On m -Subharmonic Ordering of Measures

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Abstract. We study an order-relation induced by m -subharmonic functions. We shall consider maximality with respect to this order and a related notion of minimality for certain m -subharmonic functions. This concept is then applied to the problem of convergence of measures in the weak*-topology, in particular Hessian measures.

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1. Introduction

In this paper we study an order-relation between measures on an m -hyperconvex domain Ω in $\mathbb{C}^n$. Let μ and ν be measures on Ω . We say that μ is m -subharmonically greater than ν if $\int_{\Omega}(-\varphi)d\mu \geq \int_{\Omega}(-\varphi)d\nu$, $\forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ and write $\mu \succcurlyeq \nu$, where $\mathcal{E}_{0,m}(\Omega)$ is the Cegrell class of negative m -subharmonic functions defined in Sect. 2. It is easy to see that the condition $\mu \geq \nu$ implies $\mu \succcurlyeq \nu$. But the inverse is not true (see Example 1). We also show that if u, v are functions in the Cegrell class $\mathcal{F}_m(\Omega)$ such that $u \leq v$, then their complex Hessian measures are in the relation $H_m(u) \succcurlyeq H_m(v)$ (see Proposition 2). But the inverse is not true (see Example 2).

In Sect. 4, we study maximality with respect to the $\succcurlyeq$ -ordering, and a related notion of minimality for m -subharmonic functions in the class $\mathcal{F}_m(\Omega)$. A finite measure μ on Ω is said to be maximal if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$. The Dirac measure is a maximal measure. Theorem 9 shows that each finite measure on Ω with compact support is majorized in the $\succcurlyeq$ -ordering by a maximal measure with the same total mass. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any

function $v \in \mathcal{F}_m(\Omega)$ with the same total Hessian mass, the relation $v \leq u$ implies that $v = u$. We show that if a function $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)$ is maximal measure, then u is minimal function (see Proposition 5). But the converse is still unknown. Theorem 10 shows that if $u \in \mathcal{F}_m(\Omega)$ is such that $H_m(u)$ is carried by an m -polar set, then u is a minimal function. However, there are functions in $\mathcal{F}_m(\Omega)$ whose Hessian measure are maximal and are not carried by an m -polar set. We also prove that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

In Sect. 5, we apply the m -subharmonic ordering to the problem of convergence in the weak*-topology. First, we prove that if $\{\mu_j\}$ is an m -subharmonically increasing sequence of measures on Ω with uniformly bounded total mass then μ_j converges to a measure μ in the weak*-topology. And finally, we use the notion of maximal measure to prove a sufficient condition of convergence in the weak*-topology for the class $\mathcal{F}_m(\Omega)$ (see Theorem 14).

2. Preliminaries

Let Ω be an open set in $\mathbb{C}^n$ and let m be a natural number $1 \leq m \leq n$.

As usual let $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, and let $\beta = dd^c|z|^2$ be the canonical Kähler form in $\mathbb{C}^n$. Denote by $SH_m(\Omega)$ the set of all m -subharmonic functions in Ω , and $SH_m^-(\Omega)$ for the set of all nonpositive m -subharmonic functions in Ω . For $u_1, \dots, u_m \in SH_m(\Omega) \cap L_{loc}^\infty(\Omega)$, the operator

$$\begin{aligned} H_m(u_1, \dots, u_m) &:= dd^c u_1 \wedge \dots \wedge dd^c u_m \wedge \beta^{n-m} \\ &= dd^c(u_1 dd^c u_2 \wedge \dots \wedge dd^c u_m \wedge \beta^{n-m}) \end{aligned}$$

is a nonnegative Radon measure. In particular, when $u = u_1 = \dots = u_m$, the Hessian measures

$$H_m(u) := (dd^c u)^m \wedge \beta^{n-m}$$

are well-defined for $u \in SH_m(\Omega) \cap L_{loc}^\infty(\Omega)$ (see [4]).

Definition 1. Let E be a subset of Ω . The m -relative extremal function $h_{m,E,\Omega}$ is defined by

$$h_{m,E,\Omega}(z) = \sup\{u(z) : u \in SH_m(\Omega), u \leq 0 \text{ and } u \leq -1 \text{ on } E\}.$$

By [11, Proposition 1.5], we have that $h_{m,E,\Omega}^*$ is m -subharmonic on Ω .

Definition 2. Let Ω be an open set. A function $u \in SH_m(\Omega)$ is called m -maximal if $v \in SH_m(\Omega)$, $v \leq u$ outside a compact subset of Ω implies that $v \leq u$ in Ω .

Theorem 1 [4]. Assume that $u \in SH_m(\Omega) \cap L_{loc}^\infty(\Omega)$. Then $H_m(u) = 0$ in Ω if and only if u is m -maximal.

Now let us recall the definition of m -hyperconvex domain.

Definition 3. A bounded domain $\Omega \subset \mathbb{C}^n$ is called an m -hyperconvex if there exists an m -subharmonic function $\rho: \Omega \rightarrow (-\infty, 0)$ such that the closure of the set $\{z \in \Omega: \rho(z) < c\}$ is compact in Ω for every $c \in (-\infty, 0)$. In other words, the sublevel set $\{z \in \Omega: \rho(z) < c\}$ is relatively compact in Ω . Such a function ρ is called the exhaustion function.

Theorem 2 [9, Proposition 1.4.11]. *Let Ω be an m -hyperconvex bounded domain and $K \Subset \Omega$ is compact. Then $h_{m,K,\Omega}^*$ is m -maximal in $\Omega \setminus K$.*

Let us recall the definition of m -polar sets.

Definition 4. A set $E \subset \mathbb{C}^n$ is called m -polar if for any $z \in E$ there exists a neighbourhood V of z and $v \in SH_m(V)$ such that $E \cap V \subset \{v = -\infty\}$.

The following theorem was proved by Lu.

Theorem 3 [9, Theorem 1.6.5]. *If E is m -polar set, then there exists $u \in SH_m^-(\mathbb{C}^n)$ such that $E \subset \{u = -\infty\}$.*

Throughout this paper Ω will denote a bounded m -hyperconvex domain in $\mathbb{C}^n$. Now we recall the definitions of the Cegrell classes.

Definition 5. (1) We let $\mathcal{E}_{0,m}(\Omega)$ denote the class of bounded functions in $SH_m(\Omega)$ such that

$$\lim_{z \rightarrow \partial\Omega} u(z) = 0 \text{ and } \int_{\Omega} H_m(u) < +\infty.$$

- (2) A function $u \in SH_m(\Omega)$ belongs to $\mathcal{E}_m(\Omega)$ if for each $z_0 \in \Omega$, there exists an open neighborhood $U \subset \Omega$ of z_0 and a decreasing sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ such that $u_j \downarrow u$ in U and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.
- (3) Denote $\mathcal{F}_m(\Omega)$ be the class of functions $u \in SH_m(\Omega)$ such that there exists a sequence $\{u_j\} \subset \mathcal{E}_{0,m}(\Omega)$ decreases to u in Ω and $\sup_j \int_{\Omega} H_m(u_j) < +\infty$.

We have the following inclusions

$$\mathcal{E}_{0,m} \subset \mathcal{F}_m \subset \mathcal{E}_m \text{ and } SH_m^-(\Omega) \cap L_{loc}^{\infty}(\Omega) \subset \mathcal{E}_m.$$

Below we present some of the basic properties of the Cegrell classes.

Theorem 4 [2, 9]. *For each $u \in SH_m^-(\Omega)$, there exists a sequence $\{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega})$ such that $u_j \downarrow u$ in Ω .*

Proposition 1. *Let $\mathcal{K}$ be one of the classes $\mathcal{E}_{0,m}, \mathcal{F}_m, \mathcal{E}_m$. Then $\mathcal{K}$ is a convex cone. Moreover, if $u \in \mathcal{K}$ and $v \in SH_m^-(\Omega)$ then $\max\{u, v\} \in \mathcal{K}$.*

The following lemma explains why the functions in $\mathcal{E}_{0,m}(\Omega)$ are sometimes called test functions.

Theorem 5 [2, 9]. *For $\varphi \in C_0^{\infty}(\Omega)$, there exist two functions u, v in $\mathcal{E}_{0,m} \cap C(\overline{\Omega})$ such that $\varphi(z) = u(z) - v(z), \forall z \in \Omega$.*

Following Cegrell's idea Lu proved that the Hessian operator is well-defined for the functions in the class $\mathcal{E}_m(\Omega)$.

Theorem 6 [9, Theorem 1.7.14]. *Let $u^k \in \mathcal{E}_m(\Omega)$, $k = 1, \dots, m$ and $\{u_j^k\}_j$ be sequences in $\mathcal{E}_{0,m}(\Omega)$ such that $u_j^k \downarrow u^k$, for each $1 \leq k \leq m$. Then the sequence of measures*

$$dd^c u_1^1 \wedge \dots \wedge dd^c u_j^m \wedge \beta^{n-m}$$

converge to a Radon measure in weak-topology independent to the choice of sequences $\{u_j^k\}$. We define $dd^c u^1 \wedge \dots \wedge dd^c u^m \wedge \beta^{n-m}$ to be this limit.*

Integration by parts formula is true for the function from the Cegrell class $\mathcal{F}_m(\Omega)$.

Theorem 7 [9, Theorem 1.7.18]. *Assume that $u, v, w_1, \dots, w_{m-1} \in \mathcal{F}_m(\Omega)$. Then we have*

$$\int_{\Omega} u dd^c v \wedge T = \int_{\Omega} v dd^c u \wedge T,$$

where $T = dd^c w_1 \wedge \dots \wedge dd^c w_{m-1} \wedge \beta^{n-m}$ and the equality means that if one of the two terms is finite then they are equal.

The following theorem is sometimes called the Cegrell decomposition theorem.

Theorem 8. *Let μ be a finite, positive measure on Ω . Then there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $0 \leq f \in L^1(H_m(\varphi))$ such that*

$$\mu = f H_m(\varphi) + \nu,$$

where ν is carried by a m -polar set.

Proof. By the proof of [10, Theorem 4.14], we can find a function $u \in \mathcal{E}_{1,m}(\Omega)$ and $0 \leq f \in L^1(H_m(u))$ such that $\mu = f H_m(u) + \nu$, where ν is charged by an m -polar subset of Ω . The rest of the proof goes verbatim as the proof of [10, Theorem 5.3]. $\square$

3. The m -Subharmonic Ordering

Let μ_j, μ be measures on Ω . By Theorem 5, we can see that following conditions are equivalent

- (1) $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \forall \varphi \in C_0(\Omega);$
- (2) $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \forall \varphi \in C_0^\infty(\Omega);$
- (3) $\lim_{j \rightarrow \infty} \int_{\Omega} \varphi d\mu_j = \int_{\Omega} \varphi d\mu, \forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C_0(\overline{\Omega}).$

If one of above assertion is satisfied, we say that μ_j tends to μ on Ω in the weak*-topology.

Remark 1. (1) If $\mu_j \rightarrow \mu$ in the weak*-topology on Ω , then

$$\mu(\Omega) \leq \liminf_{j \rightarrow \infty} \mu_j(\Omega).$$

(2) Assume that $\{\mu_j\}_j$ is a sequence measures on Ω and $\sup_j \mu_j(\Omega) < \infty$, then there exists a subsequence $\{\mu_{j_k}\}_k \subset \{\mu_j\}_j$ such that μ_{j_k} converges to a measure μ in the weak*-topology as $k \rightarrow \infty$.

Definition 6. Let μ and ν be measures on Ω . We write $\mu \succcurlyeq \nu$ if and only if

$$\int_{\Omega} -\varphi d\mu \geq \int_{\Omega} -\varphi d\nu, \quad \forall \varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\overline{\Omega}).$$

And we say that μ is m -subharmonically greater than ν .

Remark 2. (1) If $\mu \succcurlyeq \nu$, then $\int_{\Omega} -\varphi d\mu \geq \int_{\Omega} -\varphi d\nu$, $\forall \varphi \in SH_m^-(\Omega)$ by Theorem 4. In particular, $\mu(\Omega) \geq \nu(\Omega)$.

(2) If $\mu \geq \nu$, then $\mu \succcurlyeq \nu$. But Example 1 shows that the opposite implication is not true.

Example 1. For $a \in \Omega$, let δ_a be the Dirac measure at a . Let σ_r be the normalized measure on the sphere $\partial B(a, r)$, where r enough small such that $B(a, r) \subset \Omega$. Then for each $\varphi \in SH_m^-(\Omega)$, by the subharmonicity we have

$$\int_{\Omega} \varphi d\delta_a = \varphi(a) \leq \int_{\partial B(a,r)} \varphi d\sigma_r = \int_{\Omega} \varphi d\sigma_r.$$

Thus $\delta_a \succcurlyeq \sigma_r$, but it is clear that δ_a is not greater than σ_r even though $\delta_a(\Omega) = \sigma_r(\Omega) = 1$.

Proposition 2. If $u, v \in \mathcal{F}_m(\Omega)$ and $u \geq v$, then $H_m(v) \succcurlyeq H_m(u)$.

Proof. For $\varphi \in \mathcal{E}_{0,m}(\Omega)$, by Theorem 7

$$\begin{aligned} \int_{\Omega} -\varphi H_m(u) &= \int_{\Omega} -u dd^c \varphi \wedge dd^c u^{m-1} \wedge \beta^{n-m} \\ &\leq \int_{\Omega} -v dd^c \varphi \wedge (dd^c u)^{m-1} \wedge \beta^{n-m} \\ &= \int_{\Omega} -\varphi dd^c v \wedge (dd^c u)^{m-1} \wedge \beta^{n-m} \\ &\leq \cdots \leq \int_{\Omega} -\varphi (dd^c v)^m \wedge \beta^{n-m} = \int_{\Omega} -\varphi H_m(v). \end{aligned}$$

Thus $H_m(v) \succcurlyeq H_m(u)$. □

The following example shows that the converse implication to the statement given in Proposition 2 is not true.

Example 2. Let Ω is the unit ball $\mathbb{B}$ in $\mathbb{C}^n$, $n \geq 2$ and define the functions $v(z) = \frac{2}{3}(t^3 - 1)$, $w(z) = t^2 - 1$. Then $v, w \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\overline{\mathbb{B}})$ and $w \leq v$ on

$\mathbb{B}$, so $H_2(w) \succcurlyeq H_2(v)$ by Proposition 2. For more details, we can compute (see [12])

$$H_2(w)(z) = 4^n n! dV, \quad H_2(v)(z) = 2^{2n-1} (2n+2)(n-1)! |z|^2 dV,$$

where dV is the Lebesgue measure on $\mathbb{C}^n$. By [12] one can compute the solution u to the equation

$$H_2(u) = \frac{H_2(v) + H_2(w)}{2}, \quad u \in \mathcal{E}_{0,2}(\mathbb{B}) \cap C^2(\overline{\mathbb{B}}). \quad (1)$$

The solution is given by

$$u(z) = \frac{\sqrt{2}}{3} (|z|^2 + 1)^{\frac{3}{2}} - \frac{4}{3}.$$

We have $H_2(u) \succcurlyeq H_2(v)$ by (1). Otherwise, $u(0) > -1 = v(0)$, so $v \not\prec u$.

Remark 3. The relation $\succcurlyeq$ defines a partial order on the set of positive Borel measures on Ω . But it is not a total order. To see that consider the Dirac measures δ_z and δ_w , where $z, w \in \Omega$ and $z \neq w$. Choose $\varphi, \psi \in SH_m^-(\Omega)$ such that $\varphi(z) < \varphi(w)$ and $\psi(z) > \psi(w)$. Then $\int_{\Omega} -\varphi d\delta_z > \int_{\Omega} -\varphi d\delta_w$ and $\int_{\Omega} -\psi d\delta_z < \int_{\Omega} -\psi d\delta_w$, so δ_z and δ_w are not comparable with respect to $\succcurlyeq$.

Definition 7. For a set $E \subset \Omega$, we define the convex hull of E in Ω with respect to the family $SH_m(\Omega) \cap C(\overline{\Omega})$, denoted by $\hat{E}$ as followed

$$\hat{E} = \{z \in \Omega : \varphi(z) \leq \sup_E \varphi, \forall \varphi \in SH_m(\Omega) \cap C(\overline{\Omega})\}.$$

Remark 4. We have that $\hat{E}$ is closed in Ω . Moreover, if E is relatively compact in Ω , so is $\hat{E}$.

Proposition 3. Let μ, ν be finite regular measures on Ω such that $\mu(\Omega) = \nu(\Omega)$. If $\nu \succcurlyeq \mu$ then $\text{supp } \nu \subset \widehat{\text{supp } \mu}$.

Proof. Put $K = \text{supp } \mu$. If $\hat{K} = \Omega$ then Proposition 3 is clear. Therefore we assume that $\Omega \setminus \hat{K} \neq \emptyset$. Suppose that $\text{supp } \nu \not\subset \hat{K}$. Since $\hat{K}$ is closed in Ω , it follows that $\nu(\Omega \setminus \hat{K}) > 0$. By the regularity of ν , we can find a compact set $L \in \Omega \setminus \hat{K}$ such that $\nu(L) > 0$. From the definition of $\hat{K}$, for each $z \in L$, there exist a neighborhood $U(z)$ of z and a function $\varphi \in SH_m(\Omega) \cap C(\overline{\Omega})$ such that $\varphi(\xi) > \sup_K \varphi, \forall \xi \in U(z)$. We choose $z_1, \dots, z_k \in L$ such that $L \subset \bigcup_{i=1}^k U(z_i)$. Let $\varphi_1, \dots, \varphi_k$ be the associated functions and $M_i = \sup_K \varphi_i, M = M_1 + \dots + M_k$. Define

$$\psi = \max\{\varphi_1, M_1\} + \dots + \max\{\varphi_k, M_k\}.$$

Then we have $\psi \in SH_m(\Omega) \cap C(\overline{\Omega})$, $\psi \geq M$ on $\overline{\Omega}$, $\psi = M$ on K and $\psi > M$ on L . Define $\psi_0 = \psi - \max_{\overline{\Omega}} \psi$ and let $M_0 = M - \max_{\overline{\Omega}} \psi$. Then $\psi_0 \in SH_m^-(\Omega) \cap C(\overline{\Omega})$, $\psi_0 \geq M_0$ on Ω , $\psi_0 = M_0$ on K and $\psi_0 > M_0$ on L . Hence,

$$\int_{\Omega} -\psi_0 d\nu < -M_0\nu(\Omega) = -M_0\mu(\Omega) = \int_{\Omega} -\psi_0 d\mu.$$

Proposition 3 is proved by a contradiction. $\square$

4. Maximal Measures and Minimal Functions

We want to study the maximality with respect to the m -subharmonic ordering by using some kind of normalization.

Definition 8. A finite measure μ on Ω is said to be maximal if for any measure ν on Ω such that $\nu(\Omega) = \mu(\Omega)$, the relation $\nu \succcurlyeq \mu$ implies that $\nu = \mu$.

Example 3. For $1 \leq m < n$, we define

$$\varphi_j(z) = \max \left\{ -\frac{1}{j} |z|^{2-\frac{2n}{m}}, -1 \right\} \in SH_m^-(\mathbb{B})$$

and δ_0 is the Dirac measure defined on the unit ball $\mathbb{B}$ in $\mathbb{C}^n$. Then for each measure ν , $\nu(\Omega) = 1$ and $\nu \succcurlyeq \delta_0$ we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{B}} -\varphi_j d\nu = -\nu(\{0\})$$

and

$$-1 \leq \int_{\mathbb{B}} -\varphi_j d\delta_0 \leq \int_{\mathbb{B}} -\varphi_j d\nu \leq 1, \quad \forall j.$$

Thus we get $\nu(\{0\}) = 1$, so $\nu = \delta_0$ which implies δ_0 is maximal.

Remark 5. (1) If we can write a maximal measure as the sum $\mu = \mu_1 + \mu_2$ of two finite measures, then these are maximal too. To prove this, assume that μ_1 is not maximal. Then there is a finite measure $\nu \neq \mu_1$ such that $\nu(\Omega) = \mu_1(\Omega)$ and $\nu \succcurlyeq \mu_1$. We have $(\nu + \mu_2)(\Omega) = \mu(\Omega)$ and $\nu + \mu_2 \succcurlyeq \mu$, but $\nu + \mu_2 \neq \mu$, which is a contradiction.

(2) If μ is maximal measure, so is $c\mu$, for $c > 0$.

(3) We will show that the condition μ_1, μ_2 are maximal does not imply the maximality of $\mu_1 + \mu_2$ (see Example 5). This implies that the set of maximal measures on Ω is not a convex cone.

Definition 9. We say that a set $K \Subset \Omega$ is an interpolation set for $SH_m^-(\Omega)$ if for each $f \in C(K)$, $f < 0$ there exists a function $\varphi \in SH_m^-(\Omega)$ such that $\varphi = f$ on K .

Proposition 4. If μ is a finite measure on Ω such that $\widehat{\text{supp} \mu}$ is contained in some interpolation set K for $SH_m^-(\Omega)$, then μ is maximal.

Proof. Assume that ν is a measure on Ω such that $\nu(\Omega) = \mu(\Omega)$ and $\nu \succcurlyeq \mu$. By Proposition 3, we have $\text{supp}\nu \subset \widehat{\text{supp}\mu} \subset K$. For a given $f \in C(K)$, $f \leq 0$, there exists a function $\varphi \in SH_m^-(\Omega)$ such that $\varphi = f$ on K . We get

$$\int_{\Omega} -f d\nu = \int_{\Omega} -\varphi d\nu \leq \int_{\Omega} -\varphi d\mu = \int_{\Omega} -f d\mu.$$

This implies that $\int_{\Omega} f d\mu \geq \int_{\Omega} f d\nu$ holds for any $f \in C_0(\Omega)$, $f \leq 0$. Hence $\mu \leq \nu$, so $\mu = \nu$. $\square$

Example 4. Let $a_1, \dots, a_k \in \Omega$. For $1 \leq j \leq k$, we choose M_j such that

$$\psi_j(z) = \sum_{l \neq j}^k \ln |z - a_l| + M_j \in SH_m^-(\Omega).$$

For each value $c_j < 0$, we take $d_j > 0$ such that $d_j \psi_j(a_j) = c_j$. Define $\varphi = \max(d_1 \psi_1, \dots, d_k \psi_k)$. Then we have $\varphi \in SH_m^-(\Omega)$ and $\varphi(a_j) = c_j$. Thus the finite set $\{a_1, \dots, a_k\}$ is an interpolation set for $SH_m^-(\Omega)$. And Proposition 4 implies that the measure $\sum_{j=1}^k b_j \delta_{a_j}$ is maximal, where δ_{a_j} is the Dirac measure at the point a_j and $b_1, \dots, b_k$ are given nonnegative numbers.

We will show that each finite measure with compacted support is majorized by a maximal measure with the same total mass.

Lemma 1. Assume that μ and ν are measures on Ω such that $\nu \succcurlyeq \mu$. If $\int_{\Omega} \varphi d\mu = \int_{\Omega} \varphi d\nu > -\infty$ for some negative strictly m -subharmonic function φ . Then $\mu = \nu$.

Proof. For given $f \in C_0^\infty(\Omega)$, choose a constant $c > 0$ so that $(\pm f + c\varphi) \in SH_m^-(\Omega)$. Then we have

$$\begin{aligned} \int_{\Omega} (\pm f + c\varphi) d\mu &= \int_{\Omega} \pm f d\mu + c \int_{\Omega} \varphi d\mu \geq \int_{\Omega} (\pm f + c\varphi) d\nu \\ &= \int_{\Omega} \pm f d\nu + c \int_{\Omega} \varphi d\nu, \end{aligned}$$

which implies that $\int_{\Omega} \pm f d\mu \geq \int_{\Omega} \pm f d\nu$. So $\mu = \nu$. $\square$

Theorem 9. Let μ be a finite measure on Ω with compact support. Then there is a maximal measure μ_0 such that $\mu_0 \succcurlyeq \mu$ and $\mu_0(\Omega) = \mu(\Omega)$.

Proof. Put $K = \widehat{\text{supp}\mu}$ and

$$\mathcal{M}_{\mu} = \{\nu : \nu \succcurlyeq \mu, \nu(\Omega) = \mu(\Omega)\}.$$

Because $\mu \in \mathcal{M}_{\mu}$, so $\mathcal{M}_{\mu} \neq \emptyset$. By Proposition 3, $\text{supp}\nu \subset K$ for each $\nu \in \mathcal{M}_{\mu}$. Let ρ be the exhaustion function of Ω that is negative, continuous strictly m -subharmonic. We define

$$A = \sup_{\nu \in \mathcal{M}_{\mu}} \int_{\Omega} (-\rho) d\nu.$$

Since ρ is bounded on K , it follows that A is finite. Let $\{\nu_j\}_j$ be a sequence in $\mathcal{M}_\mu$ such that $\int_\Omega (-\rho) d\nu_j \rightarrow A$, as $j \rightarrow \infty$. By Remark 1, we may assume that ν_j tend to some measure μ_0 in the weak*-topology and $\mu_0(\Omega) \leq \mu(\Omega)$. For each $\varphi \in \mathcal{E}_{0,m} \cap C(\Omega)$,

$$\int_\Omega (-\varphi) d\mu_0 = \lim_{j \rightarrow \infty} \int_\Omega (-\varphi) d\nu_j \geq \int_\Omega (-\varphi) d\mu,$$

which implies that $\mu_0 \succcurlyeq \mu$. By Remark 2 and the fact $\mu_0 \leq \mu(\Omega)$, we get $\mu_0(\Omega) = \mu(\Omega)$. Thus $\mu_0 \in \mathcal{M}_\mu$. Take a function $f \in C_0(\Omega)$, $f = 1$ on K . We get

$$\int_\Omega (-\rho) d\mu_0 = \int_\Omega (-\rho) f d\mu_0 = \lim_{j \rightarrow \infty} \int_\Omega (-\rho) f d\nu_j = \lim_{j \rightarrow \infty} \int_\Omega (-\rho) d\nu_j = A.$$

Suppose that ν be any measure on Ω such that $\nu \geq \mu_0$ and $\nu(\Omega) = \mu(\Omega)$. Then $\nu \in \mathcal{M}_\mu$ and $A \geq \int_\Omega (-\rho) d\nu \geq \int_\Omega (-\rho) d\mu_0 = A$. Hence $\int_\Omega (-\rho) d\nu = \int_\Omega (-\rho) d\mu_0 = A$. Lemma 1 implies that $\nu = \mu_0$, so Theorem 9 is finished. $\square$

Definition 10. A function $u \in \mathcal{F}_m(\Omega)$ is said to be minimal if for any function $v \in \mathcal{F}_m(\Omega)$, the conditions $H_m(u)(\Omega) = H_m(v)(\Omega)$ and $v \leq u$ imply $v = u$.

Proposition 5. Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is a maximal measure. Then u is minimal.

To prove this proposition we need the following lemma.

Lemma 2. If $u, v \in \mathcal{F}_m(\Omega)$, $H_m(u) = H_m(v)$ and $u \leq v$ then $u = v$.

Proof. We use a method from [7]. Using integration by parts, we have

$$\begin{aligned} \int_\Omega -(u-v)(dd^c \rho)^m \wedge \beta^{n-m} &= \int_\Omega d(u-v) \wedge d^c \rho \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \\ &\leq \left[\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \right]^{\frac{1}{2}} \\ &\quad \times \left[\int_\Omega d\rho \wedge d^c \rho \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} \right]^{\frac{1}{2}}, \end{aligned}$$

where $\rho \in \mathcal{E}_{0,m}(\Omega) \cap C^\infty(\Omega)$ is a strictly m -subharmonic exhaustion function of Ω (see [2]). Hence, to prove $u = v$ it is enough to show that

$$\int_\Omega d(u-v) \wedge d^c(u-v) \wedge (dd^c \rho)^{m-1} \wedge \beta^{n-m} = 0. \quad (2)$$

If $m = 1$ then (2) is clear. For $m \geq 2$ and $j + k = m - 1$, we have

$$\begin{aligned}
 0 &\leq \int_{\Omega} -(u - v)(dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} \\
 &= \int_{\Omega} -\rho dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge \beta^{n-m} \\
 &\leq \int_{\Omega} -(u - v) \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge dd^c \rho \wedge \beta^{n-m} \\
 &= \int_{\Omega} -\rho dd^c(u - v) \wedge \sum_{a+b=m-1} (dd^c u)^a \wedge (dd^c v)^b \wedge \beta^{n-m} \\
 &= \int_{\Omega} -\rho(H_m(u) - H_m(v)) = 0.
 \end{aligned}$$

Thus, for every couple $j, k, j + k = m - 2$ we have

$$\begin{aligned}
 &\int_{\Omega} -udd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} \\
 &= \int_{\Omega} -\rho dd^c(u - v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge \beta^{n-m} = 0.
 \end{aligned}$$

Similarly, $\int_{\Omega} -vdd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0$. So

$$\begin{aligned}
 &\int_{\Omega} -(u - v)dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} \\
 &= \int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge dd^c \rho \wedge \beta^{n-m} = 0, \quad (3)
 \end{aligned}$$

for every couple $j, k, j + k = m - 2$. Assume that

$$\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} = 0 \quad (4)$$

for $j + k = m - l - 1$. By (3), (4) is true for $l = 1$. For $j + k = m - l - 2$ we have

$$\begin{aligned}
 &\int_{\Omega} d(u - v) \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k (dd^c \rho)^{l+1} \wedge \beta^{n-m} \\
 &= \int_{\Omega} -\rho(dd^c(u - v))^2 \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \\
 &= \int_{\Omega} d\rho \wedge d^c(u - v) \wedge dd^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \\
 &\leq \left| \int_{\Omega} d\rho \wedge d^c(u - v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right| \\
 &\quad + \left| \int_{\Omega} d\rho \wedge d^c(u - v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right|
 \end{aligned}$$

$$\begin{aligned}
&\leq \left[\int_{\Omega} d\rho \wedge d^c \rho \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \\
&\quad \times \left[\int_{\Omega} d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^{j+1} \wedge (dd^c v)^k \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \\
&\quad + \left[\int_{\Omega} d\rho \wedge d^c \rho \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \\
&\quad \times \left[\int_{\Omega} d(u-v) \wedge d^c(u-v) \wedge (dd^c u)^j \wedge (dd^c v)^{k+1} \wedge (dd^c \rho)^l \wedge \beta^{n-m} \right]^{\frac{1}{2}} \\
&= 0,
\end{aligned}$$

by assumption (4). So (2) is true by taking $l = m - 1$ in (4). $\square$

Proof of Proposition 5. Assume that $v \in \mathcal{F}_m(\Omega)$, $H_m(v)(\Omega) = H_m(u)(\Omega)$ and $v \leq u$. Since $v \leq u$, Proposition 2 implies that $H_m(v) \succcurlyeq H_m(u)$. From the assumption $H_m(u)$ is maximal, we get $H_m(u) = H_m(v)$. Now Proposition 5 follows from Lemma 1. $\square$

Lemma 3. Assume that $u, v \in \mathcal{E}_m(\Omega)$ and $u \geq v$. Then $\chi_{\{u=-\infty\}} H_m(u) \leq \chi_{\{v=-\infty\}} H_m(v)$.

Proof. We use a method from [1]. For $\epsilon > 0$ small enough, set $w_j = \max\{(1 - \epsilon)u - j, v\}$. Then we have $w_j = (1 - \epsilon)u - j$ on the open set $\{v < -\frac{j}{\epsilon}\}$. Therefore

$$H_m(w_j) = (1 - \epsilon)^m H_m(u) \text{ on } \{v < -\frac{j}{\epsilon}\}.$$

Hence $H_m(w_j) \geq (1 - \epsilon)^m \chi_{\{u=-\infty\}} H_m(u)$. Letting $j \rightarrow \infty$, then we get $H_m(v) \geq (1 - \epsilon)^m \chi_{\{u=-\infty\}} H_m(u)$. The proof is complete by letting $\epsilon \rightarrow 0^+$. $\square$

Lemma 4. For each $u \in \mathcal{F}_m(\Omega)$, if $H_m(u)$ is carried by an m -polar set, then $H_m(u) = \chi_{\{u=-\infty\}} H_m(u)$.

Proof. We use the same idea as in [5]. We choose a sequence $\{u_j\} \in \mathcal{E}_{0,m}(\Omega) \cap C(\Omega)$, $u_j \downarrow u$. Then $\frac{u_j}{1-u_j} \downarrow \frac{u}{1-u} \in \mathcal{F}_m(\Omega) \cap L^\infty(\Omega)$. For each $v \in C^2(\Omega)$,

$$\frac{\partial}{\partial z_l \partial \bar{z}_k} \left(\frac{v}{1-v} \right) = \frac{v_{l\bar{k}}}{(1-v)^2} + \frac{2v_l v_{\bar{k}}}{(1-v)^3}, \forall 1 \leq l, k \leq n.$$

This implies that

$$\frac{H_m(u_j)}{(1-u_j)^{2m}} \leq H_m \left(\frac{u_j}{1-u_j} \right).$$

The function $\frac{1}{(1-t)^{2m}}$ is convex on $[-\infty, 0]$, hence by [11, Proposition 2.1], $\frac{1}{(1-u)^{2m}} - 1 \in SH_m^-(\Omega)$. For every fixed k ,

$$\begin{aligned} \left(\frac{1}{(1-u_k)^{2m}} - 1 \right) H_m(u) &\geq \lim_{j \rightarrow \infty} \left(\frac{1}{(1-u_k)^{2m}} - 1 \right) H_m(u_j) \\ &\geq \lim_{j \rightarrow \infty} \left(\frac{1}{(1-u_j)^{2m}} - 1 \right) H_m(u_j) \geq \lim_{j \rightarrow \infty} \left(\frac{1}{(1-u)^{2m}} - 1 \right) H_m(u_j) \\ &= \left(\frac{1}{(1-u)^{2m}} - 1 \right) H_m(u). \end{aligned}$$

Letting $k \rightarrow \infty$, we get $\frac{H_m(u_j)}{(1-u_j)^{2m}}$ tends weakly to $\frac{H_m(u)}{(1-u)^{2m}}$. Moreover, $H_m\left(\frac{u_j}{1-u_j}\right)$ tends weakly to $H_m\left(\frac{u}{1-u}\right)$. Hence,

$$\frac{H_m(u)}{(1-u)^{2m}} \leq H_m\left(\frac{u}{1-u}\right). \quad (5)$$

Theorem 8 shows that there exist $\varphi \in \mathcal{E}_{0,m}(\Omega)$ and $f \in L^1(H_m(\varphi))$ such that

$$H_m(u) = f H_m(\varphi) + \nu,$$

where ν is carried by an m -polar set. Moreover, (5) implies that $\frac{H_m(u)}{(1-u)^{2m}}$ has no mass on m -polar sets. Hence, $\frac{\nu}{(1-u)^{2m}} = 0$, so ν is carried by the set $\{u = -\infty\}$. $\square$

Theorem 10. *Let $u \in \mathcal{F}_m(\Omega)$ be such that $H_m(u)$ is carried by an m -polar set. Then u is a minimal function.*

Proof. Assume that $v \in \mathcal{F}_m(\Omega)$, $v \leq u$ and $H_m(v)(\Omega) = H_m(u)(\Omega)$. By Lemmas 3 and 4,

$$\int_{\Omega} H_m(v) \geq \int_{\Omega} \chi_{\{v=-\infty\}} H_m(v) \geq \int_{\Omega} \chi_{\{u=-\infty\}} H_m(u) = \int_{\Omega} H_m(u).$$

Hence, $H_m(v) = \chi_{\{v=-\infty\}} H_m(v)$. By Lemma 3 again, $H_m(u) \leq H_m(v)$. Combine this with $H_m(u)(\Omega) = H_m(v)(\Omega)$, we get $H_m(u) = H_m(v)$. Lemma 2 implies that $u = v$. $\square$

Proposition 6. *Assume that μ is a finite measure on Ω such that $\widehat{\text{supp}}\mu$ is contained in a level set $\{z \in \Omega: \psi(z) = c\}$, where $c > -\infty$ and $\psi < 0$ is a strictly m -subharmonic function on Ω . Then μ is maximal.*

Proof. Suppose that $\nu \succ \mu$ and $\nu(\Omega) = \mu(\Omega)$. By Proposition 3, $\text{supp}\nu \subset \{z \in \Omega: \psi(z) = c\}$. Thus,

$$\int_{\Omega} -\psi d\nu = \int_{\Omega} -cd\nu = \int_{\Omega} -cd\mu = \int_{\Omega} -\psi d\mu < \infty.$$

Therefore, Lemma 1 implies that $\nu = \mu$, and the proof is complete. $\square$

The following example confirms Remark 5(3).

Example 5 [3, Examples 4.15, 4.16]. We consider the unit disc $\mathbb{D}$ in $\mathbb{C}$. Define the sets $S_1 = \{z = \frac{1}{2}e^{i\theta} : 0 \leq \theta \leq \pi\}$ and $S_2 = \{z = \frac{1}{2}e^{i\theta} : \pi < \theta < 2\pi\}$. Let σ be the area measure on the circle $\partial\mathbb{D}(0, \frac{1}{2})$ and define $\mu_j = \sigma|_{S_j}$, for $j = 1, 2$. We have $S_j \subset \{\psi = |z|^2 - 1 = -\frac{3}{4}\}$. Let $h_j = h_{1, S_j, \mathbb{D}}$ be the 1-relative extremal function for S_j over $\mathbb{D}$. Then $h_j \in \mathcal{E}_{0,1}(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ and $h_j = -1$ on S_j . Moreover, h_j is harmonic on the connected set $\mathbb{D} \setminus S_j$, which implies that $h > -1$ on $\mathbb{D} \setminus S_j$. Hence $\hat{S}_j = S_j$ and Proposition 6 deduces that μ_1 and μ_2 are maximal measures. But $\sigma = \mu_1 + \mu_2$ is not maximal (see Example 1).

We will show that each function in $\mathcal{F}_m(\Omega)$ is minorized by a minimal function with the same total Hessian mass.

Proposition 7. *Let $\{u_j\}$ be a decreasing sequence in $\mathcal{F}_m(\Omega)$ such that $u_j \downarrow u$ and $H_m(u_j)(\Omega) = H_m(u_{j+1})(\Omega)$ for all j . Then $u \in \mathcal{F}_m(\Omega)$ and $H_m(u)(\Omega) = H_m(u_j)(\Omega)$.*

Proof. We have $u \in SH_m^-(\Omega)$, and by Theorem 4, there exists a sequence $\{w_j\} \subset \mathcal{E}_{0,m}(\Omega) \cap C(\bar{\Omega})$ such that $w_j \downarrow u$ as $j \rightarrow \infty$. Set $v_j = \max(w_j, u_j)$. Then $v_j \geq u_j$, $v_j \in \mathcal{E}_{0,m}(\Omega)$ and $v_j \downarrow u$ as $j \rightarrow \infty$. Theorem [10, Theorem 3.22] implies that

$$\sup_j \int_{\Omega} H_m(v_j)(\Omega) \leq \sup_j H_m(u_j) = H_m(u_1) < \infty,$$

Thus, $u \in \mathcal{F}_m(\Omega)$. Since the sequence of measures $H_m(v_j)$ converges to the measure $H_m(u)$ in the weak*-topology, we get

$$\liminf_{j \rightarrow \infty} H_m(v_j)(\Omega) \geq H_m(u)(\Omega).$$

Moreover, by [10, Theorem 3.22] again, we obtain $H_m(u)(\Omega) \geq H_m(u_j)$ since $u, u_j \in \mathcal{F}_m(\Omega)$, $u \leq u_j$. $\square$

Theorem 11. *For each $u \in \mathcal{F}_m(\Omega)$, there exists a minimal function $u_0 \in \mathcal{F}_m(\Omega)$ such that $u_0 \leq u$ and $H_m(u_0)(\Omega) = H_m(u)(\Omega)$.*

Proof. Define $S = \{v \in \mathcal{F}_m(\Omega) : v \leq u, H_m(v)(\Omega) = H_m(u)(\Omega)\}$. Let T be the totally ordered subset of S and let $t(z) = \inf_{v \in T} v(z)$. We shall prove that $t \in S$. It is obvious that $t \leq u$. Let $\{K_i\}$ be a compact exhaustion sets of Ω and let $\{t_j\}$ be a sequence of continuous functions such that $t_j \geq t$ and $t_j \downarrow t$ as $j \rightarrow \infty$. For each $z \in K_i$, choose $v_z \in T$ such that $v_z(z) < t_j(z)$ and define the open set $U_z = \{w \in \Omega : v_z(w) < t_j(w)\}$. Take $z_1, \dots, z_N \in K_i$ such that $\cup_{k=1}^N U_{z_k} \supset K_i$. Since T is totally ordered, we may choose v_i^j to be the smallest of the functions $v_{z_1}, \dots, v_{z_N}$, which implies that $v_i^j < t_j$ on K_i . Now let $u_1 = v_1^1$ and u_j be the smallest of the functions $\{u_1, \dots, u_{j-1}, v_j^j\}$ if $j \geq 2$, since T is totally ordered. Then $\{u_j\}$ is a decreasing sequence of functions in T such that $u_j \leq v_j^j < t_j$ on K_j . Therefore $u_j \in \mathcal{F}_m(\Omega)$, $H_m(u_j)(\Omega) = H_m(u)(\Omega)$ and $u_j \downarrow t$, as $j \rightarrow \infty$. Proposition 7 implies $t \in \mathcal{F}_m(\Omega)$ and $H_m(t)(\Omega) = H_m(u)(\Omega)$.

Hence $t \in S$. Since T is arbitrary, Zorn's lemma deduces that there is a minimal element u_0 of S , so the proof is complete. $\square$

5. Convergence in the Weak*-Topology

We will use the m -subharmonic ordering to obtain some results on weak*-convergence of measures. If Ω is a bounded domain in $\mathbb{C}^n$ and $\{u_j\}$ is a sequence of locally bounded m -subharmonic functions on Ω which is decreasing to a function $u \in SH_m(\Omega) \cap L_{loc}^\infty(\Omega)$, then $H_m(u_j)$ converges to $H_m(u)$ in the weak*-topology (see [4]). The same conclusion holds if $SH_m(\Omega) \cap L_{loc}^\infty(\Omega)$ is replaced by the class $\mathcal{E}_m(\Omega)$, where Ω is a bounded m hyperconvex domain (see [9]).

The following example shows that Hessian operator is discontinuous with respect to the convergence in L_{loc}^1 . This example follows the idea in [8].

Example 6. For $n \geq 2$, we define

$$u_j(z_1, \dots, z_n) = \left| \sum_{k=1}^n z_k^{2j} \right|^{\frac{1}{2j}}$$

We can compute

$$\frac{\partial^2 u}{\partial z_p \partial \bar{z}_q} = \frac{1}{4} \left| \sum_{k=1}^n z_k^{2j} \right|^{\frac{1}{2j}-2} z_p^{2j-1} \bar{z}_q^{2j-1}, \quad \forall 1 \leq p, q \leq n.$$

Thus, $H_m(u_j) = 0$, for all j . We have $0 \leq u_j \leq n^{\frac{1}{2j}} u$, where $u(z_1, \dots, z_n) = \max\{|z_1|, \dots, |z_n|\}$. Hence, we get $u_j \rightarrow u$ in $L_{loc}^1(\mathbb{C}^n)$ as $j \rightarrow \infty$. We can show that $H_m(u) \neq 0$. Assume the contrary. Then $H_m(u) = 0$ on the polydisc $\Delta_n(r) = \mathbb{D}(0, r) \times \dots \times \mathbb{D}(0, r)$, i.e., u is m -maximal function on $\Delta_n(r)$. Note that $u \geq r_1$ outside the compact subset $\overline{\Delta_n(r_1)}$, where $r_1 < r$ but we do not have $u \geq r_1$ on $\Delta_n(r)$.

The following theorem give us a sufficient condition for weak*-convergence for the class $\mathcal{F}_m(\Omega)$.

Theorem 12. *If $u_j \rightarrow u$ in $L_{loc}^1(\Omega)$ and there is a strictly m -subharmonic function $v \in \mathcal{E}_{0,m}(\Omega)$ such that*

$$\int_{\Omega} v H_m(u_j) \rightarrow \int_{\Omega} v H_m(u) \text{ as } j \rightarrow \infty,$$

then $H_m(u_j)$ tends to $H_m(u)$ in the weak-topology.*

Proof. We use the idea from [6]. For $w \in \mathcal{E}_{0,m}(\Omega)$, using integration by parts (Theorem 7) we have

$$\int_{\Omega} w H_m(u_j) \leq \int_{\Omega} w H_m[(\sup_{s \geq j} u_s)^*] \downarrow \int_{\Omega} w H_m(u) \text{ as } j \rightarrow \infty.$$

Hence,

$$\limsup_{j \rightarrow \infty} \int_{\Omega} w H_m(u_j) \leq \int_{\Omega} w H_m(u). \quad (6)$$

Theorem 4 implies that (6) is true for $w \in SH_m^-(\Omega)$. Let $\varphi \in C_0^\infty(\Omega)$ be given. By assumption v is strictly m -subharmonic we can choose $A > 0$ large enough such that $(\pm\varphi + Av) \in \mathcal{E}_{0,m}(\Omega)$. By (6) we have

$$\limsup_{j \rightarrow \infty} \int_{\Omega} (\pm\varphi + Av) H_m(u_j) \leq \int_{\Omega} (\pm\varphi + Av) H_m(u).$$

Combining this with assumption $\lim_{j \rightarrow \infty} \int_{\Omega} v H_m(u_j) = \int_{\Omega} v H_m(u)$ we obtain

$$\limsup_{j \rightarrow \infty} \int_{\Omega} \pm\varphi H_m(u_j) \leq \int_{\Omega} \pm\varphi H_m(u),$$

which implies the desired result. $\square$

Definition 11. If $\{\mu_j\}$ is a sequence of measures such that $\mu_{j+1} \succ \mu_j$ for all j , then we say that $\{\mu_j\}$ is m -subharmonically increasing.

Theorem 13. Let $\{\mu_j\}$ be an m -subharmonically increasing sequence of measures on Ω such that $\sup_j \mu_j(\Omega) < \infty$. Then μ_j converges to a measure μ in the weak*-topology. Moreover, $\int_{\Omega} (-\varphi) d\mu_j \uparrow \int_{\Omega} (-\varphi) d\mu$ for each $\varphi \in SH_m^-(\Omega)$.

Proof. Let $\varphi \in SH_m^-(\Omega) \cap L^\infty(\Omega)$. Then

$$0 \leq \int_{\Omega} (-\varphi) d\mu_1 \leq \int_{\Omega} (-\varphi) d\mu_2 \leq \cdots \leq \sup_{\Omega} (-\varphi) \sup_j \mu_j(\Omega) < \infty.$$

so $\lim_{j \rightarrow \infty} \int_{\Omega} (-\varphi) d\mu_j < \infty$. Thus the limit exists for each $\varphi \in C_0(\Omega)$. It follows that this defines a measure μ on Ω that μ_j converges to μ in the weak*-topology. Moreover, we know that $\lim_{j \rightarrow \infty} \int_{\Omega} (-\varphi) d\mu_j = \int_{\Omega} (-\varphi) d\mu$ for each $\varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\bar{\Omega})$. Now, let $\varphi \in SH_m^-(\Omega)$. As above $\{\int_{\Omega} (-\varphi) d\mu_j\}$ is an increasing sequence. We always have

$$\lim_{j \rightarrow \infty} \int_{\Omega} (-\varphi) d\mu_j \geq \int_{\Omega} (-\varphi) d\mu. \quad (7)$$

To show the equality in (7), we assume the contrary, i.e.,

$$\lim_{j \rightarrow \infty} \int_{\Omega} (-\varphi) d\mu_j > \int_{\Omega} (-\varphi) d\mu.$$

Choose j_0 enough large such that $\int_{\Omega} (-\varphi) d\mu_{j_0} > \int_{\Omega} (-\varphi) d\mu$, and a sequence $\{\varphi_k\} \in \mathcal{E}_{0,m} \cap C(\bar{\Omega})$ such that $\varphi_k \downarrow \varphi$. Then we might choose k_0 such that $\int_{\Omega} (-\varphi_{k_0}) d\mu_{j_0} > \int_{\Omega} (-\varphi) d\mu$. It follows that

$$\begin{aligned}\int_{\Omega} (-\varphi_{k_0}) d\mu &= \lim_{j \rightarrow \infty} \int_{\Omega} (-\varphi_{k_0}) d\mu_j \geq \int_{\Omega} (-\varphi_{k_0}) d\mu_{j_0} \\ &> \int_{\Omega} (-\varphi) d\mu \geq \int_{\Omega} (-\varphi_{k_0}) d\mu,\end{aligned}$$

which is a contradiction. $\square$

If $\{u_j\} \subset \mathcal{F}_m(\Omega)$ converges to $u \in \mathcal{F}_m(\Omega)$ in $L^1_{loc}(\Omega)$, then we can relate the limit measure of sequence $\{H_m(u_j)\}$ in Theorem 13 to $H_m(u)$ as follows.

Corollary 1. *Assume that $\{u_j\} \subset \mathcal{F}_m(\Omega)$ such that*

- (1) u_j converges to $u \in \mathcal{F}_m(\Omega)$ in $L^1_{loc}(\Omega)$,
- (2) $\{H_m(u_j)\}$ is m -subharmonically increasing,
- (3) $\sup_j H_m(u_j) < \infty$.

Then $H_m(u_j)$ converges to a measure μ in the weak-topology such that $\mu \succcurlyeq H_m(u)$. Moreover, $\int_{\Omega} (-\varphi) H_m(u_j) \uparrow \int_{\Omega} (-\varphi) d\mu$ for each $\varphi \in SH^-_m(\Omega)$.*

Proof. By Theorem 13 it remains to show that $\mu \succcurlyeq H_m(u)$. By the proof of Theorem 12, assumption (1) implies that $\liminf_{j \rightarrow \infty} \int_{\Omega} (-\varphi) H_m(u_j) \geq \int_{\Omega} (-\varphi) H_m(u)$ for each $\varphi \in SH^-_m(\Omega)$. $\square$

The following theorem gives us a bridge between convergence in weak*-topology and the concept of maximal measures defined in Sect. 4.

Theorem 14. *Let $\{u_j\} \subset \mathcal{F}_m(\Omega)$ such that*

- (1) u_j converges to $u \in \mathcal{F}_m(\Omega)$ in $L^1_{loc}(\Omega)$,
- (2) $H_m(u)$ is a maximal measure,
- (3) $\lim_{j \rightarrow \infty} H_m(u_j)(\Omega) = H_m(u)(\Omega)$.

Then $H_m(u_j)$ converges to $H_m(u)$ in the weak-topology.*

Proof. Assumption (3) implies that there is a subsequence $\{H_m(u_{j_k})\} \subset \{H_m(u_j)\}$ which converging to a measure μ in the weak*-topology. Let $\varphi \in \mathcal{E}_{0,m}(\Omega) \cap C(\bar{\Omega})$ be given. As in the proof of Corollary 1, assumption (a) implies that $\mu \succcurlyeq H_m(u)$. Moreover, by (3) we have $\mu(\Omega) \leq \liminf_{j \rightarrow \infty} H_m(u_{j_k})(\Omega) \leq H_m(u)(\Omega)$. Thus, $\mu(\Omega) = H_m(u)(\Omega)$. By assumption (2) we can conclude that $\mu = H_m(u)$. $\square$

Open Question

One might ask if there is a converse of Proposition 5. The answer is affirmative if $n = m = 1$ (see [3, Proposition 4.11]). In higher dimension, the answer is unknown.

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