Symmetric 2-Structures, a Classification

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Dedicated to Momme Thomsen on the occasion of his 70th birthday

Abstract. We classify *symmetric 2-structures* $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$, i.e. chain structures which correspond to sharply 2-transitive permutation sets (E, Σ) satisfying the condition: “ $(*) \forall \sigma, \tau \in \Sigma : \sigma \circ \tau^{-1} \circ \sigma \in \Sigma$ ”. To every chain $K \in \mathfrak{K}$ one can associate a reflection $\tilde{K}$ in K . Then $(*)$ is equivalent to “ $(**) \forall K \in \mathfrak{K} : \tilde{K}(\mathfrak{K}) = \mathfrak{K}$ ” and one can define an orthogonality “ $\perp$ ” for chains $K, L \in \mathfrak{K}$ by “ $K \perp L \Leftrightarrow K \neq L \wedge \tilde{K}(L) = L$ ”. The classification is based on the cardinality of the set of chains which are orthogonal to a chain K and passing through a point p of K . For one of these classes (called *point symmetric 2-structures*) we prove that in each point there is a reflection and that the set of point reflections forms a regular involutory permutation set.

Mathematics Subject Classification (2000). 51B20(2000).

Keywords. Net, 2-structure, symmetric chain structure, midpoint.

1. Introduction

Let S be a non empty set, $P := S \times S$ the product set and let $\mathfrak{G}_1 := \{S \times \{x\} \mid x \in S\}$ and $\mathfrak{G}_2 := \{\{x\} \times S \mid x \in S\}$ be the sets of *generators*. Then $(P, \mathfrak{G}_1, \mathfrak{G}_2)$ is a *net*, i.e. $\forall p \in P, \forall i \in \{1, 2\} \exists_1 X \in \mathfrak{G}_i$ such that $p \in X$ (which we denote by $[p]_i := X$) and if $X \in \mathfrak{G}_1, Y \in \mathfrak{G}_2$ then $X \cap Y \neq \emptyset$. A subset $C \subset P$ is called a *chain* if $\forall X \in \mathfrak{G}_1 \cup \mathfrak{G}_2 : |C \cap X| = 1$. Let $\mathfrak{C}$ be the set of all chains of the net $(P, \mathfrak{G}_1, \mathfrak{G}_2)$, for $a, b \in P$ let $a \square b := c$ with $\{c\} = [a]_1 \cap [b]_2$ (we will also write $ab := a \square b$). Let $P^{(2)} := \{(a, b) \in P^2 \mid a \square b \neq a, b\}$ and if $A, B, C \in \mathfrak{C}$ let $\widetilde{AB} : P \rightarrow P ; x \mapsto y$ with $\{y\} = [[x]_2 \cap B]_1 \cap [[x]_1 \cap A]_2$ and $\tilde{C} := \widetilde{CC}$ (we also denote $[X]_i := \bigcup \{[x]_i \mid x \in X\}$ for $i \in \{1, 2\}$ and $X \subset P$).

Then $\tilde{C}$ is an involutory antiautomorphism of $(P, \square)$ mapping each chain onto a chain and if $D \in \mathfrak{C}$ is a further chain then the product $\tilde{C} \circ \tilde{D}$ is

an automorphism of $(P, \square)$. Two distinct chains A, B are called *orthogonal*, denoted by $A \perp B$ if $\tilde{A}(B) = B$. We have “ $A \perp B \Rightarrow B \perp A \Rightarrow \tilde{A} \circ \tilde{B} = \tilde{B} \circ \tilde{A}$ ”. If $\mathfrak{K} \subseteq \mathfrak{C}$ then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *chain structure* and the set $\mathfrak{K}^s := \{C \in \mathfrak{C} \mid \tilde{C}(\mathfrak{K}) = \mathfrak{K}\}$ the *symmetric stabilizer* of $\mathfrak{K}$. A chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *symmetric chain structure* if $\forall K \in \mathfrak{K} : \tilde{K}(\mathfrak{K}) = \mathfrak{K}$, i.e. if $\mathfrak{K} \subseteq \mathfrak{K}^s$ and *symmetric closed* if $\mathfrak{K} = \mathfrak{K}^s$, see [1–7].

A chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *2-structure* if $\forall (a, b) \in P^{(2)} : \exists_1 K \in \mathfrak{K} : a, b \in K$ - we set $\overline{a, b} := K$ -, or in other words, if $(P, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K})$ is an *incidence space*, i.e. for any two distinct points $a, b \in P$ there is exactly one block $B \in \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K}$ such that $a, b \in B$. If $(a, b) \in P^{(2)}$ then also $(a \square b, b \square a) \in P^{(2)}$ and we can form the chains $\overline{a, b}$ and $M(a, b) := \overline{a \square b, b \square a}$ called the *diagonals* of the *rectangle* $\square(a, b) := (a, a \square b, b, b \square a)$. For $M := M(a, b)$ we have $\tilde{M}(a) = b$ and therefore we call $M(a, b)$ the *midline* of a and b .

For a chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ let $\mathfrak{K}^{2\perp}$ denote the set of all pairs of orthogonal chains of $\mathfrak{K}$, $\mathfrak{K}_0^{2\perp} := \{(A, B) \in \mathfrak{K}^{2\perp} \mid A \cap B = \emptyset\}$, $\mathfrak{K}_1^{2\perp} := \{(A, B) \in \mathfrak{K}^{2\perp} \mid |A \cap B| = 1\}$, and if $p \in P$ and $K \in \mathfrak{K}$ let $(p \perp K) := \{X \in \mathfrak{K} \mid p \in X \wedge X \perp K\}$.

In this paper we consider symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$, i.e. the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is at the same time symmetric and a 2-structure. Let $\Gamma^+(\mathfrak{K}) := \text{Aut}(P, \square, \mathfrak{K})$. Then we can state:

- Theorem 1.1.** 1. The set $\mathfrak{K}^+ := \{\tilde{A} \circ \tilde{B} \mid (A, B) \in \mathfrak{K}^{2\perp}\}$ consists of involutory automorphisms of $\Gamma^+(\mathfrak{K})$, is invariant, i.e. $\forall \alpha \in \mathfrak{K}^+ : \alpha \circ \mathfrak{K}^+ \circ \alpha = \mathfrak{K}^+$, and splits into the two subsets $\mathfrak{K}_1^+ := \{\gamma \in \mathfrak{K}^+ \mid \text{Fix } \gamma \neq \emptyset\} = \{\tilde{A} \circ \tilde{B} \mid (A, B) \in \mathfrak{K}_1^{2\perp}\}$ and $\mathfrak{K}_0^+ := \{\gamma \in \mathfrak{K}^+ \mid \text{Fix } \gamma = \emptyset\} = \{\tilde{A} \circ \tilde{B} \mid (A, B) \in \mathfrak{K}_0^{2\perp}\}$. Each $\gamma \in \mathfrak{K}_1^+$ is a reflection in a point, i.e. γ is involutory, $\text{Fix } \gamma$ consists of a single point $c \in P$ and $\gamma(B) = B$ for $B \in (\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K})$ with $c \in B$ [cf. Theorem 3.1.(3)].
2. $\forall (a, b) \in P^{(2)}$ there is a $\gamma \in \mathfrak{K}^+$ with $\gamma(a) = b$ and the group $\langle \mathfrak{K}^+ \rangle$ generated by the set $\mathfrak{K}^+$ acts transitively on P (cf. Theorem 3.10).
3. $\forall p \in P, \forall K \in \mathfrak{K}$ with $p \notin K : |(p \perp K)| = 1$ [cf. Theorem 3.1.(5)].

Considering pairs $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ we will show (cf. Theorem 3.11.):

Theorem 1.2. The symmetric 2-structures split into the three classes:

- (I) There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| > 1$. Then $\mathfrak{K}_1^+ \neq \emptyset$, $|(x \perp X)| > 1$ to each $x \in P$, $X \in \mathfrak{K}$ with $x \in X$ and there are several reflections in the point x . (If $A \in (x \perp X)$ then $\tilde{A} \circ \tilde{X}$ is a reflection in x) (cf. Theorem 3.12.). For symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ of class (I) we have the bifurcation:
- (Ia) $\mathfrak{K}_1^+ \neq \mathfrak{K}^+$.
- (Ib) $\mathfrak{K}_1^+ = \mathfrak{K}^+$.

- (II) *There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K : (p \perp K) = \emptyset$. Then for each pair $(x, X) \in P \times \mathfrak{K}$ with $x \in X : (x \perp X) = \emptyset$ and each element of $\mathfrak{K}^+$ is fixed point free hence $\mathfrak{K}_1^+ = \emptyset$ (cf. Theorem 3.13.).*
- (III) *There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| = 1$. Then for each pair $(x, X) \in P \times \mathfrak{K}$ with $x \in X : |(x \perp X)| = 1$ and in $\mathfrak{K}^+$ there is exactly one reflection in the point x which we denote by $\tilde{x}$ and so $\mathfrak{K}_1^+ \neq \emptyset$ (Theorem 3.14.).*
- For symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ of class (III) we set $\tilde{P} := \{\tilde{p} \mid p \in P\}$. Then $\tilde{P} = \mathfrak{K}_1^+$ and we have the bifurcation:*
- (IIIa) $\tilde{P} \neq \mathfrak{K}^+$.
- (IIIb) $\tilde{P} = \mathfrak{K}^+$.

If $a, b \in P$ are two given points then a point m is called *midpoint* of a and b if there is $\gamma \in \mathfrak{K}_1^+$ with $\gamma(a) = b$ and $\text{Fix}\gamma = \{m\}$.

Theorem 1.3. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure and let $a, b \in P$. Then:*

1. *If $\mathfrak{K}_1^+ = \emptyset$ then a and b do not have a midpoint.*
2. *If $(a, b) \in P^{(2)}$ then a and b have at most one midpoint and there exists a midpoint if and only if $\overline{a, b} \cap M(a, b) \neq \emptyset$. If $\{m\} := \overline{a, b} \cap M(a, b) \neq \emptyset$ then m is the midpoint of a and b (cf. Theorem 3.16.).*
3. *If $\mathfrak{K}_1^+ \neq \emptyset$ and if a and b are parallel points (i.e. $[a]_1 = [b]_1$ or $[a]_2 = [b]_2$) then a and b have at least one midpoint. If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class (III) then a and b have exactly one midpoint [cf. Theorem 3.17.(3)].*
4. *If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class (IIIb) or to the class (III) and is symmetrically closed then there is exactly one $\gamma \in \mathfrak{K}^+$ with $\gamma(a) = b$, i.e. $(P, \mathfrak{K}^+)$ is a regular invariant involutory permutation set [cf. Corollary 3.5.(2) and Theorem 3.18.(2)].*
5. *If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a symmetric 2-structure such that $(P, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K})$ is an affine plane then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class (II) or (IIIb) (cf. Theorem 3.20.).*

2. Notations and Assumptions

2.1. Notations Concerning Chain Structures

In this paper we use the same notations as in [5] hence $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ denotes a chain structure, $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ the corresponding maximal chain structure. A subset $S \subseteq P$ is called *subsquare* if for all $x, y \in S$ $x \square y \in S$. Let $\Gamma := \Gamma^+ \cup \Gamma^-$ where Γ^+ , Γ^- denote the sets of all automorphisms and antiautomorphisms of $(P, \square)$ respectively. Γ^+ contains for $i = 1$ and $i = 2$ the following subgroups $\Gamma_i := \{\sigma \in \Gamma^+ \mid \forall x \in P : [\sigma(x)]_i = [x]_i\}$, whose elements are called *i-maps*. We set $\Gamma_3 := \Gamma^+ \setminus (\Gamma_1 \cup \Gamma_2)$ and if $\alpha \in \Gamma$, $P_\alpha := \{x \in P \mid (x, \alpha(x)) \in P^{(2)}\}$.

Moreover let $\Gamma^+(\mathfrak{K}) := \{\gamma \in \Gamma^+ \mid \gamma(\mathfrak{K}) = \mathfrak{K}\}$, $\Gamma^-(\mathfrak{K}) := \{\gamma \in \Gamma^- \mid \gamma(\mathfrak{K}) = \mathfrak{K}\}$, $\Gamma(\mathfrak{K}) := \Gamma^+(\mathfrak{K}) \cup \Gamma^-(\mathfrak{K})$ and $\Gamma_i(\mathfrak{K}) := \{\gamma \in \Gamma_i \mid \gamma(\mathfrak{K}) = \mathfrak{K}\}$ for $i = 1, 2, 3$. We have:

Theorem 2.1. $\Gamma = \Gamma(\mathfrak{C})$, $\Gamma^+ = \Gamma^+(\mathfrak{C})$ and $\Gamma^- = \Gamma^-(\mathfrak{C})$ and

1. $\forall \gamma \in \Gamma^- \exists_1 (A, B) \in \mathfrak{C}^2 : \gamma = \widetilde{AB}$.
2. $\widetilde{\mathfrak{C}} := \{\widetilde{A} \mid A \in \mathfrak{C}\}$ is the set of all involutions of Γ^- .
3. $\forall \gamma \in \Gamma^+, \forall C \in \mathfrak{C} \exists_1 (A, B) \in \mathfrak{C}^2 : \gamma = \widetilde{AB} \circ \widetilde{C}$ and we have:
 $\gamma \in \Gamma_1 \Leftrightarrow B = C$ and then $\text{Fix} \gamma = P \square (A \cap C)$,
 $\gamma \in \Gamma_2 \Leftrightarrow A = C$ and then $\text{Fix} \gamma = (B \cap C) \square P$.
4. For $\alpha \in \Gamma^+ : \alpha \in \Gamma_3 \Leftrightarrow P_\alpha \neq \emptyset$.
5. The identity and the set of all involutions of Γ^+ consists of all maps of the form $\gamma = \widetilde{AB} \circ \widetilde{C}$ with $\widetilde{C}(A) = A$, $\widetilde{C}(B) = B$. γ is the identity if and only if $A = B = C$. $\gamma \in \Gamma_1$ is a reflection in the 1-stripe $[A \cap C]_2$ if $B = C$ and $\widetilde{C}(A) = A$, $\gamma \in \Gamma_2$ is a reflection in the 2-stripe $[B \cap C]_1$ if $A = C$ and $\widetilde{C}(B) = B$ and $\gamma \in \Gamma_3$ if $A, B \neq C$ (then $\text{Fix} \gamma = [B \cap C]_1 \cap [A \cap C]_2$ is a subsquare).
6. $\widetilde{\mathfrak{C}} \circ \widetilde{\mathfrak{C}} \subseteq \Gamma^+$ and if $\gamma \in \Gamma^+$ then

$$\gamma \in \widetilde{\mathfrak{C}} \circ \widetilde{\mathfrak{C}} \Leftrightarrow \exists C \in \mathfrak{C} : \widetilde{C} \circ \gamma \circ \widetilde{C} = \gamma^{-1}.$$

If $\mathfrak{K}^s := \{C \in \mathfrak{C} \mid \widetilde{C}(\mathfrak{K}) = \mathfrak{K}\}$ denotes the symmetric stabilizer of $\mathfrak{K}$ then $\widetilde{\mathfrak{K}}^s := \{\widetilde{A} \mid A \in \mathfrak{K}^s\}$ is contained in $\Gamma^-(\mathfrak{K})$. Let $E \in \mathfrak{K}$ be a fixed chain and for $A, B \in \mathfrak{C}$ let $A \cdot B := \widetilde{AB}(E)$ - then $(\mathfrak{C}, \cdot)$ is a group. Two distinct chains $A, B \in \mathfrak{C}$ are called *orthogonal* and denoted by $A \perp B$ if $\widetilde{A}(B) = B$. We have $A \perp B \Rightarrow B \perp A$. Let $\mathfrak{K}^{2\perp} := \{(A, B) \in \mathfrak{K}^2 \mid A \perp B\}$ and $\mathfrak{K}^+ := \{\widetilde{A} \circ \widetilde{B} \mid (A, B) \in \mathfrak{K}^{2\perp}\}$. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called *symmetric chain structure* if $\mathfrak{K} \subseteq \mathfrak{K}^s$.

Theorem 2.2. If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is symmetric then $\Gamma(\mathfrak{K})$ contains the group $\langle \widetilde{\mathfrak{K}} \rangle$ generated by the chain reflections $\widetilde{K}$ with $K \in \mathfrak{K}$ and if $A, B \in \mathfrak{K}$ and $\alpha \in \Gamma(\mathfrak{K})$ then:

1. $\widetilde{A} \in \Gamma^-(\mathfrak{K})$ and $\text{Fix} \widetilde{A} = A$.
2. $\widetilde{A} \circ \widetilde{B} \in \Gamma^+(\mathfrak{K})$ and $\text{Fix}(\widetilde{A} \circ \widetilde{B}) = (A \cap B) \square (A \cap B)$.
3. $\widetilde{A} \circ \widetilde{B}$ is an involution $\Leftrightarrow A \perp B$.
4. $\widetilde{A}(B) = \widetilde{A} \circ \widetilde{B} \circ \widetilde{A}$
5. $\widetilde{\alpha(B)} = \alpha \circ \widetilde{B} \circ \alpha^{-1}$.

2.2. 2-Structures

In this part we assume that $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure, i.e. for any $(a, b) \in P^{(2)}$ there is exactly one chain $C \in \mathfrak{K}$ with $a, b \in C$ denoted by $\overline{a, b} := C$ (called *joining line*) and since also $(ab, ba) \in P^{(2)}$ we can form the *mid-line* $M(a, b) := ab, ba$. Therefore for any two distinct blocks $A, B \in \mathfrak{B} := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K}$

we have $|A \cap B| \leq 1$ and the pair $(P, \mathfrak{B})$ is an incidence space. We also denote $\overline{a, b} := [a]_i$ if $b \in [a]_i \setminus \{a\}$ for $i = 1, 2$.

The set $\mathfrak{K}^{2\perp}$ of pairs of orthogonal chains splits into the subclasses $\mathfrak{K}_0^{2\perp} := \{(A, B) \in \mathfrak{K}^{2\perp} \mid A \cap B = \emptyset\}$ and $\mathfrak{K}_1^{2\perp} := \{(A, B) \in \mathfrak{K}^{2\perp} \mid |A \cap B| = 1\}$ and moreover also the set $P^{(2)}$ of pairs of joinable points splits into $P_0^{(2)} := \{(a, b) \in P^{(2)} \mid \overline{a, b} \cap M(a, b) = \emptyset\}$ and $P_1^{(2)} := \{(a, b) \in P^{(2)} \mid |\overline{a, b} \cap M(a, b)| = 1\}$. Also $\mathfrak{K}^+$ falls into the two subsets $\mathfrak{K}_1^+ := \{\widetilde{A} \circ \widetilde{B} \mid (A, B) \in \mathfrak{K}_1^{2\perp}\}$ and $\mathfrak{K}_0^+ := \{\widetilde{A} \circ \widetilde{B} \mid (A, B) \in \mathfrak{K}_0^{2\perp}\}$.

Each pair $(a, b) \in P^{(2)}$ determines a *rectangle* $\square(a, b) := (a, ab, b, ba)$ with the *diagonals* $\overline{a, b}$ and $M(a, b)$ and the map $\widetilde{a, b} := \widetilde{M(a, b)} \circ \widetilde{\overline{a, b}}$ of Γ^+ interchanging the points a and b and also ab and ba . We have $|\widetilde{Fix a, b}| = 0$ if $(a, b) \in P_0^{(2)}$ and $|\widetilde{Fix a, b}| = 1$ if $(a, b) \in P_1^{(2)}$. If $(a, b) \in P_1^{(2)}$ and $\{d\} = \overline{a, b} \cap M(a, b)$ then d is called the *midpoint* of (a, b) and of the rectangle $\square(a, b)$ and we have $\widetilde{Fix a, b} = \{d\}$.

Definition 2.3. A collineation $\alpha \in \text{Aut}(P, \mathfrak{B})$ is called *point reflection* if α is involutory, has exactly one fixed point p and if for $B \in \mathfrak{B}$ with $p \in B$, $\alpha(B) = B$.

It follows that each point reflection is an element of $\Gamma^+(\mathfrak{K})$.

Theorem 2.4. Let $\alpha \in \Gamma(\mathfrak{K})$ be a reflection in a point o , $a \in P \setminus \{o\}$ and $X \in \mathfrak{B}$. Then:

1. α is completely determined by the points o, a and $a' := \alpha(a)$,
2. $o \in X \Leftrightarrow \alpha(X) = X$.

Proof. (1) There is an $i \in \{1, 2\}$ such that $a \notin [o]_i$ for instance $a \notin [o]_1$.

Then for any $x \in [a]_1$ and $x' := \alpha(x)$ we have $\{x'\} = \overline{o, x} \cap [a']_1$. Now let $p \in P \setminus [o]_2$, let $\{p_2\} := [p]_2 \cap [a]_1$ then $\{\alpha(p)\} = \overline{o, p} \cap [p'_2]_2$ if $p \notin [o]_1$ and $\{\alpha(p)\} = [o]_1 \cap [p'_2]_2$ if $p \in [o]_1$. If $p \in [o]_2$ we choose $q \in [p]_1 \setminus \{p\}$ then $\{\alpha(p)\} = [o]_2 \cap [\alpha(q)]_1$.

- (2) “ $\Leftarrow$ ” Conversely, suppose that $o \notin X$. Then $X \cap [o]_i \neq \emptyset$ for $i = 1$ or $i = 2$.

We get a contradiction with $\widetilde{Fix \alpha} = \{o\}$. $\square$

3. Symmetric 2-Structures

In this section let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure. Then by Theorem 2.2., the group generated by the line reflections $\widetilde{L}$ with $L \in \mathfrak{K}$ is contained in the group $\Gamma(\mathfrak{K})$.

3.1. Involutions

We have:

Theorem 3.1. Let $(a, b) \in P^{(2)}$, $C := \overline{a, b}$ and $D := M(a, b)$ then:

1. C and D are the diagonals of the rectangle $\square(a, b)$ and we have $C \perp D$.

2. $\widetilde{a, b}$ is an involution of $\Gamma^+(\mathfrak{K})$ interchanging the points a and b as well as ab and ba and with $\text{Fix} \widetilde{a, b} = C \cap D$.
3. If $m \in C \cap D \neq \emptyset$ then $\widetilde{a, b}$ is a point reflection in m .
4. $\mathfrak{K}^+ = \{\widetilde{a, b} \mid (a, b) \in P^{(2)}\}$, $\mathfrak{K}_0^+ = \{\widetilde{a, b} \mid (a, b) \in P_0^{(2)}\}$, $\mathfrak{K}_1^+ = \{\widetilde{a, b} \mid (a, b) \in P_1^{(2)}\}$.
5. For $K \in \mathfrak{K}$ and $p \in P \setminus K$ we have $(p \perp K) = \overline{\{p, \widetilde{K}(p)\}}$.

Proof. (1) $\widetilde{C}$ interchanges the two distinct points ab and ba and since $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is symmetric $\widetilde{C}$ is contained in $\Gamma^-(\mathfrak{K})$. Therefore $\widetilde{C}(D)$ is a chain of $\mathfrak{K}$ containing ab and ba and since $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure we obtain $\widetilde{C}(D) = D$ hence $C \perp D$.

- (2) By (1) and Theorem 2.2.(4), $\widetilde{D} = \widetilde{\widetilde{C}(D)} = \widetilde{C} \circ \widetilde{D} \circ \widetilde{C}$ hence $\widetilde{a, b} := \widetilde{C} \circ \widetilde{D}$ is an involution interchanging the points a, b and the points ab, ba . By Theorem 2.1.(2), $\text{Fix} \widetilde{a, b} = C \cap D$.
- (3) We have only to show: if $B \in \mathfrak{B}$ with $m \in B$ then $\widetilde{a, b}(B) = B$. Let $p \in B \setminus \{m\}$ then by (2), p and $p' := \widetilde{a, b}(p)$ are two distinct points and so since $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure, there is exactly one block $A \in \mathfrak{B}$ with $p, p' \in A$. Since $\widetilde{a, b}$ is an involutory collineation interchanging p and p' , A is a fixed block of $\widetilde{a, b}$. Since $\widetilde{a, b} \in \Gamma^+$ and $m \in \text{Fix} \widetilde{a, b}$ we have $\widetilde{a, b}([m]_i) = [m]_i$ (for $i = 1$ and $i = 2$). Therefore at least one of these generators is distinct of A , for instance $A \neq [m]_1$. Then $A \cap [m]_1 \subset \text{Fix} \widetilde{a, b}$ hence by (2) this is $\{m\}$ and so $m \in A$ showing $A = B$.
- (4) Follows from (1). (5) Analogously to (1). □

From Theorem 2.2, we obtain:

Theorem 3.2. Let $\sigma \in \Gamma(\mathfrak{K})$ and $(a, b) \in P^{(2)}$ then:

1. $(\sigma(a), \sigma(b)) \in P^{(2)}$
2. $\sigma(\widetilde{a, b}) = \overline{\sigma(a), \sigma(b)}$ and $\widetilde{\sigma(a), \sigma(b)} = \sigma \circ \widetilde{a, b} \circ \sigma^{-1}$.
3. $\sigma(M(a, b)) = M(\sigma(a), \sigma(b))$.
4. $(\widetilde{\sigma(a), \sigma(b)}) \in P_1^{(2)} \Leftrightarrow (a, b) \in P_1^{(2)}$.
5. $\sigma(a), \sigma(b) = \sigma \circ \widetilde{a, b} \circ \sigma^{-1}$ and $\text{Fix } \widetilde{\sigma(a), \sigma(b)} = \sigma(\text{Fix } \widetilde{a, b})$.
6. $\mathfrak{K}^+, \mathfrak{K}_1^+$ and $\mathfrak{K}_0^+$ are invariant sets of involutions and even the normalizers of each of these sets contain $\Gamma(\mathfrak{K})$.

Theorem 3.3. Let $(a, b) \in P^{(2)}$ and $K \in \mathfrak{K}$ with $\widetilde{a, b}(K) = K$ then there is exactly one $L \in \mathfrak{K}^s$ with $\widetilde{a, b} = \widetilde{K} \circ \widetilde{L}$. If $(a, b) \in P_1^{(2)}$ or if $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is symmetrically closed then we have additionally $L \in \mathfrak{K}$.

Proof. $\widetilde{a, b}(K) = K$ implies $\widetilde{K} = \widetilde{\widetilde{a, b}(K)} = \widetilde{a, b} \circ \widetilde{K} \circ \widetilde{a, b}$ Therefore $\alpha := \widetilde{K} \circ \widetilde{a, b}$ is an involution of $\Gamma^-(\mathfrak{K})$, i.e. there is an $L \in \mathfrak{K}^s$ with $\alpha = \widetilde{L}$ and so $\widetilde{a, b} = \widetilde{K} \circ \widetilde{L}$.

Let $(a, b) \in P_1^{(2)}$. Then, by Theorem 3.1.(3), $\widetilde{a, b}$ is a reflection in a point m and $m \in K$. Let $p \in K \setminus \{m\}$, $q := \widetilde{a, b}(p)$ and $L' := \overline{m, pq}$. Then $\widetilde{a, b}(L') = L'$ and $\widetilde{a, b}(pq) = qp$, hence $qp \in L'$. By Theorem 3.1.(1),(3) we have $L' \perp K$ and $\widetilde{L'} \circ \widetilde{K} = \widetilde{p, q}$ is a point reflection. By Theorem 2.4.(1) $\widetilde{L'} \circ \widetilde{K} = \widetilde{L} \circ \widetilde{K}$ hence $L = L' \in \mathfrak{K}$. $\square$

In the above proof we can replace “ $\widetilde{a, b} \in \mathfrak{K}_1^+$ ” by “ α is a point reflection” and we get:

Theorem 3.4. *Let α be a point reflection, $K \in \mathfrak{K}$ and $\alpha(K) = K$. Then there is exactly one $L \in \mathfrak{K}$ with $\alpha = \widetilde{K} \circ \widetilde{L}$.*

Corollary 3.5. 1. $\mathfrak{K}_1^+$ coincides with the set of point reflections.

2. $\forall (a, b) \in P_1^{(2)}$ there is exactly one $\gamma \in \mathfrak{K}_1^+$ with $\gamma(a) = b$.

3. Let α be a reflection in a point p and $K \in \mathfrak{K}$ then

$p \in K \Leftrightarrow \widetilde{K} \circ \alpha$ is involutory.

4. Let $p \in P$ and $K, L \in \mathfrak{K}$ with $p \in K, L$ then $|(p \perp K)| = |(p \perp L)|$ is the cardinality of the reflections in the point p .

Proof. (1) Follows from Theorem 3.1.(3) and Theorem 3.4.

(2) Let $X = \overline{a, b}$, $Y = M(a, b)$ and $\{p\} = X \cap Y = \text{Fix} \widetilde{a, b}$. Assume $\gamma'(a) = b$ for any $\gamma' = \widetilde{C} \circ \widetilde{D} \in \mathfrak{K}_1^+$ and $\{p'\} = \text{Fix} \gamma'$. We have $\gamma'(X) = X$, $\gamma'(Y) = Y$, hence $p' \in X, Y$ by Theorem 2.4.(2). Thus $p = p'$ and $\gamma' = \gamma$ by Theorem 2.4.(1).

(3) “ $\Rightarrow$ ” Since α is a reflection in a point p and $p \in K$ we have $\alpha(K) = K$ and so by Theorem 3.4., there is a $L \in \mathfrak{K}$ with $\alpha = \widetilde{K} \circ \widetilde{L}$ hence $\widetilde{K} \circ \alpha = \widetilde{K} \circ \widetilde{K} \circ \widetilde{L} = \widetilde{L}$ is an involution. “ $\Leftarrow$ ” By Theorem 2.2.(5), $\widetilde{\alpha(K)} = \alpha \circ \widetilde{K} \circ \alpha = \widetilde{K}$. Hence $\alpha(K) = K$ and $p \in K$ by Theorem 2.4.(2).

(4) Let $A \in (p \perp K)$ then $\alpha := \widetilde{K} \circ \widetilde{A}$ is a reflection in the point p hence $\alpha(L) = L$ and by Theorem 3.4, there is exactly one $B \in \mathfrak{K}$ with $\alpha := \widetilde{L} \circ \widetilde{B}$ and by (3), $p \in B$, i.e. $B \in (p \perp L)$. This shows that the cardinalities of $(p \perp K)$ and $(p \perp L)$ are equal. $\square$

Now we consider involutory elements α of $\Gamma(\mathfrak{K})$ and ask if they are contained in $\langle \widetilde{\mathfrak{K}} \rangle$. For each point $x \in P_\alpha$ we set: $X := \overline{x, \alpha(x)}$, $Y := \overline{x \square \alpha(x), \alpha(x) \square x}$ and we have: $\alpha(x) \in P_\alpha$, $\alpha(X) = X$, $\alpha(Y) = Y$ and $\square(x, \alpha(x))$ is a rectangle with the diagonals X and Y . Moreover $\widetilde{Y}(x) = \alpha(x)$ and $\widetilde{X}(x \square \alpha(x)) = \alpha(x) \square x$ and $X \perp Y$.

Theorem 3.6. *If $\alpha \in \Gamma^-$ then: $\exists_1 A \in \mathfrak{K}^s$ such that $\alpha = \widetilde{A}$ and if $x \in P \setminus A$ then $(x, \alpha(x)) \in P^{(2)}$ and $X \cap Y \subseteq A$.*

Proof. By Theorem 2.1.(2) there is exactly one chain $A \in \mathfrak{C}$ with $\alpha = \widetilde{A}$ and by $\widetilde{A}(\mathfrak{K}) = \alpha(\mathfrak{K}) = \mathfrak{K}$ we have $A \in \mathfrak{K}^s$. Since α fixes each of the points $x \square \alpha(x)$ and $\alpha(x) \square x$ we have $x \square \alpha(x), \alpha(x) \square x \in (A \cap Y)$. If we assume $X \cap Y \neq \emptyset$,

hence (since $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure) $\{z\} := X \cap Y$ consists of a single point, then $\{\bar{A}(z)\} = \{\alpha(z)\} = \alpha(X \cap Y) = X \cap Y = \{z\}$ hence $z \in A$. $\square$

Definition 3.7. Let $S \subset P$ be a subsquare with $\emptyset \neq S \neq P$. An involution $\alpha \in \Gamma(\mathfrak{K})$ is called *reflection in the subsquare S* if $\text{Fix } \alpha = S$ and if $\forall x \in P \setminus S : \overline{x, \alpha(x)} \cap S \neq \emptyset$.

Theorem 3.8. Let $\alpha \in \Gamma_3(\mathfrak{K})$ be an involution hence $P_\alpha \neq \emptyset$ and $\mathfrak{K}_\alpha := \{X \mid x \in P_\alpha\}$ then:

1. If $x \in P_\alpha$ then $\alpha(x \square \alpha(x)) = \alpha(x) \square x$ (i.e. α interchanges also the points $x \square \alpha(x)$ and $\alpha(x) \square x$).
2. $\forall K \in \mathfrak{K} : \alpha(K) = K \Leftrightarrow K \in \mathfrak{K}_\alpha$.
3. $\forall L \in \mathfrak{K}_\alpha : \alpha \circ \tilde{L}$ is an involution hence $\alpha \circ \tilde{L} \in \tilde{\mathfrak{C}}$ and $\exists_1 L^s \in \mathfrak{K}^s$ such that $\alpha = \tilde{L} \circ \tilde{L}^s$.
4. If $|\text{Fix } \alpha| > 1$ then $\mathfrak{K}_\alpha = \{\overline{x, y} \mid (x, y) \in \text{Fix } \alpha^{(2)}\}$.
5. If there is a $x \in P_\alpha$ such that $(x, \alpha(x)) \in P_1^{(2)}$ then $\text{Fix } \alpha \supseteq \widetilde{\text{Fix } x, \alpha(x)} \neq \emptyset$.
6. $\text{Fix } \alpha$ is a subsquare and for each $L \in \mathfrak{K}_\alpha : \tilde{L}(\text{Fix } \alpha) = \text{Fix } \alpha$, i.e. $L \cap \text{Fix } \alpha$ is a chain of $\text{Fix } \alpha$ if $\text{Fix } \alpha \neq \emptyset$.
7. P_α is the complement of the cross $[\text{Fix } \alpha]_1 \cup [\text{Fix } \alpha]_2$ hence $P_\alpha = P \setminus ([\text{Fix } \alpha]_1 \cup [\text{Fix } \alpha]_2)$.
8. If $\text{Fix } \alpha \neq \emptyset$ then α is a reflection in the subsquare $\text{Fix } \alpha$.
9. If $\text{Fix } \alpha$ is a point $p \in P$ and $x \in P_\alpha$ then α is a reflection in the point p and $\alpha = x, \alpha(x)$.
10. If α is a reflection in a point p then $\mathfrak{K}_\alpha = \{K \in \mathfrak{K} \mid \alpha(K) = K\} = \{K \in \mathfrak{K} \mid p \in K\}$.

Proof. (2) Let $K \in \mathfrak{K}$ with $\alpha(K) = K$. If $K \cap P_\alpha \neq \emptyset$ then $K \in \mathfrak{K}_\alpha$. Since $\alpha \in \Gamma^+$, $K \subseteq \text{Fix } \alpha$ would imply $\alpha = \text{id}$.

(3) Since α is an involution and $\alpha(L) = L$ we have by Theorem 2.2.(5), $\alpha \circ \tilde{L} \circ \alpha \circ \tilde{L} = \widetilde{\alpha(L)} \circ \tilde{L} = \tilde{L} \circ \tilde{L} = \text{id}$ hence $\alpha \circ \tilde{L}$ is an involution of Γ^- and by Theorem 2.1.(2) there is an $X' \in \mathfrak{C}$ with $\widetilde{X'} = \alpha \circ \tilde{L}$. Now $\alpha, \tilde{L} \in \Gamma(\mathfrak{K})$ imply $X' \in \mathfrak{K}^s$.

(4) Clearly, by (2), $\mathfrak{K}_\alpha \supseteq \{\overline{x, y} \mid (x, y) \in \text{Fix } \alpha^{(2)}\}$. Now let $A \in \mathfrak{K}_\alpha$ hence there is a point $p \in \text{Fix } \alpha \setminus A$ by $|\text{Fix } \alpha| > 1$ and Theorem 2.1.(5). Then we have two distinct points $x, y \in A$ with $\{x\} := [p]_1 \cap A$ and $\{y\} := [p]_2 \cap A$ ($x, y \in P^{(2)}$). Moreover $\{\alpha(x)\} = \alpha([p]_1 \cap A) = [\alpha(p)]_1 \cap \alpha(A) = [p]_1 \cap A = \{x\}$ and in the same way, $\alpha(y) = y$ hence $(x, y) \in \text{Fix } \alpha^{(2)}$ and $A = \overline{x, y}$.

(5) $(x, \alpha(x)) \in P_1^{(2)}$ implies $\widetilde{\text{Fix } x, \alpha(x)} = \overline{x, \alpha(x)} \cap \overline{x \square \alpha(x), \alpha(x) \square x} \neq \emptyset$ hence by (1), $\widetilde{\text{Fix } x, \alpha(x)} \subseteq \text{Fix } \alpha$.

(6) Let $x \in \text{Fix } \alpha$ then by (3), $\tilde{L}(x) = \tilde{L} \circ \alpha(x) = \alpha \circ \tilde{L}(x)$, i.e. $\tilde{L}(x) \in \text{Fix } \alpha$.

(8) By (6), $\text{Fix } \alpha$ is a non empty subsquare and for each $x \in P_\alpha$ we have $\overline{x, \alpha(x)} \cap \text{Fix } \alpha \neq \emptyset$. Now let for instance $x \in [\text{Fix } \alpha]_1 \setminus \text{Fix } \alpha$ and

$\{y\} := [x]_1 \cap \text{Fix } \alpha$ then $\alpha(y) = y, x \in [y]_1$ and so $\alpha(x) \in [\alpha(y)]_1 = [y]_1$ hence $x, \alpha(x) = [y]_1$, i.e. $x, \alpha(x) \cap \text{Fix } \alpha \neq \emptyset$.

- (9) The first part is a direct consequence of (8) and for the orthogonal fixed lines X and Y we have $X \cap Y = \{p\}$ and so by Theorem 3.1.(3), $\widetilde{x, \alpha(x)} = \widetilde{X} \circ \widetilde{Y}$ is a reflection in the point p interchanging, like α , the points x and $\alpha(x)$ hence by Theorem 2.4.(1), $\alpha = x, \alpha(x)$.

(10) Follows from (2), (6) and Definition 2.3. $\square$

Theorem 3.9. *Let $\alpha \in (\Gamma_1(\mathfrak{K}) \cup \Gamma_2(\mathfrak{K}))$ be an involution then:*

- $\forall A \in \mathfrak{K} \exists_1 A' \in \mathfrak{C}$ such that $A' \perp A$, $\widetilde{AA'}(\mathfrak{K}) = \mathfrak{K}$ and $\alpha = \widetilde{A'A} \circ \widetilde{\alpha}$ if $\alpha \in \Gamma_1$ (and then α is a reflection in the stripe $\text{Fix } \alpha = [A \cap A']_1$) and $\alpha = \widetilde{AA'} \circ \widetilde{\alpha}$ if $\alpha \in \Gamma_2$ (and then α is a reflection in the stripe $\text{Fix } \alpha = [A \cap A']_2$).
- If there are $(A, A') \in \mathfrak{K} \times \mathfrak{C}$ such that $A' \perp A$ and $\widetilde{AA'}(\mathfrak{K}) = \mathfrak{K}$ then $\alpha_1 := \widetilde{A'A} \circ \widetilde{\alpha}$ is an involution of $\Gamma_1(\mathfrak{K})$, $\alpha_2 := \widetilde{AA'} \circ \widetilde{\alpha}$ of $\Gamma_2(\mathfrak{K})$ and $\alpha_1 \circ \alpha_2 = \widetilde{A'A} \circ \widetilde{AA'}$ is an involution of $\Gamma_3(\mathfrak{K})$ more precisely $\alpha_1 \circ \alpha_2$ is a reflection in the subsquare $\text{Fix } (\alpha_1 \circ \alpha_2) = [A \cap A']_1 \cap [A \cap A']_2$.
- If in (2), $\{a\} := A \cap A' \neq \emptyset$ is a point then α_1 is a reflection in the generator $[a]_1$ and α_2 in the generator $[a]_2$ and moreover $\alpha_1 \circ \alpha_2$ is a reflection in the point a .

3.2. Classification

Theorem 3.10. *The group $\langle \mathfrak{K}^+ \rangle$ (and so also $\langle \widetilde{\mathfrak{K}} \rangle$) acts transitively on the set P of points and we have:*

- If in $\langle \widetilde{\mathfrak{K}} \rangle$ there is a reflection in a point o then there is a reflection in each point $p \in P$.
- For every pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ the set $(p \perp K)$ has the same cardinality.
- If $(A, B) \in \mathfrak{K}_1^{2\perp}$ then $\widetilde{A} \circ \widetilde{B}$ is a reflection in the point p with $\{p\} = A \cap B$.
- If $\alpha \in \Gamma_3(\mathfrak{K})$ is an involution with $\text{Fix } \alpha = \{o\}$ and $a \in P_\alpha$ then $\alpha = \widetilde{a, \alpha(a)}$ is a reflection in o .

Proof. Let $a, b \in P$. If $(a, b) \in P^{(2)}$ then by Theorem 3.1.(2) $\widetilde{a, b}$ is an involution of $\langle \widetilde{\mathfrak{K}} \rangle$ interchanging a and b . If for instance $[a]_1 = [b]_1$, let $c \in P \setminus ([a]_1 \cup [a]_2 \cup [b]_2)$ then $(a, c), (c, b) \in P^{(2)}$ and so $\widetilde{a, c} \circ \widetilde{c, b}$ is a map of $\langle \widetilde{\mathfrak{K}} \rangle$ taking b onto a . If α is a reflection in o and $p \in P$ let $\gamma \in \langle \widetilde{\mathfrak{K}} \rangle$ with $\gamma(o) = p$ then $\gamma \circ \alpha \circ \gamma^{-1}$ is a reflection in p .

(2) (2) is now a consequence of Corollary 3.5.(4).

(3) is a consequence of Theorem 3.1.(3) and (4) of Theorem 3.8.(9). $\square$

According to Theorem 3.10. we can classify symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ by properties of the set $(p \perp K)$ for any fixed pair (p, K) with $p \in K$.

Theorem 3.11. *The symmetric 2-structures split into the three classes:*

- (I) *There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| > 1$.*
- (II) *There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $(p \perp K) = \emptyset$.*
- (III) *There is a pair $(p, K) \in P \times \mathfrak{K}$ with $p \in K$ and $|(p \perp K)| = 1$*

Each of the classes (I) and (III) bifurcate into the subclasses (Ia) and (Ib) and into (IIIa) and (IIIb) depending whether $\mathfrak{K}_1^+ \neq \mathfrak{K}^+$ or $\mathfrak{K}_1^+ = \mathfrak{K}^+$.

Theorem 3.12. *Let the symmetric 2-structures be of class (I) Then:*

1. *To each $A \in (p \perp K)$ $\tilde{A} \circ \tilde{K}$ is a reflection in the point p and if $A, B \in (p \perp K)$ with $A \neq B$ then $\tilde{A} \circ \tilde{K}$ and $\tilde{B} \circ \tilde{K}$ are distinct reflections in the point p .*
2. *$|(x \perp X)| > 1$ to each $x \in P$, $X \in \mathfrak{K}$ with $x \in X$.*
3. *If α, β are two different reflections in the point o then $\alpha \circ \beta = \beta \circ \alpha$ and $\alpha \circ \beta$ is a reflection in the point o .*
4. *If $o \in M$ and $A, B \in (o \perp M)$ then $A \perp B$ or $A = B$.*
5. *For any $o \in P$ the set of reflections in o with identity constitute an elementary abelian group and the groups are isomorphic for any two points.*

Proof. (1) Follows from Theorem 3.1.

(2) Follows from Theorem 3.10.(2).

(3) Let $\alpha = \tilde{A} \circ \tilde{B}$. We have $o \in A, B$ and $\beta \circ \alpha \circ \beta = \beta \circ \tilde{A} \circ \tilde{B} \circ \beta = \beta \circ \tilde{A} \circ \beta \circ \beta \circ \tilde{B} \circ \beta = \tilde{A} \circ \tilde{B} = \alpha$. Hence $\alpha \circ \beta$ is an involution with fixed point o i.e. a reflection in the point o by Theorem 3.8.(9).

(4) Suppose $A \neq B$. By Theorem 3.1.(2), $\tilde{A} \circ \tilde{M}$ and $\tilde{M} \circ \tilde{B}$ are reflections in the point o . By (3), $\tilde{A} \circ \tilde{M} \circ \tilde{M} \circ \tilde{B} = \tilde{A} \circ \tilde{B}$ is an involution, hence $B \perp C$.

(5) Follows from (3) and Theorem 3.10.(1). $\square$

Theorem 3.13. *The following conditions are equivalent:*

1. *The symmetric 2-structure is of class (II).*
2. *$(x \perp X) = \emptyset$ for each pair $(x, X) \in P \times \mathfrak{K}$ with $x \in X$.*
3. *$\mathfrak{K}_1^+ = \emptyset$ i.e. each element of $\mathfrak{K}^+$ is fixed point free.*

Theorem 3.14. *The following conditions are equivalent:*

1. *The symmetric 2-structure is of class (III).*
2. *For each pair $(x, X) \in P \times \mathfrak{K}$ with $x \in X$: $|(x \perp X)| = 1$.*
3. *For every $x \in P$ there is exactly one reflection in the point x .*

For symmetric 2-structures of class (III) if $x \in P$ we denote by $\tilde{x}$ the unique reflection in the point x and we set $\tilde{P} := \{\tilde{p} \mid p \in P\}$. Then $\tilde{P} = \mathfrak{K}_1^+$. The symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ of class (IIIb) are called *point symmetric 2-structures*.

3.3. Midpoints

We recall:

Definition 3.15. If $a, b \in P$ are two given points then a point m is called *midpoint* of a and b if there is $\gamma \in \mathfrak{K}_1^+$ with $\gamma(a) = b$ and $\text{Fix} \gamma = \{m\}$.

Theorem 3.16. Let $(a, b) \in P^{(2)}$ then:

1. a and b have a midpoint $\Leftrightarrow (a, b) \in P_1^{(2)}$.
2. If $(a, b) \in P_1^{(2)}$ then $d := \overline{a, b} \cap M(a, b)$ is the only midpoint of a and b .

Theorem 3.17. If $\mathfrak{K}_1^+ \neq \emptyset$ then any two distinct parallel points $a, b \in P$ (i.e. $[a]_1 = [b]_1$ or $[a]_2 = [b]_2$) have at least one midpoint and if $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is of class (III) then they have exactly one midpoint.

Proof. If $[a]_1 = [b]_1$ let $c \in [a]_2 \setminus \{a\}$ then by Theorem 3.10.(1) there is a reflection γ in the point c and we have $a \neq d := \gamma(a) \in [a]_2$ and so $(a, d \square b) \in P^{(2)}$. Let $D := \overline{a, d \square b}$, $m := \tilde{D}(c)$ and $\mu := \tilde{D} \circ \gamma \circ \tilde{D}$. Then $\tilde{D}(d) = b$ and $\mu(m) = \tilde{D} \circ \gamma \circ \tilde{D}(\tilde{D}(c)) = \tilde{D} \circ \gamma(c) = \tilde{D}(c) = m$ hence $\mu \in \mathfrak{K}_1^+$ and μ is a reflection in the point m . Moreover $\mu(a) = \tilde{D} \circ \gamma \circ \tilde{D}(a) = \tilde{D} \circ \gamma(a) = \tilde{D}(d) = b$, i.e. m is a midpoint of a and b .

This shows that two parallel points have at least one midpoint.

Now let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be of class 3. and let m_1 and m_2 be midpoints of a and b and of a and d respectively. Then by Theorem 3.14.(3) to m_i corresponds exactly one reflection in the point m_i which we denote by $\tilde{m}_i$ hence $\tilde{m}_1(a) = b$ and $\tilde{m}_2(a) = d$. Let $F := \overline{a, m_2 m_1}$ then $\tilde{F}(a) = a$ and $\tilde{F}(m_1) = m_2$ hence $\tilde{m}_2 = \tilde{F}(m_1) = \tilde{F} \circ \tilde{m}_1 \circ \tilde{F}$ and so $\tilde{F} \circ \tilde{m}_1 = \tilde{m}_2 \circ \tilde{F}$. This implies $\tilde{F}(b) = \tilde{F} \circ \tilde{m}_1(a) = \tilde{m}_2 \circ \tilde{F}(a) = \tilde{m}_2(a) = d$ and so $\tilde{F}(db) = \tilde{F}(b) \square \tilde{F}(d) = db$ hence $db \in F$. We have proved:

Let $(b, d) \in P^{(2)}$ and $F := \overline{bd, db}$. If m_1 and m_2 are midpoints of b, bd and of bd, d respectively then $m_2 m_1 \in F$.

Assume m'_1 is a further midpoint of b, bd then $m_2 m_1, m_2 m'_1 \in F$ and $m_2 m_1, m_2 m'_1$ are parallel. Since F is a chain this implies $m_2 m_1 = m_2 m'_1$ and so $m_1 = m'_1$. $\square$

3.4. Point symmetric 2-Structures

Theorem 3.18. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a point symmetric 2-structures, $K \in \mathfrak{K}$ and let a, b be two distinct points. Then:

1. The set $K_1^\perp := \{M \in \mathfrak{K} \mid M \perp K, |K \cap M| = 1\}$ is a partition of the set of points P .
2. There is exactly one $\gamma \in \mathfrak{K}^+$ with $\gamma(a) = b$, i.e. $(P, \mathfrak{K}^+)$ is a regular invariant involutory permutation set.
3. The map $\phi: \mathfrak{K}^+ \rightarrow P; \tilde{p} \mapsto p$ is a bijection.
4. There is exactly one midpoint of a, b .

Proof. (1) By definition of class (III), there is exactly one chain orthogonal to K through any point $p \in K$ and, by $\mathfrak{K}^{2\perp} = \mathfrak{K}_1^{2\perp}$, we have exactly one such a chain through any $p \in P$.

- (2) , (4) For $(a, b) \in P^{(2)}$ it follows from $P^{(2)} = P_1^{(2)}$ and Theorem 3.5.(2). For parallel points it follows from Theorem 3.17.
 (3) Follows from $\mathfrak{K}^+ = \mathfrak{K}_1^+ = \tilde{P}$ and Theorem 3.14.(3). $\square$

3.5. Symmetric 2-Structures with Euclid's Parallel Axiom

In this section we consider symmetric 2-structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ where for $\mathfrak{B} := (\mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K})$, $(P, \mathfrak{B})$ is an affine plane. For $K, L \in \mathfrak{B}$ and $p \in P$ we denote $K \parallel L$ if $K \cap L = \emptyset$ or $K = L$ and by $(p \parallel K)$ the block of $\mathfrak{B}$ which is parallel to K and passes through p .

Lemma 3.19. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure of class (III) such that $(P, \mathfrak{B})$ is an affine plane, let $p, q \in P$ and $K, L, M \in \mathfrak{K}$.*

1. *If $M \perp K, L$ then $K \parallel L$.*
2. *If $p \in L, L \parallel K$ and $\{M\} = (p \perp K)$ then $M \perp L$.*
3. *$K \parallel L \Leftrightarrow \exists M : K, L \perp M$.*
4. *Let $p' = \tilde{K}(p)$, $p \in L, p' \in L'$ and $L \parallel K \parallel L'$. Then $L' = \tilde{K}(L)$ hence $\tilde{K}(p \parallel K) = (p' \parallel K)$.*
5. *If $K \parallel L$ and $q \in K$ then $L' := \tilde{K}(L) = \tilde{q}(L)$.*
6. *For any pair $(a, b) \in P^{(2)}$ there is exactly one midpoint of a, b .*

Proof. (1) If K, L, M have a common point then $K = L$ by Theorem 3.14.

If there is a point a with $a \in K, L$ and $a \notin M$ then $K = L$ by Theorem 3.1.(5).

- (2) Let be $\{L'\} := (p \perp M)$. By (1), $L' \parallel K$ hence $L' = L$ by Euclid axiom.
- (3) Follows from (1) and (2).
- (4) If $L'' = \tilde{K}(L)$ then we obtain $L'' = L'$ by Euclid axiom.
- (5) The statement is clear for $K = L$. Therefore let $K \neq L$ (hence $K \cap L = \emptyset$) and let $\{M\} := (q \perp L)$. Then by (2), $M \perp K, L$ and by Theorem 3.14., $\tilde{q} = \tilde{K} \circ \tilde{M}$ and so $\tilde{q}(L) = \tilde{K} \circ \tilde{M}(L) = \tilde{K}(L) = L'$.
- (6) Let $c = ab$. By Theorem 3.17., the midpoints m_1 of a, c and m_2 of b, c exist and if $L := \overline{m_1, m_2}$, $d := \tilde{L}(c)$, $C := (c \parallel L)$ and $D := (d \parallel L)$ then by (4), $D = \tilde{L}(C)$ and by (5), $\tilde{m}_1(C) = \tilde{m}_2(C) = D$. Hence $a, b, d \in D$. For the chain $M := \overline{c, d}$ we have $\tilde{M}(m_1) = m_2$, $M \perp L$ and so by (2), $M \perp D$ hence $\tilde{d} := \tilde{M} \circ \tilde{D}$ is a reflection in the point d . Finally $\tilde{d}(a) = \tilde{M} \circ \tilde{D}(a) = \tilde{M}(a) = \tilde{M} \circ \tilde{m}_1(c) = \tilde{M} \circ \tilde{m}_1 \circ \tilde{M}(c) = \tilde{M}(m_1) = \tilde{m}_2(c) = b$ and so d is a midpoint of a and b which is uniquely determined by Theorem 3.16.(2). $\square$

Theorem 3.20. *Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a symmetric 2-structure such that $(P, \mathfrak{B})$ is an affine plane then $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class (II) or (IIIb).*

Proof. If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is not in the class (II) then by Theorem 3.13., for each $(p, K) \in P \times \mathfrak{K}$ with $p \in K : (p \perp K) \neq \emptyset$. Let $L, L' \in (p \perp K)$, $q \in K \setminus \{p\}$ and $M \in (q \perp K)$. Then by Theorem 3.1.(5), $L \cap M = \emptyset$, i.e. $L = (p \parallel M) = L'$.

Thus by Theorem 3.11., $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class **(III)** and by Lemma 3.19.(6), for any $(a, b) \in P^{(2)}$ there is a midpoint. Hence by Theorem 3.16.(1), $P^{(2)} = P_1^{(2)}$ and so by Theorem 3.1.(4), $\mathfrak{K}^+ = \mathfrak{K}_1^+$, i.e. $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ belongs to the class **(IIIb)**. $\square$

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Received: October 15, 2010.

Accepted: November 2, 2012.