$\left(\int_t^{\zeta_\tau} Z_r dB_r \right) \circ \theta_h = \int_{t+h}^{\zeta_\tau} Z_{r-h} \circ \theta_h dB_r$$

$P'_{s,x}$ -a.s. on $\{\zeta \geq h\}$. Thus,

$$\begin{aligned} Y_t \circ \theta_h &= \mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_{t+h}^{\zeta_\tau} f(\mathbf{X}_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) \, dr \\ &\quad + \int_{t+h}^{\zeta_\tau} dA_r^\mu - \int_{t+h}^{\zeta_\tau} Z_{r-h} \circ \theta_h \, dB_r, \end{aligned}$$

and hence,

$$\begin{aligned} Y_{t-h} \circ \theta_h &= \mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, Y_{r-h} \circ \theta_h, Z_{r-h} \circ \theta_h) \, dr \\ &\quad + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} Z_{r-h} \circ \theta_h \, dB_r \end{aligned}$$

$P'_{s,x}$ -a.s. on $\{\zeta \geq h\}$. On the set $\{\zeta < h\}$, we have $Y_{t-h} \circ \theta_h = 0 = Y_t$ for $t \geq h$, because by the assumption, $h \leq T-s$. By what has been proved the pair $(Y_{t-h} \circ \theta_h, Z_{t-h} \circ \theta_h)$ solves BSDE (3.1) on $[h, \zeta_\tau]$, so the desired result follows from uniqueness of solutions of (3.1). $\square$

In the sequel, we say that a Borel set $N \subset D_T$ is properly exceptional if $m_1(N) = 0$ and $P'_{s,x}(\exists t \geq 0 : X_t \in N) = 0$ for every $(s, x) \in N^c$.

LEMMA 3.3. *Let $\Lambda \in \mathcal{G}'_\infty$. If $P'_{s,x}(\Lambda) = 1$ for q.e. $(s, x) \in D_T$, then there exists a properly exceptional set $N \subset D_T$ such that $P'_{s,x}(\theta_h^{-1}(\Lambda)) = 1$ for every $h > 0$ and $(s, x) \in N^c$.*

Proof. By the assumption, $P'_{s,x}(\Lambda^c) = 0$ for q.e. $(s, x) \in D_T$. It is known (see [3, Corollary 1.8.6] and [10, (6.12)]) that there exists a properly exceptional set $N \subset D_T$ such that $P'_{s,x}(\Lambda^c) = 0$ for every $(s, x) \in N^c$. Let $(s, x) \in N^c$. Then, by the Markov property, $P'_{s,x}(\theta_h^{-1}(\Lambda^c)) = E'_{s,x}(P_{\mathbf{X}_h}(\Lambda^c))$. By the assumption and the definition of N , $E'_{s,x}P_{\mathbf{X}_h}(\Lambda^c) = 0$ for every $(s, x) \in N^c$, which completes the proof. $\square$

In the case of Markov-type equations, the following hypotheses are analogues of (A1)–(A5) and (AZ).

- (H1) $f(\cdot, \cdot, y, z)$ is measurable for every $y \in \mathbb{R}, z \in \mathbb{R}^d$ and $f(t, x, \cdot, \cdot)$ is continuous for a.e. $(t, x) \in D_T$,
- (H2) There is $L > 0$ such that $|f(t, x, y, z) - f(t, x, y, z')| \leq L|z - z'|$ for every $y \in \mathbb{R}, z, z' \in \mathbb{R}^d$ and a.e. $(t, x) \in D_T$,
- (H3) There is $\kappa \in \mathbb{R}$ such that $(f(t, x, y, z) - f(t, x, y', z))(y - y') \leq \kappa|y - y'|^2$ for every $y, y' \in \mathbb{R}, z \in \mathbb{R}^d$ and a.e. $(t, x) \in D_T$,
- (H4) $f(\cdot, \cdot, 0, 0) \in L^1(D_T)$, $\mu \in \mathcal{M}_{0,b}(D_T)$, $\varphi \in L^1(D)$,
- (H5) $\forall_{r>0} (t, x) \mapsto \sup_{|y| \leq r} |f(t, x, y, 0)| \in qL^1(D_T)$.
- (HZ) There exists $\alpha \in (0, 1)$, $\gamma \geq 0$ and $\varrho \in L^1(D_T)$, $\varrho \geq 0$ such that

$$|f(t, x, y, z) - f(t, x, y, 0)| \leq \gamma(\varrho(t, x) + |y| + |z|)^\alpha$$

for every $y \in \mathbb{R}, z \in \mathbb{R}^d$ and a.e. $(t, x) \in D_T$.

LEMMA 3.4. Assume that (H1)–(H5) are satisfied. Then for q.e. $(s, x) \in D_T$ the data $\varphi(\mathbf{X}_{T-\tau(0)})\mathbf{1}_{\{\zeta > T-\tau(0)\}}$, ζ_τ , $f(\mathbf{X}, \cdot, \cdot)$, A^μ satisfy (A1)–(A5) under the measure $P'_{s,x}$.

Proof. Follows from Lemma 2.2. $\square$

PROPOSITION 3.5. Assume (H1)–(H5) and (HZ). Then, for q.e. $(s, x) \in D_T$, there exists a solution $(Y^{s,x}, Z^{s,x})$ of (3.1), and there exists a pair of processes (Y, Z) such that for q.e. $(s, x) \in D_T$,

$$Y_t^{s,x} = Y_t, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}, \quad Z^{s,x} = Z, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'. \quad (3.2)$$

Moreover, there exist Borel measurable functions $u, \psi : D_T \rightarrow \mathbb{R}$ such that for q.e. $(s, x) \in D_T$,

$$Y_t = u(\mathbf{X}_t), \quad P'_{s,x}\text{-a.s.}$$

for every $t \in [0, T - \tau(0)]$ and

$$\psi(\mathbf{X}) = Z, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'.$$

If f does not depend on z , then there is C depending only on κ, T such that

$$\begin{aligned} E'_{s,x} \int_0^{\zeta_\tau} |f(\mathbf{X}_r, u(\mathbf{X}_r))| dr &\leq C E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} |\varphi(\mathbf{X}_{T-\tau(0)})| \right. \\ &\quad \left. + \int_0^{\zeta_\tau} |f(\mathbf{X}_r, 0)| dr + \int_0^{\zeta_\tau} d|A^\mu|_r \right) \end{aligned} \quad (3.3)$$

for q.e. $(s, x) \in D_T$.

Proof. By Lemma 2.2, $E'_{s,x}|A^\mu|_{\zeta_\tau} < +\infty$ for q.e. $(s, x) \in D_T$. Moreover, for $(s, x) \in D_T$,

$$\begin{aligned} E'_{s,x} \mathbf{1}_{\{\zeta > T-\tau(0)\}} |\varphi(\mathbf{X}_{T-\tau(0)})| &\leq E'_{s,x} |\varphi(\mathbf{X}_{T-\tau(0)})| \\ &\leq \int_D p(s, x, T, y) |\varphi(y)| dy \leq c \int_D |\varphi(y)| dy. \end{aligned}$$

Therefore, by [12, Proposition 3.10], for q.e. $(s, x) \in D_T$, there exists a solution $(\bar{Y}^{s,x}, \bar{Z}^{s,x})$ of the equation

$$\begin{aligned} \bar{Y}_t^{s,x} &= \mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, 0, 0) dr + \int_t^{\zeta_\tau} dA_r^\mu \\ &\quad - \int_t^{\zeta_\tau} \bar{Z}_r^{s,x} dB_r, \quad t \geq 0 \end{aligned}$$

such that $\bar{Y}^{s,x}$ is of class (D) and $(\bar{Y}^{s,x}, \bar{Z}^{s,x}) \in \mathcal{D}^q \otimes M^q$ for any $q \in (0, 1)$. Hence,

$$\bar{Y}_t^{s,x} = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, 0, 0) dr + \int_t^{\zeta_\tau} dA_r^\mu | \mathcal{G}_t' \right),$$

so from [9, Lemma A.3.5] it follows that there exists a process $\bar{Y}$ such that $\bar{Y}_t^{s,x} = \bar{Y}_t$, $t \geq 0$, $P'_{s,x}$ -a.s. for q.e. $(s, x) \in D_T$. Since

$$\bar{Z}_t^{s,x} = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} f(\mathbf{X}_r, 0, 0) dr + \int_0^{\zeta_\tau} dA_r^\mu | \mathcal{G}'_t \right) - \bar{Y}_0,$$

using once again [9, Lemma A.3.5], we conclude that there exists a process $\bar{Z}$ such that $\bar{Z}^{s,x} = \bar{Z}$, $dt \otimes P'_{s,x}$ -a.e. on $[0, \zeta_\tau] \times \Omega'$ for q.e. $(s, x) \in D_T$. Thus, for q.e. $(s, x) \in D_T$,

$$\bar{Y}_t = \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, 0, 0) dr + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} \bar{Z}_r dB_r, \quad t \geq 0$$

$P'_{s,x}$ -a.s. From this, the method of construction of the solution of BSDE $_{s,x}(\varphi, D, f + d\mu)$ (see the proof of [12, Proposition 3.10]) and repeated application of [9, Lemma A.3.3], we deduce the existence of a pair (Y, Z) such that (3.2) is satisfied. Let $\Lambda \subset \Omega'$ be a set of those $\omega \in \Omega'$ for which Y is càdlàg and

$$Y_t = \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} f(\mathbf{X}_r, Y_r, Z_r) dr + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} Z_r dB_r, \quad t \geq 0.$$

Then, $\Lambda \in \mathcal{G}'_\infty$ and of course $P'_{s,x}(\Lambda) = 1$ for q.e. $(s, x) \in D_T$. From this and Proposition 3.2 and Lemma 3.3, it follows that if we set $u(s, x) = E'_{s,x} Y_0$ for $(s, x) \in D_T$, then for every $t \in [0, T - s]$,

$$u(\mathbf{X}_t) = E'_{\mathbf{X}_t} Y_0 = E'_{s,x}(Y_0 \circ \theta_t | \mathcal{G}'_t) = E'_{s,x}(Y_t | \mathcal{G}'_t) = Y_t, \quad P'_{s,x}\text{-a.s.}$$

for q.e. $(s, x) \in D_T$. Put $C_t = \int_0^t Z_r dr$, $t \geq 0$. Since $Z \mathbf{1}_{[0, \zeta_\tau]} = Z$, using the property of Z proved in Proposition 3.2, one can check that for every $h \geq 0$,

$$C_{(t+h) \wedge (T - \tau(0))} = C_{t \wedge (T - \tau(0))} + C_{h \wedge (T - \tau(0))} \circ \theta_{t \wedge (T - \tau(0))}.$$

Accordingly, C is a continuous AF of the process $\mathbf{X}$ killed at first exit from D_T . Therefore, it follows from [32, Theorem 66.2] that there exists a Borel measurable function ψ on D_T such that $Z = \psi(\mathbf{X})$, $dt \otimes P'_{s,x}$ -a.s. on $[0, \zeta_\tau] \times \Omega'$ for q.e. $(s, x) \in D_T$. Finally, by Tanaka's formula,

$$- \int_0^t \hat{Y}_s f(r, \mathbf{X}_r, Y_r) dr \leq |Y_t| - |Y_0| + \int_0^t \hat{Y}_{r-} dA_r^\mu - \int_0^t \hat{Y}_r dB_r, \quad 0 \leq t \leq \zeta_\tau. \quad (3.4)$$

If $\kappa \leq 0$, then by (H3), $-\hat{Y}_r(f(r, \mathbf{X}_r, Y_r)) \geq |f(r, \mathbf{X}_r, Y_r)| - |f(r, \mathbf{X}_r, 0)|$. On substituting this into (3.4) and then taking expectation on both sides of the inequality, we get (3.3) with $C = 1$. The general case can be reduced to the case $\kappa \leq 0$ by the standard change of variables. $\square$

In what follows, we denote by T_k , $k \geq 0$, the truncation operator, i.e.,

$$T_k(y) = (-k) \vee (y \wedge k), \quad y \in \mathbb{R}.$$

LEMMA 3.6. Assume that ψ_n, ψ are Borel measurable functions on D_T such that $\psi_n(\mathbf{X}) \rightarrow \psi(\mathbf{X})$, $dt \otimes P'_{s,x}$ -a.e. on $[0, \zeta_\tau] \times \Omega'$ for q.e. $(s, x) \in D_T$. Then, for some subsequence (still denoted by n), $\psi_n \rightarrow \psi$, m_1 -a.e. on D_T .

Proof. Let $(s, x) \in D_T$ be such that $\psi_n(\mathbf{X}) \rightarrow \psi(\mathbf{X})$, $dt \otimes P'_{s,x}$ -a.s. on $[0, \zeta_\tau] \times \Omega'$. Then,

$$E'_{s,x} \int_0^{\zeta_\tau} T_k(|\psi_n - \psi|)(\mathbf{X}_r) dr \rightarrow 0$$

as $n \rightarrow \infty$. Since

$$E'_{s,x} \int_0^{\zeta_\tau} T_k(|\psi_n - \psi|)(\mathbf{X}_r) dr = \int_{D_T} p_D(s, x, \theta, y) T_k(|\psi_n - \psi|)(\theta, y) d\theta dy,$$

there exists a subsequence (still denoted by n) such that $p_D(s, x, \theta, y) T_k(|\psi_n - \psi|) \rightarrow 0$, m_1 -a.e. on $[s, T] \times D$, which by the positivity of $p_D(s, x, \cdot, \cdot)$, and the definition of T_k implies that $\psi_n \rightarrow \psi$, m_1 -a.e. on $[s, T] \times D$. Since (s, x) can be chosen so that s is arbitrary close to zero, one can choose a further subsequence (still denoted by n) such that $\psi_n \rightarrow \psi$, m_1 -a.e. on D_T . $\square$

We say that a measurable function $u : D_T \rightarrow \mathbb{R}$ is of class (FD) if for q.e. $(s, x) \in D_T$ the process $[s, T] \ni t \mapsto u(t, X_t)$ is of Doob's class (D) under the measure $P_{s,x}$.

We say that a measurable function $u : D_T \rightarrow \mathbb{R}$ is quasi-càdlàg if for q.e. $(s, x) \in D_T$ the process $[s, T] \ni t \mapsto u(t, X_t)$ is càdlàg under $P_{s,x}$.

$\mathcal{FD}$ is the set of all quasi-càdlàg functions $u : D_T \rightarrow \mathbb{R}$. $\mathcal{FD}^q$ (resp. $\mathcal{FD}^q$), $q > 0$, is the set of all quasi-continuous (resp. quasi-càdlàg) functions $u : D_T \rightarrow \mathbb{R}$ such that for q.e. $(s, x) \in D_T$,

$$E_{s,x} \sup_{s \leq t \leq T} |u(t, X_t)|^q < +\infty.$$

FM^q , $q > 0$, is the set of all measurable functions $u : D_T \rightarrow \mathbb{R}$ such that for q.e. $(s, x) \in D_T$,

$$E_{s,x} \left(\int_s^T |u(t, X_t)|^2 dt \right)^{q/2} < +\infty.$$

$\mathcal{T}_2^{0,1}$ is the set of all measurable functions u on D_T such that for every $k \geq 0$, $T_k(u) \in L^2(0, T; H_0^1(D_T))$. From [4, Lemma 2.1], it follows that for every $u \in \mathcal{T}_2^{0,1}$, there exists a unique measurable function v on D_T such that $\nabla T_k(u) = \mathbf{1}_{\{|u| < k\}} v$, m_1 -a.e. We shall set $\nabla u = v$.

$W_0^{1,q}$, $q \geq 1$, is the closure of the space $C_c^\infty(D)$ in the Sobolev space $W^{1,q}$ of functions $u \in L^2(D)$ having weak derivatives in $L^q(D)$. In particular, $W_0^{1,2} = H_0^1$.

PROPOSITION 3.7. Assume that $\mu \in \mathcal{M}_{0,b}^+(D_T)$, $\varphi \in L^1(D)$, $\varphi \geq 0$. Then, $v : D_T \rightarrow \mathbb{R}$ defined as

$$v(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T - \tau(0)}) + \int_0^{\zeta_\tau} dA_r^\mu \right), \quad (s, x) \in D_T \quad (3.5)$$

is of class (FD) and belongs to $\mathcal{FD}^q$ for any $q \in (0, 1)$. Moreover, $v \in \mathcal{T}_2^{0,1}$, $v \in L^q(0, T; W_0^{1,q})$ for $q \in [1, \frac{d+2}{d+1})$, $\nabla v \in FM^q$ for $q \in (0, 1)$ and for every $k \geq 0$,

$$E'_{s,x} \int_0^{\zeta_\tau} |\sigma \nabla T_k(v)|^2(r, X_r) dr \leq 4k E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} dA_r^\mu \right) \quad (3.6)$$

for q.e. $(s, x) \in D_T$. Finally, for q.e. $(s, x) \in D_T$, the pair $(v(\mathbf{X}), \sigma \nabla v(\mathbf{X}))$ is a solution of $\text{BSDE}_{s,x}(\varphi, D, d\mu)$.

Proof. By Proposition 3.5, there exists a pair of processes (Y, Z) such that Y is of class (D), $(Y, Z) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$, and for q.e. $(s, x) \in D_T$, the pair (Y, Z) is a solution of $\text{BSDE}_{s,x}(\varphi, D, d\mu)$, i.e.,

$$Y_t = \mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} Z_r dB_r, \quad t \in [0, \zeta_\tau].$$

By Proposition 3.5 again, for $n \in \mathbb{N}$, there is a pair of processes (Y^n, Z^n) such that Y^n is of class (D), $(Y^n, Z^n) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$, and (Y^n, Z^n) is a solution of the following $\text{BSDE}_{s,x}$

$$Y_t^n = \mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} n(Y_r^n - Y_r)^- dr - \int_t^{\zeta_\tau} Z_r^n dB_r, \quad t \in [0, \zeta_\tau] \quad (3.7)$$

for q.e. $(s, x) \in D_T$. By [12, Theorem 5.7], for q.e. $(s, x) \in D_T$,

$$Y_t^n \nearrow Y_t, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}, \quad (3.8)$$

$$Z^n \rightarrow Z, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'. \quad (3.9)$$

Let $A_t^n = \int_0^t n(Y_r^n - Y_r)^- dr$. By Proposition 3.5, $Y_t = v(\mathbf{X}_t)$, $P'_{s,x}$ -a.s. for $t \in [0, T - \tau(0)]$ and $Z = \psi(\mathbf{X})$, $dt \otimes P'_{s,x}$ -a.e. on $[0, \zeta] \times \Omega'$ for some measurable function ψ on D_T . Therefore,

$$A_t^n = \int_0^t n(Y_r^n - v(\mathbf{X}_r))^- dr, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

From this, (3.7) and Proposition 3.5,

$$Y_t^n = v_n(\mathbf{X}_t), \quad P'_{s,x}\text{-a.s. for } t \in [0, \zeta_\tau] \quad (3.10)$$

and

$$Z^n = \psi_n(\mathbf{X}), \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega', \quad (3.11)$$

where

$$v_n(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} \phi_n(\mathbf{X}_r) dr \right), \quad \phi_n = n(v_n - v)^-$$

and ψ_n is a measurable function on D_T . Now fix $n \in \mathbb{N}$ and put $(\bar{Y}, \bar{Z}) = (Y^n, Z^n)$, $\bar{\phi} = \phi_n$. Then, for q.e. $(s, x) \in D_T$, the pair $(\bar{Y}, \bar{Z})$ is a solution of $\text{BSDE}_{s,x}(\varphi, D, \bar{\phi})$.

Let $\varphi_m = \varphi \wedge m$, $\bar{\varphi}_m = \bar{\varphi} \wedge m$. By Proposition 3.5, for q.e. $(s, x) \in D_T$, there exists a solution $(\bar{Y}^m, \bar{Z}^m)$ of BSDE $_{s,x}(\varphi_m, D, \bar{\varphi}_m)$ such that $\bar{Y}^m$ is of class (D) and $(\bar{Y}^m, \bar{Z}^m) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ (it is known that in fact $\bar{Y}^m$ is continuous). By [12, Theorem 5.7],

$$\bar{Y}_t^m \nearrow \bar{Y}_t, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}, \quad (3.12)$$

$$\bar{Z}^m \rightarrow \bar{Z}, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'. \quad (3.13)$$

We divide the proof that v has the desired regularity properties into three steps.

Step 1. We assume that $\varphi \in L^2(D)$ and $\mu = g \cdot m_1$ for some $g \in L^2(D_T)$, $g \geq 0$. Let $w \in \mathcal{W}(D_T)$ be a weak solution of the problem

$$\begin{cases} \frac{\partial w}{\partial t} + A_t w = -g & \text{in } D_T, \\ w(T, \cdot) = \varphi, \quad w(t, \cdot)|_{\partial D} = 0, & t \in (0, T). \end{cases}$$

Let $\tilde{w} : S_T := [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be an extension of w on S_T such that

$$\tilde{w}(t, x) = \begin{cases} w(t, x), & (t, x) \in D_T, \\ w(t, x) = 0, & (t, x) \in S_T \setminus D_T. \end{cases}$$

Then, $\tilde{w} \in \mathcal{W}(S_T)$ and hence $\frac{\partial \tilde{w}}{\partial t} + A_t \tilde{w} \in L^2(0, T; H^{-1}(\mathbb{R}^d))$. Let $g_0, G \in L^2(S_T)$ be such that $\frac{\partial \tilde{w}}{\partial t} + L_t \tilde{w} = -g_0 - \operatorname{div}(G)$. By [14, Theorem 4.3], $\tilde{w}$ has a quasi-continuous m_1 version (still denoted by $\tilde{w}$) such that $\tilde{w} \in \mathcal{F}S^2$, $\nabla \tilde{w} \in FM^2$ and for q.e. $(s, x) \in S_T$,

$$\begin{aligned} \tilde{w}(t, X_t) &= \tilde{w}(s, x) - \int_s^t g_0(r, X_r) dr - \int_s^t G(r, X_r) d^* X_r \\ &\quad + \int_s^t \sigma \nabla \tilde{w}(r, X_r) dB_{s,r}, \quad s \leq t \leq T, \quad P_{s,x}\text{-a.s.} \end{aligned}$$

Hence, by Lemma 2.1,

$$\begin{aligned} \tilde{w}(\mathbf{X}_t) &= \mathbf{1}_{\{\zeta > T - \tau(0)\}} \tilde{w}(\mathbf{X}_{T - \tau(0)}) + \int_t^{\zeta_\tau} g(\mathbf{X}_r) dr \\ &\quad - \int_t^{\zeta_\tau} \sigma \nabla \tilde{w}(\mathbf{X}_r) dB_r, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \end{aligned}$$

Observe also that $E'_{s,x} |(\tilde{w} - \varphi)(\mathbf{X}_{T - \tau(0)})| = \int_D p_D(s, x, T, y) |\tilde{w} - \varphi|(T, y) dy = 0$. Thus,

$$\tilde{w}(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T - \tau(0)}) + \int_0^{\zeta_\tau} g(\mathbf{X}_r) dr \right). \quad (3.14)$$

Therefore, $v = w$ satisfies all assertions of the proposition except from (3.6). To prove (3.6), let us fix $k > 0$ and $z \in \mathbb{R}$. Using the convention of Remark 3.1, by Tanaka's formula and taking expectations,

$$\begin{aligned} |\tilde{w}(s, x) - z| + E'_{s,x} L_{\zeta_\tau}^z(\tilde{w}(\mathbf{X})) &= E'_{s,x} |\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T - \tau(0)}) - z| \\ &\quad + E'_{s,x} \int_0^{\zeta_\tau} \operatorname{s\,gn}(\tilde{w}(\mathbf{X}_r) - z) g(\mathbf{X}_r) dr, \end{aligned}$$

where $\{L_t^z(\tilde{w}(\mathbf{X})), t \geq 0\}$ denotes the (symmetric) local time of $\tilde{w}(\mathbf{X})$ at z and $\text{s\grave{g}n}(x) = \mathbf{1}_{x \neq 0} \frac{x}{|x|}$. Hence,

$$E'_{s,x} L_{\zeta_\tau}^z(\tilde{w}(\mathbf{X})) \leq E'_{s,x} |\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) - \tilde{w}(s, x)| + E'_{s,x} \int_0^{\zeta_\tau} g(\mathbf{X}_r) dr.$$

Multiplying the above inequality by the function $i(z) = \mathbf{1}_{[-k,k]}(z)$, integrating with respect to z and applying the occupation time formula and Fubini's theorem, we get

$$E'_{s,x} \int_0^{\zeta_\tau} |\sigma \nabla T_k(\tilde{w})|^2(\mathbf{X}_r) dr \leq 2k E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \tilde{w}(s, x) + \int_0^{\zeta_\tau} g(\mathbf{X}_r) dr \right),$$

which when combined with (3.14) yields (3.6).

Step 2. We are going to show that v_n satisfies all the assertions of the proposition. To shorten notation, we write $\bar{v}$ (resp. $\bar{\psi}$) instead of v_n (resp. ψ_n). Since $\varphi_m \in L^2(D)$ and $\bar{\varphi}_m \in L^2(D_T)$, it follows from Step 1 that there exists $\bar{v}_m \in \mathcal{FS}^2$ such that $\bar{v}_m \in \mathcal{W}(D_T)$ and for q.e. $(s, x) \in D_T$,

$$\begin{aligned} \bar{v}_m(\mathbf{X}_t) &= \bar{Y}_t^m, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}, \\ \sigma \nabla \bar{v}_m(\mathbf{X}) &= \bar{Z}^m, \quad dt \otimes P'_{s,x}\text{-a.s. on } [0, \zeta_\tau] \times \Omega'. \end{aligned}$$

Put $\bar{v}'(s, x) = \limsup_{m \rightarrow \infty} \bar{v}_m(s, x)$, $(s, x) \in D_T$. Then, by (3.12), $\bar{v}'(\mathbf{X}_t) = \bar{Y}_t$, $t \in [0, \zeta_\tau]$, $P'_{s,x}$ -a.s. for q.e. $(s, x) \in D_T$. This implies that $\bar{v}' \in \mathcal{FD}^q$ for $q \in (0, 1)$, $\bar{v}'$ is of class (FD) and $\bar{v}' = \bar{v}$ q.e. The last statement implies that $\bar{v}$ belongs to the same spaces of functions as $\bar{v}'$. We have proved that (3.6) is true if $\mu = g \cdot m_1$ for some $g \in L^2(D_T)$, $g \geq 0$, and $\varphi \in L^2(D)$. Therefore, for q.e. $(s, x) \in D_T$,

$$\begin{aligned} E'_{s,x} \int_0^{\zeta_\tau} |\sigma \nabla T_k(\bar{v}_m)|(\mathbf{X}_r) dr &\leq 4k E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} \bar{\varphi}_m(\mathbf{X}_r) dr \right) \\ &\leq 4k E'_{s,x} \left(\mathbf{1}_{\{\zeta > T-\tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} \bar{\varphi}(\mathbf{X}_r) dr \right). \end{aligned} \quad (3.15)$$

From this and Lemma 2.5,

$$\int_{D_T} |\sigma \nabla (T_k(\bar{v}_m))|^2 dm_1 \leq 4k(\|\varphi\|_{L^1} + \|\bar{\varphi}_m\|_{L^1}) \leq 4k(\|\varphi\|_{L^1} + \|\bar{\varphi}\|_{L^1}). \quad (3.16)$$

Due to [24], from (3.16), it follows that $\bar{v} \in \mathcal{T}_2^{0,1} \cap L^q(0, T; W_0^{1,q}(D))$ and for some subsequence (still denoted by n), $\sigma \nabla \bar{v}_m \rightarrow \sigma \nabla \bar{v}$ weakly in $L^q(D_T)$ for $q \in [1, \frac{d+2}{d+1})$. On the other hand, by (3.13) and Lemma 3.6, $\sigma \nabla \bar{v}_m \rightarrow \bar{\psi}$, m_1 -a.e. on D_T . Hence, $\bar{\psi} = \sigma \nabla \bar{v}$, m_1 -a.e. on D_T . Summarizing, $\bar{v}$ satisfies all the assertions of the proposition.

Step 3. Using (3.8)–(3.11), one can show in the same manner as in Step 2 that v satisfies all the assertions of the proposition. The only difference from Step 2 is that in estimates of $\sigma \nabla T_k(v_n)$ of the form (3.15), (3.16) the function $\tilde{\phi}_m$ is replaced by ϕ_n , so to obtain the estimates for $\sigma \nabla T_k(v_n)$ which do not depend on n we have to show that

$$E'_{s,x} \int_0^{\zeta_\tau} \phi_n(\mathbf{X}_r) \, dr \leq E'_{s,x} \int_0^{\zeta_\tau} dA_r^\mu \quad (3.17)$$

for q.e. $(s, x) \in D_T$ and that $\|\phi_n\|_{TV} \leq \|\mu\|_{TV}$. But (3.17) follows from the fact that $v_n(s, x) \leq v(s, x)$ for q.e. $(s, x) \in D_T$ and the estimate for $\|\phi_n\|_{TV}$ follows from (3.17) and Lemma 2.5. $\square$

4. Cauchy–Dirichlet problem

It is convenient to begin the study of the obstacle problem (1.1) with the study of the Cauchy–Dirichlet problem

$$\begin{cases} \frac{\partial u}{\partial t} + A_t u = -f_u - \mu, \\ u(T, \cdot) = \varphi, \quad u(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T), \end{cases} \quad (4.1)$$

which can be regarded as problem (1.1) with $h_1 \equiv -\infty$, $h_2 \equiv +\infty$.

Let us recall that every functional $\Phi \in \mathcal{W}'(D_T)$ admits decomposition of the form

$$\Phi = (g)_t + \operatorname{div}(G) + f, \quad (4.2)$$

where $g \in L^2(0, T; H_0^1(D))$, $G = (G^1, \dots, G^d)$, $f \in L^2(D_T)$, i.e., for every $\eta \in \mathcal{W}(D_T)$,

$$\Phi(\eta) = -\left\langle g, \frac{\partial \eta}{\partial t} \right\rangle - \langle G, \nabla \eta \rangle_{L^2} + \langle f, \eta \rangle_{L^2},$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $L^2(0, T; H_0^1(D))$ and $L^2(0, T; H^{-1}(D))$. It is also known that every measure $\mu \in \mathcal{M}_{0,b}(D_T)$ admits decomposition of the form

$$\mu = \Phi + f, \quad (4.3)$$

where $\Phi \in \mathcal{W}'(D_T)$, $f \in L^1(D_T)$, i.e., for every $\eta \in C_0^\infty(D_T)$,

$$\int_{D_T} \eta \, d\mu = \Phi(\eta) + \int_{D_T} f \eta \, dm_1.$$

Accordingly, $\mu \in \mathcal{M}_{0,b}(D_T)$ can be written in the form

$$\mu = -(g)_t + \operatorname{div}(G) + f \quad (4.4)$$

for some $g \in L^2(0, T; H_0^1(D))$, $G \in L^2(D_T)$, and $f \in L^1(D_T)$. Let us stress that in general, Φ , f of the decomposition (4.3) cannot be taken nonnegative even if μ is nonnegative.

We say that a triple (g, G, f) is the decomposition of $\mu \in \mathcal{M}_{0,b}(D_T)$ if (4.4) is satisfied.

Let $\mu \in \mathcal{M}_{0,b}(D_T)$, $\varphi \in L^1(D)$ and let $f : D_T \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function.

DEFINITION. We say that a measurable function $u : D_T \rightarrow \mathbb{R}$ is a renormalized solution of the problem (4.1) if

- (a) $f_u \in L^1(D_T)$,
- (b) For some decomposition (g, G, f) of μ , $u - g \in L^\infty(0, T; L^2(D))$, $T_k(u - g) \in L^2(0, T; H_0^1(D))$ for $k \geq 0$ and

$$\lim_{k \rightarrow +\infty} \int_{\{k \leq |u-g| \leq k+1\}} |\nabla u| \, dm_1 = 0,$$

- (c) For any $S \in W^{2,\infty}(\mathbb{R})$ with compact support,

$$\begin{aligned} \frac{\partial}{\partial t}(S(u - g)) + \operatorname{div}(a \nabla u S'(u - g)) - S''(u - g) a \nabla u \cdot \nabla(u - g) \\ = -S'(u - g) f - \operatorname{div}(G S'(u - g)) + G S''(u - g) \cdot \nabla(u - g) \end{aligned}$$

in the sense of distributions,

- (d) $T_k(u - g)(T) = T_k(\varphi)$ in $L^2(D)$ for all $k \geq 0$.

Note that a different but equivalent definition of renormalized solution of (4.1) is given in [25, Definition 4.1].

Set

$$\Theta_k(s) = \int_0^s T_k(y) \, dy, \quad s \in \mathbb{R}$$

and

$$E = \left\{ \eta \in L^2(0, T; H_0^1(D)) \cap L^\infty(D_T) : \frac{\partial \eta}{\partial t} \in L^2(0, T; H^{-1}(D)) + L^1(D_T) \right\}.$$

DEFINITION. We say that a measurable function $u : D_T \rightarrow \mathbb{R}$ is an entropy solution of (4.1) if $f_u \in L^1(D_T)$, for some decomposition (g, G, f) of μ , $T_k(u - g) \in L^2(0, T; H_0^1(D))$ for any $k \geq 0$, $[0, T] \ni t \mapsto \int_D \Theta_k(u - g - \eta)(t, \cdot) \, dm$ is continuous for any $\eta \in E$, $k \geq 0$, and

$$\begin{aligned} \int_D \Theta_k(u - g - \eta)(0, \cdot) \, dm - \int_D \Theta_k(\varphi - \eta(T, \cdot)) \, dm - \langle \eta_t, T_k(u - g - \eta) \rangle \\ + \int_{D_T} a \nabla u \cdot \nabla T_k(u - g - \eta) \, dm_1 \leq \int_{D_T} f T_k(u - g - \eta) \, dm_1 \\ - \int_{D_T} G \cdot \nabla T_k(u - g - \eta) \, dm_1 + \int_{D_T} f_u T_k(u - g - \eta) \, dm_1. \end{aligned} \quad (4.5)$$

REMARK 4.1. (i) From [7, Theorem 3.1], it follows that u is a renormalized solution of (4.1) iff it is an entropy solution of (4.1).

(ii) If u is a renormalized solution of (4.1), then it is a distributional solution of (4.1) in the sense that $u, \nabla u \in L^1(D_T)$ and for any $\eta \in C_0^\infty(D_T)$,

$$\int_{D_T} u \frac{\partial \eta}{\partial t} dm_1 + \int_{D_T} a \nabla u \cdot \nabla \eta dm_1 = \int_D \varphi \eta(T, \cdot) dm + \int_{D_T} f_u \eta dm_1 + \int_{D_T} \eta d\mu \quad (4.6)$$

(see Proposition 4.5 and Theorem 4.11 in [25]).

LEMMA 4.2. Assume that $\mu_n, \mu \in \mathcal{M}_{0,b}(D_T)$ and $\|\mu_n - \mu\|_{TV} \rightarrow 0$. Then, there exist $g_n, g \in L^2(0, T; H_0^1(D))$, $G_n, G \in L^2(D_T)$, $f_n, f \in L^1(D_T)$ such that

$$\mu_n = (g_n)_t + \operatorname{div}(G_n) + f_n, \quad \mu = (g)_t + \operatorname{div}(G) + f \quad (4.7)$$

and

$$G_n \rightarrow G \text{ in } L^2(D_T), \quad f_n \rightarrow f \text{ in } L^1(D_T), \quad g_n \rightarrow g \text{ in } L^2(0, T; H_0^1(D)). \quad (4.8)$$

Proof. From the proof of [8, Theorem 2.7], it follows that each $\mu \in \mathcal{M}_{0,b}(D_T)$ admits a decomposition of the form (4.3) with Φ, f such that $\|\Phi\|_{\mathcal{W}'(D_T)} \leq 1$, $\|f\|_{L^1} \leq \|\mu\|_{TV}$. Moreover, by [8, Lemma 2.24], Φ admits decomposition (4.2) with g, G, h such that

$$\|g\|_{L^2(0,T;H_0^1(D))} + \|G\|_{L^2} + \|h\|_{L^2} \leq \|\Phi\|_{\mathcal{W}'(D_T)}.$$

Therefore, repeating arguments from the proof of [17, Corollary 3.2], we get the desired result. $\square$

LEMMA 4.3. Let $\{\mu_n\} \subset \mathcal{M}_{0,b}(D_T)$, $\mu \in \mathcal{M}_{0,b}(D_T)$, $\{\varphi_n\} \subset L^1(D)$, $\varphi \in L^1(D)$ and let u_n (resp. u) be a renormalized solution of (4.1) with terminal condition φ_n (resp. φ), $f \equiv 0$ and with $-\mu_n$ (resp. $-\mu$) on the right-hand side. If $\|\mu_n - \mu\|_{TV} \rightarrow 0$ and $\|\varphi_n - \varphi\|_{L^1} \rightarrow 0$, then $u_n \rightarrow u$, m_1 -a.e.

Proof. By Lemma 4.2, we may assume that μ_n, μ are given by (4.7) and (4.8) is satisfied. But then, the lemma follows from [24, Proposition 4]. $\square$

PROPOSITION 4.4. Let $\varphi \in L^1(D)$, $\mu \in \mathcal{M}_{0,b}(D_T)$ and let v be defined by (3.5). Then, $v \in \mathcal{T}_2^{0,1}$, $v \in L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$ and v is a renormalized solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} + A_t v = -\mu, \\ v(T, \cdot) = \varphi, \quad v(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases} \quad (4.9)$$

Proof. Without loss of generality, we may assume that $\varphi \geq 0$ and $\mu \geq 0$. Assume for a moment that $\varphi \in L^2(D_T)$ and $\mu \in \mathcal{M}_{0,b}^+ \cap \mathcal{W}'(D_T)$. From the proof of Proposition 3.7, it follows that v is the q.e. limit of v_n , where v_n is a weak solution of

$$\begin{cases} \frac{\partial v_n}{\partial t} + A_t v_n = -n(v_n - v)^-, \\ v_n(T, \cdot) = \varphi, \quad v_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases}$$

But it is known (see, e.g., [20, Theorem 1.1]) that $\{v_n\}$ converges in $L^2(D_T)$ to a unique weak solution of (4.9). Therefore, v is a weak solution of (4.9). Since a weak solution of (4.9) is a renormalized solution, this proves the proposition under the additional assumptions on φ, μ . Assume now that $\varphi \in L^1(D)$ is nonnegative and $\mu \in \mathcal{M}_{0,b}^+(D_T)$. By [21, Theorem 5.6], there exists a generalized nest, i.e., an ascending sequence $\{F_n\}$ of compact subsets of D_T such that $\text{cap}(K \setminus F_n) \rightarrow 0$ for every compact $K \subset D_T$, with the property that $\mathbf{1}_{F_n} \cdot \mu \in \mathcal{M}_{0,b}^+(D_T) \cap \mathcal{W}'(D_T)$ and $\mu(D_T \setminus \bigcup_n F_n) = 0$. Let $\varphi_n = \varphi \wedge n$. By what has already been proved, v_n defined as

$$v_n(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi_n(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta\tau} \mathbf{1}_{F_n}(\mathbf{X}_r) dA_r^\mu \right), \quad (s, x) \in D_T \quad (4.10)$$

is a renormalized solution of (4.9) with φ, μ replaced by φ_n and μ_n , respectively. Since $\|\varphi_n - \varphi\|_{L^1} \rightarrow 0$ and $\{F_n\}$ is a generalized nest, we conclude from (4.10) that $v_n(s, x) \rightarrow v(s, x)$ for q.e. $(s, x) \in D_T$. This completes the proof because by Lemma 4.3, $\{v_n\}$ converges to the renormalized solution of (4.9). $\square$

THEOREM 4.5. *Assume (H1)–(H5). Then, there exists a unique renormalized solution u of (4.1). Moreover, $u \in \mathcal{FD}$, $u \in L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1}]$ and*

$$u(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta\tau} f_u(\mathbf{X}_r) dr + \int_0^{\zeta\tau} dA_r^\mu \right) \quad (4.11)$$

for q.e. $(s, x) \in D_T$. Finally, there exists $C > 0$ depending only on κ, T such that

$$\|f_u\|_{L^1(D_T)} \leq C(\|\varphi\|_{L^1(D)} + \|f(\cdot, \cdot, 0)\|_{L^1(D_T)} + \|\mu\|_{TV}). \quad (4.12)$$

Proof. By Lemma 3.4 and [12, Proposition 3.10], for q.e. $(s, x) \in D_T$, there is a unique solution $(Y^{s,x}, Z^{s,x})$ of $\text{BSDE}_{s,x}(\varphi, D, f + d\mu)$. By Proposition 3.5, for q.e. $(s, x) \in D_T$, we have $Y_t^{s,x} = u(\mathbf{X}_t)$, $P'_{s,x}$ -a.s. for every $t \in [0, T - \tau(0)]$, where $u : D_T \rightarrow \mathbb{R}$ satisfies (4.11). Furthermore, by Lemma 2.5 and (3.3), f_u satisfies (4.12). Hence $f_u \cdot m_1 + \mu \in \mathcal{M}_{0,b}(D_T)$, and from Proposition 4.4, it follows that u is a renormalized solution of (4.1) and has the desired regularity properties. The uniqueness part of the theorem follows from the uniqueness of solutions of $\text{BSDE}_{s,x}(\varphi, D, f + d\mu)$. $\square$

REMARK 4.6. *Under the assumptions of Theorem 4.5, for every $q \in [1, \frac{d+2}{d+1}]$, there exist $C, \gamma > 0$ such that*

$$\|u\|_{L^q(0,T;W_0^{1,q}(D))} \leq C \left(\|\varphi\|_{L^1(D)} + \|f(\cdot, \cdot, 0)\|_{L^1(D_T)} + \|\mu\|_{TV} \right)^\gamma,$$

because from Remark 4.1(ii) and results proved in [5, Section 3] (see also [6]), it follows that if u is a solution of (4.1) then $\|u\|_{L^q(0,T;W_0^{1,q}(D))} \leq c \|f_u \cdot m_1 + \mu\|_{TV}^\gamma$ for some $c, \gamma > 0$.

5. Obstacle problem

We begin with a probabilistic definition of a solution of the obstacle problem.

DEFINITION. Assume (H1), (H4) and let h_1, h_2 be measurable functions on D_T such that $h_1 \leq h_2$, m_1 -a.e. We say that a pair (u, v) consisting of a measurable function $u : D_T \rightarrow \mathbb{R}$ and a measure v on D_T is a solution of the obstacle problem with terminal condition φ , right-hand side $f + d\mu$ and obstacles h_1, h_2 (OP($\varphi, f + d\mu, h_1, h_2$) for short) if

(a) $f_u \in L^1(D_T)$, $v \in \mathcal{M}_{0,b}(D_T)$, $h_1 \leq u \leq h_2$, m_1 -a.e.,

(b) For q.e. $(s, x) \in D_T$,

$$u(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} f_u(\mathbf{X}_r) dr + \int_0^{\zeta_\tau} d(A_r^\mu + A_r^v) \right),$$

(c) For every $h_1^*, h_2^* \in \mathcal{FD}$ such that $h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2$, m_1 -a.e. we have

$$\int_0^{\zeta_\tau} (u_-(\mathbf{X}_r) - h_{1-}^*(\mathbf{X}_r)) dA_r^{v+} = \int_0^{\zeta_\tau} (h_{2-}^*(\mathbf{X}_r) - u_-(\mathbf{X}_r)) dA_r^{v-} = 0, \\ P'_{s,x}\text{-a.s.}$$

for q.e. $(s, x) \in D_T$, where $g_-(\mathbf{X}_r) = \lim_{t \leq r, t \rightarrow r} g(\mathbf{X}_t)$ for $g := u, h_1^*, h_2^*$.

We say that (u, v) is a solution of the obstacle problem with one lower (resp. upper) barrier h (QP($\varphi, f + d\mu, h$) (resp. $\overline{\text{QP}}(\varphi, f + d\mu, h)$) for short) if (u, v) satisfies the conditions of the above definition with $h_1 = h, h_2 = +\infty$ (resp. $h_1 = -\infty, h_2 = h$) and $v \in \mathcal{M}_{0,b}^+(D_T)$ (resp. $-v \in \mathcal{M}_{0,b}^+(D_T)$).

REMARK 5.1. Let us note that in view of Proposition 4.4, condition (b) in the above definition says that u is a renormalized solution of (4.1) with μ replaced by $\mu + v$. Condition (c) provides a probabilistic formulation of minimality of v . In [26], minimality of v was expressed by using the notion of precise versions of functions introduced in [27]. In fact, in the linear case with L^2 data, condition (c) coincides with that introduced in [26], because for every parabolic potential h , $\hat{h}(\mathbf{X}_t) = h_-(\mathbf{X}_t)$, $t \in [0, \zeta_\tau]$, where $\hat{h}$ is the precise version of h (see [11] for details).

As in the case of the Cauchy–Dirichlet problem considered in the previous section, the proof of the existence of a solution of the obstacle problem relies heavily on the results on BSDEs proved in [12].

Suppose we are given a filtered probability space and A, B, σ, ξ as in Sect. 3.1. Moreover, suppose that we are given two progressively measurable processes U, L such that $U \leq L$ and $f : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$ such that $f(\cdot, y)$ is progressively measurable (f does not depend on z).

DEFINITION. A triple (Y, Z, R) of progressively measurable processes is a solution of the reflected backward stochastic differential equation with terminal condition ξ , right-hand side $f + dA$ and two reflecting barriers L, U (RBSDE($\xi, \sigma, f + dA, L, U$) for short) if

- (a) $Z \in M, t \mapsto f(t, Y_t, Z_t) \in L^1(0, \sigma), P\text{-a.s.},$
- (b) $L_t \leq Y_t \leq U_t$ for a.e. $t \in [0, \sigma], P\text{-a.s.},$
- (c) $Y_t = \xi + \int_t^\sigma f(s, Y_s, Z_s) ds + \int_t^\sigma dA_s - \int_t^\sigma dR_s - \int_t^\sigma Z_s dB_s, 0 \leq t \leq \sigma,$
 $P\text{-a.s.},$
- (d) $R \in \mathcal{V}$ and for any $\hat{L}, \check{U} \in \mathcal{D}$ such that $L_t \leq \hat{L}_t \leq Y_t \leq \check{U}_t \leq U_t$ for a.e. $t \in [0, \sigma], \int_0^\sigma (Y_{t-} - \hat{L}_{t-}) dR_t^+ = \int_0^\sigma (\check{U}_{t-} - Y_t) dR_t^- = 0, P\text{-a.s.}$

We say that (Y, Z, R) is a solution of RBSDE with one lower (resp. upper) barrier L (resp. U), terminal condition ξ , the right-hand side $f + dA$ ($\underline{\text{RBSDE}}(\xi, \sigma, f + dA, L)$) (resp. $\bar{\text{RBSDE}}(\xi, \sigma, f + dA, U)$) for short) if (Y, Z, R) satisfies the conditions of the above definition with $U \equiv +\infty$ (resp. $L \equiv -\infty$) and $R \in \mathcal{V}^+$ (resp. $-R \in \mathcal{V}^+$).

To shorten notation, in what follows for given μ, φ, f and h_1, h_2 we denote by $\text{RBSDE}_{s,x}(\varphi, D, f + d\mu, h_1, h_2)$ the reflected BSDE with data $\mathbf{1}_{\{\zeta > T - \tau(0)\}}\varphi$ ($\mathbf{X}_{T-\tau(0)}, \zeta_\tau, f(\mathbf{X}, \cdot, \cdot) + dA^\mu, h_1(\mathbf{X}), h_2(\mathbf{X})$) considered on the space $(\Omega', \mathcal{G}'_\infty, \{\mathcal{G}'_t\}, P'_{s,x})$.

Let $\mathcal{M}^q, q > 0$, denote the space of continuous martingales such that $E([M]_T)^{q/2} < +\infty$ for every $T > 0$.

We will need the following general growth conditions for f .

- (A6) There exists a semimartingale Γ such that Γ is of class (D), $\Gamma \in \mathcal{M}^q \oplus \mathcal{V}^1$ for $q \in (0, 1), L_t \leq \Gamma_t$ for a.e. $t \in [0, \sigma]$ and $E \int_0^\sigma f^-(t, \Gamma_t) dt < +\infty,$
- (A6') There exists a semimartingale Γ such that Γ is of class (D), $\Gamma \in \mathcal{M}^q \oplus \mathcal{V}^1$ for $q \in (0, 1), L_t \leq \Gamma_t \leq U_t$ for a.e. $t \in [0, \sigma]$ and $E \int_0^\sigma |f(t, \Gamma_t)| dt < +\infty.$

It is known that under (A1)–(A6) (resp. (A1)–(A5), (A6')), there exists a unique solution of BSDE($\xi, \sigma, f + dV, L$) (resp. RBSDE($\xi, \sigma, f + dA, L, U$)) (see Theorems 5.7, 6.6 in [12]). By Lemma 3.4, assumptions (H1)–(H5) imply that (A1)–(A5) hold under the measure $P'_{s,x}$ for q.e. $(s, x) \in D_T$.

Analytic analogue of conditions (A6), (A6') is as follows.

- (H6) There exists a measurable function $v : D_T \rightarrow \mathbb{R}$, a measure $\lambda \in \mathcal{M}_{0,b}(D_T)$ and $\phi \in L^1(D), \phi \geq \varphi$, such that v is a renormalized solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} + A_t v = -\lambda, \\ v(T, \cdot) = \phi, \quad v(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T) \end{cases} \quad (5.1)$$

and $f_v^- \in L^1(D_T), v \geq h_1, m_1\text{-a.e. on } D_T,$

- (H6') There exists a measurable function $v : D_T \rightarrow \mathbb{R}$, a measure $\lambda \in \mathcal{M}_{0,b}(D_T)$ and $\phi \in L^1(D), \phi \geq \varphi$, such that v is a renormalized solution of (5.1) and $f_v \in L^1(D_T), h_1 \leq v \leq h_2, m_1\text{-a.e. on } D_T.$

LEMMA 5.2. *Let $L = h_1(\mathbf{X}), U = h_2(\mathbf{X})$. If v satisfies (H6) (resp. (H6')), then $\Gamma = v(\mathbf{X})$ satisfies (A6) (resp. (A6')) under the measure $P'_{s,x}$ for q.e. $(s, x) \in D_T$.*

Proof. Follows immediately from Propositions 3.7 and 4.4. $\square$

We first prove the comparison and uniqueness results for solutions of the obstacle problem.

PROPOSITION 5.3. *Let (u_i, v_i) be a solution of $\text{OP}(\varphi_i, f^i + d\mu_i, h_1^i, h_2^i)$, $i = 1, 2$. If $\varphi_1 \leq \varphi_2$, m -a.e., $d\mu_1 \leq d\mu_2$, $h_1^1 \leq h_1^2$, $h_2^1 \leq h_2^2$, m_1 -a.e. and either*

$$f^2 \text{ satisfies (H3) and } \mathbf{1}_{\{u_1 > u_2\}} \left(f_{u_1}^1 - f_{u_2}^2 \right) \leq 0, \quad m_1\text{-a.e.}$$

or

$$f^1 \text{ satisfies (H3) and } \mathbf{1}_{\{u_1 > u_2\}} \left(f_{u_2}^1 - f_{u_2}^2 \right) \leq 0, \quad m_1\text{-a.e.}$$

then $u_1 \leq u_2$ q.e.

Proof. The desired result follows from [12, Corollary 6.2], because by the definition of a solution of the obstacle problem and Proposition 3.7, for q.e. $(s, x) \in D_T$ the triple $(u_i(\mathbf{X}), \sigma \nabla u_i(\mathbf{X}), A^{v_i})$ is a solution of $\text{RBSDE}_{s,x}(\varphi_i, D, f^i + d\mu_i, h_1^i, h_2^i)$. $\square$

COROLLARY 5.4. *Let assumption (H3) hold. Then, there exists at most one solution of $\text{OP}(\varphi, f + d\mu, h_1, h_2)$.*

Proof. Follows immediately from Proposition 5.3. $\square$

Before proving our main result on existence and approximation by the penalization method of solutions of the obstacle problem with one barrier, let us recall that a function v on D_T is a supersolution of $\text{PDE}(\varphi, f + d\mu)$ if there exists a measure $\lambda \in \mathcal{M}_{0,b}^+(D_T)$ such that v is a renormalized solution of the problem

$$\begin{cases} \frac{\partial v}{\partial t} + A_t v = -f_v - \mu - \lambda, \\ v(T, \cdot) = \varphi, \quad v(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases}$$

THEOREM 5.5. *Assume (H1)–(H6).*

- (i) *There exists a solution (u, v) of $\text{OP}(\varphi, f + d\mu, h_1)$ such that u is of class (FD), $u \in \mathcal{F}\mathcal{D}^q$ for $q \in (0, 1)$, $\nabla u \in FM^q$ for $q \in (0, 1)$, $u \in \mathcal{T}_2^{0,1}$, $u \in L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$.*
- (ii) *For $n \in \mathbb{N}$ let u_n be a renormalized solution of the problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} + A_t u_n = -f_{u_n} - n(u_n - h_1)^- - \mu, \\ u_n(T, \cdot) = \varphi, \quad u_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases} \quad (5.2)$$

Then, $u_n \nearrow u$ q.e. on D_T , $u_n \rightarrow u$ in $L^q(0, T; W_0^{1,q}(D))$ for any $q \in [1, \frac{d+2}{d+1})$ and $v_n \rightarrow v$ weakly, where $dv_n = n(u_n - h_1)^- dm_1$.

Proof. (i) By Lemmas 3.4, 5.2 and [12, Theorem 5.7], for q.e. $(s, x) \in D_T$, there exists a unique solution $(Y^{s,x}, Z^{s,x}, R^{s,x})$ of $\text{RBSDE}_{s,x}(\varphi, D, f + d\mu, h_1)$ such that $Y^{s,x}$ is of class (D), $(Y^{s,x}, Z^{s,x}) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and $R^{s,x} \in \mathcal{V}^{+,1}$. By [12, Theorem 5.7], for q.e. $(s, x) \in D_T$,

$$Y_t^{n,s,x} \nearrow Y_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}, \quad (5.3)$$

$$Z_t^{n,s,x} \rightarrow Z_t^{s,x}, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega', \quad (5.4)$$

where $(Y^{n,s,x}, Z^{n,s,x})$ is a solution of the equation

$$Y_t^{n,s,x} = \mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_t^{\zeta_\tau} \left(f(r, \mathbf{X}_r, Y_r^{n,s,x}) + n(Y_r^{n,s,x} - h_1(\mathbf{X}_r))^- \right) dr \\ + \int_t^{\zeta_\tau} dA_r^\mu - \int_t^{\zeta_\tau} Z_r^{n,s,x} dB_r, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}$$

By Proposition 3.7, $u_n(\mathbf{X}_t) = Y_t^{n,s,x}$, $t \in [0, \zeta_\tau]$, $Z^{n,s,x} = \sigma \nabla u_n(\mathbf{X})$, $dt \otimes P'_{s,x}$ -a.e. on $[0, \zeta_\tau] \times \Omega'$ for q.e. $(s, x) \in D_T$. For $(s, x) \in D_T$ set $u(s, x) = \limsup_{n \rightarrow +\infty} u_n(s, x)$. Then by (5.3), for q.e. $(s, x) \in D_T$,

$$u(\mathbf{X}_t) = Y_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \quad (5.5)$$

Moreover, by (5.4) and Lemma 3.6, there exists a Borel measurable function ψ on D_T such that for q.e. $(s, x) \in D_T$,

$$\psi(\mathbf{X}) = Z^{s,x}, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'. \quad (5.6)$$

Set

$$A_t = u(\mathbf{X}_0) - u(\mathbf{X}_t) - \int_0^t f_u(\mathbf{X}_r) dr - \int_0^t dA_r^\mu + \int_0^t \psi(\mathbf{X}_r) dB_r, \quad t \in [0, \zeta_\tau].$$

By (5.5) and (5.6),

$$A_t = R_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \quad (5.7)$$

for q.e. $(s, x) \in D_T$. It is an elementary check that A is an AF of $\mathbb{X}'$. In fact, by (5.7), it is a positive functional. Therefore, by Proposition in Section II.1 of [28] and [21, Theorem 5.6], there exists a positive smooth measure ν on D_T such that $A = A^\nu$. From this, the definition of A and (5.5), it follows that

$$u(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} f_u(\mathbf{X}_r) dr + \int_0^{\zeta_\tau} d(A_r^\mu + A_r^\nu) \right) \quad (5.8)$$

for q.e. $(s, x) \in D_T$. Let $\bar{v}$ denote the function from condition (H6). By Lemma 5.2, $\bar{v}(\mathbf{X})$ satisfies (A6) (with $L = h_1(\mathbf{X})$) under $P'_{s,x}$ for q.e. $(s, x) \in D_T$. Therefore, arguing as at the beginning of the proof of [12, Theorem 5.7] and using Theorem 4.5, one can show that there exists a supersolution $\bar{v}$ of $\text{PDE}(\varphi \vee \phi, f + d\mu)$ such that $u_n \leq \bar{v}$ q.e. on D_T for $n \in \mathbb{N}$. From this, the fact that $u_1 \leq u$ q.e. on D_T and (H3), it follows that $f_{\bar{v}} - \kappa \bar{v} \leq f_u - \kappa u \leq f_{u_1} - u_1$, m_1 -a.e. Hence, $f_u \in L^1(D_T)$, because $u_1, \bar{v}, f_{u_1}, f_{\bar{v}} \in L^1(D_T)$. Since $u \leq \bar{v}$ q.e. on D_T , it follows from (5.8) that

$$E'_{s,x} \int_0^{\zeta_\tau} dA_r^\nu \leq 2E'_{s,x} |\varphi(\mathbf{X}_{T-\tau(0)})| + E'_{s,x} |\phi(\mathbf{X}_{T-\tau(0)})| \\ + E'_{s,x} \int_0^{\zeta_\tau} (|f_u(\mathbf{X}_r)| + |f_{\bar{v}}(\mathbf{X}_r)|) dr + 2E'_{s,x} \int_0^{\zeta_\tau} d(|A_r^\mu| + |A_r^\lambda|)$$

for q.e. $(s, x) \in D_T$. Hence, by Lemma 2.5, $v \in \mathcal{M}_{0,b}^+(D_T)$. From (5.8) and Proposition 4.4, it follows now that u is a renormalized solution of the problem (4.1) with μ replaced by $\mu + v$, $u \in \mathcal{T}_2^{0,1}$ and $u \in L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$. Actually, from the definition of a solution of reflected BSDE and (5.5), (5.7), it follows that the pair (u, v) is a solution of $\text{OP}(\varphi, f + d\mu, h_1)$. Moreover, from (5.5) and the fact that for q.e. $(s, x) \in D_T$, the process $Y^{s,x}$ is of class (D) and $Y^{s,x} \in \mathcal{D}^q$ for $q \in (0, 1)$, we conclude that u is of class (FD) and $u \in \mathcal{FD}^q$ for $q \in (0, 1)$. Furthermore, since $(u(\mathbf{X}), \psi(\mathbf{X}))$ is a solution of $\text{BSDE}_{s,x}(\varphi, D, f_u + d(\mu + v))$, it follows from Proposition 3.7 that for q.e. $(s, x) \in D_T$,

$$\sigma \nabla u(\mathbf{X}) = \psi(\mathbf{X}), \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega'. \quad (5.9)$$

Hence, $\nabla u \in FM^q$ for $q \in (0, 1)$, because $Z^{s,x} \in M^q$, $q \in (0, 1)$, for q.e. $(s, x) \in D_T$.

(ii) From (5.3) and (5.5), it follows that $u_n \nearrow u$ q.e. on D_T , whereas from (5.4), (5.6), (5.9) and Lemma 3.6, it follows that $\nabla u_n \rightarrow \nabla u$ in measure m_1 . Since $u_n \leq u$ q.e. on D_T , we have

$$\begin{aligned} E'_{s,x} \int_0^{\zeta_\tau} dA_r^{v_n} &\leq C(\kappa, T) \left(E'_{s,x} |\varphi(\mathbf{X}_{T-\tau(0)})| \right. \\ &\quad \left. + E'_{s,x} \int_0^{\zeta_\tau} |f_u(\mathbf{X}_r)| dr + E'_{s,x} \int_0^{\zeta_\tau} d(|A^\mu|_r + |A^\lambda|_r) \right). \end{aligned}$$

Hence, $\sup_{n \geq 1} \|v_n\|_{TV} < +\infty$ by Lemma 2.5, and consequently, by Remark 4.6 and Vitali's theorem, $u_n \rightarrow u$ in $L^q(0, T; W_0^{1,q}(D))$ for every $q \in [1, \frac{d+2}{d+1})$. Suppose that for some subsequence, still denoted by n , $\{v_n\}$ converges weakly on D_T to some measure v' . By Remark 4.1, u_n is a distributional solution of (5.2) and u is a solution of (4.1) with μ replaced by $\mu + v$, i.e., for any $\eta \in C_0^\infty(D_T)$,

$$\int_{D_T} \left(u_n \frac{\partial \eta}{\partial t} + a \nabla u_n \cdot \nabla \eta \right) dm_1 = \int_D \varphi \eta(T, \cdot) dm + \int_{D_T} f_{u_n} \eta dm_1 + \int_{D_T} \eta d(\mu + v_n)$$

and (4.6) is satisfied with μ replaced by $\mu + v$. Since $f_{u_n} \rightarrow f_u$, m_1 -a.e. and by (H3), $f_v + \kappa(u_1 - v) \leq f_{u_n} \leq f_{u_1} + \kappa(u - u_1)$, applying the Lebesgue dominated convergence theorem, we conclude that $f_{u_n} \rightarrow f_u$ in $L^1(D_T)$. Since we also know that $u_n \rightarrow u$ and $\nabla u_n \rightarrow \nabla u$ in $L^1(D_T)$, it follows that $\int_{D_T} \eta dv = \int_{D_T} \eta dv'$, and hence that $v = v'$. Thus, $v_n \rightarrow v$ weakly on D_T , and the proof is complete. $\square$

COROLLARY 5.6. Assume (H1)–(H4). Let (u, μ) be a solution of $\text{OP}(\varphi, f + d\mu, h_1)$. Then,

$$u = \text{quasi-essinf}\{v \geq h_1, m_1\text{-a.e.} : v \text{ is a supersolution of } \text{PDE}(\varphi, f + d\mu)\}$$

q.e. on D_T .

Proof. Follows from Theorem 5.5, Proposition 3.7, Theorem 4.5 and [12, Lemma 4.9]. $\square$

COROLLARY 5.7. *Let (u, v) be a solution of $\text{OP}(\varphi, f + d\mu, h_1)$ and let $h^* \in \mathcal{FD}$ be such that $h \leq h^* \leq u$, m_1 -a.e. Then, for q.e. $(s, x) \in D_T$,*

$$u(s, x) = \sup_{\sigma \in \mathcal{T}'} E'_{s,x} \left(\int_0^{\sigma \wedge \zeta_\tau} f_u(\mathbf{X}_r) dr + \int_0^{\sigma \wedge \zeta_\tau} dA_r^\mu + h^*(\mathbf{X}_\sigma) \mathbf{1}_{\{\sigma < \zeta_\tau\}} \mathbf{1}_{\{\sigma < T - \tau(0)\}} + \varphi(\mathbf{X}_{T-\tau(0)}) \mathbf{1}_{\{\sigma = T - \tau(0)\}} \right),$$

where $\mathcal{T}'$ denotes the set of all $\{\mathcal{G}_t'\}$ -stopping times.

Proof. Follows from Theorem 5.5 and [12, Lemma 4.9]. $\square$

We now turn to the obstacle problem with two barriers.

THEOREM 5.8. *Assume (H1)–(H5), (H6').*

- (i) *There exists a solution (u, v) of $\text{OP}(\varphi, f + d\mu, h_1, h_2)$ such that u is of class (FD), $u \in \mathcal{FD}^q$ for $q \in (0, 1)$, $u \in \mathcal{T}_2^{0,1}$, $L^q(0, T, W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$ and $\nabla u \in FM^q$ for $q \in (0, 1)$.*
- (ii) *If A^μ is continuous, h_1, h_2 are quasi-continuous and $h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot)$, m -a.e. then u is quasi-continuous.*
- (iii) *Let u_n be a renormalized solution of the problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} + A_t u_n = -f_{u_n} - \mu - n(u_n - h_1)^- + n(u_n - h_2)^-, \\ u_n(T, \cdot) = \varphi, \quad u_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T). \end{cases}$$

Then, $u_n \rightarrow u$ q.e. on D_T and $u_n \rightarrow u$ in $L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$.

- (iv) *Let $(\underline{u}_n, \underline{\beta}_n)$ be a solution of $\underline{\text{OP}}(\varphi, \underline{f}_n + d\mu, h_1)$ with*

$$\underline{f}_n(t, x, y) = f(t, x, y) - n(y - h_2(t, x))^+.$$

Then $\underline{u}_n \searrow u$ q.e. on D_T , $\underline{u}_n \rightarrow u$ in $L^q(0, T; W_0^{1,q}(D))$ for $q \in [1, \frac{d+2}{d+1})$, $\underline{\beta}_n \geq \underline{\beta}_{n+1}$, $\underline{\beta}_n \searrow v^+$ setwise, $\underline{\gamma}_n \rightarrow v^-$ weakly on D_T , where $\underline{\gamma}_n = n(\underline{u}_n - h_2)^+ \cdot m_1$.

Proof. By Lemmas 3.4, 5.2 and [12, Theorem 6.6], for q.e. $(s, x) \in D_T$, there exists a solution $(Y^{s,x}, Z^{s,x}, R^{s,x})$ of $\text{RBSDE}_{s,x}(\varphi, D, f + d\mu, h_1, h_2)$ such that $Y^{s,x}$ is of class (D), $(Y^{s,x}, Z^{s,x}) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and $R^{s,x} \in \mathcal{V}^1$. Moreover, by [12, Theorem 6.6] again, for q.e. $(s, x) \in D_T$,

$$\underline{Y}_t^{n,s,x} \searrow Y_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}, \quad (5.10)$$

$$\underline{Z}^{n,s,x} \rightarrow Z^{s,x}, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega', \quad (5.11)$$

$$\underline{K}_t^{n,s,x} \searrow R_t^{s,x,+}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.}, \quad (5.12)$$

where $(\underline{Y}_t^{n,s,x}, \underline{Z}_t^{n,s,x}, \underline{K}_t^{n,s,x})$ is a solution of $\underline{\text{RBSDE}}_{s,x}(\varphi, D, \underline{f}_n + d\mu, h_1)$. By Theorem 5.5, $\underline{u}_n(\mathbf{X}_t) = \underline{Y}_t^{n,s,x}$, $t \in [0, \zeta_\tau]$, $P'_{s,x}$ -a.s. $\sigma \nabla \underline{u}_n(\mathbf{X}) = \underline{Z}_t^{n,s,x}$, $dt \otimes P'_{s,x}$ -a.e. on $[0, \zeta_\tau] \times \Omega'$ and $\underline{K}_t^{n,s,x} = A_t^{\underline{\beta}_n}$, $t \in [0, \zeta_\tau]$, $P'_{s,x}$ -a.s. Put $u(s, x) = \limsup_{n \rightarrow +\infty} \underline{u}_n(s, x)$, $(s, x) \in D_T$. Then, by (5.10),

$$u(\mathbf{X}_t) = Y_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \quad (5.13)$$

for q.e. $(s, x) \in D_T$. By Lemma 3.6 and (5.11), there exists a Borel measurable function ψ on D_T such that

$$\psi(\mathbf{X}) = Z^{s,x}, \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega' \quad (5.14)$$

for q.e. $(s, x) \in D_T$. Since $\underline{\beta}_n - \underline{\beta}_{n+1}$ is the Revuz measure of the AF $A_t^{\underline{\beta}_n} - A_t^{\underline{\beta}_{n+1}}$ and by (5.12) the functional is positive, it follows that the measure $\underline{\beta}_n - \underline{\beta}_{n+1}$ is positive. Therefore, $\{\underline{\beta}_n\}$ is a nonincreasing sequence and $R_t^{s,x,+} = A_t^{\nu_1}$, $t \in [0, \zeta_\tau]$, $P'_{s,x}$ -a.s., where ν_1 is a setwise limit of $\{\underline{\beta}_n\}$. Put

$$A_t = u(\mathbf{X}_0) - u(\mathbf{X}_t) - \int_0^t f_u(\mathbf{X}_r) dr + A_t^{\nu_1} + \int_0^t \psi(\mathbf{X}_r) dB_r, \quad t \geq 0.$$

As in the proof of Theorem 5.5, one can show that there exists a nonnegative smooth measure ν_2 on D_T such that $A = A^{\nu_2}$. Let $\nu = \nu^1 - \nu^2$. From the construction of ν^1, ν^2 , it follows that $\nu^1 = \nu^+$, $\nu^2 = \nu^-$ and

$$A_t^{\nu} = R_t^{s,x}, \quad t \in [0, \zeta_\tau], \quad P'_{s,x}\text{-a.s.} \quad (5.15)$$

for q.e. $(s, x) \in D_T$. By (5.13) and (5.15),

$$u(s, x) = E'_{s,x} \left(\mathbf{1}_{\{\zeta > T - \tau(0)\}} \varphi(\mathbf{X}_{T-\tau(0)}) + \int_0^{\zeta_\tau} f_u(\mathbf{X}_r) dr + \int_0^{\zeta_\tau} d(A_r^{\mu} + A_r^{\nu}) \right) \quad (5.16)$$

for q.e. $(s, x) \in D_T$. By (6.8) in [12] (see the beginning of the proof of [12, Theorem 6.6]) and Theorems 4.5, 5.5 and Lemma 2.5,

$$\|\underline{\beta}_n\|_{TV} \leq \|\underline{\beta}_1\|_{TV}, \quad \|\underline{\gamma}_n\|_{TV} \leq \|\delta_2^n\|_{TV}, \quad (5.17)$$

where $\delta_2^n = n(\nu_2^n - h_2)^+ \cdot m_1$, ν_2^n is a renormalized solution of $\text{PDE}(\phi \vee \varphi, \underline{f}_{\underline{\beta}_n} + d\lambda_2 + d\mu)$ with $\lambda_2 = f_v^- \cdot m_1 + \lambda^- + \mu^-$ (λ is the measure from condition (H6')). From the first inequality in (5.17), it follows that $\|\nu^+\|_{TV} < +\infty$. Furthermore, from Theorem 5.5, we know that $\sup_{n \geq 1} \|\delta_2^n\|_{TV} < +\infty$. Therefore, $\{\underline{\gamma}_n\}$ is tight, which in fact implies that $\underline{\gamma}_n \rightarrow \nu^-$ weakly on D_T (see the reasoning at the end of the proof of Theorem 5.5). Hence, $\|\nu^-\|_{TV} < +\infty$, and consequently $\nu \in \mathcal{M}_{0,b}(D_T)$. By (6.32) in [12] and Theorem 5.5, $\underline{u}_n \geq \bar{v}$, where $\bar{v}$ is the first component of the solution of $\overline{\text{OP}}(\varphi, f + d\mu, h_2)$. Using (5.13)–(5.16) and Proposition 4.4, one can deduce now in much the same way as in the proof of Theorem 5.5 that (u, ν) is a solution of $\text{OP}(\varphi, f + d\mu, h_1, h_2)$, u has the regularity properties stated in (i) and

$$\sigma \nabla u(\mathbf{X}) = \psi(\mathbf{X}), \quad dt \otimes P'_{s,x}\text{-a.e. on } [0, \zeta_\tau] \times \Omega' \quad (5.18)$$

for q.e. $(s, x) \in D_T$. From what has already been proved, (5.11), (5.14), (5.18), and Lemma 3.6, we get (iv). Finally, assertions (ii) and (iii) follow from [12, Theorem 6.6(ii),(v)], Lemma 3.6, Remark 4.6 and the stochastic representation (5.13)–(5.15), (5.18) of the solution (u, ν) . $\square$

COROLLARY 5.9. *Assume that A^μ is continuous, the barriers h_1, h_2 are quasi-continuous and $h_1(T, \cdot) \leq \varphi \leq h_2(T, \cdot)$, m -a.e. Then, the minimality condition (c) in the definition of a solution of the obstacle problem is equivalent to (1.5).*

Proof. Since u, h_1, h_2 are quasi-continuous, from condition (c), it follows that

$$E'_{s,x} \int_0^{\zeta_\tau} (u - h_1)(\mathbf{X}_r) dA_r^{v^+} = E'_{s,x} \int_0^{\zeta_\tau} (h_2 - u)(\mathbf{X}_r) dA_r^{v^-} = 0. \quad (5.19)$$

Hence,

$$\begin{aligned} & \int_s^T \int_D (u - h_1)(t, y) p_D(s, x, t, y) dv^+(t, y) \\ &= \int_s^T \int_D (h_2 - u)(t, y) p_D(s, x, t, y) dv^-(t, y) = 0, \end{aligned} \quad (5.20)$$

which implies (1.5), because $p_D(s, x, \cdot, \cdot)$ is positive on $(s, T] \times D$. Conversely, assume that (1.5) is satisfied. Then, (5.20), and consequently (5.19) is satisfied. Of course, (5.19) implies (c). $\square$

Let us note that in general, even in the case of one quasi-l.s.c. or quasi-u.s.c. reflecting barrier, the integrals in (1.5) may be strictly positive (see [11, Example 5.5]).

COROLLARY 5.10. *Assume (H1)–(H4). Let (u, v) be a solution of $\text{OP}(\varphi, f + d\mu, h_1, h_2)$. Then*

$u = \text{quasi-ess inf}\{v \geq h_1, m_1\text{-a.e.} : v \text{ is a supersolution of } \text{PDE}(\varphi, f + d\mu - dv^-)\}$
 $q.e. \text{ on } D_T.$

Proof. Follows from Theorem 5.8 and [12, Lemma 4.9]. $\square$

COROLLARY 5.11. *Let (u, v) be a solution of $\text{OP}(\varphi, f + d\mu, h_1, h_2)$ and let $h_1^*, h_2^* \in \mathcal{FD}$ be such that $h_1 \leq h_1^* \leq u \leq h_2^* \leq h_2$, m_1 -a.e. Then,*

$$\begin{aligned} u(s, x) = & \text{ess sup}_{\sigma \in \mathcal{T}'} \text{ess inf}_{\delta \in \mathcal{T}'} E'_{s,x} \left(\int_0^{\sigma \wedge \delta \wedge \zeta_\tau} f_u(\mathbf{X}_r) dr + \int_0^{\sigma \wedge \delta \wedge \zeta_\tau} dA_r^\mu \right. \\ & + h_1^*(\mathbf{X}_\delta) \mathbf{1}_{\{\delta \leq \sigma < T - \tau(0)\}} \mathbf{1}_{\{\delta < \zeta_\tau\}} + h_2^*(\mathbf{X}_\sigma) \mathbf{1}_{\{\sigma < \delta\}} \mathbf{1}_{\{\sigma < \zeta_\tau\}} \\ & \left. + \varphi(\mathbf{X}_{T - \tau(0)}) \mathbf{1}_{\{\sigma = \delta = T - \tau(0)\}} \right) \end{aligned}$$

$q.e. \text{ on } D_T$, where $\mathcal{T}'$ is the set of all $\{\mathcal{G}_t'\}$ -stopping times.

Proof. Follows from Theorem 5.8 and [18, Proposition 3.1]. $\square$

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