J -Tangent Affine Hyperspheres

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Abstract. In this paper we study J -tangent affine hyperspheres. Under some additional conditions we give a local characterization of 3-dimensional J -tangent affine hyperspheres.

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1. Introduction

Centro-affine real hypersurfaces with a J -tangent transversal vector field were first studied by Cruceanu in [1]. He proved that such hypersurfaces $f: M^{2n+1} \rightarrow \mathbb{C}^{n+1}$ can be locally expressed in the form

$$f(x_1, \dots, x_{2n}, z) = Jg(x_1, \dots, x_{2n}) \cos z + g(x_1, \dots, x_{2n}) \sin z,$$

where g is some smooth function defined on an open subset of $\mathbb{R}^{2n}$. He also showed that if the induced almost contact structure is Sasakian then a hypersurface must be a hyperquadric. The latter result was generalized in [3] to arbitrary hypersurfaces with J -tangent transversal vector field.

Since the class of centro-affine hypersurfaces with a J -tangent transversal vector field is quite large, the question arises whether there are affine hyperspheres with a J -tangent Blaschke normal field. A nontrivial 3-dimensional example was provided in [4]. The main purpose of this paper is to give a local characterization of 3-dimensional J -tangent affine hyperspheres with involutive contact distribution $\mathcal{D}$.

In Sect. 2 we briefly recall basic formulas of affine differential geometry and recall the notion of an affine hypersphere.

In Sect. 3 we recall the notion of a J -tangent transversal vector field, a definition of the induced almost contact structure as well as some results obtained in [3].

Section 4 contains the main results of this paper. We prove that there are no improper J -tangent affine hyperspheres and we give a local representation of 3-dimensional J -tangent affine hyperspheres under additional condition that the contact distribution is involutive.

2. Preliminaries

We briefly recall the basic formulas of affine differential geometry. For more details, we refer to [2]. Let $f: M \rightarrow \mathbb{R}^{n+1}$ be an orientable connected differentiable n -dimensional hypersurface immersed in the affine space $\mathbb{R}^{n+1}$ equipped with its usual flat connection D . Then for any transversal vector field C we have

$$D_X f_* Y = f_*(\nabla_X Y) + h(X, Y)C \quad (2.1)$$

and

$$D_X C = -f_*(SX) + \tau(X)C, \quad (2.2)$$

where X, Y are vector fields tangent to M . It is known that ∇ is a torsion-free connection, h is a symmetric bilinear form on M , called the second fundamental form, S is a tensor of type $(1, 1)$, called the shape operator, and τ is a 1-form, called the transversal connection form.

We assume that h is nondegenerate so that h defines a semi-Riemannian metric on M . If h is nondegenerate, then we say that the hypersurface or the hypersurface immersion is nondegenerate. In this paper we assume that f is always nondegenerate. We have the following

Theorem 2.1 ([2], Fundamental equations). *For an arbitrary transversal vector field C the induced connection ∇ , the second fundamental form h , the shape operator S , and the 1-form τ satisfy the following equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \quad (2.3)$$

$$(\nabla_X h)(Y, Z) + \tau(X)h(Y, Z) = (\nabla_Y h)(X, Z) + \tau(Y)h(X, Z), \quad (2.4)$$

$$(\nabla_X S)(Y) - \tau(X)SY = (\nabla_Y S)(X) - \tau(Y)SX, \quad (2.5)$$

$$h(X, SY) - h(SX, Y) = 2d\tau(X, Y). \quad (2.6)$$

The Eqs. (2.3), (2.4), (2.5), and (2.6) are called the equations of Gauss, Codazzi for h , Codazzi for S and Ricci, respectively.

For a hypersurface immersion $f: M \rightarrow \mathbb{R}^{n+1}$ a transversal vector field C is said to be equiaffine (resp. locally equiaffine) if $\tau = 0$ (resp. $d\tau = 0$).

When f is nondegenerate, there exists a canonical transversal vector field C , called the affine normal (or the Blaschke normal field). The affine normal is uniquely determined up to sign by the following conditions: the metric volume

form ω_h of h is ∇ -parallel and coincides with the induced volume form Θ , where ω_h is defined by $|\omega_h(X_1, \dots, X_n)| = |\det[h(X_i, X_j)]|^{1/2}$ and Θ is defined by $\Theta(X_1, \dots, X_n) = \det[f_*X_1, \dots, f_*X_n, C]$ for tangent vectors X_i ($i=1, \dots, n$). The affine immersion f with the Blaschke normal field C is called a Blaschke hypersurface. In this case fundamental equations can be rewritten as follows

Theorem 2.2 ([2], Fundamental equations). *For a Blaschke hypersurface f , we have the following fundamental equations:*

$$R(X, Y)Z = h(Y, Z)SX - h(X, Z)SY, \quad (2.7)$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \quad (2.8)$$

$$(\nabla_X S)(Y) = (\nabla_Y S)(X), \quad (2.9)$$

$$h(X, SY) = h(SX, Y). \quad (2.10)$$

A Blaschke hypersurface is called an affine hypersphere if $S = \lambda I$, where $\lambda = \text{const}$.

If $\lambda = 0$ f is called an improper affine hypersphere, if $\lambda \neq 0$ f is called a proper affine hypersphere.

For simplicity we shall omit f_* in front of vector fields in most cases.

3. Induced Almost Contact Structures

Let $\dim M = 2n + 1$ and $f: M \rightarrow \mathbb{R}^{2n+2}$ be a nondegenerate affine hypersurface. We always assume that $\mathbb{R}^{2m} \simeq \mathbb{C}^m$ is endowed with the standard complex structure J . In particular, if $m = n + 1$ we have

$$J(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) = (-y_1, \dots, -y_{n+1}, x_1, \dots, x_{n+1}).$$

Let C be a transversal vector field on M . We say that C is J -tangent if $JC_x \in f_*(T_x M)$ for every $x \in M$. We also define a distribution $\mathcal{D}$ on M as the biggest J invariant distribution on M , that is,

$$\mathcal{D}_x = f_*^{-1}(f_*(T_x M) \cap J(f_*(T_x M)))$$

for every $x \in M$. It is clear that $\dim \mathcal{D} = 2n$. A vector field X is called a $\mathcal{D}$ -field if $X_x \in \mathcal{D}_x$ for every $x \in M$. We use the notation $X \in \mathcal{D}$ for vectors as well as for $\mathcal{D}$ -fields. We say that the distribution $\mathcal{D}$ is nondegenerate if h is nondegenerate on $\mathcal{D}$.

Recall that a $(2n + 1)$ -dimensional manifold M is said to have an almost contact structure if there exist on M a tensor field φ of type $(1,1)$, a vector field ξ and a 1-form η which satisfy

$$\varphi^2(X) = -X + \eta(X)\xi, \quad (3.1)$$

$$\eta(\xi) = 1 \quad (3.2)$$

for every $X \in TM$.

Let $f: M \rightarrow \mathbb{R}^{2n+2}$ be a nondegenerate hypersurface with a J -tangent transversal vector field C . Then we can define a vector field ξ , a 1-form η and a tensor field φ of type $(1,1)$ as follows:

$$\xi := JC, \quad (3.3)$$

$$\eta|_{\mathcal{D}} = 0 \text{ and } \eta(\xi) = 1, \quad (3.4)$$

$$\varphi|_{\mathcal{D}} = J|_{\mathcal{D}} \text{ and } \varphi(\xi) = 0. \quad (3.5)$$

It is easy to see that (φ, ξ, η) is an almost contact structure on M . This structure is called the almost contact structure on M induced by C (or simply induced almost contact structure).

For an induced almost contact structure we have the following theorem

Theorem 3.1 ([3]). *If (φ, ξ, η) is the induced almost contact structure on M then the following equations hold:*

$$\eta(\nabla_X Y) = -h(X, \varphi Y) + X(\eta(Y)) + \eta(Y)\tau(X), \quad (3.6)$$

$$\varphi(\nabla_X Y) = \nabla_X \varphi Y + \eta(Y)SX - h(X, Y)\xi, \quad (3.7)$$

$$\begin{aligned} \eta([X, Y]) &= -h(X, \varphi Y) + h(Y, \varphi X) + X(\eta(Y)) - Y(\eta(X)) \\ &\quad + \eta(Y)\tau(X) - \eta(X)\tau(Y), \end{aligned} \quad (3.8)$$

$$\varphi([X, Y]) = \nabla_X \varphi Y - \nabla_Y \varphi X - \eta(X)SY + \eta(Y)SX, \quad (3.9)$$

$$\eta(\nabla_X \xi) = \tau(X), \quad (3.10)$$

$$\eta(SX) = h(X, \xi) \quad (3.11)$$

for every $X, Y \in \mathcal{X}(M)$.

4. J -Tangent Affine Hyperspheres

An affine hypersphere with a transversal J -tangent Blaschke normal field we call a J -tangent affine hypersphere.

It is obvious that the standard hypersphere $S^{2n+1}(r)$ in $\mathbb{R}^{2n+2}$

$$x_1^2 + x_2^2 + \dots + x_{2n+2}^2 = r^2$$

is a J -tangent affine hypersphere, since it is an affine hypersphere and the affine normal field is orthogonal to it. The next example shows that there are also other J -tangent affine hyperspheres:

Example 4.1 ([4]). Let us consider the affine immersion f defined as follows:

$$f: \mathbb{R}^3 \ni (x, y, z) \mapsto \begin{bmatrix} \sin x & \sinh y \\ -\cos x & \sinh y \\ \cos x & \cosh y \\ \sin x & \cosh y \end{bmatrix} \cos z + \begin{bmatrix} \cos x & \cosh y \\ \sin x & \cosh y \\ -\sin x & \sinh y \\ \cos x & \sinh y \end{bmatrix} \sin z \in \mathbb{R}^4$$

with the transversal vector field

$$C: \mathbb{R}^3 \ni (x, y, z) \mapsto -f(x, y, z) \in \mathbb{R}^4.$$

f is a J -tangent affine hypersphere, since C is the affine normal field (what can be shown by straightforward computations) and $JC = f_z \in f_*TM$. Moreover, in the canonical coordinates on $\mathbb{R}^3$ we have

$$h = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, h is not positive definite.

As an immediate consequence of Theorem 3.1 we have the following:

Theorem 4.1. *There are no improper J -tangent affine hyperspheres.*

Proof. From Theorem 3.1 (formula (3.11)) we have $\eta(SX) = h(X, \xi)$ for all $X \in \mathcal{X}(M)$. Thus, if $S = 0$ then, $h(X, \xi) = 0$ for every $X \in \mathcal{X}(M)$, which contradicts nondegeneracy of h (since $\xi \neq 0$). $\square$

Now we can state the main result of this paper:

Theorem 4.2. *Let $f: M \mapsto \mathbb{R}^4$ be a J -tangent affine hypersphere with involutive distribution $\mathcal{D}$. Then f can be locally expressed in the form:*

$$\begin{aligned} f(x, y, z) = & \lambda^{-\frac{5}{8}} \begin{bmatrix} \sin \sqrt{\lambda}x & \sinh \sqrt{\lambda}y \\ -\cos \sqrt{\lambda}x & \sinh \sqrt{\lambda}y \\ \cos \sqrt{\lambda}x & \cosh \sqrt{\lambda}y \\ \sin \sqrt{\lambda}x & \cosh \sqrt{\lambda}y \end{bmatrix} \cos \lambda z \\ & + \lambda^{-\frac{5}{8}} \begin{bmatrix} \cos \sqrt{\lambda}x & \cosh \sqrt{\lambda}y \\ \sin \sqrt{\lambda}x & \cosh \sqrt{\lambda}y \\ -\sin \sqrt{\lambda}x & \sinh \sqrt{\lambda}y \\ \cos \sqrt{\lambda}x & \sinh \sqrt{\lambda}y \end{bmatrix} \sin \lambda z \in \mathbb{R}^4 \end{aligned} \quad (4.1)$$

for some $\lambda > 0$.

We split the proof of Theorem 4.2 into several lemmas.

Lemma 4.1. *For every $X, Y \in \mathcal{D}$ we have*

$$h(\varphi X, \varphi Y) = -h(X, Y).$$

Proof. Since $\mathcal{D}$ is involutive and $\ker \eta = \mathcal{D}$ we have $\eta([X, Y]) = 0$ for all $X, Y \in \mathcal{D}$. Now using the formula (3.8) we obtain

$$h(X, \varphi Y) = h(Y, \varphi X) \quad (4.2)$$

for every $\mathcal{D}$ -fields X and Y . Setting $Y := \varphi Y$ in (4.2) and using the fact that $\varphi^2 = -I$ on $\mathcal{D}$ we immediately get

$$-h(X, Y) = h(\varphi X, \varphi Y).$$

$\square$

Lemma 4.2. *For every $x \in M$ there exists a neighbourhood U of x and a $\mathcal{D}$ -field $X \in \mathcal{X}(U)$, $X \neq 0$ such that $h(X, X) = 1$, $h(\varphi X, \varphi X) = -1$, $h(X, \varphi X) = 0$.*

Proof. First observe that h is nondegenerate on $\mathcal{D}$, it means that for every $x \in M$ h_x is nondegenerate on $\mathcal{D}_x$. We will prove it by contradiction, namely, suppose there exists $x \in M$ such that h_x is degenerate on $\mathcal{D}_x$. Now, we can find $w \in \mathcal{D}_x$, $w \neq 0$ such that $h_x(v, w) = 0$ for every $v \in \mathcal{D}_x$. From formula (3.11) we also have $h_x(\xi_x, w) = 0$. Since every vector $t \in T_x M$ can be expressed in the form

$$t = \alpha v + \beta \xi_x,$$

where $\alpha, \beta \in \mathbb{R}$, $v \in \mathcal{D}_x$, we obtain

$$h_x(w, t) = \alpha h_x(w, v) + \beta h_x(w, \xi_x) = 0$$

for all $t \in T_x M$. We have that h_x is nondegenerate on $T_x M$ so it follows that $w = 0$, which contradicts the assumption.

Now we show that for every $x \in M$ we can find a $\mathcal{D}$ -field Z such that

$$h_x(Z_x, \varphi Z_x) \neq 0.$$

Assume that there exists $x \in M$ such that for all $Z_x \in \mathcal{D}_x$ we have

$$h_x(Z_x, \varphi Z_x) = 0.$$

Then, for any $v, w \in \mathcal{D}_x$ we have:

$$\begin{aligned} 0 &= h_x(v + w, \varphi v + \varphi w) = h_x(v, \varphi v) + h_x(w, \varphi v) + h_x(v, \varphi w) \\ &\quad + h_x(w, \varphi w) = h_x(w, \varphi v) + h_x(v, \varphi w). \end{aligned}$$

Applying Lemma 4.1 we obtain

$$h_x(v, w) = 0$$

for all $v, w \in \mathcal{D}_x$, which contradicts nondegeneracy of h on $\mathcal{D}$. Let x be an arbitrary point of M and let $Z \in \mathcal{D}$ be such that $h_x(Z_x, \varphi Z_x) \neq 0$. Then there exists a neighbourhood U of x such that $h(Z, \varphi Z) \neq 0$ on U . Without loss of generality we can assume that $h(Z, \varphi Z) > 0$ on U (if $h(Z, \varphi Z) < 0$ we can replace Z by φZ). Now, we can define another vector field Y by the formula

$$Y := \alpha Z + \beta \varphi Z,$$

where

$$\alpha = \sqrt{\sqrt{h(Z, Z)^2 + h(Z, \varphi Z)^2} + h(Z, Z)}$$

and

$$\beta = \sqrt{\sqrt{h(Z, Z)^2 + h(Z, \varphi Z)^2} - h(Z, Z)}.$$

It is obvious that α and β are smooth and positive functions on U . Moreover

$$\begin{aligned} h(Y, \varphi Y) &= (\alpha^2 - \beta^2)h(Z, \varphi Z) - 2\alpha\beta h(Z, Z) \\ &= 2h(Z, Z)h(Z, \varphi Z) - 2h(Z, \varphi Z)h(Z, Z) = 0 \end{aligned}$$

and

$$\begin{aligned} h(Y, Y) &= \alpha^2 h(Z, Z) + \beta^2 h(\varphi Z, \varphi Z) + 2\alpha\beta h(Z, \varphi Z) \\ &= 2(h(Z, Z))^2 + 2\alpha\beta h(Z, \varphi Z) > 0 \end{aligned}$$

since α, β and $h(Z, \varphi Z)$ are positive functions on U . It is easy to verify that $X := \frac{Y}{\sqrt{h(Y, Y)}}$ has the required properties. $\square$

Lemma 4.3. For X from Lemma 4.2 the following equalities hold:

$$\begin{aligned} \nabla_\xi X &= -\lambda\varphi X, \quad \nabla_\xi \varphi X = \lambda X, \quad \nabla_\xi \xi = 0, \quad \nabla_X \xi = -\lambda\varphi X, \\ \nabla_{\varphi X} \xi &= \lambda X, \quad \nabla_X X = 0, \quad \nabla_X \varphi X = \xi, \quad \nabla_{\varphi X} \varphi X = 0, \quad \nabla_{\varphi X} X = \xi, \end{aligned}$$

where λ is some positive constant.

Proof. From Theorem 3.1 we easily get

$$\nabla_\xi X, \nabla_\xi \xi, \nabla_X \xi, \nabla_{\varphi X} \xi, \nabla_X X, \nabla_{\varphi X} \varphi X \in \mathcal{D}.$$

Since $\dim \mathcal{D} = 2$, there are two smooth functions α, β defined on U such that

$$\nabla_\xi X = \alpha X + \beta \varphi X. \quad (4.3)$$

Now (3.7) implies

$$\nabla_\xi \varphi X = \alpha \varphi X - \beta X. \quad (4.4)$$

Moreover (3.7) and (3.11) imply

$$\nabla_\xi \xi = 0.$$

Since f is an affine hypersphere, we have $S = \lambda I$, where λ is a constant. We can assume that $\lambda > 0$ (otherwise we can change the sign of the Blaschke normal field). Let ω_h be the volume form for h . Then (since f is a Blaschke hypersurface) we have in particular

$$\begin{aligned} 0 &= (\nabla_\xi \omega_h)(X, \varphi X, \xi) = \xi(\omega_h(X, \varphi X, \xi)) - \omega_h(\nabla_\xi X, \varphi X, \xi) - \omega_h(X, \nabla_\xi \varphi X, \xi) \\ &\quad - \omega_h(X, \varphi X, \nabla_\xi \xi) = -\omega_h(\nabla_\xi X, \varphi X, \xi) - \omega_h(X, \nabla_\xi \varphi X, \xi) \end{aligned}$$

because $\nabla_\xi \xi = 0$ and $\omega_h(X, \varphi X, \xi) = \sqrt{\lambda} = \text{const.}$ Now, using (4.3) and (4.4), we obtain

$$\begin{aligned} (\nabla_\xi \omega_h)(X, \varphi X, \xi) &= -\omega_h(\alpha X, \varphi X, \xi) - \omega_h(\beta \varphi X, \varphi X, \xi) \\ &\quad - \omega_h(X, \alpha \varphi X, \xi) + \omega_h(X, \beta X, \xi) = -2\alpha\sqrt{\lambda}. \end{aligned}$$

Thus $\alpha = 0$.

We also have

$$\nabla_X \xi = -\varphi SX = -\lambda\varphi X$$

and

$$\nabla_{\varphi X} \xi = \lambda X.$$

From the Coddazi equation for h (2.8) we have

$$\begin{aligned}(\nabla_\xi h)(X, \varphi X) &= -h(\nabla_\xi X, \varphi X) - h(X, \nabla_\xi \varphi X) \\ &= -h(\nabla_\xi X, \varphi X) - h(\xi, \nabla_X \varphi X) = (\nabla_X h)(\xi, \varphi X),\end{aligned}$$

so $h(\xi, \nabla_X \varphi X) = -\lambda - 2\beta$. Again from (3.7) we get

$$\nabla_X \varphi X = \varphi(\nabla_X X) + \xi.$$

Thus $-\lambda - 2\beta = h(\xi, \nabla_X \varphi X) = h(\xi, \varphi(\nabla_X X) + \xi) = \lambda$, that is, $\beta = -\lambda$. Summarizing the above consideration, we obtain

$$\nabla_\xi X = -\lambda \varphi X, \quad \nabla_\xi \varphi X = \lambda X, \quad \nabla_\xi \xi = 0, \quad \nabla_X \xi = -\lambda \varphi X, \quad \nabla_{\varphi X} \xi = \lambda X.$$

Since $\nabla_X X \in \mathcal{D}$ and $\nabla_{\varphi X} \varphi X \in \mathcal{D}$, we have

$$\nabla_X X = pX + q\varphi X, \quad \nabla_{\varphi X} \varphi X = aX + b\varphi X,$$

where p, q, a, b are smooth functions on U . Hence

$$\begin{aligned}\nabla_X \varphi X &= \varphi(\nabla_X X) + \xi = p\varphi X - qX + \xi, \\ \nabla_{\varphi X} X &= \xi - \varphi(\nabla_{\varphi X} \varphi X) = \xi - a\varphi X + bX.\end{aligned}$$

Now, using the fact that

$$(\nabla_X \omega_h)(X, \varphi X, \xi) = 0,$$

we get $p = 0$. Similarly from

$$(\nabla_{\varphi X} \omega_h)(X, \varphi X, \xi) = 0$$

we get $b = 0$. Again using the Coddazi equation for h (2.8) we have

$$(\nabla_X h)(X, \varphi X) = (\nabla_{\varphi X} h)(X, X).$$

Thus $q = 0$. In a similar way, using the equality

$$(\nabla_{\varphi X} h)(\varphi X, X) = (\nabla_X h)(\varphi X, \varphi X),$$

we can show that $a = 0$. Thus we have

$$\nabla_X X = 0, \quad \nabla_{\varphi X} \varphi X = 0, \quad \nabla_X \varphi X = \xi, \quad \nabla_{\varphi X} X = \xi.$$

The proof of lemma is completed. $\square$

Now we can return to the proof of the main theorem

Proof of Theorem 4.2. From Lemma 4.3, since ∇ is a torsion free connection, we immediately get

$$[\xi, X] = [\xi, \varphi X] = [X, \varphi X] = 0.$$

Now, Frobenius' theorem implies that there exists a local coordinates system (x, y, z) on U such that $\frac{\partial}{\partial x} = X$, $\frac{\partial}{\partial y} = \varphi X$, $\frac{\partial}{\partial z} = \xi$. In these coordinates f satisfies the following differential equations:

$$f_{xx} = C = -\lambda f = -J\xi = -Jf_z, \quad (4.5)$$

$$f_{xy} = f_z, \quad (4.6)$$

$$f_{xz} = -\lambda f_y, \quad (4.7)$$

$$f_{yy} = -C = \lambda f = Jf_z, \quad (4.8)$$

$$f_{yz} = \lambda f_x, \quad (4.9)$$

$$f_{zz} = -\lambda Jf_z = -\lambda^2 f. \quad (4.10)$$

The Eq. (4.9) can be easily obtained from (4.7) and (4.10). Moreover the equation (4.6) can be determined from the remaining, as well.

From (4.10) we get

$$f(x, y, z) = c_1(x, y) \cos \lambda z + c_2(x, y) \sin \lambda z,$$

where c_1, c_2 are smooth functions with values in $\mathbb{R}^4$. Now, from (4.5), (4.7) and (4.8) we obtain

$$\begin{cases} c_{1xx} = -\lambda c_1 \\ c_{1yy} = \lambda c_1 \\ Jc_{1x} = c_{1y} \\ Jc_2 = c_1 \end{cases}$$

Solving this system of equations we get

$$c_1(x, y) = (ae^{\sqrt{\lambda}y} + be^{-\sqrt{\lambda}y}) \cos \sqrt{\lambda}x + (-Jae^{\sqrt{\lambda}y} + Jbe^{-\sqrt{\lambda}y}) \sin \sqrt{\lambda}x$$

and $c_2(x, y) = -Jc_1(x, y)$, where $a = (a_1, a_2, a_3, a_4)^T$, $b = (b_1, b_2, b_3, b_4)^T \in \mathbb{R}^4$. Since f must be an affine hypersphere with $S = \lambda I$, the affine normal field C must have a form $C = -\lambda f$. It is obvious that $\tau = 0$. By straightforward computations we obtain:

$$\theta\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = 4\lambda^3 \det[a \, b \, J a \, J b] \quad \text{and} \quad \omega_h\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \sqrt{\lambda},$$

so f is an affine hypersphere if and only if

$$\det[a \, b \, J a \, J b] = \frac{1}{4} \lambda^{-\frac{5}{2}}.$$

Now, it is sufficient to find an affine J -invariant transformation A such that $\det A = 1$. Let

$$A = \lambda^{\frac{5}{8}} \begin{bmatrix} a_3 + b_3 & -a_1 + b_1 & a_1 + b_1 & a_3 - b_3 \\ a_4 + b_4 & -a_2 + b_2 & a_2 + b_2 & a_4 - b_4 \\ -a_1 - b_1 & -a_3 + b_3 & a_3 + b_3 & -a_1 + b_1 \\ -a_2 - b_2 & -a_4 + b_4 & a_4 + b_4 & -a_2 + b_2 \end{bmatrix}.$$

A is J -invariant and $\det A = 4\lambda^{\frac{5}{2}} \det[a \, b \, Ja \, Jb] = 1$. It is not difficult to verify that $A^{-1} \circ f$ has the form (4.1), what completes the proof of the theorem. $\square$

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References

- [1] Cruceanu, V.: Real hypersurfaces in complex centro-affine spaces. Results Math. **13**, 224–234 (1988)
- [2] Nomizu, K., Sasaki, T.: Affine Differential Geometry. Cambridge University Press, Cambridge (1994)
- [3] Szancer, M., Szancer, Z.: Real hypersurfaces with an induced almost contact structure. Colloquium Math. **114**, 41–51 (2009)
- [4] Szancer, Z.: Real hypersurfaces with a special transversal vector field. Ann. Polonici Math. **105.3**, 239–252 (2012)

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