Remarks on Farah's Theorems

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Abstract. The aim of this paper is to prove two Farah's Theorems concerning approximate group homomorphisms, without some assumptions present in Farah's Theorems. Farah's approach to the Ulam's type stability of homomorphisms between finite groups seems appropriate and interesting.

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1. Introduction

Farah [1] investigated mappings between a finite group and a product of groups which are approximately homomorphisms, with respect to some metric on the range. His main positive result says that if the metric corresponds to a nonpathological submeasure, then the mapping can be approximated by a homomorphism within a constant error.

These results can be thought of as belonging to the general “stability program,” initiated by a question of Ulam (see [2, 3]), however, Farah's motivation was of a different kind, see Introduction and References in [1].

Let me remind his Theorems. But first we set some notations and remind some notions. In the whole paper let H_i , for $i \in I$ be some groups, φ is a strictly positive submeasure on the index-set I . Then

$$d_\varphi(x, y) = \varphi(\{i \in I : x(i) \neq y(i)\}), \quad x, y \in \prod_{i \in I} H_i,$$

is a metric on $\prod_{i \in I} H_i$ which induces the product topology on it (where each H_i is taken with the discrete topology), in particular the group operations

of $\prod_{i \in I} H_i$ are continuous with respect to d_φ . Recall that a submeasure φ is pathological if

$$\varphi \neq \sup_{\phi \leq \varphi} \phi,$$

where the supremum is taken over all measures ϕ dominated by φ . For a finite set A let $|A|$ mean the number of elements of A . Let G be a finite set, $\nu: 2^G \rightarrow [0, 1]$, $\mu: 2^{G \times G} \rightarrow [0, 1]$ be the measures defined by $\nu(A) = \frac{|A|}{|G|}$, $\mu(B) = \frac{|B|}{|G|^2}$, for $A \subset G$, $B \subset G \times G$.

Theorem 1.1 ([1]). *Suppose that G and H are groups, G is finite, $\delta \leq \frac{1}{11}$, $f: G \rightarrow H$ satisfies*

$$\begin{aligned} \nu(\{(a, b) \in G^2 : f(a)f(b) \neq f(ab)\}) &\leq \delta, \\ \nu(\{a \in G : f(a) \neq f(a^{-1})^{-1}\}) &\leq \delta. \end{aligned} \quad (1.1)$$

Then there is a homomorphism $g: G \rightarrow H$, such that

$$\nu(\{x \in G : f(x) \neq g(x)\}) \leq \frac{\delta}{1 - 3\delta}.$$

Theorem 1.2 ([1]). *Suppose that G is a finite group, $H = \prod_{i \in I} H_i$, $\varepsilon \in (0, \infty)$, If*

$$\begin{aligned} d_\varphi(f(a)f(b), f(ab)) &\leq \varepsilon; \quad a, b \in G, \\ d_\varphi(f(a), f(a^{-1})^{-1}) &\leq \varepsilon; \quad a \in G, \end{aligned} \quad (1.2)$$

then there is a homomorphism $F: G \rightarrow H$ such that

$$d_\varphi(f(x), F(x)) \leq 24\varepsilon, \quad x \in G.$$

Since for every $f: G \rightarrow H$, where G, H are groups, the condition

$$f(ab) = f(a)f(b), \quad a, b \in G,$$

implies

$$f(a)^{-1} = f(a^{-1}), \quad a \in G,$$

the assumptions (1.1) and (1.2) seems unnatural, and in many papers concerning stability of homomorphisms only approximated version of Cauchy equation, $f(xy) = f(x)f(y)$, is being assumed. Therefore, in the present paper, we want to present the versions of Theorems 1.1 and 1.2, without these assumptions. However, our versions are not the generalizations of Farah's Theorems, since we add the assumptions about commutativity in groups G and H .

2. The Other Versions of Farah's Theorems

The analogue of Theorem 1.1 is the following

Theorem 2.1. *Let $(G, +)$ be a finite abelian group. Let $(H, +)$ be an abelian group, $f: G \rightarrow H$, $\delta \in (0, 1 - \frac{\sqrt{3}}{2})$. Suppose that*

$$\mu(\{(x, y) : f(x) + f(y) \neq f(x + y)\}) \leq \delta.$$

Then there is a homomorphism $F: G \rightarrow H$ such that

$$\nu(\{x : f(x) \neq F(x)\}) \leq \frac{\delta}{1 - 2\delta}.$$

We don't give a proof of this result, since it is a direct corollary of the next theorem (it is enough to put $\varepsilon = 0$ in Theorem 2.2 to get Theorem 2.1). Let us only notice that, as follows from the proof of Theorem 2.2 (see (2.6) with $\varepsilon = 0$),

$$\nu(\{x \in G : f(a + x) - f(x) = F(a)\}) \geq 1 - 2\delta. \quad (2.1)$$

Theorem 2.2. *Let $(G, +)$ be a finite abelian group, $\varepsilon \geq 0$. Let $(H, +)$ be an abelian group with translation-invariant metric $d: H \times H \rightarrow [0, \infty)$, $f: G \rightarrow H$, $\delta \in (0, 1 - \frac{\sqrt{3}}{2})$. Suppose that*

$$\mu(\{(x, y) : d(f(x) + f(y), f(x + y)) > \varepsilon\}) \leq \delta. \quad (2.2)$$

Then there is a 20ε -approximate homomorphism $F: G \rightarrow H$ (i.e.,

$$d(F(a + b), F(a) + F(b)) \leq 20\varepsilon, \quad a, b \in G)$$

such that

$$\nu(\{x : d(f(x), F(x)) > 7\varepsilon\}) \leq \frac{\delta}{1 - 2\delta}. \quad (2.3)$$

Proof. Put $Z := \{(x, y) : d(f(x) + f(y), f(x + y)) > \varepsilon\}$ and

$$D_a := \{(x, y) : d(f(a + x) - f(x), f(a + y) - f(y)) > 2\varepsilon\}$$

for $a \in G$.

First notice that for every $a \in G$ we have the implication

$$(x, y) \in D_a \Rightarrow ((a + x, y) \in Z \vee (a + y, x) \in Z).$$

Indeed, if $(a + x, y) \notin Z$ and $(a + y, x) \notin Z$ then

$$\begin{aligned} & d(f(a + x) + f(y), f(a + y) + f(x)) \\ & \leq d(f(a + x) + f(y), f(a + x + y)) + d(f(a + y + x), f(a + y) + f(x)) \leq 2\varepsilon, \end{aligned}$$

whence $(x, y) \notin D_a$. Therefore,

$$D_a \subset \{(x, y) : (a + x, y) \in Z\} \cup \{(x, y) : (a + y, x) \in Z\}.$$

Since $\mu(\{(x, y) : (a + x, y) \in Z\}) \leq \delta$ and $\mu(\{(x, y) : (a + y, x) \in Z\}) \leq \delta$, we infer that

$$\mu(D_a) \leq 2\delta. \quad (2.4)$$

Now, we will show that for every $a \in G$ there exists $p \in H$ such that

$$\nu(\{x \in G : d(f(a+x) - f(x), p) \leq 2\varepsilon\}) > \frac{1}{2}. \quad (2.5)$$

Suppose that such p does not exist. Then, for every $y \in G$ we have

$$\nu(\{x \in G : d(f(a+x) - f(x), f(a+y) - f(y)) \leq 2\varepsilon\}) \leq \frac{1}{2},$$

whence

$$\mu(\{(x, y) \in G^2 : (x, y) \notin D_a\}) \leq \frac{1}{2}.$$

Since $\delta < \frac{1}{4}$ this contradicts (2.4). Hence we proved the existence of such p that (2.5) holds. For every $a \in H$ choose an $p_a \in \{p \in H : (2.5) \text{ holds}\}$. Thus, we can define $F: G \rightarrow H$ by the formula $F(a) = p_a$.

We will show that

$$\nu(\{x \in G : d(f(a+x) - f(x), F(a)) \leq 6\varepsilon\}) \geq 1 - 2\delta. \quad (2.6)$$

Notice that the supposition

$$\forall_{y \in G} \nu(\{x \in G : d(f(a+x) - f(x), f(a+y) - f(y)) > 2\varepsilon\}) > 2\delta,$$

leads to a contradiction. Indeed, it means that

$$\forall_{y \in G} \nu(\{x \in G : (x, y) \in D_a\}) > 2\delta,$$

which implies $\mu(D_a) > 2\delta$ contrary to (2.4). Therefore, there exists a $y_0 \in G$ such that

$$\nu(\{x \in G : d(f(a+x) - f(x), f(a+y_0) - f(y_0)) > 2\varepsilon\}) \leq 2\delta. \quad (2.7)$$

Put $p^* := f(a+y_0) - f(y_0)$ and notice that by (2.7) we have

$$\nu(\{x \in G : d(f(a+x) - f(x), p^*) \leq 2\varepsilon\}) \geq 1 - 2\delta > \frac{1}{2}. \quad (2.8)$$

Since $\nu(\{x \in G : d(f(a+x) - f(x), F(a)) \leq 2\varepsilon\}) > \frac{1}{2}$, the sets

$$\{x \in G : d(f(a+x) - f(x), F(a)) \leq 2\varepsilon\}$$

and

$$\{x \in G : d(f(a+x) - f(x), p^*) \leq 2\varepsilon\}$$

have nonempty intersection. Therefore,

$$d(F(a), p^*) \leq d(F(a), f(a+x_0) - f(x_0)) + d(f(a+x_0) - f(x_0), p^*) \leq 4\varepsilon,$$

where x_0 is an arbitrary element of this intersection. By (2.8) we infer that (2.6) is satisfied.

Now we are going to show that F is an approximate homomorphism. Fix $a, b \in G$. As a consequence of (2.6) note that

$$\nu(\{x : d(F(a+b), f(a+b+x) - f(x)) \leq 6\varepsilon\}) \geq 1 - 2\delta, \quad (2.9)$$

$$\nu(\{y : d(F(b), f(b+y) - f(y)) \leq 6\varepsilon\}) \geq 1 - 2\delta. \quad (2.10)$$

By (2.6) and (2.10) we get

$$\mu(U) \geq (1 - 2\delta)^2,$$

where

$$U := \{(x, y) : d(F(a) + F(b), f(a + x) - f(x) + f(b + y) - f(y)) \leq 12\varepsilon\}.$$

Put

$$V := \{(x, y) \in G^2 : (a + x, b + y) \in Z\}$$

and

$$W := \{(x, y) \in G^2 : (x, a + b + y) \in Z\}.$$

We have $\mu(V) \leq \delta$ and $\mu(W) \leq \delta$, moreover it is easy to notice that

$$U \setminus (V \cup W) \subset \{(x, y) \in G^2 : d(F(a) + F(b), f(a + b + y) - f(y)) \leq 14\varepsilon\}.$$

Hence,

$$\begin{aligned} \mu(\{(x, y) \in G^2 : d(F(a) + F(b), f(a + b + y) - f(y)) \leq 14\varepsilon\}) \\ \geq \mu(U \setminus (V \cup W)) \geq (1 - 2\delta)^2 - 2\delta. \end{aligned}$$

We infer that

$$\nu(\{y \in G : d(F(a) + F(b), f(a + b + y) - f(y)) \leq 14\varepsilon\}) \geq (1 - 2\delta)^2 - 2\delta. \quad (2.11)$$

Since $\delta < 1 - \frac{\sqrt{3}}{2}$, we have $((1 - 2\delta)^2 - 2\delta) + (1 - 2\delta) > 1$. Therefore the sets

$$\{x \in G : d(F(a) + F(b), f(a + b + x) - f(x)) \leq 14\varepsilon\}$$

and

$$\{x \in G : d(F(a + b), f(a + b + x) - f(x)) \leq 6\varepsilon\}$$

have nonempty intersection [see (2.9) and (2.11)]. This means that

$$d(F(a) + F(b), F(a + b)) \leq 20\varepsilon.$$

Finally, we will show (2.3). Suppose on the contrary, that

$$\nu(\{x \in G : d(f(x), F(x)) > 7\varepsilon\}) > \frac{\delta}{1 - 2\delta}.$$

Taking that into account and also (2.2) and (2.6), we have

$$\begin{aligned} \delta &\geq \mu(\{(x, y) \in G^2 : d(f(x + y) - f(y), f(x)) > \varepsilon\}) \\ &\geq \mu(\{(x, y) \in G^2 : d(f(x + y) - f(y), F(x)) \leq 6\varepsilon \wedge d(F(x), f(x)) > 7\varepsilon\}) \\ &> (1 - 2\delta) \cdot \frac{\delta}{1 - 2\delta} = \delta, \end{aligned}$$

a contradiction. □

Notice the following corollary from Theorem 2.1, which states that f cannot have too few pairs (x, y) for which $f(x) + f(y) \neq f(x + y)$.

Corollary 2.3. *Under the assumptions of Theorem 2.1 if moreover*

$$|\{(x, y) : f(x) + f(y) \neq f(x + y)\}| < \frac{|G|^2}{|G| + 2},$$

then

$$f(x) + f(y) = f(x + y), \quad x, y \in G.$$

Proof. For $\delta < \frac{1}{|G|+2}$ we have

$$\frac{\delta}{1 - 2\delta} \cdot |G| < 1.$$

Hence the set $\{x : f(x) \neq F(x)\}$ is empty. $\square$

The question arises: if $\{(x, y) \in G^2 : f(x) + f(y) \neq f(x + y)\} \neq \emptyset$, then how many elements does this set contain at least?

Now, similarly as in [1], from Theorem 2.1 we deduce an analogue of Theorem 1.2, i.e., the following

Theorem 2.4. *Let G be a finite abelian group, $H = \prod_{i \in I} H_i$, where H_i are abelian groups, $f: G \rightarrow H$, $\varepsilon \in (0, \infty)$, If*

$$d_\varphi(f(a) + f(b), f(a + b)) \leq \varepsilon; \quad a, b \in G, \quad (2.12)$$

then there is a homomorphism $F: G \rightarrow H$ such that

$$d_\varphi(f(x), F(x)) \leq 9\varepsilon, \quad x \in G.$$

Proof. Put $\delta := \frac{133}{1000} \in (0, 1 - \frac{\sqrt{3}}{2})$. Let $f_i: G \rightarrow H_i$ be defined by the formula $f_i(x) = f(x)(i)$. Put

$$Z := \{(i, (a, b)) : i \in I \times G^2 : f_i(a + b) \neq f_i(a) + f_i(b)\}.$$

From the assumption (2.12) we get

$$(\varphi \times \mu)(Z) \leq \varepsilon.$$

By Fubini's Theorem we infer that

$$\varphi(\{i \in I : \mu(\{(a, b) \in G^2 : (i, (a, b)) \in Z\}) > \delta\}) \leq \frac{\varepsilon}{\delta}. \quad (2.13)$$

Let $I_0 := \{i \in I : \mu(\{(a, b) \in G^2 : (i, (a, b)) \in Z\}) \leq \delta\}$. Notice that (2.13) means that

$$\varphi(I \setminus I_0) \leq \frac{\varepsilon}{\delta}. \quad (2.14)$$

For every $i \in I_0$ we can use Theorem 2.1 to deduce the existence of a homomorphism $F_i: G \rightarrow H_i$ such that

$$\nu(\{x \in G : f_i(x) \neq F_i(x)\}) \leq \frac{\delta}{1 - 2\delta}$$

and, moreover [see (2.1)]

$$\nu(\{x \in G : f_i(a + x) - f_i(x) = F_i(a)\}) \geq 1 - 2\delta, \quad a \in G. \quad (2.15)$$

For $i \in I \setminus I_0$ let $F_i: G \rightarrow H_i$ be defined by $F_i(x) = 0_{H_i}$. Finally, put $F: G \rightarrow H$, by $F(x)(i) = F_i(x)$. It is obvious that F is a homomorphism.

Now, we will show that

$$d_\varphi(f(a), F(a)) \leq \frac{\varepsilon}{1-2\delta} + \frac{\varepsilon}{\delta}, \quad a \in G. \quad (2.16)$$

Suppose, on the contrary that there is an $a_0 \in G$ such that

$$\varphi(\{i \in I : f_i(a_0) \neq F_i(a_0)\}) > \frac{\varepsilon}{1-2\delta} + \frac{\varepsilon}{\delta}.$$

By (2.14) we get

$$\varphi(\{i \in I \setminus I_0 : f_i(a_0) \neq F_i(a_0)\}) \leq \frac{\varepsilon}{\delta}.$$

Theefore,

$$\varphi(\{i \in I_0 : f_i(a_0) \neq F_i(a_0)\}) > \frac{\varepsilon}{1-2\delta}.$$

Taking that into account and (2.15) we obtain

$$(\varphi \times \nu)(\{(i, x) \in I \times G : f_i(a_0 + x) - f_i(x) = F_i(a_0) \neq f_i(a_0)\}) > \varepsilon.$$

Therefore there exists a $b \in G$ such that

$$\varphi(\{i \in I : f_i(a_0 + b) - f_i(b) = F_i(a_0) \neq f_i(a_0)\}) > \varepsilon.$$

But it means that

$$d_\varphi(f(a_0 + b), f(a_0) + f(b)) > \varepsilon,$$

contrary to the assumption (2.12). This ends the proof of (2.16).

Finally, notice that

$$\frac{\varepsilon}{1-2\delta} + \frac{\varepsilon}{\delta} \leq 9\varepsilon.$$

□

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