Inclusions Characterizing Polynomial-Type Multifunctions

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Abstract. This work is motivated by some earlier papers concerning a pair of functional inequalities characterizing polynomials. This system is also related to the notion of microperiodic function. We study multifunctions satisfying two simultaneous conditional functional inclusions. An explicit formula for the solution to this system of inclusions is given. Applying this result we obtain its counterpart for single-valued functions, and moreover we get a characterization of real polynomial-type measurable multifunctions. Some properties of measurable microperiodic multifunctions are also given.

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1. Introduction

According to [12], a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is called a polynomial of order $p - 1 \in \mathbb{N}_0$ if it satisfies for every $h \in \mathbb{R}$ the functional equation

$$\Delta_h^p(f)(x) = 0, \quad x \in \mathbb{R},$$

where $\Delta_h(f)(x) = f(x + h) - f(x)$, $x \in \mathbb{R}$ is the difference operator, Δ_h^{p+1} is the p th iterate of Δ_h .

In [15] it was proved that every polynomial $f : \mathbb{R} \rightarrow \mathbb{R}$ can be characterized by the system of only two functional inequalities

$$\Delta_a(f)(x) \leq P(x), \quad \Delta_b(f)(x) \leq Q(x), \quad x \in \mathbb{R},$$

where $a, b \in \mathbb{R}$ and the polynomials P, Q satisfy some special conditions. For the convenience of the reader we repeat this result [15, Th. 2], which generalizes earlier outcomes obtained for constant functions P, Q (cf. [11, 14, 16]).

Lemma 1. [15, Th. 2] *Let $k \in \mathbb{N}$, $a, b, \alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$ such that $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$ satisfy conditions*

$$(-1)^j \left(\sum_{i=j}^k B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1} - \sum_{i=j}^k B_{i-j} \binom{i}{j} \beta_i b^{i-j-1} \right) \geq 0, \quad j \in \{0, 1, \dots, k\}.$$

If a function $f : \mathbb{R} \rightarrow \mathbb{R}$ continuous at least at one point satisfies

$$\begin{cases} f(x+a) \leq f(x) + \sum_{j=0}^k \alpha_j x^j, & x \in \mathbb{R} \\ f(x+b) \leq f(x) + \sum_{j=0}^k \beta_j x^j, & x \in \mathbb{R}, \end{cases}$$

then

$$f(x) - f(0) = \sum_{j=0}^k \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}}{j+1} x^{j+1} = \sum_{j=0}^k \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \beta_i b^{i-j-1}}{j+1} x^{j+1}.$$

Some further generalizations can be found in [3], where a similar problem was considered for measurable function f .

On the other hand the system of functional inequalities

$$\Delta_a(f)(x) \leq 0, \quad \Delta_b(f)(x) \leq 0, \quad x \in \mathbb{R},$$

is strongly connected with the notion of microperiodic function. It is known that a real microperiodic function f continuous at some point is constant. An analogous result concerning Lebesgue measurable microperiodic functions has been obtained by A. Łomnicki in [9] (see also [8, 17]), for Baire measurable functions can be found in [18]. Afterwards M. Kuczma in [13] proved that a microperiodic function defined on a topological group with values in a separable metric space which is Haar or Baire measurable must be constant almost everywhere in G . Further generalizations of the above mentioned results are due to J. Brzdęk [4–6].

The aim of this paper is to prove a counterpart to the above mentioned results in the case of measurable multifunctions.

2. Preliminaries

Denote by 2^X the family of all subsets of $X \neq \emptyset$ and write $P(X) := 2^X \setminus \{\emptyset\}$. If X is a Hausdorff topological space (linear or metric space, when appropriate) $P_{bf}(X)$, $P_{bc}(X)$ the family of bounded closed and bounded convex members of $P(X)$, respectively.

If X is a metric space so is $(P_{bf}(X), h)$ (cf. [7, 10]), where h is the Hausdorff distance defined as follows

$$h(A, B) := \max\{e(A, B), e(B, A)\},$$

$$e(A, B) := \sup\{d(a, B) : a \in A\}, \quad d(a, B) := \inf\{d(a, b) : b \in B\}.$$

We will need the following lemma (cf. [10, Th. I.1.15, Rem. I.1.16]).

Lemma 2. *If (X, d) is a metric space and $A, B \in P(X)$, then*

$$\begin{aligned} h(A, B) &:= \sup\{|d(x, A) - d(x, B)| : x \in X\}, \\ e(A, B) &:= \sup\{d(x, B) - d(x, A) : x \in X\}. \end{aligned}$$

Given a measurable space $(X, \mathcal{M})$ and a separable metric space (Y, d) we say that a multifunction $F : X \rightarrow 2^Y$ is weakly measurable (measurable for short) if the lower inverse image $F^-(U) := \{x \in X : F(x) \cap U \neq \emptyset\}$ is measurable for every open set $U \subset Y$. F is strongly measurable if the lower inverse image $F^-(C) := \{x \in X : F(x) \cap C \neq \emptyset\}$ is measurable for every closed set $C \subset Y$. A single-valued function $f : X \rightarrow \mathbb{R}$ is measurable if $f^{-1}(U) \in \mathcal{M}$ for every open set $U \subset \mathbb{R}$.

According to [10, Prop. II.1.3] a strongly measurable multifunction is measurable. Both notions of measurability are equivalent for compact valued multifunctions. Moreover we will need the following characterization of measurable multifunctions (see [10, Prop. II.1.4]).

Lemma 3. *Multifunction $F : X \rightarrow 2^Y$ is measurable iff for every $y \in Y$ $x \mapsto d(y, F(x))$ is a measurable function with values in $[0, +\infty]$.*

We say that a multifunction or a function is Haar (Lebesgue, Baire) measurable if it is measurable with respect to $\mathcal{H}$ the σ -algebra of Haar measurable sets (Lebesgue measurable sets $\mathcal{L}$, sets with Baire property $\mathcal{B}$, respectively).

If $(X, \mathcal{M}, \mu)$ is a measure space, denote $\mathcal{M}_0$ the σ -ideal of sets of the measure zero. We say that a property holds almost everywhere with respect to $\mathcal{M}$ ($\mathcal{M}$ -a.e. for abbreviation) if there exists a set $E \in \mathcal{M}_0$ such that the property holds for every $x \in X \setminus E$. In the case of σ -algebra of the sets with the Baire property, the above mentioned set $E \in \mathcal{B}_0$, which is the σ -ideal of sets of the first category.

If Y is a linear topological space we denote $\bar{A}$ the closure of $A \subset Y$. Given a multifunction $F : X \rightarrow P(Y)$ we will use the symbol $\bar{F}$ for the multifunction $\bar{F}(x) := \overline{F(x)}$, $x \in X$.

3. Measurable Microperiodic Multifunctions

According to [13], we say a function f defined on topological group X is microperiodic if there exists a dense set $P \subset X$ with

$$f(px) = f(x), \quad x \in X, p \in P. \quad (1)$$

Replacing in (1) the equality by an inclusion, we introduce the notion of microperiodic multifunction.

Definition 1. Let $(X, \cdot)$ be a topological group, Y a nonempty set. We say that a multifunction $F : X \rightarrow 2^Y$ is microperiodic if there exists a dense subset P of X such that

$$F(px) \subset F(x), \quad x \in X, p \in P. \quad (\text{M})$$

Observe that a multifunction F satisfying

$$F(px) \supset F(x), \quad x \in X, p \in P,$$

satisfies the inclusion (M) with the set $P^{-1} := \{p^{-1} : p \in P\}$, where p^{-1} stands for the inverse of p . Moreover, if the set P is dense in X , then P^{-1} is dense as well. Therefore without loss of generality we consider functions fulfilling (M).

Example 1. Let $A, B \subset Y$, $P \subset X$ and

$$F(x) = \begin{cases} A & \text{for } x \in P, \\ B & \text{for } x \in X \setminus P. \end{cases}$$

$F : X \rightarrow 2^Y$ is microperiodic provided P is dense and for every $p \in P$ the following condition holds

$$px \in P, \text{ for } x \in P \quad \text{and} \quad px \in X \setminus P, \text{ for } x \in X \setminus P.$$

It is strongly measurable iff $P \in \mathcal{M}$.

In particular, we can take $X = \mathbb{R}$ with σ -algebra of Lebesgue measurable sets on $\mathbb{R}$ and $P = \mathbb{Q}$. Note that in this case F is constant almost everywhere.

On the other hand we have an example of nonmeasurable microperiodic multifunction.

Example 2. According to [12, Th. 12.4.1] there exists an additive function $f : \mathbb{R} \rightarrow \mathbb{R}$ with small graph, which means that $\text{card } f(\mathbb{R}) = \aleph_0$. Then f is discontinuous and therefore it is not Lebesgue measurable (see [12, Th. 9.4.2]). By [12, Lem. 12.7.2], f is microperiodic with dense $P = \ker f$ (see [12, Lem. 12.2.1]).

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an additive function with small graph. Assume that $C \in P(\mathbb{R})$ such that $d(0, C) > 0$. Since f is additive, the multifunction $F : \mathbb{R} \rightarrow P(\mathbb{R})$ given by

$$F(x) = |f(x)| \cdot C = \{|f(x)| \cdot c : c \in C\}, \quad x \in \mathbb{R}$$

is microperiodic.

Write $g(x) := d(0, F(x)) = |f(x)|d(0, C)$, $x \in \mathbb{R}$. Then by [12, Th. 12.6.6], the set $g^{-1}(\{0\}) = f^{-1}(\{0\})$ is not Lebesgue measurable, consequently g is not measurable and thus F is not measurable as well. Obviously F is not strongly measurable. Obviously, it is not constant a.e..

Under some measurability assumptions, microperiodic single-valued function is constant almost everywhere. Analogous result can be obtained for measurable multifunctions. First we prove a multivalued counterpart of more general result obtained by J. Brzdęk in [4], which is recalled below for the convenience of the reader.

Lemma 4. [4, Th.1] *Let $(X, \cdot)$ be a topological group, P a dense subset of X , $f : X \rightarrow \mathbb{R}$. Suppose that X is locally compact (of the second category, respectively), f is $\mathcal{H}$ -measurable ($\mathcal{B}$ -measurable) and one of the following two conditions holds*

- (i) $f(px) \geq f(x)$, $x \in X, p \in P$,
- (ii) P is countable and $f(px) \geq f(x)$ a.e. for every $p \in P$.

Then f is constant a.e..

On account of Example 2, a measurability assumption in the following theorem is essential.

Theorem 1. *Let X be a topological group, (Y, d) a separable metric space, P a dense subset of X . Assume that X is locally compact (of the second category, respectively), $F : X \rightarrow P(Y)$ is $\mathcal{H}$ -measurable ($\mathcal{B}$ -measurable) satisfying one of the conditions*

- (i) F is microperiodic multifunction,
- (ii) P is countable and $F(px) \subset F(x)$ a.e. for every $p \in P$.

Then $\overline{F}$ is constant a.e..

Proof. We prove the theorem in case of Haar measurability under the assumption (i), similar considerations apply in other cases.

Let μ denote the Haar measure on X and let $D = \{y_n : n \in \mathbb{N}\}$ be a dense subset of Y . Fix $n \in \mathbb{N}$ and define $f_n(x) := d(y_n, F(x))$, $x \in X$ which is Haar measurable, by Lemma 3. According to (M)

$$f_n(px) \geq f_n(x), \quad x \in X, p \in P.$$

Therefore by Proposition 4, f_n is constant almost everywhere.

It follows that for every $n \in \mathbb{N}$ there exists a set E_n of measure zero and $c_n \in \overline{\mathbb{R}}$ such that

$$f_n(x) = c_n \quad \text{for } x \in X \setminus E_n.$$

Set $E := \bigcup_{n \in \mathbb{N}} E_n$, hence $\mu(E) = 0$ and $E \neq X$. Take $x_0 \in X \setminus E$. Accordingly

$$d(y_n, F(x)) = d(y_n, F(x_0)), \quad x \in X \setminus E, n \in \mathbb{N}. \quad (2)$$

Fix $x \in X \setminus E$. By density of D , $d(y, F(x)) = d(y, F(x_0))$ for every $y \in Y$. Therefore by Lemma 2, $h(F(x), F(x_0)) = 0$ and consequently $\overline{F}(x) = \overline{F}(x_0)$. $\square$

Applying analysis similar to that in the above theorem, we obtain the following counterpart of [3, Prop.2.2] for real multifunctions.

Theorem 2. *Let $I \subset \mathbb{R}$ be a nontrivial interval, P a dense subset of $\mathbb{R}$. If $F : I \rightarrow P(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable, respectively) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that*

$F(x+p) \subset F(x)$ for every $x \in I \setminus E, p \in P$ such that $x+p \in I$,
then $\overline{F}$ is constant a.e..

It is known that the set $P = \{na+mb : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$, provided $a < 0 < b$ and $\frac{a}{b} \notin \mathbb{Q}$ (cf. [4, 13]). Therefore directly from Theorem 2 we have what follows.

Corollary 1. Assume that $I \subset \mathbb{R}$ be a nontrivial interval $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$. If $F : I \rightarrow P(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable, respectively) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that

$$\begin{cases} F(x+a) \subset F(x), & x \in I, x+a \in I, \\ F(x+b) \subset F(x), & x \in I, x+b \in I, \end{cases}$$

then $\overline{F}$ is constant a.e..

The case of multifunctions with closed values is analogous to the case of single valued functions. Under the assumptions of each of the above theorems, if F has closed values, it is constant a.e..

4. Approximately Microperiodic Multifunctions

Similarly to single-valued functions, we have stability results in the multivalued case. Namely we show that a multifunction which approximately satisfies inclusion (M) is close (in some sense) to almost constant multifunction.

An easy consequence of [6, Cor. 3] is the following result concerning approximately microperiodic multifunctions with compact values.

Theorem 3. Let X be a topological group, (Y, d) a separable metric space, P a dense subset of X , $\varepsilon > 0$. Assume that X is locally compact (of the second category, respectively). If a $\mathcal{H}$ -measurable ($\mathcal{B}$ -measurable) multifunction $F : X \rightarrow P_k(Y)$ fulfills

$$h(F(px), F(x)) \leq \varepsilon, \quad x \in X, p \in P,$$

then there exists a set $E \in \mathcal{H}_0$ ($E \in \mathcal{B}_0$) such that

$$h(F(x), F(y)) \leq 2\varepsilon, \quad x, y \in X \setminus E.$$

Moreover, if every closed ball in $P_k(Y)$ is compact then there is $A \in P_k(Y)$ with $h(F(x), A) \leq \varepsilon$ a.e..

Proof. Observe that in the case, measurable multifunction $F : X \rightarrow P_k(Y)$ is measurable as a single-valued function with values in the metric space $(P_k(Y), h)$. Since Y is a separable metric space, $P_k(Y)$ is separable as well (see [7, Th. II-8]). So we can apply [6, Cor. 3]. $\square$

Applying Proposition 4, after analysis similar to that in the proof of Theorem 1 we can drop the assumption on compactness of values of F .

Theorem 4. *Let X be a topological group, (Y, d) a separable metric space, P a dense subset of X , $\varepsilon > 0$. Assume that X is locally compact (of the second category, respectively). If a $\mathcal{H}$ -measurable ($\mathcal{B}$ -measurable) multifunction $F : X \rightarrow P(Y)$ fulfills*

$$e(F(x), F(px)) \leq \varepsilon, \quad x \in X, p \in P, \quad (3)$$

then there exists a set $E \in \mathcal{H}_0$ ($E \in \mathcal{B}_0$) and $x_0 \in X$ such that

$$h(F(x), F(x_0)) \leq 2\varepsilon, \quad x \in X \setminus E.$$

Proof. We prove the theorem in case of Haar measurability, similar considerations we apply in case of Baire measurable multifunction.

Let μ denote the Haar measure on X , set $D = \{y_n : n \in \mathbb{N}\}$ a dense subset of Y and define $f_n(x) := d(y_n, F(x))$, $x \in X$, $n \in \mathbb{N}$. According to (3) and Lemma 2

$$d(y, F(px)) - d(y, F(x)) \leq \varepsilon, \quad x \in X, p \in P,$$

and consequently for every $n \in \mathbb{N}$

$$f_n(px) - f_n(x) \leq \varepsilon, \quad x \in X, p \in P.$$

Moreover, by Lemma 3, functions f_n are Haar measurable. Therefore by Proposition 4, for every $n \in \mathbb{N}$ there exists $c_n \in \mathbb{R}$ and a set $E_n \subset X$ of measure zero such that

$$|f_n(x) - c_n| \leq \varepsilon, \quad x \in X \setminus E_n.$$

Set $E := \bigcup_{n \in \mathbb{N}} E_n$, hence $\mu(E) = 0$ and $E \neq X$. Take $x, x_0 \in X \setminus E$. Then for every $n \in \mathbb{N}$

$$|d(y_n, F(x)) - d(y_n, F(x_0))| \leq |f_n(x) - c_n| + |c_n - f_n(x_0)| \leq 2\varepsilon. \quad (4)$$

Since D is dense in Y the above inequality holds for every $y \in Y$ and finally by Lemma 2

$$h(F(x), F(x_0)) = \sup\{|d(y, F(x)) - d(y, F(x_0))| : y \in Y\} \leq 2\varepsilon.$$

□

5. Conditional Simultaneous Inclusions

Now we are in position to investigate properties of real multifunctions satisfying the simultaneous conditional inclusions. Assume that I is a nontrivial interval, $F : I \rightarrow P(\mathbb{R})$, $k \in \mathbb{N}$, $a, b, \alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$, $E \subset I$.

$$\begin{cases} F(x+a) \subset F(x) + \sum_{j=0}^k \alpha_j x^j, & x \in I \setminus E, x+a \in I \\ F(x+b) \subset F(x) + \sum_{j=0}^k \beta_j x^j, & x \in I \setminus E, x+b \in I, \end{cases} \quad (5)$$

We first study the case $k = 0$. As an immediate consequence of [3, Prop. 3.1], provided below, we get its counterpart for multifunctions with convex bounded values in $\mathbb{R}$, namely bounded intervals denoted by $\langle a, b \rangle$.

Lemma 5. [3, Prop. 3.1] Assume that $a, b, \alpha, \beta \in \mathbb{R}$, $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$, $\frac{\alpha}{a} \geq \frac{\beta}{b}$. Let $I \subset \mathbb{R}$ be a nontrivial interval with the length greater than $b - a$. Then a function $f : I \rightarrow \mathbb{R}$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable respectively) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that

$$\begin{cases} f(x+a) - f(x) \leq \alpha, & x \in I \setminus E, x+a \in I \\ f(x+b) - f(x) \leq \beta, & x \in I \setminus E, x+b \in I \end{cases} \quad (6)$$

if and only if $\frac{\alpha}{a} = \frac{\beta}{b}$ and there exists $d \in \mathbb{R}$ such that $f(x) = \frac{\alpha}{a}x + d$ $\mathcal{L}$ -a.e. ($\mathcal{B}$ -a.e.).

Theorem 5. Assume that $a, b, \alpha, \beta \in \mathbb{R}$, $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$, $\frac{\alpha}{a} \geq \frac{\beta}{b}$. Let $I \subset \mathbb{R}$ be a nontrivial interval with the length greater than $b - a$. A multifunction $F : I \rightarrow P_{bc}(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable respectively) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that

$$\begin{cases} F(x+a) \subset F(x) + \alpha, & x \in I \setminus E, x+a \in I \\ F(x+b) \subset F(x) + \beta, & x \in I \setminus E, x+b \in I, \end{cases} \quad (7)$$

if and only if $\frac{\alpha}{a} = \frac{\beta}{b}$ and there exists $c, d \in \mathbb{R}$ such that $F(x) = \frac{\alpha}{a}x + \langle c, d \rangle$ $\mathcal{L}$ -a.e. ($\mathcal{B}$ -a.e.).

Proof. We give the proof for the case of σ -algebra $\mathcal{L}$. Assume that F is measurable and has bounded convex values. Then it is of the form $F(x) = \langle f(x), g(x) \rangle$ where $f, g : I \rightarrow \mathbb{R}$ are measurable single-valued functions and there exists $E \in \mathcal{L}_0$ such that

$$\begin{cases} f(x) + \alpha \leq f(x+a) \leq g(x+a) \leq g(x) + \alpha, & x \in I \setminus E, x+a \in I \\ f(x) + \beta \leq f(x+b) \leq g(x+b) \leq g(x) + \beta, & x \in I \setminus E, x+b \in I. \end{cases}$$

Applying Proposition 5 for the function g we conclude that $\frac{\alpha}{a} = \frac{\beta}{b}$ and there exists $d \in \mathbb{R}$ such that $f(x) = \frac{\alpha}{a}x + d$ $\mathcal{L}$ -a.e.. Therefore $\frac{-\alpha}{a} = \frac{-\beta}{b}$ and application of Proposition 5 for the function $-f$ yields $-f(x) = -\frac{-\alpha}{a}x + e$ for some $e \in \mathbb{R}$. Consequently $F(x) = \frac{\alpha}{a}x + \langle -e, d \rangle$ for almost all $x \in I$.

The proof of the opposite implication is left to the reader. $\square$

Example 3. If a function $f : I \rightarrow \mathbb{R}$ fulfills assumptions of the above theorem, then the multifunction $F : I \rightarrow P(\mathbb{R})$ of the form $F(x) := (-\infty, f(x))$ (values of F are open or closed intervals) satisfies for $x \in I$

$$F(x+a) \subset F(x) + \alpha, \quad F(x+b) \subset F(x) + \beta \quad (8)$$

and

$$F(x) = (-\infty, \frac{\alpha}{a}x + d) = \frac{\alpha}{a}x + (-\infty, d).$$

Example 4. Let I be a nontrivial interval, $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$, $\alpha, \beta \in \mathbb{R}$. Put $B := \{m\alpha + n\beta : m, n \in \mathbb{Z}\}$. Then the multifunction $F : I \rightarrow P(\mathbb{R})$ defined by

$F(x) := B$ satisfies (8) but it is not of the form $\frac{\alpha}{a}x + D$ with any nonempty set $D \subset \mathbb{R}$.

On the contrary, suppose that there exist $c \in \mathbb{R}$, $D \in P(\mathbb{R})$ and $E \in \mathcal{L}_0$ such that $F(x) = cx + D$ for $x \in I \setminus E$. Since $d + c(I \setminus E) \subset B$ for $d \in D$, the set $I \setminus E$ is countable and consequently $I \in \mathcal{L}_0$.

Remark 1. If a multifunction $F : I \rightarrow P(\mathbb{R})$ with bounded values given by $F(x) := cx + A$ fulfills (8), then $c = \frac{\alpha}{a} = \frac{\beta}{b}$.

We are thus led to the general case of multifunctions with nonempty values.

Theorem 6. Assume that $a, b, \alpha, \beta \in \mathbb{R}$, $a < 0 < b$, $\frac{a}{b} \notin \mathbb{Q}$, $\frac{\alpha}{a} = \frac{\beta}{b}$. Let $I \subset \mathbb{R}$ be a nontrivial interval with the length greater than $b - a$, $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$ respectively).

A multifunction of the form $F(x) = \frac{\alpha}{a}x + A$, $x \in I \setminus E$ for some $A \in P(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) and (7) holds true with equalities.

If a multifunction $F : I \rightarrow P(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) satisfies (7), then there exists $A \in P(\mathbb{R})$ such that $\overline{F}(x) = \frac{\alpha}{a}x + A$ $\mathcal{L}$ -a.e. ($\mathcal{B}$ -a.e.).

Proof. We give the proof in case of $\mathcal{L}$ -measurability. Assume that there exists $E \in \mathcal{L}_0$ such that $F : \mathbb{R} \rightarrow P(\mathbb{R})$ is of the form $F(x) = \frac{\alpha}{a}x + A$ for $x \in I \setminus E$. It is obvious that (7) holds true with equalities. If $\alpha = 0$, then F is constant almost everywhere and therefore it is measurable. If $\alpha \neq 0$, take an open set $U \subset \mathbb{R}$. Then the lower inverse image of U is measurable, since

$$\begin{aligned} F^-(U) &= \{x \in I : F(x) \cap U \neq \emptyset\} = E \cup \{x \in I \setminus E : (\frac{\alpha}{a}x + A) \cap U \neq \emptyset\} \\ &= E \cup \bigcup_{y \in A} \{x \in I \setminus E : \frac{\alpha}{a}x + y \in U\} \\ &= E \cup \bigcup_{y \in A} \{x \in I \setminus E : x \in \frac{a}{\alpha}(-y + U)\} \\ &= E \cup I \setminus \bigcup_{y \in A} \frac{a}{\alpha}(-y + U). \end{aligned}$$

Consequently F is measurable.

Now assume that (7) is satisfied with a set $E \in \mathcal{L}_0$ and define $G(x) = \overline{F}(x) - cx$, $x \in I$, where $c = \frac{\alpha}{a} = \frac{\beta}{b}$. On account of [10, Prop. II.1.8], $\overline{F}$ is measurable and has closed values and by [10, Prop. II.2.3] there exists a sequence $\{f_n\}_{n \in \mathbb{N}}$ of $\mathcal{L}$ -measurable selections such that $F(x) = \overline{\{f_n(x) : n \in \mathbb{N}\}}$. Therefore G is $\mathcal{L}$ -measurable, since $G(x) = \overline{\{f_n(x) - cx : n \in \mathbb{N}\}}$ and every function $x \mapsto f_n(x) - cx$ is measurable. Moreover for every $x \in I \setminus E$

$$\begin{cases} G(x+a) \subset G(x), & x+a \in I \\ G(x+b) \subset G(x), & x+b \in I. \end{cases} \quad (9)$$

Define $P_k := \{ma + nb : m, n \in \mathbb{N}, m + n \leq k\}$, $k \in \mathbb{N}$ and let $P := \bigcup_{k \in \mathbb{N}} P_k$ which is dense in $\mathbb{R}$ (see e.g. [15]). Observe that $H := \bigcup_{m,n \in \mathbb{Z}} (E + ma + nb)$ is a set of measure zero and moreover satisfies the following condition

$$\forall_{m,n \in \mathbb{N}_0} \forall_{x \in I \setminus H} \quad x + ma + nb \in I \implies x + ma + nb \in I \setminus H. \quad (10)$$

We prove by induction that

$$\forall_{k \in \mathbb{N}} \forall_{p \in P_k} \forall_{x \in I \setminus H} \quad x + p \in I \implies G(x + p) \subset G(x). \quad (11)$$

The case $k = 1$ is just (9). So assume that (11) is satisfied for $k \in \mathbb{N}$. Choose $q \in P_{k+1}$ such that $x + q \in I$, then by (10) $x + q \in I \setminus H$.

Since the length of the interval I is greater than $b - a$, it is easy to show that at least one of points $x + q - a, x + q - b$ belongs to I . Assume that $x + q - a \in I$ and put $p := q - a \in P_k$. According to (10), $x + q - a \in I \setminus H$ and by the inductive hypothesis

$$G(x + q) = G(x + q - a + a) \subset G(x + q - a) \subset G(x).$$

It follows that assumptions of Theorem 2 are satisfied, consequently there exists $A \in P_f(\mathbb{R})$ such that $G(x) = A$ almost everywhere in I . Hence $\overline{F}(x) = cx + A$ for almost all $x \in I$. $\square$

Denote by

$$B_j = \sum_{k=0}^j \frac{1}{k+1} \sum_{i=0}^k (-1)^i \binom{k}{i} i^j, \quad j \in \mathbb{N}_0$$

the Bernoulli numbers.

Theorem 7. Let $k \in \mathbb{N}$, $a < 0 < b$ with $\frac{a}{b} \notin \mathbb{Q}$ and $\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$ satisfy conditions

$$\sum_{i=j}^k B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1} = \sum_{i=j}^k B_{i-j} \binom{i}{j} \beta_i b^{i-j-1}, \quad j \in \{0, 1, \dots, k\}. \quad (12)$$

Let $I \subset \mathbb{R}$ be a nontrivial interval with the length greater than $b - a$. If a multifunction $F : I \rightarrow P(\mathbb{R})$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that (5) holds true, then there exists $A \in P(\mathbb{R})$ such that

$$F(x) = \sum_{j=0}^k \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}}{j+1} x^{j+1} + A, \quad \mathcal{L}\text{-a.e. } (\mathcal{B}\text{-a.e.}). \quad (13)$$

If $F : I \rightarrow P(\mathbb{R})$ is of the form (13), then it is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) and fulfills (5) with equalities for some $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$).

Proof. The proof is by induction on k . The case $k = 0$ is proved in Theorem 6. Assume that (5) holds true for $E \in \mathcal{L}_0$, $a, b, \alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$ and define the multifunction

$$G(x) := F(x) - \frac{\alpha_k}{a(k+1)} x^{k+1}, \quad x \in I. \quad (14)$$

It is easily seen that G is measurable. We will show that it satisfies (5) with the polynomials $\sum_{j=0}^{k-1} \alpha_j^* x^j$, $\sum_{j=0}^{k-1} \beta_j^* x^j$ where

$$\alpha_j^* = \alpha_j - \frac{\alpha_k}{k+1} \binom{k+1}{j} a^{k-j}, \quad \beta_j^* = \beta_j - \frac{\beta_k}{k+1} \binom{k+1}{j} b^{k-j}.$$

Take $x \in I \setminus E$, such that $x + a \in I$.

$$\begin{aligned} G(x+a) &= F(x+a) - \frac{\alpha_k}{a(k+1)} (x+a)^{k+1} \\ &\subset F(x) + \sum_{j=0}^k \alpha_j x^j - \frac{\alpha_k}{a(k+1)} \sum_{j=0}^{k+1} \binom{k+1}{j} a^{k+1-j} x^j \\ &= F(x) - \frac{\alpha_k}{a(k+1)} x^{k+1} + \sum_{j=0}^{k-1} \alpha_j x^j + \alpha_k x^k \\ &\quad - \frac{\alpha_k}{a(k+1)} \sum_{j=0}^{k-1} \binom{k+1}{j} a^{k+1-j} x^j - \frac{\alpha_k}{a(k+1)} (k+1) a x^k \\ &= G(x) + \sum_{j=0}^{k-1} \left[\alpha_j - \frac{\alpha_k}{k+1} \binom{k+1}{j} a^{k-j} \right] x^j. \end{aligned}$$

Similarly

$$G(x+b) \subset G(x) + \sum_{j=0}^{k-1} \left[\beta_j - \frac{\beta_k}{k+1} \binom{k+1}{j} b^{k-j} \right] x^j.$$

Moreover, by [15, Lemma 2]

$$\sum_{i=j}^{k-1} B_{i-j} \binom{i}{j} \alpha_i^* a^{i-j-1} = \sum_{i=j}^k B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}.$$

Consequently, by the induction hypothesis there exists $A \in P(\mathbb{R})$ such that

$$\begin{aligned} G(x) &= \sum_{j=0}^{k-1} \sum_{i=j}^{k-1} \frac{B_{i-j} \binom{i}{j} \alpha_i^* a^{i-j-1}}{j+1} x^{j+1} + A \\ &= \sum_{j=0}^{k-1} \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}}{j+1} x^{j+1} + A, \end{aligned}$$

hence according to (14)

$$F(x) = \frac{B_0 \binom{k}{k} \alpha_k a^{-1}}{k+1} x^{k+1} + \sum_{j=0}^{k-1} \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}}{j+1} x^{j+1} + A$$

and the proof of the first assertion is complete. The proof of the second one follows similarly and is left to the reader. $\square$

The above theorem yields an analogon of [15, Th. 2] for measurable single-valued function f satisfying

$$\begin{cases} f(x+a) \leq f(x) + \sum_{j=0}^k \alpha_j x^j, & x \in I \setminus E, x+a \in I \\ f(x+b) \leq f(x) + \sum_{j=0}^k \beta_j x^j, & x \in I \setminus E, x+b \in I. \end{cases} \quad (15)$$

It suffices to use it for the multifunction $F(x) := (-\infty, f(x)]$.

Corollary 2. *Let $k \in \mathbb{N}$, $a < 0 < b$ with $\frac{a}{b} \notin \mathbb{Q}$ and $\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$ satisfy (12). Assume that $I \subset \mathbb{R}$ be a nontrivial interval with the length greater than $b-a$. If a function $f: I \rightarrow \mathbb{R}$ is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) and there exists $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$) such that (15) holds true, then there exists $c \in \mathbb{R}$ such that*

$$f(x) = \sum_{j=0}^k \sum_{i=j}^k \frac{B_{i-j} \binom{i}{j} \alpha_i a^{i-j-1}}{j+1} x^{j+1} + c, \quad \mathcal{L}\text{-a.e. } (\mathcal{B}\text{-a.e.}). \quad (16)$$

If $f: I \rightarrow \mathbb{R}$ is of the form (16), then it is $\mathcal{L}$ -measurable ($\mathcal{B}$ -measurable) and fulfills (15) with equalities for some $E \in \mathcal{L}_0$ ($E \in \mathcal{B}_0$).

Assume that $F: I \rightarrow P(\mathbb{R})$ is a polynomial-type multifunction, i.e. a multifunction satisfying $F(x) = \sum_{j=0}^k c_j x^{j+1} + A$ $\mathcal{L}$ -a.e. ($\mathcal{B}$ -a.e.). According to [15, Rem. 3], we can find $a < 0 < b$ with $\frac{a}{b} \notin \mathbb{Q}$ and $\alpha_0, \dots, \alpha_k, \beta_0, \dots, \beta_k \in \mathbb{R}$ such that (12) holds and the polynomial $f(x) := \sum_{j=0}^k c_j x^{j+1}$ is of the form (16) with $c = 0$. Then by the above corollary f satisfies (15) with equalities. Since we have a representation $F(x) = \{\sum_{j=0}^k c_j x^{j+1} + d : d \in A\}$, (5) is fulfilled (with equalities). Therefore, similarly to the single-valued case, polynomial-type multifunctions can be characterized by Theorem 7.

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