Maximal and Minimal Triangular Matrices

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Abstract. We call a nonscalar matrix maximal (or minimal) if its centralizer is maximal (respectively minimal) in the poset of all centralizers of matrices. We discuss the form of maximal and minimal matrices in the algebra of upper triangular matrices.

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1. Introduction

The interest in commuting matrices has quite long history. First results in this field are due to Frobenius [9]. It is known that the commutativity of two matrices a , b is connected to the eigenvalues of a , b and ab [7], eigenvectors of a and b [8, 12], but also to partitioning of matrices a , b into Jordan blocks [10]. Also maximal commutative subalgebras of $\mathcal{M}_{n \times n}(F)$ are investigated by many (see for instance [4, 11]).

In this work we will focus on a very similar topic.

Let $\mathcal{A}$ be a matrix algebra and let a be a nonscalar matrix. Denote by $C(a)$ the centralizer of a in $\mathcal{A}$. We can define a preorder $\preceq$ on $\mathcal{A}$ as follows.

$$a \preceq b \quad \text{if and only if} \quad C(a) \subseteq C(b)$$

If $a \in \mathcal{A}$ is a maximal, or respectively minimal, element in $(\mathcal{A}, \preceq)$, then we call a a maximal, or respectively minimal, matrix.

Such elements were described in [6, 15], but they are also considered in various contexts (e.g. in [5]).

In this article we study maximal and minimal elements of $\mathcal{T}_k(F)$ —the algebra of all $k \times k$ upper triangular matrices over a field F . First we wish to describe its maximal elements.

Before we formulate our results, let us note that the symbol e_k stands for the $k \times k$ identity matrix and e_{nm} for the matrix with 1 in the position (n, m) and 0 in every other position. Moreover, $\sigma(a)$ will denote the spectrum of a .

Theorem 1. *Let F be a field and $k \geq 3$. Then $a \in \mathcal{T}_k(F)$ is maximal if and only if a is of the form $\alpha e_k + \beta b$, where either*

1. b is similar to e_{nm} for some $n < m$,
or
2. b is idempotent different from 0 and e_k .

Analogous result holds for $\mathcal{T}_\infty(F)$ —the algebra of $\mathbb{N} \times \mathbb{N}$ upper triangular matrices over F .

Before we present a theorem, let us note that we write e_∞ for infinite identity matrix.

Theorem 2. *Let F be a field. The matrix $a \in \mathcal{T}_\infty(F)$ is maximal if and only if a is of the form $\alpha e_\infty + \beta b$, where either*

1. b is similar to e_{nm} for some $n \leq m$,
or
2. b is idempotent different from 0 and e_∞ .

In the algebra of all $n \times n$ matrices over a field F of at least n elements a matrix is minimal if and only if it is nonderogatory (see [6, 15]). This means that the minimality may have something to do with the Jordan form of a matrix. The case of triangular matrices is similar. However, before presenting this result we need to introduce some notions.

Definition 1. Let F be a field and $k \in \mathbb{N}$. If $a, a' \in \mathcal{T}_k(F)$ and there exist an invertible matrix $t \in \mathcal{T}_k(F)$ such that $t^{-1}at = a'$, then we will say that a, a' are T-similar.

Definition 2 (see [19]). Let $I_1 \sqcup I_2 \sqcup \dots \sqcup I_s = \{1, 2, \dots, k\}$ be a decomposition of $\{1, 2, \dots, k\}$ (that is $I_1, \dots, I_s$ are subsets with $I_i \cap I_j = \emptyset$ for $i \neq j$ and $I_1 \cup \dots \cup I_s = \{1, 2, \dots, k\}$). If $A_j = (a_{xy}^j)_{x,y=1}^{|I_j|}$ are matrices of order $|I_j|$, $j = 1, 2, \dots, s$, then the generalized direct sum is defined by

$$A = \left(a_{i,j=1}^k \right) = (A_1)_{I_1} \oplus (A_2)_{I_2} \oplus \dots \oplus (A_s)_{I_s},$$

where

$$a_{ij} = \begin{cases} 0 & \text{if } i \in I_n, j \in I_m, n \neq m \\ a_{xy}^n & \text{if } i = i_x, j = i_y, i, j \in I_n = \{i_1, i_2, \dots, i_{|I_n|}\}, i_p < i_{p+1}. \end{cases}$$

An example of a matrix that is a generalized direct sum is given below.

Example 1. Suppose that

$$a = \begin{pmatrix} \alpha & 0 & 1 & 0 \\ 0 & \beta & 0 & \gamma \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta' \end{pmatrix},$$

then $a = (a_1)_{I_1} \oplus (a_2)_{I_2}$, where

$$a_1 = \begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}, \quad I_1 = \{1, 3\}, \quad a_2 = \begin{pmatrix} \beta & \gamma \\ 0 & \beta' \end{pmatrix}, \quad I_2 = \{2, 4\}.$$

We give some description of minimal triangular matrices in the following theorem.

Theorem 3. *Let F be a field and let k be a natural number. The matrix $a \in \mathcal{T}_k(F)$ is minimal if and only if a is T-similar to a matrix that is a generalized direct sum of matrices $a_1 \in \mathcal{T}_{k_1}(F)$, $a_2 \in \mathcal{T}_{k_2}(F)$, $\dots$, $a_n \in \mathcal{T}_{k_n}(F)$ that satisfy the following conditions.*

- Each of the matrices a_i is minimal in $\mathcal{T}_{k_i}(F)$.
- We have

$$|\sigma(a)| = \min(n, |F|). \quad (1)$$

It can be easily seen that the number on the left side of Eq. (1) is equal to the number of different entries that appear on the main diagonal of a .

Also in this case, analogous result holds.

Theorem 4. *Suppose F is a field. The matrix $a \in \mathcal{T}_\infty(F)$ is minimal in $\mathcal{T}_\infty(F)$ if and only if it is a matrix $\oplus_{n \in I} (a_n)_{I_n}$ such that*

- each of the matrices a_n is minimal in $\mathcal{T}_{|I_n|}(F)$,
- we have $|\{a_{nn} : n \in \mathbb{N}\}| = \min(|I|, |F|)$.

2. Proofs of Theorems 1 and 3

2.1. Notation

First we introduce the notation.

By $j_k(\alpha)$ we will understand $k \times k$ Jordan cell with α on the main diagonal. The matrix transposed to x will be denoted by x^T .

We will use the symbol $\text{Diag}(a_1, a_2, \dots, a_r)$ for the block-diagonal matrix with the blocks $a_1, a_2, \dots, a_r$ on the main diagonal.

We will write $\mathcal{M}_{k \times k'}(F)$ for the vector space of all $k \times k'$ matrices over F . By $\mathcal{D}_k(F)$ we will understand the algebra of $k \times k$ diagonal matrices, and by $\mathcal{NT}_k(F)$ the algebra of $k \times k$ triangular matrices that are nilpotent.

By $Z(\mathcal{A})$ we will denote the center of the algebra $\mathcal{A}$, and by $C_{\mathcal{A}}(a)$ the centralizer of a in $\mathcal{A}$. If $\mathcal{A}$ will be fixed we will simply write $C(a)$.

If a, b are elements of some algebra $\mathcal{A}$ and b is invertible in $\mathcal{A}$, then we will write a^b for the element $b^{-1}ab$.

We end this subsection with the following obvious remark.

Remark 1. Let $\mathcal{A}$ be a matrix algebra over a field F containing the identity matrix e_k .

1. The matrix $a \in \mathcal{A}$ is maximal if and only if for any $\alpha, \beta \in F$, $\beta \neq 0$, the matrix $\alpha e_k + \beta a$ is maximal.

2. The matrix $a \in \mathcal{A}$ is minimal if and only if for any $\alpha, \beta \in F$, $\beta \neq 0$, the matrix $\alpha e_k + \beta a$ is minimal.
3. The matrix $a \in \mathcal{A}$ is maximal if and only if for any invertible $c \in \mathcal{A}$ the matrix a^c is maximal.
4. The matrix $a \in \mathcal{A}$ is minimal if and only if for any invertible $c \in \mathcal{A}$ the matrix a^c is minimal.

2.2. Maximal Triangular Matrices

We start with some easy remarks. The first one does not require a proof.

Remark 2. Let $\mathcal{A}$ be any algebra over a field F and $a \in \mathcal{A}$. Then for any $\alpha \in F$ we have $C(a) \subseteq C(a^2 - \alpha a)$.

We also have

Lemma 1. *For any field F and any number $k \in \mathbb{N}$ we have $Z(\mathcal{T}_k(F)) = \{\alpha e_k : \alpha \in F\}$.*

Proof. This follows easily from the fact that if x is in the center of $\mathcal{T}_k(F)$, then it must commute with all the matrices e_{nm} for $1 \leq n \leq m \leq k$. $\square$

We will also need some facts about diagonalization of matrices.

Lemma 2 ([16], Lemma 2.3). *Let F be any field. If $x \in \mathcal{T}_\infty(F)$ is an idempotent, then there exists an invertible matrix $t \in \mathcal{T}_\infty(F)$ such that x^t is a diagonal matrix.*

From the above lemma we can easily derive

Corollary 1. *Let F be any field and $k \in \mathbb{N}$. If $x \in \mathcal{T}_k(F)$ is an idempotent, then there exists an invertible matrix $t \in \mathcal{T}_k(F)$ such that x^t is a diagonal matrix.*

Lemma 3. *Let F be a field and let $k \in \mathbb{N}$. If $x \in \mathcal{T}_k(F)$ is similar to e_{nm} for some $n < m$, then x is T-similar to $e_{n'm'}$ for some $n' < m'$.*

Proof. As x is similar to e_{nm} , it must be a matrix of rank equal to 1. Thus, there exist $x_1, x_2, \dots, x_{n'} \in F$, $x_{n'} \neq 0$, and $y_{m'}, y_{m'+1}, \dots, y_k \in F$, $y_{m'} \neq 0$, such that

$$x = \sum_{\substack{1 \leq i \leq n' \\ m' \leq j \leq k}} x_i y_j e_{ij}.$$

Since x is triangular, $n' < m'$.

Define t_1 as

$$e_k - \sum_{j=m'+1}^k \frac{y_j}{y_{m'}} e_{m'j}.$$

Then we have

$$x^{t_1} = \sum_{1 \leq i \leq n'} x_i y_{m'} e_{im'}.$$

Let's put now

$$t_2 = e_k + \sum_{i=1}^{n'-1} \frac{x_i}{x_{n'}} e_{in'}.$$

For this matrix we obtain

$$(x^{t_1})^{t_2} = x_{n'} y_{m'} e_{n'm'}.$$

Now it suffices to use

$$t_3 = e_k + (x_{n'} y_{m'} - 1) e_{n'n'}$$

to get $x^{t_1 t_2 t_3} = e_{n'm'}$. $\square$

The above facts let us prove our first theorem.

Proof of Theorem 1. First we prove the sufficiency and next the necessity of holding conditions 1 or 2 for a to be maximal.

1. The sufficiency.

Suppose that $a = \alpha e_k + \beta b$, where b has one of the forms from Theorem 1. First we will consider b of the form given in point 1, then we will move to the b from point 2.

Case 1 condition 1 holds.

According to Remark 1 and Lemma 3, we can consider b instead of a . Moreover, by the same remark, we can focus on e_{nm} instead of b . Consider then e_{nm} .

From $x e_{nm} = e_{nm} x$, we get that x must satisfy the conditions

$$x_{nn} = x_{mm}, \quad x_{in} = 0 \quad \text{for } 1 \leq i < n, \quad x_{mj} = 0 \quad \text{for } j > m. \quad (2)$$

Suppose that $C(c) \supseteq C(e_{nm})$. Then c commutes with every x satisfying (2).

From the fact that for any $l \in \{1, 2, \dots, k\} \setminus \{n, m\}$ we have $ce_{ll} = e_{ll}c$ we obtain that $c = \sum_{i=1}^k c_{ii} e_{ii} + c_{nm} e_{nm}$.

Since for any pair l_1, l_2 such that $l_1 \neq n, l_2 \neq m$ we have $ce_{l_1 l_2} = e_{l_1 l_2} c$, we must have $c_{l_1 l_1} = c_{l_2 l_2}$. As $k \geq 3$ we can choose l_1, l_2 in such way that $l_1 \neq n, m$, and $l_2 \neq l_1$. From this we get $c_{ii} = c_{jj}$ for all i, j . Thus $c = \alpha e_k + c_{nm} e_{nm}$.

Case 2 condition 2 holds.

Again, by Remark 1, we can focus on matrices of the form $\sum_{i \in I} e_{ii}$, $\emptyset \subsetneq I \subsetneq \{1, 2, \dots, k\}$. The centralizer of such a matrix consists of the matrices x satisfying the following conditions.

$$x_{im} = 0, \quad x_{nj} = 0 \quad \text{for all } i, j \in I, \quad n, m \notin I \quad (3)$$

Suppose that c commutes with all matrices satisfying (3).

First we notice that since c commutes with $\sum_{i \in I} e_{ii}$ and $\sum_{n \notin I} e_{nn}$ we must have $c_{nj} = c_{im} = 0$ for all $i, j \in I, n, m \notin I$. Hence c can be written

as a sum of some c_1, c_2 satisfying conditions $(c_1)_{ij} \neq 0$ if and only if $i, j \in I$, and $(c_2)_{nm} \neq 0$ if and only if $n, m \notin I$.

As c commutes with every matrix of the form

$$\sum_{i \leq j, i, j \in I} x_{ij} e_{ij}$$

and every matrix of the form

$$\sum_{n \leq m, n, m \notin I} x_{nm} e_{nm},$$

from Lemma 1 we obtain that $c_1 = \alpha_1 \sum_{i \in I} e_{ii}$ and $c_2 = \alpha_2 \sum_{n \notin I} e_{nn}$. Hence we can write that $c = \alpha e_k + \beta \sum_{i \in I} e_{ii}$.

2. The necessity.

Let a be maximal.

Suppose first that $|\sigma(a)| = p$. If $p \geq 3$, we perform as follows.

Assume that (i_1, i_1) is the position in which the first nonzero entry on the main diagonal appears.

From Remark 2 we know that $C(a) \subseteq C(a^2 - \alpha a)$ for any $\alpha \in F$. Hence $C(a) \subseteq C(a^2 - a_{i_1 i_1} a)$. The matrix $a^2 - a_{i_1 i_1} a$ has at least $p - 1 \geq 2$ different entries on the main diagonal, so it is not a scalar matrix. Thus, as a was maximal, the matrix $a^2 - a_{i_1 i_1} a$ must be maximal as well.

If $a^2 - a_{i_1 i_1} a$ has only two different entries on the main diagonal, put $a' = a^2 - a_{i_1 i_1} a$. Otherwise, repeat the above procedure, until we do not get the matrix with two different entries on the main diagonal. Then, denote the obtained matrix by a' .

By the above arguments, if a is maximal, then its centralizer coincides with the centralizer of some a' such that $|\sigma(a')| \leq 2$. Hence, we will focus now on such matrices.

We make use of Remark 1 and obtain that instead of a' we can investigate a'' such that

- a'' has only zeros on the main diagonal – in the case when $a' = a$ has the same entry in all the positions (i, i) ($1 \leq i \leq k$),
- a'' has only zeros and ones on the main diagonal – in the case when a' has two different entries in the positions (i, i) ($1 \leq i \leq k$).

Consider then a'' .

Just as above, we can write that $C(a'') \subseteq C((a'')^2 - a'')$. We have the two possibilities: either $C((a'')^2 - a'') = \mathcal{T}_k(F)$ or $C((a'')^2 - a'')$ coincides with $C(a'')$ and is maximal. We discuss these cases separately.

Case 1 $C((a'')^2 - a'') = \mathcal{T}_k(F)$.

Since for all $1 \leq i \leq k$ we have $((a'')^2 - a'')_{ii} = 0$, from $C((a'')^2 - a'') = \mathcal{T}_k(F)$ and Lemma 1, it follows that $(a'')^2 - a'' = 0$, i.e. a'' is idempotent. By Corollary 1, for some $t \in \mathcal{T}_k(F)$ $a''' := (a'')^t$ is a diagonal matrix, so $a''' := \sum_{i \in I} e_{ii}$ for some $I \subseteq \{1, 2, \dots, k\}$ and it is of the form from

the second case of the theorem. As we have checked this a''' is maximal. Moreover, since $C(a) = C(a''')$, so a must be of the form $\alpha e_k + \beta \sum_{i \in I} e_{ii}$.

Case 2 $C((a'')^2 - a'') = C(a'')$.

Note that as $((a'')^2)_{ii} = a''_{ii}$ for all $1 \leq i \leq k$ and k is finite, the matrix $(a'')^2 - a''$ is nilpotent.

Consider now some matrix $b \in \mathcal{T}_k(F)$ that is nilpotent and maximal. Let us find a conjugate of b that is easier to work with.

Consider the indices n, m for which $b_{nm} \neq 0$ and the difference $m - n$ is minimal. If there is more than one such pair, choose the one with the minimal n .

Define t_1 as $e_k + b_{nm}^{-1} \sum_{i=1}^{n-1} b_{im} e_{in}$ and b_1 as b^{t_1} . We have

$$(b_1)_{im} = 0 \quad \text{for all } 1 \leq i < m.$$

Next, define t_2 as $e_k - (b_1)_{nm}^{-1} \sum_{j=m+1}^k (b_1)_{nj} e_{mj}$ and b_2 as $b_1^{t_2}$. Then

$$(b_2)_{nj} = 0 \quad \text{for all } j > n$$

and also

$$(b_2)_{im} = 0 \quad \text{for all } 1 \leq i < m.$$

Finally, let $t_3 = e_k + (b_{nm} - 1)e_{nm}$ and $b_3 = b_2^{t_3}$. Then

$$(b_3)_{nm} = 1 \tag{4}$$

and

$$(b_3)_{im} = 0 \quad \text{for all } 1 \leq i < m, \tag{5}$$

$$(b_3)_{nj} = 0 \quad \text{for all } j > n. \tag{6}$$

Suppose that $x \in C(b_3)$. Then from Eqs. (4), (5), (6) we obtain that

$$x_{nn} = x_{mm}, \quad x_{in} = 0 \quad \text{for } 1 \leq i < n, \quad x_{mj} = 0 \quad \text{for } j > m. \tag{7}$$

However, in the first part of the proof we have learned that the latter means that $x \in C(e_{nm})$. Thus, as b_3 is maximal, we must have $C(b_3) = C(e_{nm})$ and consequently $C(b) = C(y)$, where y is similar to e_{nm} .

Summing up, in point 1. we have checked that the matrices of the given forms are, in fact, maximal, whereas in point 2. we have shown that they are also the only matrices that satisfy the necessary condition for maximality. Thus, the proof is complete.

2.3. Minimal Triangular Matrices

In this subsection we will prove Theorem 3. We start with some calculations. More precisely, we have

Remark 3. Let F be a field, $k_1, k_2 \in \mathbb{N}$, $k = k_1 + k_2$. Assume that $a \in \mathcal{T}_{k_1}(F)$, $b \in \mathcal{T}_{k_2}(F)$, then

$$C_{\mathcal{M}_{k \times k}(F)} \left(\left(\frac{a|0}{0|b} \right) \right) = \left\{ \left(\frac{x_1|x_2}{x_3|x_4} \right) : \right. \\ \left. x_1 \in C_{\mathcal{M}_{k_1 \times k_1}(F)}(a), x_4 \in C_{\mathcal{M}_{k_2 \times k_2}(F)}(b), \right. \\ \left. ax_2 - x_2b = 0, bx_3 - x_3a = 0 \right\}, \quad (8)$$

and

$$C_{\mathcal{T}_k(F)} \left(\left(\frac{a|0}{0|b} \right) \right) \\ = \left\{ \left(\frac{x_1|x_2}{0|x_4} \right) : x_1 \in C_{\mathcal{T}_{k_1}(F)}(a), x_4 \in C_{\mathcal{T}_{k_2}(F)}(b), ax_2 - x_2b = 0 \right\}. \quad (9)$$

Proof. This follows from calculations on block matrices. $\square$

Let us consider the equation $ax_2 - x_2b = 0$ that appears in (8) and (9). It is known (see [13] or [3]) that it can be replaced by the equation

$$(e_{k_2} \otimes a - b^T \otimes e_{k_1}) \text{vec}(x_2) = 0, \quad (10)$$

where $\otimes$ is the Kronecker product and $\text{vec}(x_2)$ is a matrix from $\mathcal{M}_{k_1 k_2 \times 1}(F)$ defined by the rule $(x_2)_{ij} = (\text{vec}(x_2))_{(j-1)k_1+i}$.

Clearly, if $a_{ii} \neq b_{jj}$ for all $1 \leq i \leq k_1$, $1 \leq j \leq k_2$, the above system has exactly one solution, namely $x_2 = 0$, whereas if $a_{ii} = b_{jj}$ for all $1 \leq i \leq k_1$, $1 \leq j \leq k_2$, then system (10) has more than one solution, in particular $x_2 = e_{1k_2}$. Thus we obtain

Corollary 2. Suppose F is a field, $\alpha, \beta \in F$, $\alpha \neq \beta$, $k_1, k_2 \in \mathbb{N}$, $k = k_1 + k_2$. If $a \in \mathcal{NT}_{k_1}(F)$, $b \in \mathcal{NT}_{k_2}(F)$, then

$$C_{\mathcal{M}_{k \times k}(F)} \left(\left(\frac{\alpha e_{k_1} + a | 0}{0 | \beta e_{k_1} + b} \right) \right) \subsetneq C_{\mathcal{M}_{k \times k}(F)} \left(\left(\frac{\alpha e_{k_1} + a | 0}{0 | \alpha e_{k_2} + b} \right) \right)$$

and

$$C_{\mathcal{T}_k(F)} \left(\left(\frac{\alpha e_{k_1} + a | 0}{0 | \beta e_{k_1} + b} \right) \right) \subsetneq C_{\mathcal{T}_k(F)} \left(\left(\frac{\alpha e_{k_1} + a | 0}{0 | \alpha e_{k_2} + b} \right) \right)$$

Theorem 3 states that the minimality of a matrix is connected to T-similarity and generalized direct sums of matrices. Therefore we cite now some results in which these two notions appear.

Proposition 1 (Prop.1.2, [19]). Let $A \in \mathcal{T}_n(F)$ have the different eigenvalues $\alpha_1, \dots, \alpha_k$ and set $E_j = \{i : a_{ii} = \alpha_j\}$. Then there exist $A_1, \dots, A_k$, $A_j \in \mathcal{T}_{|E_j|}(F)$, $\sigma(A_j) = \{\alpha_j\}$ such that

$$A \text{ is T-similar to } (A_1)_{E_1} \oplus \dots \oplus (A_k)_{E_k};$$

moreover, if A is T-similar to $(\hat{A}_1)_{E_1} \oplus \dots \oplus (\hat{A}_k)_{E_k}$, then A_j is T-similar to $\hat{A}_j$, $j = 1, 2, \dots, k$.

In the same article it was proved

Theorem 5 (Thm.1.5, [19]). *Let $A \in \mathcal{T}_k(F)$, and assume that one of the following conditions is met*

1. *A is nonderogatory,*
2. *$\dim \ker(A - \lambda I)^2 = \dim \ker(A - \lambda I)^3$ for each $\lambda \in F$.*

Then there exists a matrix J which is a generalized direct sum of Jordan blocks and which is T-similar to A .

It should be noted that this theorem was formulated for matrices defined over $\mathbb{C}$, but the proof works also for the fields different from $\mathbb{C}$.

From the above considerations we get

Corollary 3. *Let F be a field and $k_1, \dots, k_n \in \mathbb{N}$, $k = \sum_{i=1}^n k_i$. Suppose also that $\alpha_1, \alpha_2, \dots, \alpha_n \in F$ are pairwise distinct.*

If $a_i \in \mathcal{T}_{k_i}(F)$ are such that $\sigma(a_i) = \{\alpha_i\}$ for all $1 \leq i \leq n$, then

$$\begin{aligned} C_{\mathcal{T}_k(F)}((a_1)_{I_1} \oplus \dots \oplus (a_n)_{I_n}) \\ = \left\{ (x_1)_{I_1} \oplus \dots \oplus (x_n)_{I_n} : \forall 1 \leq i \leq n \ x_i \in C_{\mathcal{T}_{k_i}(F)}(a_i) \right\}. \end{aligned}$$

Proof. It can be observed that $(a_1)_{I_1} \oplus (a_2)_{I_2} \oplus \dots \oplus (a_n)_{I_n}$ is similar (please note that it does not have to be T-similarity) to the matrix $\text{Diag}(a_1, \dots, a_n)$. From Remark 3 and discussion of solutions of Eq. (10) we know that if $\alpha_i \neq \alpha_j$ for all $i \neq j$, then

$$C_{\mathcal{M}_{k \times k}(F)}(\text{Diag}(a_1, \dots, a_n)) = \{ \text{Diag}(x_1, \dots, x_n) : \forall i=1, 2, \dots, n \ x_i \in C(a_i) \}.$$

Hence

$$\begin{aligned} C_{\mathcal{M}_{k \times k}(F)}((a_1)_{I_1} \oplus (a_2)_{I_2} \oplus \dots \oplus (a_n)_{I_n}) \\ = \{ (x_1)_{I_1} \oplus (x_2)_{I_2} \oplus \dots \oplus (x_n)_{I_n} : \forall 1 \leq i \leq n \ x_i \in C(a_i) \}. \end{aligned}$$

and

$$\begin{aligned} C_{\mathcal{T}_k(F)}((a_1)_{I_1} \oplus (a_2)_{I_2} \oplus \dots \oplus (a_n)_{I_n}) \\ = \left\{ (x_1)_{I_1} \oplus (x_2)_{I_2} \oplus \dots \oplus (x_n)_{I_n} : \forall 1 \leq i \leq n \ x_i \in C_{\mathcal{M}_{k_i \times k_i}(F)}(a_i) \right\} \cap \mathcal{T}_k(F) \\ = \left\{ (x_1)_{I_1} \oplus (x_2)_{I_2} \oplus \dots \oplus (x_n)_{I_n} : \forall 1 \leq i \leq n \ x_i \in C_{\mathcal{T}_{k_i}}(a_i) \right\} \end{aligned}$$

□

Now we can prove our second main result.

Proof of Theorem 3. We divide the proof into two steps. First we consider the matrices defined over a field containing at least k elements. Next we consider the second case for an arbitrary field.

1. *The case when $|F|$ is greater or equal to n .*

The sufficiency.

As it was written above there exists $t \in \mathcal{T}_k(F)$ such that a^t is equal to $(a_1)_{I_1} \oplus \dots \oplus (a_n)_{I_n}$ for some $I_1, \dots, I_n$, where each a_i has exactly one

eigenvalue α_i , $\alpha_1, \dots, \alpha_n \in F$. Eigenvalues are pairwise distinct, each a_i is minimal in $\mathcal{T}_{k_i}(F)$, and none of the matrices a_i is T-similar to a generalized direct sum of more than one matrix. Again, since t is upper triangular, we have $(\mathcal{T}_k(F))^t = \mathcal{T}_k(F)$, so it suffices to consider a^t and its centralizer in $\mathcal{T}_k(F)$. From Corollary 3 we know that this centralizer consists of matrices of the form

$$(x_1)_{I_1} \oplus \cdots \oplus (x_n)_{I_n}, \quad \text{where } x_i \in C(a_i) \text{ for all } 1 \leq i \leq n. \quad (11)$$

Moreover, each $C(a_i)$ is minimal in $\mathcal{T}_{|I_i|}(F)$, so the centralizer of a^t is minimal, and so is the centralizer of a .

The necessity.

Suppose now that a does not satisfy the condition from the claim. By Proposition 1 there exists a matrix t such that $a' := a^t$ is a generalized direct sum of some $a_1, a_2, \dots, a_n$, $a_i \in \mathcal{T}_{k_i}(F)$, having disjoint spectra. Again consider a' . From Corollary 3 we know that the centralizer of a' consists of matrices of the form $(x_1)_{I_1} \oplus (x_2)_{I_2} \oplus \cdots \oplus (x_n)_{I_n}$, where $c_i \in C(a_i)$ for each $1 \leq i \leq n$.

For a fixed eigenvalue α of a consider now these $a_{j_1}, a_{j_2}, \dots, a_{j_p}$ that correspond with this eigenvalue. Define c as

$$c = \left\{ [\oplus_{i \neq j_1, \dots, j_p} (a_i)_{I_i}] \oplus (a_{j_1})_{I_{j_1}} [\oplus_{2 \leq i \leq p} (a_{j_i} + (\beta_i - \alpha)e_{|I_{j_i}|})_{I_{j_i}}] \right\}^{t^{-1}},$$

where β_i are different from all eigenvalues of a . From Corollaries 2 and 3 we get that $C(c) \subsetneq C(a)$, so a is not minimal.

2. The case when $|F|$ is less than n .

The sufficiency.

This proof is a modification of the previous case. Suppose that a has eigenvalues $\alpha_1, \alpha_2, \dots, \alpha_n$ and that for some t

$$a^t = [(a_1^{(1)})_{I_1^{(1)}} \oplus (a_2^{(1)})_{I_2^{(1)}} \oplus \cdots \oplus (a_{m_1}^{(1)})_{I_{m_1}^{(1)}}] \\ \oplus \cdots \oplus [(a_1^{(n)})_{I_1^{(n)}} \oplus (a_2^{(n)})_{I_2^{(n)}} \oplus \cdots \oplus (a_{m_n}^{(n)})_{I_{m_n}^{(n)}}],$$

where $a_j^{(i)}$ corresponds with the eigenvalue α_i and $a_j^{(i)}$ is not T-similar to any generalized direct sum. Then

$$C(a^t) = \left\{ (x_1)_{\cup_{p=1}^{m_1} I_p} \oplus \cdots \oplus (x_n)_{\cup_{p=1}^{m_n} I_p} : x_i \in C((a_1^{(i)})_{I_1^{(i)}} \oplus \cdots \oplus (a_{m_i}^{(i)})_{I_{m_i}^{(i)}}) \right\}. \quad (12)$$

Assume that $m_i > 1$ for some i . With no loss of generality, let i be equal to n . Suppose first that $C(c) \subseteq C(a^t)$. This implies that $c = (c_1)_{\cup_{p=1}^{m_1} I_p} \oplus \cdots \oplus (c_n)_{\cup_{p=1}^{m_n} I_p}$ and for some j

$$C(c_j) \subseteq C((a_1^{(j)})_{I_1^{(j)}} \oplus \cdots \oplus (a_{m_1}^{(j)})_{I_{m_1}^{(j)}}). \quad (13)$$

As $a_{m_p}^{(j)}$ are all minimal this means that to satisfy (13) c must be of the same block form as a^t . However, this means that

$$c = [(c_1^{(1)})_{I_1^{(1)}} \oplus (c_2^{(1)})_{I_2^{(1)}} \oplus \cdots \oplus (c_{m_1}^{(1)})_{I_{m_1}^{(1)}}] \\ \oplus \cdots \oplus [(c_1^{(n)})_{I_1^{(n)}} \oplus (c_2^{(n)})_{I_2^{(n)}} \oplus \cdots \oplus (c_{m_n}^{(n)})_{I_{m_n}^{(n)}}],$$

where $c_j^{(i)}$ corresponds with some eigenvalue α'_i , where $\alpha'_i = \alpha_l$ for some l , i.e. we can only change the eigenvalues of the components. However, at least one of them would be then equal to some other eigenvalue of a . Hence, for some $t' \in \mathcal{T}_k(F)$ the centralizer $C(c^{t'})$ would be of the same form as $C(a^t)$ in (12) but with different sets of indices $I_j^{(i)}$. Thus, we would not have $C(c) \subseteq C(a)$.

The necessity.

Suppose now that a is minimal. For some $t \in \mathcal{T}_k(F)$, a^t is of form (12), and its centralizer is of form (12). From the latter it follows that a has exactly $|F|$ eigenvalues, and each of the matrices $a_j^{(i)}$ is minimal in $\mathcal{T}_{|I_j^{(i)}|}(F)$.

Obviously, it does not take much effort to prove

Corollary 4. *Suppose F is a field and $k \in \mathbb{N}$. If $a \in \mathcal{T}_k(F)$ is nonderogatory, then it is minimal in $\mathcal{T}_k(F)$.*

Proof. This is a consequence of the proof of Theorem 3. □

Example 2. Consider a matrix

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 1 & 0 \\ & & & 0 & 0 \\ & & & & 1 \end{pmatrix} = (j_2(0)) \oplus (j_2(0)) \oplus (1)$$

in two contexts.

Suppose first that $a \in \mathcal{T}_5(\mathbb{Z}_2)$. Then, both conditions given in Theorem 3 are fulfilled, so a is minimal.

Next, assume that $a \in \mathcal{T}_5(\mathbb{Z}_3)$. Then $|\sigma(a)| = 2 \leq 3 = |\mathbb{Z}_3|$. Therefore, a is not minimal. In particular, $C((j_2(0)) \oplus (j_2(2)) \oplus (1)) \subsetneq C(a)$.

It can be easily noticed that Theorem 3 does not give the full description of all minimal triangular matrices—it does not say how to check whether a matrix with one eigenvalue that is not T-similar to some generalized sum is minimal. This problem seems to be more complex. At the moment we point out only one simple necessary condition for minimality of such matrix.

Lemma 4. *Let F be a field and $k \in \mathbb{N}$. If $a \in \mathcal{T}_k(F)$ is nilpotent and minimal, then it is not a power (of exponent greater than 1) of any nilpotent matrix.*

Proof. Let $a \in \mathcal{T}_k(F)$ be nilpotent. It suffices to show that $C(a) \subsetneq C(a^2)$. Clearly $C(a) \subseteq C(a^2)$.

Consider the matrices of the form

$$x = \left(\begin{array}{c|c} 0 & y \\ \hline 0 & 0 \end{array} \right) \in \mathcal{T}_k(F), \quad \text{where } y \in \mathcal{M}_{k-1 \times 1}(F). \quad (14)$$

Such x is in $C(a)$ if and only if $ax = 0$. This equation is equivalent to $\tilde{a}y = 0$, where $\tilde{a} \in \mathcal{T}_{k-1}(F)$ is such that $\tilde{a}_{nm} = a_{nm}$ for $1 \leq n, m \leq k-1$. The equation $\tilde{a}y = 0$ has solutions that depend on $k-1 - \text{rank}(\tilde{a})$ parameters.

Analogously x of form (14) is in $C(a^2)$ if and only if $a^2x = 0$ which is equivalent to $\tilde{a}^2y = 0$ and the solutions of this equation depend on $k-1 - \text{rank}(\tilde{a}^2)$ parameters. As a is nilpotent, $\tilde{a}$ is nilpotent as well, and in the case when $\tilde{a} \neq 0$, we have $\text{rank}(\tilde{a}^2) < \text{rank}(\tilde{a})$. If $\tilde{a} = 0$, but $a \neq 0$, one can notice that $a^2 = 0$, so a can not be a power of any nilpotent matrix. Hence, there exist solutions of $\tilde{a}^2y = 0$ that do not satisfy $\tilde{a}y = 0$. Consequently, there exists x satisfying $a^2x = 0$ that do not satisfy $ax = 0$. Thus $C(a) \subsetneq C(a^2)$. $\square$

3. Results for Infinite Triangular Matrices

3.1. More Notation and a Remark About Centralizer Theorem

In this section some new notation appears. We explain it here.

As in the finite case by e_{nm} we will understand the matrix with 1 in the position (n, m) and 0 in every other position.

The symbol $\mathcal{D}_\infty(F)$ denotes the algebra of all $\mathbb{N} \times \mathbb{N}$ diagonal matrices.

The meaning of $\text{Diag}(a_1, a_2, a_3, \dots)$ is analogous to the finite $\text{Diag}(a_1, \dots, a_n)$.

If $x \in \mathcal{T}_\infty(F)$ has nonzero entries only in the positions (i, j) where $i \in I$, $j \in J$, then we will write

$$x = \sum_{i \in I, j \in J} x_{ij} e_{ij}.$$

In particular we define $j_\infty(\alpha)$ as follows.

$$j_\infty(\alpha) = \alpha e_\infty + \sum_{i=1}^{\infty} e_{i, i+1} = \begin{pmatrix} \alpha & 1 & 0 & 0 & \cdots \\ & \alpha & 1 & 0 & \\ & & \alpha & 1 & \\ & & & \alpha & \\ & & & & \ddots \end{pmatrix}$$

The matrix $j_\infty(\alpha)$ is simply an infinite analog of a Jordan block (some more information about infinite Jordan blocks and infinite Jordan forms can be found in [20]).

Before we prove our next result, we would like to pay some attention to the centralizer theorem (called also double centralizer theorem). This classical

result (see [14, 21]) states that for any $a \in \mathcal{M}_{k \times k}(F)$ we have

$$C_{\mathcal{M}_{k \times k}(F)}(C_{\mathcal{M}_{k \times k}(F)}(F)) = F[a],$$

where by $F[a]$ we understand the algebra of all polynomials in a with the entries in F . The two examples below show that this theorem does not hold in algebras of infinite matrices.

Example 3. Let us consider $j_\infty(0)$ and its centralizer in $\mathcal{T}_\infty(R)$, where R is any associative, commutative ring. It can be easily computed that

$$C_{\mathcal{T}_\infty(R)}(j_\infty(0)) = \{t \in \mathcal{T}_\infty(R) : \forall i, j, k \in \mathbb{N} \quad t_{i+k, j+k} = t_{ij}\}. \quad (15)$$

In particular, $j_\infty(0)$ commutes with

$$x = \sum_{\substack{n, m \in \mathbb{N} \\ m > n}} e_{nm} \quad (16)$$

that is not a polynomial of $j_\infty(0)$. Nevertheless, it should be mentioned that all the elements of $C_{\mathcal{T}_\infty(F)}(j_\infty(0))$ are simply the upper triangular Toeplitz matrices that can be written as the formal power series $\sum_{i=0}^{\infty} \alpha_i j_\infty^i(0)$ (see [1], but also [2]). Moreover, the centralizer of the algebra of upper triangular Toeplitz matrices is the same algebra, so we have $C_{\mathcal{T}_\infty(F)}(C_{\mathcal{T}_\infty(F)}) = F[[j_\infty(0)]]$.

Example 4. Consider now $\mathcal{M}_{C_f}(R)$ – the algebra of all $\mathbb{N} \times \mathbb{N}$ matrices with the finite number of nonzero entries in each column. Also in this case the centralizer of $j_\infty(0)$ is different from $R[j_\infty(0)]$. More precisely, it can be checked that if x commutes with $j_\infty(0)$, then $x \in \mathcal{T}_\infty(R)$ and it is of form (15). Hence $C_{\mathcal{M}_{C_f}}(C_{\mathcal{M}_{C_f}}(j_\infty(0)))$ contains, in particular, $C_{\mathcal{T}_\infty}(C_{\mathcal{T}_\infty}(j_\infty(0)))$, so, again, it is not equal to $R[j_\infty(0)]$.

3.2. Maximal Infinite Triangular Matrices

The proof of Theorem 2 is very similar to the proof of Theorem 1. However, we also need one more fact from [17] (see also [18]). Namely, this is an infinite dimensional analog of the first point of Proposition 1.

Lemma 5 (Proposition 2.1, [17]). *If F is any field and $a \in \mathcal{T}_\infty(F)$, then there exists $x \in \mathcal{T}_\infty(F)$ such that*

$$a^x = \oplus_{n \in I} (a_n)_{I_n}$$

with a_n satisfying

- $(a_n)_{ii} \neq (a_m)_{jj}$ for all $n \neq m$,
- $(a_n)_{ii} = (a_n)_{jj}$ for all n .

Now we prove Theorem 2.

Proof of Theorem 2. The proof of sufficiency is the same as in the case of finite matrices.

To prove the necessity we consider two cases.

Case 1 The set $\{a_{nn} : n \in \mathbb{N}\}$ is finite.

In this case the claim can also be proved as in the case of finite matrices.

Case 2 The set $\{a_{nn} : n \in \mathbb{N}\}$ is infinite.

In this case $a^x = \oplus_{n \in I} (a_n)_{I_n}$, where $\{(a_n)_{ii} : i \in I_n\} = \{\alpha_n\}$ with $\alpha_n \neq \alpha_m$ for $n \neq m$. It suffices to notice that for $b = (a_1 + (\alpha_2 - \alpha_1)e_{|I_1|})_{I_1} \oplus_{n \in I \setminus \{1\}} (a_n)_{I_n}$, we have $C(a^c) \subsetneq C(b)$. Thus a with infinite spectrum can not be maximal.

3.3. Minimal Infinite Triangular Matrices

The proof of Theorem 4 is analogous to the proof of Theorem 3, so we let ourselves to omit it.

It should be noted that we do not know the description of minimal matrices a satisfying condition $|\{a_{nn} : n \in \mathbb{N}\}| = 1$. Nevertheless, we would like to present two sufficient conditions for minimality. Let's notice first that the following holds.

Lemma 6. *For any field F and $\alpha \in F$ the matrix $j_\infty(\alpha)$ is minimal in $\mathcal{T}_\infty(F)$.*

Proof. As in the previous proof, with no loss of generality we can assume that $\alpha = 0$.

In Example 3 we have found $C(j_\infty(0))$. Suppose that $C(c) \subseteq C(j_\infty(0))$. Then c commutes with $j_\infty(0)$, so it is of the form given by Eq. (15). However, such c commutes with every other matrix of the form as given in Eq. (15). Hence $C(c) \supseteq C(j_\infty(0))$. $\square$

Lemma 7. *Suppose F is a field and $a \in \mathcal{T}_\infty(F)$ is such that for some $\alpha \in F$ we have $\{a_{nn} : n \in \mathbb{N}\} = \{\alpha\}$ and $a_{n,n+1} \neq 0$ for $n \in \mathbb{N}$. Then a is T-similar to $j_\infty(\alpha)$.*

Proof. With no loss of generality we can assume that we have $a_{nn} = 0$ for all $n \in \mathbb{N}$.

First we define $d \in \mathcal{D}_\infty(F)$ by the rule

$$d_{nn} = \begin{cases} 1 & \text{for } n = 1 \\ d_{n-1,n-1}^{-1} a_{n-1,n}^{-1} & \text{for } n > 1. \end{cases}$$

Then $a' := a^d$ satisfies $a'_{n,n+1} = 1$ for all $n \in \mathbb{N}$.

Suppose now that $t j_\infty(0) = a' t$. Then the entries of t must satisfy

$$t_{n,m-1} = \sum_{i=n+1}^m a'_{ni} t_{im} \quad \text{for } m \geq n+1. \quad (17)$$

Hence we find these entries as follows.

From (17) for $m = n+1$ we have $t_{nn} = a'_{n,n+1} t_{n+1,n+1}$, so we put $t_{11} = 1$ and $t_{n+1,n+1} = (a'_{n,n+1})^{-1} t_{nn}$ for $n \geq 1$. Observe that since $a'_{n,n+1} \neq 0$, all t_{nn} are invertible.

Next, from (17) for $m = n+2$ we have

$$t_{n,n+1} = a'_{n,n+1} t_{n+1,n+2} + a'_{n,n+2} t_{n+2,n+2},$$

so we put (for instance) $t_{12} = 0$ and

$$t_{n+1,n+2} = (a'_{n,n+2})^{-1}(t_{n,n+1} - a'_{n,n+2}t_{n+2,n+2}).$$

As $a'_{n,n+1} \neq 0$ the latter formula is well-defined.

Analogously, having found first m diagonals we put $t_{1,m+1} = 0$ and then

$$t_{n,n+m} = (a'_{n,n+1})^{-1} \left(t_{n-1,n+m-1} - \sum_{i=n+2}^m a'_{ni} t_{im} \right).$$

This way we can construct all the diagonals of t , and consequently t itself. $\square$

Note that from the last part of the proof of the above lemma we can not conclude that $C(c) = C(j_\infty(0))$ for an arbitrary c of the form (15), we only know that $C(c) \supseteq C(j_\infty(0))$.

Example 5. Let $a = \sum_{i=1}^{\infty} e_{i,i+2}$. Then

$$C(a) = \{t \in \mathcal{T}_\infty(F) : t_{n+2,m+2} = t_{nm} \text{ for } 1 \leq n \leq m\},$$

what implies $C(a) \supsetneq C(j_\infty(0))$.

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