Paley-Wiener Isomorphism Over Infinite-Dimensional Unitary Groups

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Abstract. An analog of the Paley-Wiener isomorphism for the Hardy space with an invariant measure over infinite-dimensional unitary groups is described. This allows us to investigate on such space the shift and multiplicative groups, as well as, their generators and intertwining operators. We show applications to the Gauss-Weierstrass semigroups and to the Weyl–Schrödinger irreducible representations of complexified infinite-dimensional Heisenberg groups.

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1. Introduction

The work deals with the Hardy space H_χ^2 of square-integrable $\mathbb{C}$ -valued functions with respect to a probability measure χ over the infinite-dimensional unitary group $U(\infty) := \bigcup \{U(m) : m \in \mathbb{N}\}$, extended by unit $\mathbb{1}$, which irreducibly acts on a separable complex Hilbert space E with an orthonormal basis $\{e_m\}$. Here, $U(m)$ is the subgroup of unitary $m \times m$ -matrices endowed with Haar's measure χ_m .

In what follows, $U(\infty)$ is densely embedded via a universal mapping π into the space of virtual unitary matrices $\mathfrak{U} = \varprojlim U(m)$ defined as the projective limit under Livšic's mappings $\pi_m^{m+1} : U(m+1) \rightarrow U(m)$. The projective limit $\chi = \varprojlim \chi_m$, such that each image-measure $\pi_m^{m+1}(\chi_{m+1})$ is equal to χ_m , is concentrated on the range $\pi(U(\infty))$ consisting of stabilized sequences (see [18, 20]). The measure χ is invariant under right actions [20, n.4].

We refer to [5, 26] for applications of χ to stochastic processes. Needed properties of Hardy spaces H_χ^2 can be found in [15]. Various cases of Hardy spaces in infinite-dimensional settings were considered in [9, 17].

Now, we briefly describe results. Using a unitarily weighted symmetric Fock space $(\Gamma_w, \langle \cdot | \cdot \rangle_w)$ with a canonical orthogonal basis of symmetric tensor products $\{e_i^{\odot \lambda}\}$ of basis elements $\{e_m\} \subset E$ indexed by Young diagrams λ and normalized by measure χ , we find an orthogonal basis of polynomial $\{\phi_i^\lambda\}$ in H_χ^2 such that the conjugate-linear mapping

$$\Phi: \Gamma_w \rightarrow H_\chi^2$$

is a surjective isometry with one-to-one correspondence $e_i^{\odot \lambda} \rightleftharpoons \phi_i^\lambda$. This allows us to establish in Theorem 2 an integral formula for a Fock-symmetric $\mathcal{F}$ -transform

$$\mathcal{F}: H_\chi^2 \ni f \mapsto \hat{f} \in H_w^2$$

where the Hilbert space H_w^2 , uniquely determined by Γ_w , consists of Hilbert–Schmidt analytic entire functions on E . Thus, the $\mathcal{F}$ -transform acts as an analog of the Paley–Wiener isomorphism over infinite-dimensional groups.

Furthermore, we investigate two different representations of the additive group $(E, +)$ over the Hardy space H_χ^2 by shift and multiplicative groups. Theorem 3 states that the $\mathcal{F}$ -transform is an intertwining operator between the multiplication group $M_a^\dagger$ on H_χ^2 and the shift group T_a on H_w^2 . On the other hand, Theorem 4 shows that $\mathcal{F}$ is the same between the shift group $T_a^\dagger$ on H_χ^2 and the multiplication group M_{a^*} on H_w^2 . Integral formulas describing interrelations between their generators are established. In Theorem 5 suitable commutation relations are stated.

Applications to the Gauss–Weierstrass-type semigroups on H_χ^2 are shown in Theorem 6. Another application to linear representations of complexified infinite-dimensional Heisenberg groups on H_χ^2 in a Weyl–Schrödinger form is given in Theorem 7.

Infinite-dimensional Heisenberg groups was considered in [16] by using reproducing kernel Hilbert spaces. The Schrödinger representation of infinite-dimensional Heisenberg groups on L_γ^2 with respect to a Gaussian measure γ over a real Hilbert space is described in [3] (see also earlier publications [1, 2]).

In conclusion, we note that a motivation for this study was the following simple relations in the Hardy space H_χ^2 over 1-dimensional group $U(1) = \{u = \exp(i\vartheta) : \vartheta \in [-\pi, \pi]\}$. In this case, $\{u^n : n \in \mathbb{Z}_+\}$ is an orthonormal basis and $\Gamma_w \simeq \ell^2$, since $\Phi^* f = (f_n) \in \ell^2$ for any $f \in H_\chi^2$ with Fourier coefficients $f_n = \int f(u) \bar{u}^n du = \langle u^n | \Phi^* f \rangle_w$. While, $\hat{f}(x) = \int f(u) \exp(x\bar{u}) du = \sum f_n x^n / n! = \langle \varepsilon(x) | \Phi^* f \rangle_w$, where $\varepsilon(x) = (x^n / n!) \in \ell^2$ for all $x \in \mathbb{C}$.

Moreover, the equalities $T_a \hat{f}(x) = \int f(u) \exp[(x+a)\bar{u}] du = \mathcal{F}(M_a^\dagger f)(x)$ with $M_a^\dagger f(u) = \exp(a\bar{u}) f(u)$ hold for all $x, a \in \mathbb{C}$. On the other hand, $(M_a \hat{f})(x) = \exp(x\bar{a}) \sum f_n x^n / n! = \sum (T_a^\dagger f_n) x^n / n!$ with $T_a^\dagger f_n = \sum_{k=0}^n \binom{n}{k} \bar{a}^k f_{n-k}$. Hence,

$M_{\bar{a}}\hat{f} = \mathcal{F}(T_{\bar{a}}^{\dagger}f)$, where $(T_{\bar{a}}^{\dagger}f)(u) = \sum (T_{\bar{a}}^{\dagger}f_n)u^n$. In result, $T_a M_{\bar{b}} = \exp(a\bar{b}) M_{\bar{b}} T_a$ for all $a, b \in \mathbb{C}$ and the Weyl-Schrödinger representation of Heisenberg's group from Theorem 7 retains a classic form.

The case H_{χ}^2 over m -dimensional group $U(m)$ is similar with a proviso that the weighted Fock space $\Gamma_{\mathbf{w}}$ is normalized by $\|e_i^{\odot \lambda}\|_{\mathbf{w}} = \binom{n+m-1}{n}^{-1/2}$ where $n = |\lambda|$ is a homogeneity degree of the basis polynomial ϕ_i^{λ} in H_{χ}^2 . Note that the normalization $\|e_i^{\odot \lambda}\|_{\mathbf{w}} = n!^{-1/2}$ with $n = |\lambda|$ leads to the case of Segal-Bargmann's space H_{γ}^2 with standard centered probability Gaussian measure γ on $\mathbb{C}^m$.

2. Hilbert-Schmidt Analyticity

Let E stand for a separable complex Hilbert space with scalar product $\langle \cdot | \cdot \rangle$ norm $\|\cdot\|$ and a fixed orthonormal basis $\{e_k : k \in \mathbb{N}\}$. Denote by $E_{\text{alg}}^{\otimes n} = E \otimes \cdots \otimes E$ ($n \in \mathbb{N}$) its algebraic tensor power consisted of the linear span of elements $\psi_n = x_1 \otimes \cdots \otimes x_n$ with $x_i \in E$ ($i = 1, \dots, n$). Set $x^{\otimes n} := x \otimes \cdots \otimes x$. The symmetric algebraic tensor power $E_{\text{alg}}^{\odot n} = E \odot \cdots \odot E$ is defined to be the range of the projector $\mathfrak{s}_n : E_{\text{alg}}^{\otimes n} \ni \psi_n \mapsto x_1 \odot \cdots \odot x_n$ with $x_1 \odot \cdots \odot x_n := (n!)^{-1} \sum_{\sigma \in \sigma(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$ where $\sigma : \{1, \dots, n\} \mapsto \{\sigma(1), \dots, \sigma(n)\}$ runs through all permutations. The symmetric algebraic Fock space is defined as the algebraic direct sum $\Gamma_{\text{alg}} = \sum_{n \in \mathbb{Z}_+} E_{\text{alg}}^{\odot n}$ with $E_{\text{alg}}^{\odot 0} = \mathbb{C}$.

Let $E_{\text{h}}^{\otimes n} := E \otimes_{\text{h}} \cdots \otimes_{\text{h}} E$ be the completion of $E_{\text{alg}}^{\otimes n}$ by Hilbertian norm $\|\psi_n\|_{\text{h}} = \langle \psi_n | \psi_n \rangle_{\text{h}}^{1/2}$ with $\langle \psi_n | \psi'_n \rangle_{\text{h}} = \langle x_1 | x'_1 \rangle \cdots \langle x_n | x'_n \rangle$. Denote by $E_{\text{h}}^{\odot n}$ the range of continuous extension of $\mathfrak{s}_n$ on $E_{\text{h}}^{\otimes n}$. As usual, the symmetric Fock space is defined to be the Hilbertian direct sum $\Gamma_{\text{h}} = \bigoplus_{n \in \mathbb{Z}_+} E_{\text{h}}^{\odot n}$.

Denote by $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Z}_+^m$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$ a partition of $n \in \mathbb{N}$, that is, $n = |\lambda|$ where $|\lambda| := \lambda_1 + \cdots + \lambda_m$. Any λ may be identified with Young's diagram of length $l(\lambda) = m$. Let $\mathbb{Y}$ denote all diagrams and $\mathbb{Y}_n = \{\lambda \in \mathbb{Y} : |\lambda| = n\}$. Assume that $\mathbb{Y}_0 = \{\emptyset \in \mathbb{Y} : |\emptyset| = 0\}$ and $l(\emptyset) = 1$. Let $\mathbb{N}_*^m := \{\iota = (\iota_1, \dots, \iota_m) \in \mathbb{N}^m : \iota_l \neq \iota_k, \forall l \neq k\}$. For each $\lambda \in \mathbb{Y}$ we assign the constant

$$C_{|\lambda|, l(\lambda)} := \frac{(l(\lambda) - 1)! |\lambda|!}{(l(\lambda) - 1 + |\lambda|)!} \leq 1. \quad (1)$$

The spaces $E_{\text{alg}}^{\odot n}$ and Γ_{alg} may be generated by the basis of symmetric tensors

$$\begin{aligned} e^{\odot \mathbb{Y}_n} &= \bigcup \left\{ e_i^{\odot \lambda} := e_{\iota_1}^{\otimes \lambda_1} \odot \cdots \odot e_{\iota_{l(\lambda)}}^{\otimes \lambda_{l(\lambda)}} : (\lambda, \iota) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)} \right\}, \\ e^{\odot \mathbb{Y}} &= \bigcup \{ e^{\odot \mathbb{Y}_n} : n \in \mathbb{Z}_+ \} \quad \text{with} \quad e_i^{\odot \emptyset} = 1, \end{aligned}$$

respectively. As is known [4, Sect. 2.2.2], norm of basis element in $\Gamma_{\mathfrak{h}}$ is equal to

$$\|e_i^{\odot \lambda}\|_{\mathfrak{h}}^2 = \frac{\lambda!}{|\lambda|!}, \quad \lambda! := \lambda_1! \cdots \lambda_m!. \quad (2)$$

Let us define a new Hilbertian norm on Γ_{alg} by the equality $\|\cdot\|_{\mathfrak{w}} = \langle \cdot | \cdot \rangle_{\mathfrak{w}}^{1/2}$ where scalar product $\langle \cdot | \cdot \rangle_{\mathfrak{w}}$ is determined via the orthogonal relations

$$\langle e_i^{\odot \lambda} | e_{i'}^{\odot \lambda'} \rangle_{\mathfrak{w}} = \begin{cases} C_{|\lambda|, l(\lambda)} \|e_i^{\odot \lambda}\|_{\mathfrak{h}}^2 & : \lambda = \lambda' \text{ and } i = i', \\ 0 & : \lambda \neq \lambda' \text{ or } i \neq i'. \end{cases}$$

Denote by $E_{\mathfrak{w}}^{\odot n}$ and $\Gamma_{\mathfrak{w}}$ the appropriate completions of $E_{\text{alg}}^{\odot n}$ and Γ_{alg} , respectively. For any $i \in \mathbb{N}_*^{l(\lambda)}$ there corresponds in $E_{\mathfrak{w}}^{\odot n}$ the d -dimension subspace with $d = C_{|\lambda|, l(\lambda)}^{-1}$, spanned by elements $\{e_i^{\odot \lambda} : \lambda \in \mathbb{Y}_n\}$. The Hilbertian orthogonal sum

$$\Gamma_{\mathfrak{w}} = \bigoplus_{n \in \mathbb{Z}_+} E_{\mathfrak{w}}^{\odot n}$$

endowed with $\langle \cdot | \cdot \rangle_{\mathfrak{w}}$ we will call *unitarily weighted* symmetric Fock space.

Let $x = \sum e_k x_k$ be the Fourier series of $x \in E$ with coefficients $x_k = \langle x | e_k \rangle$. We assign to any $(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}$ the n -homogenous Hilbert-Schmidt polynomial defined via the Fourier coefficients

$$x_i^{\lambda} := \langle x^{\otimes n} | e_i^{\odot \lambda} \rangle_{\mathfrak{w}} = x_{i_1}^{\lambda_1} \cdots x_{i_{l(\lambda)}}^{\lambda_{l(\lambda)}}, \quad x \in E.$$

Using the tensor multinomial theorem, we define in $\Gamma_{\mathfrak{w}}$ the Fourier decomposition of exponential vectors (or coherent state vectors)

$$\begin{aligned} \varepsilon(x) &:= \bigoplus_{n \in \mathbb{Z}_+} \frac{x^{\otimes n}}{n!} = \bigoplus_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\sum_{k \in \mathbb{N}} e_k x_k \right)^{\otimes n} \\ &= \bigoplus_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \frac{n!}{\lambda!} e_i^{\odot \lambda} x_i^{\lambda} \end{aligned} \quad (3)$$

with respect to the basis $e^{\odot \mathbb{Y}}$. It is convergent in $\Gamma_{\mathfrak{w}}$ in view of (1) and

$$\begin{aligned} \|\varepsilon(x)\|_{\mathfrak{w}}^2 &= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!^2} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \left(\frac{n!}{\lambda!} \right)^2 \|e_i^{\odot \lambda}\|_{\mathfrak{w}}^2 |x_i^{\lambda}|^2 \\ &= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!^2} \sum_{(\lambda, i)} \frac{n!}{\lambda!} C_{|\lambda|, l(\lambda)} |x_i^{\lambda}|^2 \leq \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i)} \frac{n!}{\lambda!} |x_i^{\lambda}|^2 \\ &= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\sum_{k \in \mathbb{N}} |x_k|^2 \right)^n = e^{\|x\|^2}. \end{aligned} \quad (4)$$

Particularly, (4) implies that the function $E \ni x \mapsto \varepsilon(x) \in \Gamma_{\mathfrak{w}}$ is entire analytic.

Definition 1. The space of $\mathbb{C}$ -valued *Hilbert–Schmidt entire analytic functions* in variable $x \in E$, associating with the unitarily weighted symmetric Fock space $\Gamma_{\mathbb{W}}$, is defined to be

$$H_{\mathbb{W}}^2 := \{\psi^*(x) := \langle \varepsilon(x) \mid \psi \rangle_{\mathbb{W}} : \psi \in \Gamma_{\mathbb{W}}\} \quad \text{with the norm} \quad \|\psi^*\| = \|\psi\|_{\mathbb{W}}.$$

Every function ψ^* is entire analytic as the composition of $\varepsilon(\cdot)$ with $\langle \cdot \mid \psi \rangle_{\mathbb{W}}$. The subspace in $H_{\mathbb{W}}^2$ of n -homogenous Hilbert–Schmidt polynomials is defined to be

$$H_{\mathbb{W}}^{2,n} = \{\psi_n^*(x) = \langle x^{\otimes n} \mid \psi_n \rangle_{\mathbb{W}} : \psi_n \in E_{\mathbb{W}}^{\odot n}\}.$$

Evidently, $H_{\mathbb{W}}^2 = \mathbb{C} \oplus H_{\mathbb{W}}^{2,1} \oplus H_{\mathbb{W}}^{2,2} \oplus \dots$

It is important that $H_{\mathbb{W}}^2$ is uniquely determined by $\Gamma_{\mathbb{W}}$ since $\{\varepsilon(x) : x \in E\}$ is total in $\Gamma_{\mathbb{W}}$. Similarly, for the subspace $H_{\mathbb{W}}^{2,n}$ which is uniquely determined by $E_{\mathbb{W}}^{\odot n}$, since $\{x^{\otimes n} : x \in E\}$ is total in $E_{\mathbb{W}}^{\odot n}$. The last totality follows from the polarization formula for symmetric tensor products

$$e_i^{\odot \lambda} = \frac{1}{2^n n!} \sum_{\theta_1, \dots, \theta_n = \pm 1} \theta_1 \dots \theta_n a^{\otimes n} \quad \text{with} \quad a = \sum_{i=1}^{l(\lambda)} \theta_i e_{i_i}^{\otimes \lambda_i} \quad (5)$$

which is valid for all $e_i^{\odot \lambda} \in e^{\odot \mathbb{Y}_n}$ (see e.g. [11, Sect. 1.5]) Thus, the conjugate-linear isometries $\psi \mapsto \psi^*$ from $\Gamma_{\mathbb{W}}$ onto $H_{\mathbb{W}}^2$ and from $E_{\mathbb{W}}^{\odot n}$ onto $H_{\mathbb{W}}^{2,n}$ hold.

In conclusion, we can notice that every analytic function $\psi^* \in H_{\mathbb{W}}^2$ determined by $\psi = \bigoplus \psi_n \in \Gamma_{\mathbb{W}}$, ($\psi_n \in E_{\mathbb{W}}^{\odot n}$) has the Taylor expansion at zero

$$\psi^*(x) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \frac{\langle e_i^{\odot \lambda} \mid \psi_n \rangle_{\mathbb{W}}}{\|e_i^{\odot \lambda}\|_{\mathbb{W}}^2} x_i^\lambda, \quad x \in E$$

that follows from (3). The function ψ^* is entire Hilbert–Schmidt analytic [15, n.5].

Note that analytic functions of Hilbert–Schmidt types were also considered in [10, 14, 21]. More general classes of analytic functions associated with coherent sequences of polynomial ideals were described in [8].

3. Hardy Space Over $U(\infty)$

In what follows, we endow each group $U(m)$ with the probability Haar measure χ_m and assume that $U(m)$ is identified with its range with respect to the embedding $U(m) \ni u_m \mapsto \begin{bmatrix} u_m & 0 \\ 0 & \mathbb{1} \end{bmatrix} \in U(\infty)$. The Livšic transform from $U(m+1)$ onto $U(m)$ is described in [18, Proposition 0.1] and [20, Lemma 3.1] as the surjective Borel mapping

$$\pi_m^{m+1} : u_{m+1} := \begin{bmatrix} z_m & a \\ b & t \end{bmatrix} \longmapsto u_m := \begin{cases} z_m - [a(1+t)^{-1}b] & : t \neq -1 \\ z_m & : t = -1. \end{cases}$$

The projective limit $\mathfrak{U} := \varprojlim U(m)$ under π_m^{m+1} has surjective Borel projections $\pi_m: \mathfrak{U} \ni u \mapsto u_m \in U(m)$ such that $\pi_m = \pi_m^{m+1} \circ \pi_{m+1}$.

Consider a universal dense embedding $\pi: U(\infty) \hookrightarrow \mathfrak{U}$ which to every $u_m \in U(m)$ assigns the stabilized sequence $u = (u_k)$ such that (see [20, n.4])

$$\pi: U(m) \ni u_m \mapsto (u_k) \in \mathfrak{U}, \quad u_k = \begin{cases} \pi_k^m(u_m) : k < m \\ u_m : k \geq m, \end{cases} \quad (6)$$

where $\pi_k^m := \pi_k^{k+1} \circ \dots \circ \pi_{m-1}^m$ for $k < m$ and π_k^m is identity mapping for $k \geq m$. On its range $\pi(U(\infty))$, endowed with the Borel structure from $\mathfrak{U}$, we consider the inverse mapping

$$\pi^{-1}: \mathfrak{U}_\pi \rightarrow U(\infty) \quad \text{where} \quad \mathfrak{U}_\pi := \pi(U(\infty)).$$

The right action $\mathfrak{U}_\pi \ni u \mapsto u.g \in \mathfrak{U}_\pi$ with $g = (v, w) \in U(\infty) \times U(\infty)$ is defined by $\pi_m(u.g) = w^{-1}\pi_m(u)v$ where m is so large that $g = (v, w) \in U(m) \times U(m)$.

Following [18, n.3.1], [20, Lemma 4.8] via the Kolmogorov consistency theorem (see e.g. [19, Theorem 1], [24, Corollary 4.2]) we uniquely define on $\mathfrak{U} = \varprojlim U(m)$ the probability measure $\chi := \varprojlim \chi_m$ such that each image-measure $\pi_m^{m+1}(\chi_{m+1})$ is equal to χ_m . For any Borel subset $A \subset \mathfrak{U}_\pi$ we have $\pi_{m+1}(A) \subseteq (\pi_m^{m+1})^{-1}[\pi_m(A)]$, because $\pi_m = \pi_m^{m+1} \circ \pi_{m+1}$. It follows that $(\chi_m \circ \pi_m)(A) = \pi_m^{m+1}(\chi_{m+1})[\pi_m(A)] = \chi_{m+1}[(\pi_m^{m+1})^{-1}[\pi_m(A)]] \geq (\chi_{m+1} \circ \pi_{m+1})(A)$. Hence, χ satisfies the condition

$$\chi(A) = \inf(\chi_m \circ \pi_m)(A) = \lim \chi_m(A) \quad (7)$$

and therefore the projective limit $\varprojlim \chi_m$ exists on $\mathfrak{U}_\pi$ via the well known Prohorov theorem [6, Theorem IX.52]. Moreover, it is a Radon probability measure concentrated on $\mathfrak{U}_\pi$ [24, Theorem 4.1]. By the known portmanteau theorem [13, Theorem 13.16] and Fubini's theorem the invariance of Haar measures χ_m together with (7) yield the following invariance properties under the right action

$$\begin{aligned} \int f(u.g) d\chi(u) &= \int f(u) d\chi(u), \quad g \in U(\infty) \times U(\infty), \quad f \in L_\chi^\infty, \\ \int f d\chi &= \int d\chi(u) \int_{U(m) \times U(m)} f(u.g) d(\chi_m \otimes \chi_m)(g), \end{aligned}$$

where L_χ^∞ stands for the space of all χ -essentially bounded complex-valued functions defined on $\mathfrak{U}_\pi$ and endowed with norm $\|f\|_\infty = \text{ess sup}_{u \in \mathfrak{U}_\pi} |f(u)|$.

Let L_χ^2 be the space of square-integrable $\mathbb{C}$ -valued functions f on $\mathfrak{U}_\pi$ with norm

$$\|f\|_\chi = \langle f | f \rangle_\chi^{1/2} \quad \text{where} \quad \langle f | f \rangle_\chi := \int f_1 \bar{f}_2 d\chi.$$

The embedding $L_\chi^\infty \hookrightarrow L_\chi^2$ holds, moreover, $\|f\|_\chi \leq \|f\|_\infty$ for all $f \in L_\chi^\infty$.

To given the E -valued mapping $\mathfrak{U}_\pi \ni u \mapsto \pi^{-1}(u)e_1$, we can well-define the Borel χ -essentially bounded functions in the variable $u \in \mathfrak{U}_\pi$,

$$\phi_k := \phi_{e_k}, \quad \phi_{e_k}(u) = \langle \pi^{-1}(u)e_1 \mid e_k \rangle, \quad k \in \mathbb{N},$$

which do not depend on the choice of e_1 in $\bigcup S(m)$ where $S(m)$ is the m -dimensional unit sphere in E [15, n.3]. The uniqueness of $\phi_x(u) = \langle \pi^{-1}(u)e_1 \mid x \rangle$ with $x \in E$ results from the total embedding $\pi: U(\infty) \hookrightarrow \mathfrak{U}$. From (6) it follows that $\pi^{-1} \circ \pi_m^{-1}$ coincides with the embedding $U(m) \hookrightarrow U(\infty)$. Hence, by (7) and the portmanteau theorem there exist the limit

$$\int \phi_x d\chi = \lim_{m \rightarrow \infty} \int_{U(m)} \phi_x d(\chi_m \circ \pi_m) = \lim_{m \rightarrow \infty} \int_{U(m)} (\phi_x \circ \pi_m^{-1}) d\chi_m,$$

i.e., $\phi_x \in L_\chi^\infty$ for any $\phi_x(u) = \langle \pi^{-1}(u)e_1 \mid x \rangle$ with $x \in E$.

By formula (5) to every $e_i^{\odot \lambda} \in e^{\odot \mathbb{Y}_n}$ there uniquely corresponds the Borel function from L_χ^∞

$$\phi_i^\lambda(u) := \langle [\pi^{-1}(u)e_1]^{\otimes n} \mid e_i^{\odot \lambda} \rangle_{\mathbb{W}} = \phi_{i_1}^{\lambda_1}(u) \dots \phi_{i_{l(\lambda)}}^{\lambda_{l(\lambda)}}(u)$$

in the variable $u \in \mathfrak{U}_\pi$. It follows that the orthogonal basis $e^{\odot \mathbb{Y}}$ of elements $e_i^{\odot \lambda} = e_{i_1}^{\otimes \lambda_1} \odot \dots \odot e_{i_m}^{\otimes \lambda_m}$, indexed by $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{Y}$ and $i = (i_1, \dots, i_m) \in \mathbb{N}_*^m$ with $m = l(\lambda)$, uniquely determines the systems of Borel χ -essentially bounded functions in the variable $u \in \mathfrak{U}_\pi$,

$$\begin{aligned} \phi^{\mathbb{Y}_n} &= \bigcup \{ \phi_i^\lambda := \phi_{i_1}^{\lambda_1} \dots \phi_{i_m}^{\lambda_m} : (\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^m, m = l(\lambda) \}, \\ \phi^{\mathbb{Y}} &= \bigcup \{ \phi^{\mathbb{Y}_n} : n \in \mathbb{Z}_+ \} \quad \text{with} \quad \phi_i^\emptyset \equiv 1. \end{aligned}$$

Definition 2. The Hardy space H_χ^2 is defined as the closed complex linear span of $\phi^{\mathbb{Y}}$ endowed with L_χ^2 -norm.

The following assertion is proved in [15, Theorem 3.2].

Theorem 1. The system of Borel functions $\phi^{\mathbb{Y}}$ forms an orthogonal basis in H_χ^2 such that

$$\|\phi_i^\lambda\|_\chi = C_{|\lambda|, l(\lambda)}^{1/2} \|e_i^{\odot \lambda}\|_{\mathbb{H}}, \quad \lambda \in \mathbb{Y}, \quad i \in \mathbb{N}_*^{l(\lambda)}.$$

Define the subspace $H_\chi^{2,n} \subset H_\chi^2$ for any $n \in \mathbb{N}$ to be the closed linear span of the subsystem $\phi^{\mathbb{Y}_n}$. Theorem 1 implies that $H_\chi^{2,n} \perp H_\chi^{2,m}$ in L_χ^2 for any $n \neq m$. This provides the orthogonal decomposition

$$H_\chi^2 = \mathbb{C} \oplus H_\chi^{2,1} \oplus H_\chi^{2,2} \oplus \dots$$

4. Fock-Symmetric $\mathcal{F}$ -Transform

The one-to-one correspondence $e_i^{\odot\lambda} \rightleftharpoons \phi_i^\lambda$ allows us to define via the change of orthonormal bases

$$\Phi: \Gamma_{\mathbb{W}} \ni e_i^{\odot\lambda} \|e_i^{\odot\lambda}\|_{\mathbb{W}}^{-1} \mapsto \phi_i^\lambda \|\phi_i^\lambda\|_{\chi}^{-1} \in H_{\chi}^2, \quad \lambda \in \mathbb{Y}, \quad i \in \mathbb{N}_*^{l(\lambda)}$$

the isometric conjugate-linear mapping $\Phi: \Gamma_{\mathbb{W}} \rightarrow H_{\chi}^2$. The adjoint mapping $\Phi^*: H_{\chi}^2 \rightarrow \Gamma_{\mathbb{W}}$ is defined by $\langle \Phi e_i^{\odot\lambda} | f \rangle_{\chi} = \langle e_i^{\odot\lambda} | \Phi^* f \rangle_{\mathbb{W}}$ with $f \in H_{\chi}^2$. The suitable Fourier decomposition has the form

$$\Phi\psi = \sum_{(\lambda, i) \in \mathbb{Y} \times \mathbb{N}_*^{l(\lambda)}} \hat{\psi}_{(\lambda, i)} \phi_i^\lambda \|\phi_i^\lambda\|_{\chi}^{-1}, \quad \hat{\psi}_{(\lambda, i)} := \langle e_i^{\odot\lambda} | \psi \rangle_{\mathbb{W}} \|e_i^{\odot\lambda}\|_{\mathbb{W}}^{-1}$$

for any $\psi \in \Gamma_{\mathbb{W}}$. In particular, the equality $\Phi x = \sum x_k \phi_k$ is valid for all $x \in E$. This gives the equalities

$$\|\Phi x\|_{\chi}^2 = \sum |x_k|^2 = \|x\|^2, \quad x \in E.$$

Using this, we will examine the composition of Φ with the $\Gamma_{\mathbb{W}}$ -valued function $\varepsilon: E \ni x \mapsto \varepsilon(x)$. Its correctness justifies the following assertion that substantially uses the L_{χ}^{∞} -valued function

$$\phi_x: \mathfrak{U}_{\pi} \ni u \mapsto (\Phi x)(u) = \sum x_k \phi_k(u)$$

which is linear in the variable $x \in E$.

Similarly to the known case of Wiener spaces, the function Φx can be seen as a group analog of the Paley-Wiener map (see e.g. [12, n.4.4] or [23]).

Lemma 1. *The composition $\Phi\varepsilon(x)$, which is understood as the function*

$$[\Phi\varepsilon(x)](u): \mathfrak{U}_{\pi} \ni u \mapsto \exp(\phi_x(u)),$$

takes values in L_{χ}^{∞} for all $x \in E$.

Proof. Applying Φ to the Fourier decomposition (3), we obtain

$$\Phi\varepsilon(x) = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \frac{n!}{\lambda!} x_i^\lambda \phi_i^\lambda = \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\sum_{k \in \mathbb{N}} x_k \phi_k \right)^n = \exp(\phi_x).$$

It directly follows that $\|\Phi\varepsilon(x)\|_{\infty} \leq \exp \|\phi_x\|_{\infty}$. $\square$

Theorem 2. *For every $f = \sum f_n \in H_{\chi}^2$, ($f_n \in H_{\chi}^{2,n}$) the entire analytic function $\hat{f}(x) := \langle \varepsilon(x) | \Phi^* f \rangle_{\mathbb{W}}$ in the variable $x \in E$ and its Taylor coefficients at origin have the integral representations*

$$\hat{f}(x) = \int \exp(\bar{\phi}_x) f d\chi \quad \text{and} \quad d_0^n \hat{f}(x) = \int \bar{\phi}_x^n f_n d\chi, \quad (8)$$

respectively. The mapping $\mathcal{F}: H_{\chi}^2 \ni f \mapsto \hat{f} \in H_{\mathbb{W}}^{2,n}$ (regarded as a Fock-symmetric $\mathcal{F}$ -transform) provides the isometries

$$H_{\chi}^2 \simeq H_{\mathbb{W}}^2 \quad \text{and} \quad H_{\chi}^{2,n} \simeq H_{\mathbb{W}}^{2,n}.$$

Proof. First recall that the $\Gamma_{\mathbb{W}}$ -valued function $\varepsilon(\cdot)$ is entire analytic on E , therefore $\hat{f}$ is the same, as the composition of $\varepsilon(\cdot)$ with $\langle \cdot | \Phi^* f \rangle_{\mathbb{W}}$. Farther on, consider the Fourier decomposition with respect to the basis $\phi^{\mathbb{Y}}$,

$$f = \sum_{n \in \mathbb{Z}_+} f_n = \sum_{\substack{n \in \mathbb{Z}_+ \\ (\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}}} \frac{\hat{f}_{\lambda, i, n} \bar{\phi}_i^\lambda}{\|\phi_i^\lambda\|_\chi}, \quad \hat{f}_{\lambda, i, n} = \frac{1}{\|\phi_i^\lambda\|_\chi} \int f \bar{\phi}_i^\lambda d\chi.$$

Applying Φ^* to f in this decomposition and substituting $\hat{f}_{\lambda, i, n}$ into $\hat{f}$, we obtain

$$\begin{aligned} \hat{f}(x) &= \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \frac{n!}{\lambda!} \frac{\hat{f}_{\lambda, i, n} \langle e_i^{\odot \lambda} | e_i^{\odot \lambda} \rangle_{\mathbb{W}} x_i^\lambda}{\|e_i^{\odot \lambda}\|_{\mathbb{W}}} \\ &= \int \sum_{n \in \mathbb{Z}_+} \frac{1}{n!} \left(\sum_{(\lambda, i) \in \mathbb{Y}_n \times \mathbb{N}_*^{l(\lambda)}} \frac{n!}{\lambda!} x_i^\lambda \bar{\phi}_i^\lambda \right) f d\chi = \int \exp(\bar{\phi}_x) f d\chi \end{aligned}$$

where the last equality is valid by Lemma 1. It particularly follows that for $y = \alpha x$,

$$\hat{f}(y) = \int \exp(\bar{\phi}_{\alpha x}) f d\chi = \sum \alpha^n \int \frac{\bar{\phi}_x^n}{n!} f_n d\chi, \quad \alpha \in \mathbb{C}.$$

Differentiating $\hat{f}$ at $y = 0$ and using the n -homogeneity of derivatives, we obtain

$$d_0^n \hat{f}(x) = \frac{d^n}{d\alpha^n} \sum \alpha^n \int \frac{\bar{\phi}_x^n}{n!} f_n d\chi \Big|_{\alpha=0} = \int \bar{\phi}_x^n f_n d\chi.$$

Finally, we notice that the isometry $H_\chi^2 \simeq H_{\mathbb{W}}^2$ holds, since the isometry Φ^* is surjective. In the case of polynomials we similarly get $H_\chi^{2,n} \simeq H_{\mathbb{W}}^{2,n}$. $\square$

Note that a different integral formula for analytic functions employing Wiener measures on infinite-dimensional Banach spaces was presented in [22].

5. Exponential Creation and Annihilation Groups

Let us define the linear mapping $j_n: E_{\mathbb{W}}^{\odot n} \rightarrow E_{\mathbb{H}}^{\odot n}$ to be the continuous extension of identity mapping acting on the dense subspace $E_{\text{alg}}^{\odot n} \subset E_{\mathbb{W}}^{\odot n} \cap E_{\mathbb{H}}^{\odot n}$. Such continuous extension j_n is a contractive injection with dense range. In fact, it suffices to expand elements from $E_{\mathbb{W}}^{\odot n}$ and $E_{\mathbb{H}}^{\odot n}$ into the Fourier series with respect to orthogonal basis $e^{\odot \mathbb{Y}_n}$ and apply the inequality

$$\|e_i^{\odot \lambda}\|_{\mathbb{W}}^2 = C_{|\lambda|, l(\lambda)} \|e_i^{\odot \lambda}\|_{\mathbb{H}}^2 \leq \|e_i^{\odot \lambda}\|_{\mathbb{H}}^2, \quad \lambda \in \mathbb{Y}_n \quad (9)$$

which follows from Theorem 1, taking into account the inequality (1). Using subsequently that $E_{\mathbb{H}}^{\odot n}$ is reflexive, we obtain that its adjoint operator

$j_n^*: E_{\mathfrak{h}}^{\odot n} \rightarrow E_{\mathfrak{w}}^{\odot n}$ is a contractive injection with dense range. Thus, the mapping j_n is also injective. Moreover, $E_{\mathfrak{h}}^{\odot n} \xrightarrow{j_n^*} E_{\mathfrak{w}}^{\odot n} \xrightarrow{j_n} E_{\mathfrak{h}}^{\odot n}$ forms a Gelfand triple. Particularly, the operator $\mathfrak{s}_n$ possesses continuous extension on $E_{\mathfrak{w}}^{\odot n}$.

Using this, we consider the linear operator

$$\mathfrak{s}_{n/m} := \mathfrak{s}_n \circ (j_m \otimes j_{n-m}) \quad \text{with} \quad m \leq n$$

defined to be $\phi_m \odot \psi_{n-m} = \mathfrak{s}_{n/m}(\phi_m \otimes \psi_{n-m}) \in E_{\mathfrak{w}}^{\odot n}$ for all $\phi_m \in E_{\mathfrak{w}}^{\odot m}$, $\psi_{n-m} \in E_{\mathfrak{w}}^{\odot(n-m)}$.

Lemma 2. *The mapping $\mathfrak{s}_{n/m}$ from $E_{\mathfrak{w}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{w}}^{\odot(n-m)}$ to $E_{\mathfrak{w}}^{\odot n}$ is a contractive injection with dense range.*

Proof. Expand elements of $E_{\mathfrak{w}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{w}}^{\odot(n-m)}$ with respect to $e_i^{\odot \lambda} \otimes e_j^{\odot \mu}$ for all $\lambda, \mu \in \mathbb{Y}$, $i \in \mathbb{N}^{l(\lambda)}$, $j \in \mathbb{N}^{l(\mu)}$ such that $|\lambda| = m$, $|\mu| = n - m$. Using (9), we have

$$\begin{aligned} \|e_i^{\odot \lambda} \otimes e_j^{\odot \mu}\|_{E_{\mathfrak{w}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{w}}^{\odot(n-m)}} &= \|e_i^{\odot \lambda}\|_{\mathfrak{w}} \|e_j^{\odot \mu}\|_{\mathfrak{w}} \\ &\leq \|e_i^{\odot \lambda}\|_{\mathfrak{h}} \|e_j^{\odot \mu}\|_{\mathfrak{h}} = \|e_i^{\odot \lambda} \otimes e_j^{\odot \mu}\|_{\mathfrak{h}}. \end{aligned}$$

As above, it implies that the mapping $j_m \otimes j_{n-m}: E_{\mathfrak{w}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{w}}^{\odot(n-m)} \rightarrow E_{\mathfrak{h}}^{\otimes n}$, defined to be the continuous extension of identity mapping on $E_{\text{alg}}^{\odot m} \otimes E_{\text{alg}}^{\odot(n-m)}$, is a contractive injection. Using subsequently that $E_{\mathfrak{h}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{h}}^{\odot(n-m)}$ is reflexive, we get the Gelfand triple

$$E_{\mathfrak{w}}^{\odot m} \otimes_{\mathfrak{h}} E_{\mathfrak{w}}^{\odot(n-m)} \xrightarrow{\mathfrak{s}_{n/m}} E_{\mathfrak{h}}^{\odot n} \xrightarrow{j_n^*} E_{\mathfrak{w}}^{\odot n}$$

where injections are contractive and have dense ranges. $\square$

Lemma 3. *The $\Gamma_{\mathfrak{w}}$ -valued function, defined on $\{\varepsilon(x): x \in E\}$ by*

$$\mathcal{T}_{\mathfrak{a}}\varepsilon(x) = \varepsilon(x + \mathfrak{a}),$$

has a unique linear extension $\mathcal{T}_{\mathfrak{a}}: \Gamma_{\mathfrak{w}} \ni \psi \mapsto \mathcal{T}_{\mathfrak{a}}\psi \in \Gamma_{\mathfrak{w}}$ such that

$$\|\mathcal{T}_{\mathfrak{a}}\psi\|_{\mathfrak{w}}^2 \leq \exp(\|\mathfrak{a}\|^2) \|\psi\|_{\mathfrak{w}}^2 \quad \text{and} \quad \mathcal{T}_{\mathfrak{a}+\mathfrak{b}} = \mathcal{T}_{\mathfrak{a}}\mathcal{T}_{\mathfrak{b}} = \mathcal{T}_{\mathfrak{b}}\mathcal{T}_{\mathfrak{a}} \quad \text{for all} \quad \mathfrak{a}, \mathfrak{b} \in E.$$

Proof. Let us define the creation operators $\delta_{\mathfrak{a},n}^m: E_{\mathfrak{w}}^{\odot(n-m)} \rightarrow E_{\mathfrak{w}}^{\odot n}$ ($m \leq n$) as

$$\delta_{\mathfrak{a},n}^m x^{\otimes(n-m)} := \mathfrak{s}_{n/m} [\mathfrak{a}^{\otimes m} \otimes x^{\otimes(n-m)}] = \frac{(n-m)!}{n!} \frac{d^m(x + t\mathfrak{a})^{\otimes n}}{dt^m} \Big|_{t=0} \quad (10)$$

for all $\mathfrak{a}, x \in E$. Note that the second equality in (10) follows from the binomial formula for symmetric tensor elements $(x + t\mathfrak{a})^{\otimes n} = \sum_{m=0}^n \binom{n}{m} (t\mathfrak{a})^{\otimes m} \odot x^{\otimes(n-m)}$. Put $\delta_{\mathfrak{a},n}^0 = 1$. If $\mathfrak{a} = 0$ then $\delta_{0,n}^m = 0$. Summing over $n \geq m$ with coefficients $1/(n-m)!$, we get

$$\delta_{\mathfrak{a}}^m \varepsilon(x) = \frac{d^m \varepsilon(x + t\mathfrak{a})}{dt^m} \Big|_{t=0} = \bigoplus_{n \geq m} \frac{\mathfrak{s}_{n/m} [\mathfrak{a}^{\otimes m} \otimes x^{\otimes(n-m)}]}{(n-m)!}, \quad t \in \mathbb{C}. \quad (11)$$

This series is convergent, since by Lemma 2 and (4) the inequality

$$\|\delta_a^m \varepsilon(x)\|_{\mathbb{W}} \leq \|a\|^m \left\| \bigoplus_{n \geq m} \frac{x^{\otimes(n-m)}}{(n-m)!} \right\|_{\mathbb{W}} = \|a\|^m \|\varepsilon(x)\|_{\mathbb{W}} \quad (12)$$

holds. From (11) and the tensor binomial formula mentioned above it follows that

$$\bigoplus_{m=0}^n \frac{1}{m!} \delta_{a,n}^m \frac{x^{\otimes(n-m)}}{(n-m)!} = \bigoplus_{m=0}^n \frac{a^{\otimes m} \odot x^{\otimes(n-m)}}{m!(n-m)!} = \frac{(x+a)^{\otimes n}}{n!}.$$

Summing over $n \in \mathbb{Z}_+$ with coefficients $1/n!$ and using (11), we obtain

$$\begin{aligned} \mathcal{T}_a \varepsilon(x) &= \bigoplus_{n \in \mathbb{Z}_+} \sum_{m=0}^n \frac{1}{m!} \delta_{a,n}^m \frac{x^{\otimes(n-m)}}{(n-m)!} \\ &= \sum_{m \in \mathbb{Z}_+} \frac{1}{m!} \bigoplus_{n \geq m} \delta_{a,n}^m \frac{x^{\otimes(n-m)}}{(n-m)!} = \exp(\delta_a) \varepsilon(x). \end{aligned}$$

The inequalities (4) and (12) yield $\|\mathcal{T}_a \varepsilon(x)\|_{\mathbb{W}}^2 \leq \exp(\|a\|^2) \|\varepsilon(x)\|_{\mathbb{W}}^2$. Taking into account the totality of $\{\varepsilon(x) : x \in E\}$, this inequality implies the required inequality on $\Gamma_{\mathbb{W}}$. It also follows that $\mathcal{T}_{a+b} = \mathcal{T}_a \mathcal{T}_b = \mathcal{T}_b \mathcal{T}_a$, since $\delta_{a+b} = \delta_a + \delta_b$ for all $a, b \in E$ by linearity of creation operators. This ends the proof. $\square$

We define the adjoint operators $\delta_{a,n}^{*m} : E_{\mathbb{W}}^{\odot n} \ni \psi_n \rightarrow \delta_{a,n}^{*m} \psi_n \in E_{\mathbb{W}}^{\odot(n-m)}$ as

$$\langle \delta_{a,n}^m x^{\otimes(n-m)} \mid \psi_n \rangle_{\mathbb{W}} = \langle x^{\otimes(n-m)} \mid \delta_{a,n}^m \psi_n \rangle_{\mathbb{W}}, \quad a, x \in E$$

for $n \geq m$. It immediately follows that for every $\psi_{n-m} \in E_{\mathbb{W}}^{\odot(n-m)}$ and $x \in E$,

$$\begin{aligned} \langle \delta_{a,n}^{*m} x^{\otimes n} \mid \psi_{n-m} \rangle_{\mathbb{W}} &= \langle x^{\otimes n} \mid \delta_{a,n}^m \psi_{n-m} \rangle_{\mathbb{W}} = \langle x^{\otimes n} \mid a^{\otimes m} \odot \psi_{n-m} \rangle_{\mathbb{W}} \\ &= \langle x \mid a \rangle^m \langle x^{\otimes(n-m)} \mid \psi_{n-m} \rangle_{\mathbb{W}}. \end{aligned} \quad (13)$$

Using $\delta_{a,n}^{*m}$, we can uniquely define a $\Gamma_{\mathbb{W}}$ -valued function $\mathcal{T}_a^*$ by the equalities

$$\mathcal{T}_a^* \varepsilon(x) = \exp(\delta_a^*) \varepsilon(x) = \sum_{m \in \mathbb{Z}_+} \frac{\delta_{a,n}^{*m} \varepsilon(x)}{m!}, \quad \delta_a^{*m} \varepsilon(x) := \bigoplus_{n \geq m} \frac{\delta_{a,n}^{*m} x^{\otimes n}}{n!} \quad (14)$$

for all $a, x \in E$. Taking into account Lemma 3, we obtain the following claim.

Lemma 4. *The $\Gamma_{\mathbb{W}}$ -valued function $\mathcal{T}_a^*$, defined by (14), possesses a unique linear extension $\mathcal{T}_a^* : \Gamma_{\mathbb{W}} \ni \psi \mapsto \mathcal{T}_a^* \psi \in \Gamma_{\mathbb{W}}$ such that*

$$\|\mathcal{T}_a^* \psi\|_{\mathbb{W}}^2 \leq \exp(\|a\|^2) \|\psi\|_{\mathbb{W}}^2 \quad \text{and} \quad \mathcal{T}_{a+b}^* = \mathcal{T}_a^* \mathcal{T}_b^* = \mathcal{T}_b^* \mathcal{T}_a^* \quad \text{for all } a, b \in E.$$

Definition 3. We will call the $\Gamma_{\mathbb{W}}$ -valued functions $\mathcal{T}_a$ and $\mathcal{T}_a^*$ in variable $a \in E$ the exponential creation and annihilation groups, respectively.

6. Intertwining Properties of $\mathcal{F}$ -Transform

Let us define on the space H_χ^2 the multiplicative group $M_a^\dagger: E \ni a \mapsto M_a^\dagger$ to be

$$M_a^\dagger f(u) = \exp[\bar{\phi}_a(u)] f(u), \quad f \in H_\chi^2, \quad u \in \mathfrak{U}_\pi.$$

It can be considered as a linear representation of the additive group $(E, +)$. By Lemma 1 the function $u \mapsto \exp[\bar{\phi}_a(u)]$ with a fixed a belongs to L_χ^∞ . Hence, $M_a^\dagger$ is continuous on H_χ^2 . The generator of the 1-parameter group $\mathbb{C} \ni t \mapsto M_{ta}^\dagger$ coincides with the operator of multiplication by the L_χ^∞ -valued function

$$\bar{\phi}_a: \mathfrak{U}_\pi \ni u \mapsto \bar{\phi}_a(u) \quad \text{where} \quad dM_{ta}^\dagger/dt|_{t=0} = \bar{\phi}_a.$$

The continuity of $E \ni a \mapsto \exp(\bar{\phi}_a)$ implies that this 1-parameter group $M_{ta}^\dagger$ is strongly continuous on H_χ^2 . As a consequence, its generator $(\bar{\phi}_a f)(u) = \bar{\phi}_a(u) f(u)$ with domain $\mathfrak{D}(\bar{\phi}_a) = \{f \in H_\chi^2: \bar{\phi}_a f \in H_\chi^2\}$ is closed and densely-defined. As well, its power $\bar{\phi}_a^m$ defined on $\mathfrak{D}(\bar{\phi}_a^m) = \{f \in H_\chi^2: \bar{\phi}_a^m f \in H_\chi^2\}$ for any $m \in \mathbb{N}$ is the same (see, e.g. [7] for details).

The additive group $(E, +)$ may be also linearly represented on H_w^2 as the shift group

$$T_a \hat{f}(x) = \hat{f}(x + a), \quad f \in H_\chi^2, \quad x, a \in E.$$

The directional derivative on the space H_w^2 along a nonzero $a \in E$ coincides with the generator of the 1-parameter shift subgroup $\mathbb{C} \ni t \mapsto T_{ta}$, that is,

$$\mathfrak{d}_a \hat{f} = \lim_{t \rightarrow 0} t^{-1} (T_{ta} \hat{f} - \hat{f}) \quad \text{with domain} \quad \mathfrak{D}(\mathfrak{d}_a) := \{\hat{f} \in H_w^2: \mathfrak{d}_a \hat{f} \in H_w^2\}.$$

Note that the 1-parameter shift group T_{ta} , which is intertwined with $M_{ta}^\dagger$ by the $\mathcal{F}$ -transform

$$T_{ta} \hat{f}(x) = \int \exp[\bar{\phi}_{x+ta}] f d\chi = \int \exp(\bar{\phi}_x) M_{ta}^\dagger f d\chi, \quad (15)$$

is strongly continuous on H_w^2 . Since $\mathfrak{D}(\mathfrak{d}_a^m)$ contains all polynomials from H_w^2 , each operator $\mathfrak{d}_a^m$ with domain $\mathfrak{D}(\mathfrak{d}_a^m) = \{\hat{f} \in H_w^2: \mathfrak{d}_a^m \hat{f} \in H_w^2\}$ is closed and densely-defined. From (15) it directly follows

$$\mathfrak{d}_a^m \hat{f}(x) = \int \exp(\bar{\phi}_x) \frac{d^m M_{ta}^\dagger}{dt^m} \Big|_{t=0} f d\chi = \int \exp(\bar{\phi}_x) \bar{\phi}_a^m f d\chi \quad (16)$$

for all $f \in \mathfrak{D}(\bar{\phi}_a^m)$ and $x \in E$. On the other hand, by Theorem 2 we have

$$T_a \hat{f}(x) = \langle T_a \varepsilon(x) | \Phi^* f \rangle = \langle \varepsilon(x) | T_a^* \Phi^* f \rangle = \int \exp(\bar{\phi}_x) \Phi T_a^* \Phi^* f d\chi. \quad (17)$$

Theorem 2 together with (15) and (17) imply that $M_a^\dagger$ is connected with the exponential annihilation group T_a^* by the intertwining operator Φ . This can be written as $M_a^\dagger = \Phi T_a^* \Phi^*$. Thus, the $\mathcal{F}$ -transform serves as an intertwining

operator for the groups $M_a^\dagger$ on H_χ^2 . Moreover, using (15), (16) and (17), we obtain

$$d^m T_{ta} \widehat{f}(x) / dt^m|_{t=0} = \langle \varepsilon(x) \mid \delta_a^{*m} \Phi^* f \rangle_w = \mathfrak{D}_a^m \widehat{f}(x).$$

As a result, we have proved the following statement.

Theorem 3. *For every $f \in H_\chi^2$ the following equalities hold,*

$$T_a \mathcal{F}(f) = \mathcal{F}(M_a^\dagger f), \quad M_a^\dagger f = \Phi T_a^* \Phi^* f, \quad a \in E,$$

Moreover, for every $f \in \mathfrak{D}(\bar{\phi}_a^m)$ ($m \in \mathbb{N}$) and a nonzero $a \in E$,

$$\mathfrak{D}_a^m \widehat{f}(x) = \langle \varepsilon(x) \mid \delta_a^{*m} \Phi^* f \rangle_w = \int \exp(\bar{\phi}_x) \bar{\phi}_a^m f d\chi, \quad x \in E.$$

Let us consider on H_w^2 the multiplicative group with a nonzero $a \in E$,

$$M_{a^*} \widehat{f}(x) = \widehat{f}(x) \exp\langle x \mid a \rangle, \quad \widehat{f} \in H_w^2.$$

The generator on H_w^2 of the appropriate 1-parameter subgroup $\mathbb{C} \ni t \mapsto M_{ta^*}$ is

$$dM_{ta^*}/dt|_{t=0} = \langle \cdot \mid a \rangle := a^*, \quad a \in E.$$

Hence, it coincides with the following linear operator of multiplication

$$(a^* \widehat{f})(x) = \langle x \mid a \rangle \widehat{f}(x) \quad \text{with domain} \quad \mathfrak{D}(a^*) = \{\widehat{f} \in H_w^2 : a^* \widehat{f} \in H_w^2\}.$$

Its power a^{*m} is densely-defined on $\mathfrak{D}(a^{*m}) = \{\widehat{f} \in H_w^2 : a^{*m} \widehat{f} \in H_w^2\}$ which contains all polynomials from H_w^2 .

Using Lemma 3 we can represent the additive group $(E, +)$ over the space H_χ^2 by the shift group

$$T_a^\dagger = \Phi T_a \Phi^* \quad \text{with the generator} \quad \delta_a^\dagger = \Phi \delta_a \Phi^*$$

defined on $\mathfrak{D}(\delta_a^\dagger) = \{f \in H_\chi^2 : \delta_a^\dagger f \in H_\chi^2\}$. This means that $T_a^\dagger$ is connected via the intertwining operator Φ with the exponential creation group T_a .

Theorem 4. *For every $f \in H_\chi^2$ the following equality holds,*

$$M_{a^*} \mathcal{F}(f) = \mathcal{F}(T_a^\dagger f), \quad a \in E,$$

that is, the $\mathcal{F}$ -transform is an intertwining operator for the groups M_{a^} on H_w^2 and $T_a^\dagger$ on H_χ^2 . Moreover, for every $f \in \mathfrak{D}(\delta_a^{\dagger m}) = \{f \in H_\chi^2 : \delta_a^{\dagger m} f \in H_\chi^2\}$ ($m \in \mathbb{N}$) and a nonzero $a \in E$,*

$$(a^{*m} \widehat{f})(x) = \langle \varepsilon(x) \mid \delta_a^{*m} \Phi^* f \rangle_w = \int \exp(\bar{\phi}_x) \delta_a^{\dagger m} f d\chi, \quad x \in E. \quad (18)$$

Proof. The equality (13) yields $\langle x \mid a \rangle^m \psi_{n-m}^*(x) = \langle \delta_{a,n}^{*m} x^{\otimes n} \mid \psi_{n-m} \rangle_w$ for all $n \geq m$. By Theorem 2 for any $f = \sum_n f_n \in H_\chi^2$ there exists a unique

$\psi = \bigoplus_n \psi_n$ in Γ_w with $\psi_n \in E^{\odot n}$ such that $\Phi^* f = \psi$ and $f_n = \psi_n^*$. Summing over all $m \in \mathbb{Z}_+$ and $n \geq m$ and using (14), we obtain that

$$\begin{aligned} M_{a^*} \widehat{f}(x) &= \exp\langle x \mid a \rangle \langle \varepsilon(x) \mid \Phi^* f \rangle_w = \sum_{m \in \mathbb{Z}_+} \frac{\langle x \mid a \rangle^m}{m!} \sum_{n \geq m} \psi_{n-m}^*(x) \\ &= \langle T_a^* \varepsilon(x) \mid \Phi^* f \rangle_w = \langle \varepsilon(x) \mid T_a \Phi^* f \rangle_w. \end{aligned}$$

By Theorem 2 and Lemma 3 it follows that the equalities

$$M_{ta^*} \widehat{f}(x) = \langle \varepsilon(x) \mid T_{ta} \Phi^* f \rangle_w = \int \exp(\bar{\phi}_x) T_{ta}^\dagger f \, d\chi, \quad t \in \mathbb{C} \quad (19)$$

hold for all $\widehat{f} \in H_w^2$. On the other hand, the equalities (14) and (19) yield

$$\left. \frac{d^m M_{ta^*} \widehat{f}(x)}{dt^m} \right|_{t=0} = \int \exp(\bar{\phi}_x) \left. \frac{d^m T_{ta}^\dagger}{dt^m} \right|_{t=0} f \, d\chi = \int \exp(\bar{\phi}_x) \delta_a^{\dagger m} f \, d\chi$$

for all $f \in \mathfrak{D}(\delta_a^{\dagger m})$. This in turn yields (18). $\square$

7. Commutation Relations

Describe the commutation relations between $M_a^\dagger$ and $T_b^\dagger$ on the Hardy space H_χ^2 .

Theorem 5. *For any nonzero $a, b \in E$ the commutation relations*

$$M_a^\dagger T_b^\dagger = \exp\langle a \mid b \rangle T_b^\dagger M_a^\dagger, \quad (\bar{\phi}_a \delta_b^\dagger - \delta_b^\dagger \bar{\phi}_a) f = \langle a \mid b \rangle f$$

hold, wherein f belongs to the dense subspace $\mathfrak{D}(\bar{\phi}_b^2) \cap \mathfrak{D}(\delta_a^{\dagger 2}) \subset H_\chi^2$.

Proof. Let us prove that the following equalities hold,

$$T_a M_{b^*} = \exp\langle a \mid b \rangle M_{b^*} T_a, \quad (\mathfrak{d}_a b^* - b^* \mathfrak{d}_a) \widehat{f} = \langle a \mid b \rangle \widehat{f} \quad (20)$$

where $\widehat{f} \in \mathfrak{D}(b^{*2}) \cap \mathfrak{D}(\mathfrak{d}_a^2)$. First property follows from the direct calculations:

$$\begin{aligned} M_{b^*} T_a \widehat{f}(x) &= \exp\langle x \mid b \rangle \widehat{f}(x + a), \\ T_a M_{b^*} \widehat{f}(x) &= \widehat{f}(x + a) \exp\langle x \mid b \rangle \exp\langle a \mid b \rangle = \exp\langle a \mid b \rangle M_{b^*} T_a \widehat{f}(x) \end{aligned}$$

for all $\widehat{f} \in H_w^2$ and $x \in E$. For any $\widehat{f} \in \mathfrak{D}(b^{*2}) \cap \mathfrak{D}(\mathfrak{d}_a^2)$ and $t \in \mathbb{C}$, we have

$$\begin{aligned} \left. \frac{d^2}{dt^2} T_{ta} M_{tb^*} \widehat{f} \right|_{t=0} &= [\mathfrak{d}_a^2 T_{ta} M_{tb^*} \widehat{f} + 2\mathfrak{d}_a T_{ta} b^* M_{tb^*} \widehat{f} + T_{ta} b^{*2} M_{tb^*} \widehat{f}]_{t=0} \\ &= (\mathfrak{d}_a^2 + 2\mathfrak{d}_a b^* + b^{*2}) \widehat{f}. \end{aligned}$$

On the other hand, differentiating again, we have

$$\begin{aligned} \frac{d}{dt} T_{ta} M_{tb^*} \widehat{f} \Big|_{t=0} &= \left[\frac{d}{dt} \exp\langle ta \mid \bar{t}b \rangle M_{tb^*} T_{ta} \widehat{f} + \exp\langle ta \mid \bar{t}b \rangle \frac{d}{dt} M_{tb^*} T_{ta} \widehat{f} \right]_{t=0}, \\ (\mathfrak{d}_a^2 + 2\mathfrak{d}_a \mathfrak{b}^* + \mathfrak{b}^{*2}) \widehat{f} &= \frac{d}{dt} \left[\frac{d}{dt} T_{ta} M_{tb^*} \widehat{f} \right]_{t=0} = \left[\frac{d^2}{dt^2} \exp\langle ta \mid \bar{t}b \rangle M_{tb^*} T_{ta} \widehat{f} \right. \\ &\quad \left. + 2 \frac{d}{dt} \exp\langle ta \mid \bar{t}b \rangle \frac{d}{dt} M_{tb^*} T_{ta} \widehat{f} \right. \\ &\quad \left. + \exp\langle ta \mid \bar{t}b \rangle \frac{d^2}{dt^2} M_{tb^*} T_{ta} \widehat{f} \right]_{t=0} \\ &= 2\langle a \mid b \rangle \widehat{f} + (\mathfrak{d}_a^2 + 2\mathfrak{b}^* \mathfrak{d}_a + \mathfrak{b}^{*2}) \widehat{f}. \end{aligned}$$

This yields (20) where $\mathfrak{D}(\mathfrak{b}^{*2}) \cap \mathfrak{D}(\mathfrak{d}_a^2)$ contains the dense subspace in $H_{\mathfrak{w}}^2$ of all polynomials $\widehat{f}$ generating by finite sums $\Phi^*(f) = \bigoplus_n \psi_n \in \Gamma_{\mathfrak{w}}$.

From $M_{\mathfrak{b}^*} \widehat{f}(x) = \langle \varepsilon(x) \mid \mathcal{T}_{\mathfrak{b}} \Phi^* f \rangle_{\mathfrak{w}}$ it follows that $\widehat{I} \mathcal{T}_{\mathfrak{b}} = M_{\mathfrak{a}^*} \widehat{I}$ with $\widehat{I} := \mathcal{F} \Phi$. Thus, $T_{\mathfrak{b}}^\dagger = \Phi \mathcal{T}_{\mathfrak{b}} \Phi^* = \Phi \widehat{I}^{-1} M_{\mathfrak{b}^*} \widehat{I} \Phi^* = \mathcal{F}^{-1} M_{\mathfrak{b}^*} \mathcal{F}$. Using that $M_{\mathfrak{a}}^\dagger = \mathcal{F}^{-1} T_{\mathfrak{a}} \mathcal{F}$ with $\mathcal{F}^{-1}: H_{\mathfrak{w}}^2 \rightarrow H_\chi^2$ and applying (20), we obtain

$$\begin{aligned} M_{\mathfrak{a}}^\dagger T_{\mathfrak{b}}^\dagger &= \mathcal{F}^{-1} T_{\mathfrak{a}} M_{\mathfrak{b}^*} \mathcal{F} = \exp\langle x \mid b \rangle \mathcal{F}^{-1} M_{\mathfrak{b}^*} T_{\mathfrak{a}} \mathcal{F} = \exp\langle x \mid b \rangle T_{\mathfrak{b}}^\dagger M_{\mathfrak{a}}^\dagger, \\ (\bar{\phi}_{\mathfrak{a}} \delta_{\mathfrak{b}}^\dagger - \delta_{\mathfrak{b}}^\dagger \bar{\phi}_{\mathfrak{a}}) f &= \mathcal{F}^{-1} (\mathfrak{d}_a \mathfrak{b}^* - \mathfrak{b}^* \mathfrak{d}_a) \mathcal{F} f = \langle a \mid b \rangle f \end{aligned}$$

for all $f \in \mathfrak{D}(\bar{\phi}_{\mathfrak{b}}^2) \cap \mathfrak{D}(\delta_{\mathfrak{a}}^{\dagger 2})$. For any $f = \sum_n f_n \in H_\chi^2$ there exists a unique $\psi = \bigoplus_n \psi_n$ in $\Gamma_{\mathfrak{w}}$ with $\psi_n \in E_{\mathfrak{w}}^{\odot n}$ such that the equalities $\Phi^* f = \psi$ and $f_n = \psi_n^*$ hold. Hence, the following embedding $\mathfrak{D}(\bar{\phi}_{\mathfrak{b}}^2) \cap \mathfrak{D}(\delta_{\mathfrak{a}}^{\dagger 2}) \subset H_\chi^2$ is dense. $\square$

8. Gauss-Weierstrass Semigroups

Next we show that the 1-parameter Gauss-Weierstrass semigroups on the Hardy space H_χ^2 can be well described by shift and multiplicative groups (a classic case can be found in [7, n.4.3.2]). For this purpose we use the Gaussian kernel

$$\mathfrak{g}_r(\tau) = \frac{1}{\sqrt{4\pi r}} \exp\left(-\frac{\tau^2}{4r}\right), \quad \tau \in \mathbb{R}, \quad r > 0.$$

Theorem 6. *The 1-parameter Gauss-Weierstrass semigroups $\{W_r^{\delta_{\mathfrak{a}}^\dagger} : r > 0\}$ and $\{W_r^{\bar{\phi}_{\mathfrak{a}}} : r > 0\}$, defined on the Hardy space H_χ^2 for any nonzero $\mathfrak{a} \in E$ as*

$$W_r^{\delta_{\mathfrak{a}}^\dagger} f = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) T_{\tau \mathfrak{a}}^\dagger f \, d\tau \quad \text{and} \quad W_r^{\bar{\phi}_{\mathfrak{a}}} f = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) M_{\tau \mathfrak{a}}^\dagger f \, d\tau, \quad f \in H_\chi^2, \quad (21)$$

are generated by $\delta_{\mathfrak{a}}^{\dagger 2}$ and $\bar{\phi}_{\mathfrak{a}}^2$, respectively.

Proof. First it is sufficient to prove that the axillary 1-parameter families of linear operators over $H_{\mathbb{W}}^2$

$$G_r^{a^*} \hat{f} = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) M_{\tau a^*} \hat{f} d\tau \quad \text{and} \quad G_r^{\partial_a} \hat{f} = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) T_{\tau a} \hat{f} d\tau, \quad \hat{f} \in H_{\mathbb{W}}^2 \quad (22)$$

can be generated by $\mathbf{a}^{*2}$ and $\mathfrak{d}_a^2$ and satisfy the semigroup property. Properties of Gaussian kernel yield

$$\begin{aligned} \int_{\mathbb{R}} \mathfrak{g}_r(\tau) \tau^{2k} d\tau &= \frac{1}{2\sqrt{\pi r}} \int_{\mathbb{R}} e^{-\frac{\tau^2}{4r}} \tau^{2k} d\tau \Big|_{\tau=2\sqrt{r}v} = \frac{(2\sqrt{r})^{2k}}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-v^2} v^{2k} dv \\ &= \frac{2^{2k} r^k}{\sqrt{\pi}} \Gamma\left(\frac{2k+1}{2}\right) = \frac{2(2k-1)!}{(k-1)!} r^k, \quad k \in \mathbb{N}. \end{aligned}$$

We can rewrite $G_r^{a^*} \hat{f}$ on the dense subspace $\{\hat{f} \in H_{\mathbb{W}}^2 : \exp(\tau a^*) \hat{f} \in H_{\mathbb{W}}^2\}$ as

$$\begin{aligned} G_r^{a^*} \hat{f} &= \int_{\mathbb{R}} \mathfrak{g}_r(\tau) \exp(\tau a^*) \hat{f} d\tau = \sum_{l \in \mathbb{Z}_+} \frac{\mathbf{a}^{*l} \hat{f}}{l!} \int_{\mathbb{R}} \mathfrak{g}_r(\tau) \tau^l d\tau \\ &= \sum_{k \in \mathbb{Z}_+} \frac{2(2k-1)!}{(k-1)!} \frac{r^k \mathbf{a}^{*2k} \hat{f}}{(2k)!} = \sum_{k \in \mathbb{Z}_+} \frac{r^k \mathbf{a}^{*2k} \hat{f}}{k!} = \exp(r \mathbf{a}^{*2}) \hat{f} \end{aligned}$$

By first equality in (22) the family $G_r^{a^*}$ can be extended to the convolution

$$\mathfrak{g}_r \circledast \hat{f} := \int_{\mathbb{R}} \mathfrak{g}_r(\tau) M_{\tau a^*} \hat{f} d\tau, \quad \hat{f} \in H_{\mathbb{W}}^2$$

(dependent on $\mathbf{a}$) over the whole space $H_{\mathbb{W}}^2$. Thus, to show that the semigroup property holds, it suffices to show that

$$\mathfrak{g}_{r+s} \circledast \hat{f} = G_{r+s}^{a^*} \hat{f} = (G_r^{a^*} \circ G_s^{a^*}) \hat{f} = \mathfrak{g}_r \circledast (\mathfrak{g}_s \circledast \hat{f}) = (\mathfrak{g}_r * \mathfrak{g}_s) \circledast \hat{f}.$$

But this straightly follows from the known convolution equality $\mathfrak{g}_{r+s} = \mathfrak{g}_r * \mathfrak{g}_s$.

Further, using the equality $T_a^\dagger = \mathcal{F}^{-1} M_{a^*} \mathcal{F}$ we obtain that

$$W_r^{\delta_a^\dagger} f = \int_{\mathbb{R}} \mathfrak{g}_r(\tau) \mathcal{F}^{-1} M_{\tau a^*} \mathcal{F} f d\tau = \mathcal{F}^{-1} G_r^{a^*} \mathcal{F} f$$

for all $f \in H_{\chi}^2$. By Theorem 4 it follows that

$$\frac{dW_r^{\delta_a^\dagger} f}{dr} \Big|_{r=0} = \mathcal{F}^{-1} \frac{G_r^{a^*} \hat{f}}{dr} \Big|_{r=0} = \mathcal{F}^{-1} \mathbf{a}^{*2} \hat{f} = \delta_a^{\dagger 2} f$$

for all $f \in \mathfrak{D}(\delta_a^{\dagger 2})$, since $\hat{f} \in \mathfrak{D}(\mathbf{a}^{*2})$ and $\delta_a^{\dagger 2} = \mathcal{F}^{-1} \mathbf{a}^{*2} \mathcal{F}$. Hence, the case of semigroup $W_r^{\delta_a^\dagger}$ is proven.

Similar reasonings can be applied to the semigroup $G_r^{\partial_a}$. As a result, we obtain that the equalities $W_r^{\bar{\phi}_a} = \mathcal{F}^{-1} G_r^{\partial_a} \mathcal{F}$ and $\bar{\phi}_a^2 = \mathcal{F}^{-1} \mathfrak{d}_a^2 \mathcal{F}$ hold. $\square$

9. Complexified Infinite-Dimensional Heisenberg Group

Let us give yet another application. Consider an infinite-dimensional analog of the Heisenberg group over $\mathbb{C}$. Namely, let us define the group $\mathfrak{G}$ of upper triangular matrix-type elements

$$X(\mathbf{a}, \mathbf{b}, t) = \begin{bmatrix} 1 & \mathbf{a} & t \\ 0 & 1 & \mathbf{b} \\ 0 & 0 & 1 \end{bmatrix}, \quad t \in \mathbb{C}, \quad \mathbf{a}, \mathbf{b} \in E$$

with unit $X(0, 0, 0)$ and multiplication

$$\begin{bmatrix} 1 & \mathbf{a} & t \\ 0 & 1 & \mathbf{b} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathbf{a}' & t' \\ 0 & 1 & \mathbf{b}' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{a} + \mathbf{a}' & t + t' + \langle \mathbf{a} | \mathbf{b}' \rangle \\ 0 & 1 & \mathbf{b} + \mathbf{b}' \\ 0 & 0 & 1 \end{bmatrix}.$$

Obviously, $X(\mathbf{a}, \mathbf{b}, t)^{-1} = X(-\mathbf{a}, -\mathbf{b}, -t + \langle \mathbf{a} | \mathbf{b} \rangle)$.

We will now describe an irreducible linear representation of the group $\mathfrak{G}$. For this purpose we will use the algebra $\mathbb{H}$ of quaternions $\gamma = \alpha_1 + \alpha_2 \mathbf{i} + \beta_1 \mathbf{j} + \beta_2 \mathbf{k} = (\alpha_1 + \alpha_2 \mathbf{i}) + (\beta_1 + \beta_2 \mathbf{i})\mathbf{j} = \alpha + \beta \mathbf{j}$ as pairs of complex numbers $(\alpha, \beta) \in \mathbb{C}^2$ with $\alpha = \alpha_1 + \alpha_2 \mathbf{i}, \beta = \beta_1 + \beta_2 \mathbf{i} \in \mathbb{C}$ and $\alpha_i, \beta_i \in \mathbb{R}$ ($i = 1, 2$) where basis elements in $\mathbb{R}^4$ satisfy the relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\mathbf{j}\mathbf{k} = -1$, $\mathbf{k} = \mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i}$, $\mathbf{k}\mathbf{i} = \mathbf{i}\mathbf{k} = \mathbf{j}$. Thus, $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$ is a vector space over $\mathbb{C}$ [25]. Denote $\beta := \Im \gamma$ where $\gamma = \alpha + \beta \mathbf{j}$.

Let $E_{\mathbb{H}} = E \oplus E\mathbf{j}$ be the Hilbert space with $\mathbb{H}$ -valued scalar product

$$\langle \mathbf{p} | \mathbf{p}' \rangle = \langle \mathbf{a} + \mathbf{b}\mathbf{j} | \mathbf{a}' + \mathbf{b}'\mathbf{j} \rangle = \langle \mathbf{a} | \mathbf{a}' \rangle + \langle \mathbf{b} | \mathbf{b}' \rangle + [\langle \mathbf{a}' | \mathbf{b} \rangle - \langle \mathbf{a} | \mathbf{b}' \rangle]\mathbf{j}$$

where $\mathbf{p} = \mathbf{a} + \mathbf{b}\mathbf{j}$ with $\mathbf{a}, \mathbf{b} \in E$ (similarly, for $\mathbf{p}' = \mathbf{a}' + \mathbf{b}'\mathbf{j}$). Hence,

$$\Im \langle \mathbf{p} | \mathbf{p}' \rangle = \langle \mathbf{a}' | \mathbf{b} \rangle - \langle \mathbf{a} | \mathbf{b}' \rangle, \quad \Im \langle \mathbf{p} | \mathbf{p} \rangle = 0.$$

The following theorem describes a representation of the above infinite-dimensional Heisenberg group which can be seen as an analog of the Weyl-Schrödinger representation

Theorem 7. *The linear representation of $\mathfrak{G}$ over H_{χ}^2*

$$\mathscr{W}^{\dagger}: \mathfrak{G} \ni X(\mathbf{a}, \mathbf{b}, t) \longmapsto \exp \left[t + \frac{1}{2} \langle \mathbf{a} | \mathbf{b} \rangle \right] T_{\mathbf{a}}^{\dagger} M_{\mathbf{b}}^{\dagger}$$

is well defined and irreducible.

Proof. First we prove that the following operator representation

$$\mathscr{W}: \mathfrak{G} \ni X(\mathbf{a}, \mathbf{b}, t) \longmapsto \exp \left[t + \frac{1}{2} \langle \mathbf{a} | \mathbf{b} \rangle \right] M_{\mathbf{a}} * T_{\mathbf{b}}$$

into the algebra of all bounded linear operator on $H_{\mathbf{w}}^2$ is well defined and irreducible. Consider the auxiliary group $\mathbb{C} \times E_{\mathbb{H}}$ with the multiplication

$$(t, \mathbf{p})(t', \mathbf{p}') = \left(t + t' - \frac{1}{2} \Im \langle \mathbf{p} | \mathbf{p}' \rangle, \mathbf{p} + \mathbf{p}' \right)$$

for all $\mathbf{p} = \mathbf{a} + \mathbf{b}\mathbf{j}$, $\mathbf{p}' = \mathbf{a}' + \mathbf{b}'\mathbf{j} \in E_{\mathbb{H}}$. It is related to $\mathfrak{G}$ via the mapping

$$\mathcal{G}: X(\mathbf{a}, \mathbf{b}, t) \longmapsto \left(t - \frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle, \mathbf{a} + \mathbf{b}\mathbf{j} \right).$$

Check that $\mathcal{G}$ is a group isomorphism. In fact,

$$\begin{aligned} \mathcal{G}(X(\mathbf{a}, \mathbf{b}, t)X(\mathbf{a}', \mathbf{b}', t')) &= \mathcal{G}(X(\mathbf{a} + \mathbf{a}', \mathbf{b} + \mathbf{b}', t + t' + \langle \mathbf{a} \mid \mathbf{b}' \rangle)) \\ &= \left(t + t' + \langle \mathbf{a} \mid \mathbf{b}' \rangle - \frac{1}{2}[\langle \mathbf{a} + \mathbf{a}' \mid \mathbf{b} + \mathbf{b}' \rangle], (\mathbf{a} + \mathbf{a}') + (\mathbf{b} + \mathbf{b}')\mathbf{j} \right) \\ &= \left(t + t' - \frac{1}{2}[\langle \mathbf{a} \mid \mathbf{b} \rangle + \langle \mathbf{a}' \mid \mathbf{b}' \rangle] + \frac{1}{2}[\langle \mathbf{a} \mid \mathbf{b}' \rangle - \langle \mathbf{a}' \mid \mathbf{b} \rangle], (\mathbf{a} + \mathbf{a}') + (\mathbf{b} + \mathbf{b}')\mathbf{j} \right) \\ &= \left(t - \frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle, \mathbf{a} + \mathbf{b}\mathbf{j} \right) \left(t' - \frac{1}{2}\langle \mathbf{a}' \mid \mathbf{b}' \rangle, \mathbf{a}' + \mathbf{b}'\mathbf{j} \right) \\ &= \mathcal{G}(X(\mathbf{a}, \mathbf{b}, t))\mathcal{G}(X(\mathbf{a}', \mathbf{b}', t')). \end{aligned}$$

Now let us check that the Weyl-like operator

$$W(\mathbf{p}) = \exp \left[\frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle \right] M_{\mathbf{a}*} T_{\mathbf{b}}, \quad \mathbf{p} = \mathbf{a} + \mathbf{b}\mathbf{j}$$

on the space $H_{\mathbf{w}}^2$ satisfies the commutation relation

$$W(\mathbf{p} + \mathbf{p}') = \exp \left[-\frac{1}{2}\Im \langle \mathbf{p} \mid \mathbf{p}' \rangle \right] W(\mathbf{p})W(\mathbf{p}').$$

In fact, using (20), we obtain

$$\begin{aligned} &\exp \left[\frac{1}{2}\langle \mathbf{a} \mid \mathbf{b}' \rangle - \frac{1}{2}\langle \mathbf{a}' \mid \mathbf{b} \rangle \right] W(\mathbf{p})W(\mathbf{p}') \\ &= \exp \left[\frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle + \frac{1}{2}\langle \mathbf{a}' \mid \mathbf{b}' \rangle \right] \exp \left[\frac{1}{2}\langle \mathbf{a} \mid \mathbf{b}' \rangle - \frac{1}{2}\langle \mathbf{a}' \mid \mathbf{b} \rangle \right] M_{\mathbf{a}*} T_{\mathbf{b}} M_{\mathbf{a}'*} T_{\mathbf{b}'} \\ &= \exp \left[\frac{1}{2}\langle \mathbf{a} + \mathbf{a}' \mid \mathbf{b} + \mathbf{b}' \rangle \right] M_{\mathbf{a} + \mathbf{a}'*} T_{\mathbf{b} + \mathbf{b}'} = W(\mathbf{p} + \mathbf{p}'). \end{aligned}$$

As a consequence, the mapping $\mathcal{S}: \mathbb{C} \times E_{\mathbb{H}} \ni (t, \mathbf{p}) \longmapsto \exp(t)W(\mathbf{p})$ is a group isomorphism. So, $\mathcal{W}$ is also a group isomorphism as a composition of the group isomorphisms $\mathcal{S}$ and $\mathcal{G}$.

Let us check irreducibility. If there exists an element $x_0 \neq 0$ in E and an integer $n > 0$ such that

$$\exp \left[t + \frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle \right] e^{\langle \mathbf{a} \mid \mathbf{x} \rangle} [x_0^*(x + \mathbf{b})]^n = 0 \quad \text{for all } x, \mathbf{a}, \mathbf{b} \in E$$

then $x_0 = 0$. This gives a contradiction. Hence the representation $\mathcal{W}$ is irreducible. Finally, using that

$$\exp \left[t + \frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle \right] T_{\mathbf{a}}^{\dagger} M_{\mathbf{b}}^{\dagger} = \mathcal{F}^{-1} \left(\exp \left[t + \frac{1}{2}\langle \mathbf{a} \mid \mathbf{b} \rangle \right] M_{\mathbf{a}*} T_{\mathbf{b}} \right) \mathcal{F},$$

we conclude that the group representation $\mathcal{W}^{\dagger} = \mathcal{F}^{-1}\mathcal{W}\mathcal{F}$ is irreducible. $\square$

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