A Representation of a Point Symmetric 2-Structure by a Quasi-Domain

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Dedicated to Momme Thomsen on the occasion of his 70th birthday

Abstract. In a symmetric 2-structure $\Sigma = (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ we fix a chain $E \in \mathfrak{K}$ and define on E two binary operations “+” and “•”. Then $(E, +)$ is a K-loop and for $E^* := E \setminus \{o\}$, $(E^*, \cdot)$ is a Bol loop. If Σ is even point symmetric then $(E, +, \cdot)$ is a quasidomain and one has the set $Aff(E, +, \cdot) := \{a^+ \circ b^\bullet \mid a \in E, b \in E^*\}$ of affine permutations. From $Aff(E, +, \cdot)$ one can reproduce via a “chain derivation” the point symmetric 2-structure Σ .

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1. Introduction and notations

There is a one to one correspondence between chain structures $\Sigma = (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ and permutation sets (E, Ω) via a germ operation Π_E and a chain derivation κ . If a chain $E \in \mathfrak{K}$ is fixed then Π_E maps each chain $K \in \mathfrak{K}$ onto a permutation of $Sym E$ and then the pair (E, Ω) with $\Omega := \Pi_E(\mathfrak{K})$ is called the *germ* of the chain structure Σ . Then Σ is symmetric i.e. for each $K \in \mathfrak{K}$ the reflection $\tilde{K}$ in K maps any chain $L \in \mathfrak{K}$ onto a chain of $\mathfrak{K}$ - if and only if

$$(S) \quad \forall \omega \in \Omega, \omega^{-1} \in \Omega, \omega \circ \Omega \circ \omega \subseteq \Omega.$$

And Σ is a 2-structure iff (E, Ω) is a sharply 2-transitive permutation set. By [1], we know: If the set E is provided with two binary operations “+” and “•” such that $(E, +, \cdot)$ is a field or a near field or even a near domain and if for $a \in E$, $b \in E \setminus \{o\}$ we set “ $a^+ : E \rightarrow E; x \mapsto a + x$ ” and “ $b^\bullet : E \rightarrow E; x \mapsto b \cdot x$ ” and

$Aff(E, +, \cdot) := \{a^+ \circ b^\bullet \mid a \in E, b \in E \setminus \{o\}\}$ - called the *corresponding affine permutation set* - then (E, Ω) with $\Omega := Aff(E, +, \cdot)$ is a sharply 2-transitive permutation group. Vice versa if (E, Ω) is a sharply 2-transitive permutation group then E can be provided with two binary operations “+” and “ $\cdot$ ” such that $(E, +, \cdot)$ is a near-domain and $Aff(E, +, \cdot)$ coincides with Ω . The notion *quasi-domain* $(Q, +, \cdot)$, introduced by Kist [2], includes both, quasi-fields and near-domains and for a quasi-domain $(Q, +, \cdot)$ the pair $(Q, Aff(Q, +, \cdot))$ is a set of permutations which is sharply 2-transitive under certain conditions (cf. Theorem 1.5). Here we show: If (E, Ω) with $\Omega := \Pi_E(\mathfrak{K})$ is the germ of a point symmetric 2-structure Σ and if $\mathfrak{I}_E := \{\omega \in \Omega \mid \omega^2 = id \neq \omega\}$ then (E, Ω) satisfies besides (S) the condition

$$(S'') \quad \Omega \circ \mathfrak{I}_E \subseteq \Omega$$

and E can be provided with binary operations “+” and “ $\cdot$ ” such that $(E, +, \cdot)$ is a quasi domain with $\Omega = Aff(E, +, \cdot)$. Moreover $(E, +)$ is a K-loop and $(E \setminus \{o\}, \cdot)$ is a Bol loop (see [3]).

1.1. Chain Structures

We recall some facts about chain structures $\Sigma = (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ (cf. [4, 5]). Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be the corresponding maximal chain structure hence $\mathfrak{K} \subseteq \mathfrak{C}$. If $A, B \in \mathfrak{C}$, $p, q \in P$ let $pq := [p]_1 \cap [q]_2$, $pA := [p]_1 \cap A$, $Ap := [p]_2 \cap A$,

$$\widetilde{AB} : P \rightarrow P ; p \mapsto (Bp)(pA).$$

and $\widetilde{A} := \widetilde{AA}$. Moreover let $P^{(2)} := \{(a, b) \in P^2 \mid \{a, b\} \neq \{ab, ba\}\}$.

If $E \in \mathfrak{C}$ is fixed then for $A \cdot B := \widetilde{AB}(E)$, $(\mathfrak{C}, \cdot) = (\mathfrak{C}, \cdot)_E$ becomes a group called the *maximal chain group*.

We set “ $A \perp B \Leftrightarrow \widetilde{A}(B) = B \wedge A \neq B$ ” (then A and B are called *orthogonal*), $\mathfrak{C}^{\perp\perp} := \{A, B\} \in \mathfrak{C}^2 \mid A \perp B\}$, $\mathfrak{K}^{\perp\perp} := \mathfrak{C}^{\perp\perp} \cap \mathfrak{K}^2$, $A^\perp := \{C \in \mathfrak{C} \mid C \perp A\}$,

$$(p \perp A) := \{K \in \mathfrak{C} \mid K \perp A \wedge p \in K\}$$

and if $A \in \mathfrak{K}$,

$$(p \perp A)_{\mathfrak{K}} := (p \perp A) \cap \mathfrak{K}.$$

The chain structures $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is called

$$\text{symmetric} \Leftrightarrow \forall K \in \mathfrak{K} : \widetilde{K}(\mathfrak{K}) = \mathfrak{K}$$

$$2\text{-structure} \Leftrightarrow \forall a, b \in P, \{a, b\} \neq \{ab, ba\} \exists_1 K \in \mathfrak{K} : a, b \in K.$$

In [6] we introduced a classification of symmetric 2-structures Σ depending whether for $K \in \mathfrak{K}$ and $p \in K$ the cardinality $|(p \perp K)_{\mathfrak{K}}|$ is > 1 (class I), is $= 0$ (class II) or is $= 1$ (class III). In [7] we showed if Σ is of class III then any two orthogonal chains intersect (in exactly one point) and called Σ *point symmetric*.

Now we add some formulas concerning maximal chain structures:

Theorem 1.1. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ be a maximal chain structure, $E \in \mathfrak{C}$ fixed and let $\mathfrak{J}$ denote the set of all involutions of the group $(\mathfrak{C}, \cdot)_E$ then:

- (1) $\mathfrak{J} = \{C \in \mathfrak{C} \mid C \perp E\} = E^\perp$.
- (2) $\forall A, B \in \mathfrak{C} : A \perp B \Leftrightarrow A \cdot B^{-1} \in \mathfrak{J}$.
- (3) $\forall A, B, C \in \mathfrak{C} : \widetilde{AB}(C) = A \cdot C^{-1} \cdot B$.

1.2. The Germ of a Chain Structure

Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain structure and $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{C})$ the corresponding maximal chain structure. For $A, B \in \mathfrak{C}$ the map $\widetilde{AB}$ is a permutation of P mapping the chain B bijectively onto the chain A and A onto B . Therefore the restriction $\widetilde{AB} \circ \widetilde{AB}|_B$ of the product $\widetilde{AB} \circ \widetilde{AB}$ onto the chain B is a permutation of the set of points of B . Fixing a chain $E \in \mathfrak{K}$ we obtain by

$$\Pi_E(\mathfrak{K}) := \{\widetilde{KE} \circ \widetilde{KE}|_E \mid K \in \mathfrak{K}\}$$

a set of permutations of the elements of E . The permutation set

$$\Pi_E((P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})) := (E, \Pi_E(\mathfrak{K}))$$

is called the *germ* of the chain structure $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ in E .

We remark $\Pi_E(\mathfrak{C}) = \text{Sym } E$.

Conversely let (E, Ω) , $\Omega \subseteq \text{Sym } E$ be a permutation set. We set:

$$P := E \times E, \quad \mathfrak{G}_1 := \{\{x\} \times E \mid x \in E\}, \quad \mathfrak{G}_2 := \{E \times \{x\} \mid x \in E\},$$

$$\forall \omega \in \Omega, \quad \kappa(\omega) := \{(x, \omega(x)) \mid x \in E\}, \quad \kappa(\Omega) := \{\kappa(\omega) \mid \omega \in \Omega\}.$$

Then $\kappa((E, \Omega)) := (P, \mathfrak{G}_1, \mathfrak{G}_2, \kappa(\Omega))$ is a chain structure called *chain derivation* of the permutation set (E, Ω) . We have:

Theorem 1.2. (1) If $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a chain structure, $E \in \mathfrak{K}$ then

$$\kappa(\Pi_E(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})) = (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$$

(2) If (E, Ω) is a permutation set then $\Pi_E(\kappa(E, \Omega)) = (E, \Omega)$.

(3) The map Π_E is an isomorphism from the group $(\mathfrak{C}, \cdot)_E$ onto the symmetric group $(\text{Sym } E, \circ)$.

Now we apply the germ operation Π_E on certain sets of chains contained in $\mathfrak{K}$ or in $\mathfrak{C}$.

Theorem 1.3. Let $E \in \mathfrak{K}$ and $o \in E$ be fixed, let $\mathfrak{J}_E$ denote the set of all involutions of $(\text{Sym } E, \circ)$ and let $\Omega := \Pi_E(\mathfrak{K})$. Then:

- (1) $\Pi_E(\mathfrak{J}) = \mathfrak{J}_E$ and $\Pi_E(E) = \text{id} \in \Omega$.
- (2) If $\mathfrak{K}_o := \{K \in \mathfrak{K} \mid o \in K\}$ and $\Omega_o := \{\omega \in \Omega \mid \omega(o) = o\}$ then $\Pi_E(\mathfrak{K}_o) = \Omega_o$.
- (3) Let $\mathfrak{N}_{\mathfrak{C}}(E^\perp) := \{C \in \mathfrak{C} \mid C \cdot E^\perp \cdot C^{-1} = E^\perp\}$ be the normalizer of $E^\perp$ in the group $(\mathfrak{C}, \cdot)$ and $\mathfrak{N}_{\text{Sym } E}(\mathfrak{J}_E) := \{\sigma \in \text{Sym } E \mid \sigma \circ \mathfrak{J}_E \circ \sigma^{-1} = \mathfrak{J}_E\}$ then $\Pi_E(\mathfrak{N}_{\mathfrak{C}}(E^\perp)) = \mathfrak{N}_{\text{Sym } E}(\mathfrak{J}_E)$.
- (4) If $A, B \in \mathfrak{C}$ and $\alpha := \Pi_E(A)$, $\beta := \Pi_E(B)$ then “ $A \perp B \Leftrightarrow \alpha^{-1} \circ \beta \in \mathfrak{J}_E$ ”
- (5) Let $p \in P$, $A \in \mathfrak{C}$ and $\alpha := \Pi_E(A)$ then: “ $p \in A \Leftrightarrow \alpha(pE) = Ep$ ” and “ $\text{Fix } \alpha = A \cap E$ ”.

(6) Let $p \in P$, $A \in \mathfrak{K}$ and $\alpha := \Pi_E(A)$ then

$$\begin{aligned}\Pi_E((p \perp A)_{\mathfrak{K}}) &= \{\beta \in \Omega \mid \beta(pE) = Ep \wedge \alpha^{-1} \circ \beta \in \mathfrak{J}_E\} \\ \Pi_E((p \perp E)_{\mathfrak{K}}) &= \{\beta \in \Omega \cap \mathfrak{J}_E \mid \beta(pE) = Ep\} \\ \Pi_E((o \perp E)_{\mathfrak{K}}) &= \{\beta \in \Omega \cap \mathfrak{J}_E \mid o \in \text{Fix}\beta\}\end{aligned}$$

(7) In the case of a symmetric 2-structure we have the following translations:
Let $A, B \in \mathfrak{K}$ and $\alpha := \Pi_E(A), \beta := \Pi_E(B)$, then:

$$A \perp B \Leftrightarrow A \cdot B^{-1} \in \mathfrak{J} \Leftrightarrow \alpha \circ \beta^{-1} \in \mathfrak{J}_E$$

$$A \cap B \neq \emptyset \Leftrightarrow \text{Fix } \alpha \circ \beta^{-1} \neq \emptyset$$

(Dropping and erecting the perpendicular) $\forall \omega \in \Omega \forall a, b \in E \exists \lambda \in \mathfrak{J}_E \circ \omega$
with $\lambda(a) = b$.

1.3. Translation of Properties

By Theorem 1.2. each property of a chain structure is rendered to a property of the germ and vice versa. So we have:

Theorem 1.4. Let $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a chain structure, $E \in \mathfrak{K}$ and $(E, \Omega) := \Pi_E(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ the germ. Then:

- (1) $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a 2-structure $\Leftrightarrow (E, \Omega)$ is a sharply 2-transitive permutation set.
- (2) $(P, \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K})$ is an affine plane $\Leftrightarrow (E, \Omega)$ is a sharply 2-transitive permutation set and $\forall \alpha \in \Omega, \forall a, b \in E$ with $\alpha(a) \neq b, \exists \beta \in \Omega$ such that $\beta(a) = b$ and $\text{Fix}(\beta \circ \alpha^{-1}) = \emptyset$.
- (3) $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is symmetric (i.e. $\forall K \in \mathfrak{K} : \tilde{K}(\mathfrak{K}) = \mathfrak{K}$) $\Leftrightarrow$ (cf. [4])

$$\begin{aligned}\forall K \in \mathfrak{K} : K \cdot \mathfrak{K} \cdot K &= \mathfrak{K} \wedge \mathfrak{K}^{-1} = \mathfrak{K} \Leftrightarrow \\ (S) \Omega &= \Omega^{-1} \wedge \forall \omega \in \Omega : \omega \circ \Omega \circ \omega = \Omega.\end{aligned}$$

- (4) $(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ is a symmetric 2-structure $\Leftrightarrow (E, \Omega)$ is a sharply 2-transitive permutation set satisfying (S) and $\text{id} \in \Omega$.

A permutation set (E, Ω) satisfying the condition (S) shall be also called *symmetric*.

1.4. Quasi-Domains and Their Affine Permutations

In order to have a common concept which includes both, near-domains and quasi-fields, Kist (cf. [2]) introduced the notion *quasi-domain*. That is a set $(Q, +, \cdot)$ provided with two binary operations such that the following conditions are satisfied:

- (1) $(Q, +)$ is a loop (o denotes the neutral element ; let $Q^* = Q \setminus \{o\}$).
- (2) $\forall a, b \in Q : a + b = o \Rightarrow b + a = o$.
- (3) $(Q^*, \cdot)$ is a loop (let e denote the neutral element of $(Q^*, \cdot)$).
- (4) $\forall a, b, c \in Q : a(b + c) = ab + ac$.

(5) $\forall a \in Q : o \cdot a = o$.

(6) $\forall a, b \in Q \exists d_{a,b} \in Q$ such that $\forall x \in Q : a + (b + x) = (a + b) + d_{a,b}x$.

By (Q.4), " $\forall x \in Q : x + x = o \Leftrightarrow e + e = o$ ".

Therefore we say: " $(Q, +, \cdot)$ has *characteristic* 2 if $e + e = o$ " (expressed by $\text{char}(Q) = 2$) and " $(Q, +, \cdot)$ has *characteristic* $\neq 2$ if $e + e \neq o$ " (expressed by $\text{char}(Q) \neq 2$).

Remark 1. If $(Q, +, \cdot)$ is a quasi-domain then by (6), $(Q, +)$ is an A_l -loop (cf. [3] p.35). $(Q, +, \cdot)$ is a *quasi-field* if $(Q, +)$ is a group, a *near-domain* if $(Q^*, \cdot)$ is a group and a *near-field* if $(Q, +)$ and $(Q^*, \cdot)$ are groups.

If $(Q, +, \cdot)$ is a quasi-domain then for each $a \in Q$ and for each $b \in Q^*$ the maps

$$"a^+ : Q \rightarrow Q ; x \mapsto a + x" \text{ and } "b^\bullet : Q \rightarrow Q ; x \mapsto b \cdot x"$$

are permutations of $\text{Sym } Q$ - we set $Q^+ := \{a^+ \mid a \in Q\}$ and $(Q^*)^\bullet := \{b^\bullet \mid b \in Q^*\}$ - and the elements of the set

$$\Gamma := \text{Aff}(Q, +, \cdot) := \{a^+ \circ b^\bullet \mid a \in Q, b \in Q^*\} = Q^+ \circ (Q^*)^\bullet$$

are called *affine permutations of the quasi-domain* $(Q, +, \cdot)$.

By [2] Theorems (1.18), (1.20) and (1.21) we have:

Theorem 1.5 [Kist [2]]. *Let $(Q, +, \cdot)$ be a quasi-domain, $\Gamma := \text{Aff}(Q, +, \cdot)$ the corresponding affine permutation set, $\Gamma_o := \{\gamma \in \Gamma \mid \gamma(o) = o\}$ and $\Lambda := Q^+ \circ \Gamma_o \circ Q^+$. Then:*

- (1) $Q^+ \circ \text{Aut}(Q, +)$ is a group.
- (2) $\Gamma = Q^+ \circ \Gamma_o = \Gamma_o \circ Q^+$ and $(Q^+)^{-1} = Q^+$.
- (3) $\{(\gamma(o), \gamma(e)) \mid \gamma \in \Gamma\} = \binom{Q}{2}$.
- (4) $\forall p \in Q^*, \forall (a, b) \in \binom{Q}{2} \exists \gamma \in \Gamma$ with $\gamma(o) = a$ and $\gamma(p) = b$.
- (5) Λ acts 2-transitive on Q .
- (6) The following three statements are equivalent:
 "(a) $\forall a, b, c, x \in Q : c(d_{a,b}x) = (cd_{a,b})x$ ", "(b) $\Lambda = \Gamma$ " and
 "(c) (Q, Λ) is a sharply 2-transitive permutation set".
- (7) The following statements are equivalent:
 "(a) $ab = e \Rightarrow \forall x \in Q : a(bx) = x$ ", "(b) $\Gamma = \Gamma^{-1}$ ".
- (8) The following statements are equivalent ([2, page 35]):
 "(a) $\forall x \in Q : (-e)x = x(-e)$ ", "(b) $\forall a, b \in Q : (-a)b = -ab$ ".
- (9) (7) implies (8).

Proof. (1) Let $a, b \in Q, \alpha, \beta \in \text{Aut}(Q, +)$. Then $(a^+ \circ \alpha) \circ (b^+ \circ \beta) = (a + \alpha(b))^+ \circ d_{a, \alpha(b)}^\bullet \circ \alpha \circ \beta$. By (Q.4), $d_{a, \alpha(b)}^\bullet \in \text{Aut}(Q, +)$ hence $(a^+ \circ \alpha) \circ (b^+ \circ \beta) \in Q^+ \circ \text{Aut}(Q, +)$ and $(a^+ \circ \alpha)^{-1} = (\alpha^{-1}(-a))^+ \circ (\alpha^{-1} \circ ((d_{a, -a})^\bullet)^{-1}) \in Q^+ \circ \text{Aut}(Q, +)$. $\square$

Remark 2. Kühlbrandt used for his investigations [8] a more restricted concept of quasi-domain by claiming also the property (6 (a)) of Theorem 1.5. as an axiom.

He considered the following condition for sharply 2-transitive permutation set (E, Ω) :

- (*) $\mathfrak{J}_E \cdot \Omega = \Omega \cdot \mathfrak{J}_E = \Omega$ and every $\omega \in \Omega$ which interchanges two distinct elements of E is an involution.

We recall the following result of Kühlbrandt:

Theorem 1.6 (Kühlbrandt [8]).

- (1) Let (Ω, E) be a sharply 2-transitive permutation set satisfying condition (*). Then an addition and multiplication can be defined in E such that $(E, +, \cdot)$ is a quasi-domain satisfying the conditions (6)(a) and (8)(b) of Theorem 1.5. Furthermore Ω is the set of affine mappings of $(E, +, \cdot)$.
- (2) Let $(E, +, \cdot)$ be a quasi-domain satisfying the condition (6)(a) of Theorem 1.5 and let Ω be the set of affine mappings of $(E, +, \cdot)$. Then:
 - (a) (*) is equivalent to the condition (8)(b) of Theorem 1.5.
 - (b) (S) is equivalent to the condition

$$\forall a, b, x \in E : a(b(ax)) = (a(ba))x \text{ (Bol-identity)}.$$

Comparing the results of Kist and Kühlbrandt we get:

Corollary 1.7. If a quasi-domain $(E, +, \cdot)$ satisfies the property (6 (a)) of Theorem 1.5 and the Bol-identity then $(E, +, \cdot)$ satisfies (7 (a)) of Theorem 1.5.

Proof. By Theorem 1.6(2)(b) the set of affine mappings of the quasi-domain satisfies (S) hence in particular it satisfies condition (7)(b) of Theorem 1.5. $\square$

2. Sharply 2-Transitive Permutation Sets

In this part let (E, Ω) be a sharply 2-transitive permutation set. Then for $\omega \in \Omega$ also $(E, \omega^{-1} \circ \Omega)$ is a sharply 2-transitive permutation set. Therefore we assume for (E, Ω) without loss of generality, $id \in \Omega$. For $(a, b) \in \binom{E}{2}$ let $(a \leftrightarrow b)$ denote the uniquely determined permutation of Ω interchanging a and b and let $\mathfrak{J}'_E := \{(a \leftrightarrow b) \mid (a, b) \in \binom{E}{2}\}$. If $\mathfrak{J}_E$ denotes the set of all involutions in Ω then $\mathfrak{J}_E \subseteq \mathfrak{J}'_E$. For $a \in E$ let $E^a := E \setminus \{a\}$, $(a \leftrightarrow E^a) := \{(a \leftrightarrow x) \mid x \in E^a\}$, $\tilde{a}' := \{\alpha \in \mathfrak{J}'_E \mid \alpha(a) = a\}$ and $\tilde{a} := \{\alpha \in \mathfrak{J}_E \mid \alpha(a) = a\}$.

2.1. Two Binary Operations

We set $E_f := \{p \in E \mid \tilde{p} \neq \emptyset\}$. If $E_f \neq \emptyset$ let $o \in E_f$ be fixed and let $o^\circ \in \tilde{o}$, if $E_f = \emptyset$ let any $o \in E$ be fixed and let $o^\circ := id$. For each $p \in E \setminus \{o\}$ let $p^\circ := (o \leftrightarrow p)$ and $p^+ := p^\circ \circ o^\circ$. Then (E, E°) with $E^\circ := \{p^\circ \mid p \in E\}$ is a reflection structure (cf. e.g. [9] Sect. 6.) and in E we define an addition “+” by:

$$+ : E \times E \rightarrow E ; (a, b) \mapsto a + b := a^+(b).$$

Then by [9] 2.3., $(E, +)$ is a left loop - called the *loop-derivation* of $(E, \mathfrak{J}'_E)$ in o^- .

If $\mathfrak{J}'_E = \mathfrak{J}_E = E^\circ$ and if $\mathfrak{J}_E$ is invariant, i.e. $\forall \alpha \in \mathfrak{J}_E : \alpha \circ \mathfrak{J}_E \circ \alpha = \mathfrak{J}_E$, then $(E, +)$ is a K-loop.

Now we fix a further element $e \in E^* := E \setminus \{o\}$ and for each $a \in E^*$ let $a^\bullet \in \Omega$ be the element uniquely determined by $a^\bullet(o) = o$ and $a^\bullet(e) = a$. In E we define a multiplication “ $\cdot$ ” by:

$$\cdot : E \times E \rightarrow E ; (a, b) \mapsto a \cdot b := \begin{cases} a^\bullet(b), & a \neq o \\ o, & a = o \end{cases}.$$

Then e is the neutral element of $(E, \cdot)$ and if $a, b \in E^*$ then the equation $a \cdot x = b$ has the unique solution $x = (a^\bullet)^{-1}(b)$ and for $x \cdot a = b$ let $\xi \in \Omega$ uniquely determined by $\xi(o) = o$ and $\xi(a) = b$. Then $x := \xi(e)$ is the unique solution of $x \cdot a = b$. Therefore $(E^*, \cdot)$ is a loop.

The triple $(E, +, \cdot)$ we call the *double-loop-derivation* of the sharply 2-transitive permutation set (E, Ω) in the points $(o, e) \in \binom{E}{2}$.

2.2. Symmetric Permutation Sets

From now on let (E, Ω) be a sharply 2-transitive symmetric permutation set and let $(E, +, \cdot)$ be the corresponding double-loop-derivation in two points $(o, e) \in \binom{E}{2}$.

Theorem 2.1. *If (E, Ω) is symmetric then for each $\omega \in \Omega$ the group $\langle \omega \rangle$ generated by the element ω is a subset of Ω and we have:*

- (1) $\mathfrak{J}_E = \mathfrak{J}'_E$.
- (2) $\mathfrak{J}_E$ is invariant, i.e. $\forall \alpha \in \mathfrak{J}_E : \alpha \circ \mathfrak{J}_E \circ \alpha = \mathfrak{J}_E$, i.e. $\mathfrak{J}_E \subseteq \mathfrak{N}_{\text{Sym } E}(\mathfrak{J}_E)$.
- (3) $\forall \alpha \in \mathfrak{J}_E : |\text{Fix } \alpha| \leq 1$.
- (4) $\forall (a, b) \in \binom{E}{2}$ the map

$$(a \leftrightarrow b)^i : \tilde{a} \rightarrow \tilde{b} ; \xi \mapsto (a \leftrightarrow b) \circ \xi \circ (a \leftrightarrow b)$$

is a bijection hence $|\tilde{a}| = |\tilde{b}|$.

- (5) If $\exists \alpha \in \mathfrak{J}_E$ with $\text{Fix } \alpha \neq \emptyset$ then: $\forall b \in E : \tilde{b} \neq \emptyset$.
- (6) If (E, Ω) satisfies the condition
- (S') $\forall \omega \in \Omega : \omega \circ \mathfrak{J}_E \circ \omega^{-1} \subseteq \Omega$ then:
 - (a) $\forall a \in E : |\tilde{a}| \leq 1$.
 - (b) All elements of $\mathfrak{J}_E$ are conjugated in Ω , i.e. $\forall \alpha, \beta \in \mathfrak{J}_E \exists \omega \in \Omega : \beta = \omega \circ \alpha \circ \omega^{-1}$.
 - (c) $\forall (a, b) \in \binom{E}{2}, \forall \omega \in \Omega : (\omega(a) \leftrightarrow \omega(b)) = \omega \circ (a \leftrightarrow b) \circ \omega^{-1}$ hence $\forall \alpha, \beta \in \mathfrak{J}_E : |\text{Fix } \alpha| = |\text{Fix } \beta|$ and “ $\forall \omega \in \Omega : \omega \circ \mathfrak{J}_E \circ \omega^{-1} = \mathfrak{J}_E$ ”.

Proof. (1) Let $(a, b) \in \binom{E}{2}$ and $\omega := (a \leftrightarrow b)$ then $\omega^3 \in \Omega$ and $\omega^3(a) = b$, $\omega^3(b) = a$ hence $\omega^3 = (a \leftrightarrow b) = \omega$ and so $\omega^2 = id$. Therefore $\mathfrak{J}_E = \mathfrak{J}'_E$.

- (2) Let $\alpha, \beta \in \mathfrak{J}_E$ then $\alpha \circ \beta \circ \alpha$ is an involution and by $\omega \circ \Omega \circ \omega = \Omega$ we have $\alpha \circ \beta \circ \alpha \in \mathfrak{J}_E$.
- (3) Since (E, Ω) is sharply 2-transitive and since $id \in \Omega$, $|\text{Fix } \alpha| \geq 2$ would imply $\alpha = id$.

- (4) Let $\alpha \in \tilde{a}$. By (1), $(a \leftrightarrow b)$ and $\beta := (a \leftrightarrow b) \circ \alpha \circ (a \leftrightarrow b)$ are involutions contained in $\mathfrak{J}_E$ and $\text{Fix } \beta = b$ hence $\beta \in \tilde{b}$.
- (5) By (3), $\text{Fix } \alpha$ consists of a single point a hence $\alpha \in \tilde{a}$. By (4), $|\tilde{b}| = |\tilde{a}| \geq 1$.
- (6) A proof of (a) and (b) is given in [8] §3 (3.1), (3.2), (3.4). $\square$

According to Theorem 2.1.(4), there are three types of sharply 2-transitive symmetric permutation sets (E, Ω) possible, depending whether for $o \in E$ we have: $|\tilde{o}| > 1$ (Class I), $|\tilde{o}| = 0$ (Class II) or $|\tilde{o}| = 1$ (Class III) and from Theorem 1.3. (4) we obtain: If $(E, \Omega) := \Pi_E(\Sigma)$ is the germ of a symmetric 2-structure Σ then Σ and (E, Ω) are in the same class. If (E, Ω) belongs to class III then $\forall \alpha \in \Omega \cap \mathfrak{J}_E$, $|\text{Fix } \alpha| = 1$, so symmetric 2-structures of class III we also called *point symmetric*.

According to Theorem 2.1. (1) and (2), we have:

Theorem 2.2. *If (E, Ω) is of class II or III then $(E, +)$ is a K-loop and:*

- (1) *If (E, Ω) is of class III then:*
- (a) $\forall a \in E, |\tilde{a}| = 1$ and the map $\sim: E \rightarrow \tilde{E} := \{\tilde{x} | x \in E\}; x \mapsto \tilde{x}$ is a bijection.
 - (b) $\tilde{E} = \mathfrak{J}_E$.
 - (c) Any two points $a, b \in E$ have exactly one midpoint $m \in E$, i.e. there is exactly one involution $\mu \in \mathfrak{J}_E$ with $\text{Fix } \mu = m$ and $\mu(a) = b$.
 - (d) The K-loop $(E, +)$ is uniquely 2-divisible.
- (2) *If (E, Ω) satisfies (S') then*
 $\Omega_o := \{\omega \in \Omega \mid \omega(o) = o\} \subseteq \text{Aut}(E, +)$.

Proof. (1) Follows from [7, Theorem 2.4] and [9, 6.1].

- (2) Let $\gamma \in \Omega_o$ and $a, b \in E$. Then by $\gamma(o) = o$ and Theorem 2.1.(6)(c), $\gamma \circ a^+ = \gamma \circ (o \leftrightarrow a) \circ o^\circ = \gamma \circ (o \leftrightarrow a) \circ \gamma^{-1} \circ \gamma \circ o^\circ \circ \gamma^{-1} \circ \gamma = (\gamma(o) \leftrightarrow \gamma(a)) \circ \gamma(o)^\circ \circ \gamma = (o \leftrightarrow \gamma(a)) \circ o^\circ \circ \gamma = (\gamma(a))^+ \circ \gamma$. Therefore $\gamma(a + b) = \gamma \circ a^+(b) = (\gamma(a))^+ \circ \gamma(b) = \gamma(a) + \gamma(b)$, i.e. γ is an automorphism of $(E, +)$. $\square$

Remark 3. If (E, Ω) is of class I then $E^\circ \neq \mathfrak{J}_E$ and E° is not invariant. Thus $(E, +)$ is only a left loop.

Together with Theorem 2.2. we have the result:

Theorem 2.3. *Let (E, Ω) be of class II or III, then $(E, +, \cdot)$ has the properties:*

- (1) *$(E, +)$ is a K-loop and $(E^*, \cdot)$ is a Bol loop, i.e. $\forall a, b \in E^*, a^\bullet \circ b^\bullet \circ a^\bullet = (a \cdot (b \cdot a))^\bullet$.*
- (2) *If (E, Ω) satisfies the condition*
 (S°) *For $\alpha, \beta, \gamma, \delta \in \mathfrak{J}_E \cup \{id\}$ with $\text{Fix}(\alpha \circ \beta \circ \gamma \circ \delta) \neq \emptyset : \alpha \circ \beta \circ \gamma \circ \delta \in \Omega$ then*
 $\forall a, b \in E \exists d_{a,b} \in E$ *such that $\forall x \in E : a + (b + x) = (a + b) + d_{a,b} \cdot x$.*

- (3) If (E, Ω) satisfies (S') then $(E, +, \cdot)$ is left-distributive, i.e. $\forall a, b, c \in E : a \cdot (b + c) = a \cdot b + a \cdot c$.
- (4) If (E, Ω) satisfies the condition (S'') $\Omega \circ \mathfrak{J}_E \subseteq \Omega$

then:

- (a) (E, Ω) satisfies also (S') , (S^o) and $\mathfrak{J}_E \circ \Omega \subseteq \Omega$.
- (b) The group $\langle E^+ \rangle$ generated by $E^+ := \{p^+ \mid p \in E\}$ is a subset of Ω .
- (c) $\forall a, b \in E \exists d_{a,b} \in E$ such that $\forall x \in E : a + (b + x) = (a + b) + d_{a,b} \cdot x$ and moreover $c^\bullet \circ d_{a,b}^\bullet = d_{ca,cb}^\bullet \circ c^\bullet$ for $c \in E^*$.
- (d) $\forall a \in E, \forall b \in E^* : a^+ \circ b^\bullet \in \Omega$.
- (e) $\forall \omega \in \Omega \exists a \in E, b \in E^*$ with $\omega = a^+ \circ b^\bullet$.
- (f) $\forall a, b, c, x \in E : c(d_{a,b}x) = (cd_{a,b})x$.
- (g) $ab = e \Rightarrow \forall x \in E : a(bx) = x$.

Proof. (1) $a^\bullet \circ b^\bullet \circ a^\bullet \in a^\bullet \circ \Omega \circ a^\bullet \subseteq \Omega$, by (S), $a^\bullet \circ b^\bullet \circ a^\bullet(o) = o$ and $a^\bullet \circ b^\bullet \circ a^\bullet(e) = a \cdot (b \cdot a)$. Hence $a^\bullet \circ b^\bullet \circ a^\bullet = (a \cdot (b \cdot a))^\bullet$.

- (2) Since $(E, +)$ is a K-loop we have $a + (b + x) = a^+ \circ b^+(x) = (a + b)^+ \circ \delta_{a,b}(x)$ where $\delta_{a,b} = ((a + b)^+)^{-1} \circ a^+ \circ b^+ \in \text{Aut}(E, +)$ hence $\delta_{a,b}(o) = o$ and $\delta_{a,b} = o^\circ \circ (a + b)^\circ \circ a^\circ \circ o^\circ \circ b^\circ \circ o^\circ$ with $o^\circ, (a + b)^\circ, a^\circ, (o^\circ \circ b^\circ \circ o^\circ) \in (\mathfrak{J}_E \cup \{id\})$. Therefore by (S^o) , $\delta_{a,b} \in \Omega_o$. If $d_{ab} := \delta_{a,b}(e)$ then $d_{ab}^\bullet = \delta_{a,b}$ and $a + (b + x) = (a + b) + d_{ab} \cdot x$.

- (3) We may assume, $a \neq o$ then $a^\bullet \in \Omega_o$ and by Theorem 2.2. (3), $a^\bullet \in \text{Aut}(E, +)$ hence $a \cdot (b + c) = a^\bullet(b + c) = a^\bullet(b) + a^\bullet(c) = a \cdot b + a \cdot c$.

- (4) a) Let $\omega \in \Omega$ hence by Theorem 2.1., $\omega^2 \in \Omega$ and so $\omega^2 \circ \mathfrak{J}_E \subseteq \Omega$. This is equivalent to $\omega \circ \mathfrak{J}_E \circ \omega^{-1} \subseteq \omega^{-1} \circ \Omega \circ \omega^{-1} \subseteq \Omega$.

- (4) b) Since $p^+ = p^\circ \circ o^\circ \in \mathfrak{J}_E \circ \mathfrak{J}_E \subseteq \Omega \circ \mathfrak{J}_E \subseteq \Omega$ and $a^+ \circ b^+ \in \Omega \circ b^\circ \circ o^\circ \subseteq (\Omega \circ \mathfrak{J}_E) \circ \mathfrak{J}_E \subseteq \Omega \circ \mathfrak{J}_E \subseteq \Omega$ we obtain by induction, $\langle E^+ \rangle \subseteq \Omega$.

- (4) c) Since $(E, +)$ is a K-loop we have $a + (b + x) = a^+ \circ b^+(x) = (a + b)^+ \circ \delta_{a,b}(x)$ where by b) $\delta_{a,b} = ((a + b)^+)^{-1} \circ a^+ \circ b^+ \in \Omega$. Since $\delta_{a,b}(o) = o$, $\delta_{a,b} \in \Omega_o \subseteq \text{Aut}(E, +)$ (cf. Theorem 2.2. (3)) and therefore if $d_{ab} := \delta_{a,b}(e)$ then $d_{ab}^\bullet = \delta_{a,b}$ and $a + (b + x) = (a + b) + d_{ab} \cdot x$. Moreover by (3) and (4) a), $c^\bullet \circ a^+ \circ b^+(x) = c \cdot (a + (b + x)) = c \cdot a + (c \cdot b + c \cdot x) = (c \cdot a + c \cdot b) + d_{c \cdot a, c \cdot b} \cdot (c \cdot x)$ and $c \cdot (a + (b + x)) = c \cdot ((a + b) + d_{a,b} \cdot x) = (c \cdot a + c \cdot b) + c \cdot (d_{a,b} \cdot x)$ hence $c^\bullet \circ d_{a,b}^\bullet = d_{ca,cb}^\bullet \circ c^\bullet$.

- (4) d) By (4) a), $\mathfrak{J}_E \circ \Omega \subseteq \Omega$ hence $a^+ \circ b^\bullet = a^\circ \circ o^\circ \circ b^\bullet \in a^\circ \circ (\mathfrak{J}_E \circ \Omega_o) \subseteq \mathfrak{J}_E \circ \Omega \subseteq \Omega$.

- (4) e) Let $a := \omega(o)$ and $b := o^\circ \circ a^\circ \circ \omega(e)$ then $b \neq o$ hence by (4) d), $a^+ \circ b^\bullet \in \Omega$ and since $a^+ \circ b^\bullet(o) = a = \omega(o)$ and $a^+ \circ b^\bullet(e) = a + b = \omega(e)$ we have $\omega = a^+ \circ b^\bullet$ because Ω is sharply 2-transitive.

- (4) f) From (4)(d), (e) it follows that $\text{Aff}(E, +, \cdot) = \Omega$ hence (f) is a consequence of Theorem 1.5 (6). (4)(g) follows from Corollary 1.7. $\square$

Corollary 2.4. *If (E, Ω) satisfies (S'') then (E, Ω) is of class II or III.*

Proof. The proof of (4)(a) did not use the assumption about the class of (E, Ω) , hence it follows from Theorem 2.1(6)(a). $\square$

Remark 4. If in Theorem 2.3 (E, Ω) satisfies (S'') then $(E, +, \cdot)$ is a quasi-domain called the *associated* quasi-domain of (E, Ω) .

From Theorems 2.3. and 1.5. we obtain:

Corollary 2.5. *Let (E, Ω) be of class II or III then:*

- (1) *If (E, Ω) satisfies (S^o) and (S') then $(E, +, \cdot)$ is a quasi-domain where $(E, +)$ is a K-loop and $(E^*, \cdot)$ is a Bol loop.*
- (2) *If (E, Ω) satisfies (S'') then $(E, +, \cdot)$ is a quasi-domain satisfying the condition (6)(a) and (7)(a) of Theorem 1.5 and therefore $\text{Aff}(E, +, \cdot) = \Omega = \Omega^{-1}$.*

Corollary 2.6. *If in a quasi-domain $(Q, +, \cdot)$, $(Q, \cdot)$ satisfies the Bol-identity and the condition (6)(a) of Theorem 1.5 then $(Q, +, \cdot)$ satisfies the condition (7)(a) of Theorem 1.5 and $(Q, +)$ is a K-loop.*

Proof. By Theorem 1.6 (2)(b), $(Q, \text{Aff}(Q, +, \cdot))$ satisfies (S) and by Corollary 1.7 the conditions (7)(a),(b) of Theorem 1.5. By Theorem 1.5(7),(9), $(Q, +, \cdot)$ also satisfies (8)(b). Hence, by Theorem 1.6(2)(a), $(Q, \text{Aff}(Q, +, \cdot))$ satisfies (*). Thus, by Corollary 2.4, $(Q, \text{Aff}(Q, +, \cdot))$ is a sharply 2-transitive permutation set of class II or III and $(Q, +)$ is a K-loop, by Theorem 2.3. $\square$

3. Properties of the Germ of a Symmetric 2-Structure

By Theorems 1.2. and 1.4., the theory of symmetric 2-structures is equivalent with the theory of sharply 2-transitive symmetric permutation sets and by [6] there are three classes of symmetric 2-structure $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ depending whether for $K \in \mathfrak{K}$ and $p \in K$ we have $|(p \perp K)_{\mathfrak{K}}| > 1$ (Class I) or $(p \perp K)_{\mathfrak{K}} = \emptyset$ (Class II) or $|(p \perp K)_{\mathfrak{K}}| = 1$ (Class III). By [7], each symmetric 2-structure of class III is *point-symmetric*, i.e. $\forall (A, B) \in \mathfrak{K}^{2\perp} : A \cap B \neq \emptyset$.

We have the corresponding classification of sharply 2-transitive symmetric permutation sets (E, Ω) : Let again $o \in E$ be fixed and let $\Omega_o := \{\omega \in \Omega \mid \omega(o) = o\}$ be the stabilizer of the element o . Then (E, Ω) belongs to class I if $|\Omega_o \cap \mathfrak{J}_E| > 1$, to class II if $\Omega_o \cap \mathfrak{J}_E = \emptyset$ and to class III if $|\Omega_o \cap \mathfrak{J}_E| = 1$. If (E, Ω) belongs to class III then $\forall \alpha \in \Omega \cap \mathfrak{J}_E, |\text{Fix } \alpha| = 1$. This classification agrees with that given after Theorem 2.1.

If (E, Ω) is a sharply 2-transitive permutation group then the condition (S) is satisfied. By [10, Satz (11.10)] there are two types of such permutation groups characterized by

“ $\forall \alpha \in \mathfrak{J}_E : |\text{Fix } \alpha| = 1$ ” hence (E, Ω) belongs to class III and by

“ $\forall \alpha \in \mathfrak{J}_E : \text{Fix } \alpha = \emptyset$ ” hence (E, Ω) belongs to class II.

By [1, § 5], to (E, Ω) there corresponds a neardomain $(E, +, \cdot)$ such that Ω is the affine group of the neardomain and (E, Ω) belongs to class III if $\text{Char}(E, +, \cdot) \neq 2$ and to class II if $\text{Char}(E, +, \cdot) = 2$.

We can summarize:

Theorem 3.1. *Let $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a 2-structure, $E \in \mathfrak{K}$ and $(E, \Omega) := \Pi_E(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ the germ. Then:*

- (A) *Let $o \in E$ be fixed, $\Omega_o := \{\omega \in \Omega \mid \omega(o) = o\}$, $\mathfrak{J}_E := \{\omega \in \Omega \mid \omega^2 = \text{id} \neq \omega\}$ and let Σ be symmetric then*
 - (1) Σ *belongs to class I* $\Leftrightarrow |\Omega_o \cap \mathfrak{J}_E| > 1$.
 - (2) Σ *belongs to class II* $\Leftrightarrow \Omega_o \cap \mathfrak{J}_E = \emptyset$.
 - (3) Σ *belongs to class III* $\Leftrightarrow |\Omega_o \cap \mathfrak{J}_E| = 1$.
- (B) *If Σ is point symmetric then (E, Ω) has the following properties:*
 - (1) $\forall a \in E \exists_1 a^\circ \in \mathfrak{J}_E$ *with* $\text{Fix } a^\circ = \{a\}$.
 - (2) $\forall a, b \in E \exists_1 c \in E$ *with* $c^\circ(a) = b$.
 - (3) $\forall \omega \in \Omega, \forall a \in E$ *and* $a' := \omega(a) : \omega \circ a^\circ = a'^\circ \circ \omega$ *and* $\omega \circ a^\circ \in \Omega$.
- (C) *The following statements are equivalent:*
 - (1) Σ *is double symmetric.*
 - (2) (E, Ω) *is a sharply 2-transitive permutation group.*
 - (3) *There is a neardomain $(E, +, \cdot)$ such that (E, Ω) is the affine group of the neardomain.*
- (D) *If Σ is double symmetric then Σ belongs to class II or III and the following statements are equivalent:*
 - (1) $(P, \mathfrak{B})$ *with* $\mathfrak{B} := \mathfrak{G}_1 \cup \mathfrak{G}_2 \cup \mathfrak{K}$ *is an affine plane.*
 - (2) $(E, +, \cdot)$ *is a planar nearfield*
- (E) *If Σ is finite then $(P, \mathfrak{B})$ is an affine plane.*

Because of Theorem 3.1. we have now to focus our interest on symmetric 2-structures which are not double symmetric.

3.1. Some Properties of a Point Symmetric 2-Structure

Let $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a point symmetric 2-structure, i.e. $\forall (A, B) \in \mathfrak{K}^{2\perp} : |A \cap B| = 1$, let $E \in \mathfrak{K}$ be fixed, let $(\mathfrak{C}, \cdot) := (\mathfrak{C}, \cdot)_E$, let $\mathfrak{J}_{\mathfrak{K}} := \mathfrak{J} \cap \mathfrak{K}$ and for $A \in \mathfrak{K}$ let here $A_{\mathfrak{K}}^\perp := A^\perp \cap \mathfrak{K}$. Then we have (cf. [7]):

- Theorem 3.2.**
- (1) $\forall p \in P, \forall K \in \mathfrak{K} : |(p \perp K)_{\mathfrak{K}}| = 1$.
 - (2) $\forall (A, B), (C, D) \in \mathfrak{K}^{2\perp}$ *with* $A \cap B = C \cap D : \tilde{A} \circ \tilde{B} = \tilde{C} \circ \tilde{D}$ *and if* $p := A \cap B$ *then* $\tilde{p} := \tilde{A} \circ \tilde{B}$ *is the reflection in the point* p .
 - (3) *For* $a \in P$ *let “* $\alpha_1 : P \rightarrow P ; p \mapsto \tilde{a}p(p)$ *” and “* $\alpha_2 : P \rightarrow P ; p \mapsto \tilde{p}a(p)$ *” then* α_i *is the reflection in the generator* $[a]_i$ *also denoted by* $[\tilde{a}]_i$ *and* $\alpha_i \in \text{Aut}(P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$.
 - (4) *We have :* $\tilde{a} = \alpha_1 \circ \alpha_2 = \alpha_2 \circ \alpha_1$.
 - (5) *If* $K \in \mathfrak{K}$ *and* $k := K \cap [a]_i$ *then* $\alpha_i(K) = (k \perp K)_{\mathfrak{K}}$.

Theorem 3.3. *Let $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ be a point symmetric 2-structure. Then:*

- (1) $\forall A, B \in \mathfrak{K} : \widetilde{AB}(A_{\mathfrak{K}}^{\perp}) = B_{\mathfrak{K}}^{\perp}$.
- (2) $\forall A, B \in \mathfrak{K} \ \forall C \in A_{\mathfrak{K}}^{\perp} : A \cdot C^{-1} \cdot B \in \mathfrak{K}$.
- (3) $\mathfrak{K} \cdot \mathfrak{J}_{\mathfrak{K}} = \mathfrak{K}$ and $\mathfrak{J}_{\mathfrak{K}} \cdot \mathfrak{K} = \mathfrak{K}$.

Proof. (1) Let $x \in P$ and $y := \widetilde{AB}(x) = (Bx)(xA) = [Bx]_1 \cap [xA]_2$.

Let $a := (x \perp A)_{\mathfrak{K}} \cap A$ (hence $(x \perp A)_{\mathfrak{K}} = (a \perp A)_{\mathfrak{K}}$), let $\alpha_2 := \overline{[a]}_2$ be the reflection in the generator $[a]_2$ (hence by Theorem 3.2 (5), $\alpha_2((a \perp A)_{\mathfrak{K}}) = A$) and let $b := [a]_2 \cap B$. Then $\widetilde{AB}(a) = b$ and again by Theorem 3.2 (5), $\alpha_2(B) = (b \perp B)_{\mathfrak{K}}$. Since $x = [x]_1 \cap (a \perp A)_{\mathfrak{K}}$, $\alpha_2(x) = [x]_1 \cap A = xA$ hence $\alpha_2([x]_2) = [xA]_2$ and $\alpha_2(Bx) = [Bx]_1 \cap [xA]_2 = y$ implying $\alpha_2(B) = \alpha_2(b, Bx) = \overline{b, y} \perp B$. Thus $y \in (b \perp B)_{\mathfrak{K}}$, i.e. $\widetilde{AB}(A_{\mathfrak{K}}^{\perp}) = B_{\mathfrak{K}}^{\perp}$.

- (2) Let $C \in A_{\mathfrak{K}}^{\perp}$ hence $A \cdot C^{-1} \cdot A = C$ then by (1), $D := \widetilde{AB}(C) = A \cdot C^{-1} \cdot B \in B_{\mathfrak{K}}^{\perp} \subseteq \mathfrak{K}$.
- (3) Let $A \in \mathfrak{K}$ and $B \in \mathfrak{J}_{\mathfrak{K}}$ hence $B = B^{-1} \in E^{\perp}$. Then by (2), $A \cdot B^{-1} \cdot E = A \cdot B \in \mathfrak{K}$. By Theorem 1.4.(3), $\mathfrak{K} = \mathfrak{K}^{-1}$ and so $\mathfrak{K} \cdot \mathfrak{J}_{\mathfrak{K}} = \mathfrak{K}$ implies $\mathfrak{J}_{\mathfrak{K}} \cdot \mathfrak{K} = \mathfrak{K}$. $\square$

3.2. A Representation of a Point Symmetric 2-Structure by a Quasi-Domain

In the point symmetric 2-structure $\Sigma := (P, \mathfrak{G}_1, \mathfrak{G}_2, \mathfrak{K})$ let $E \in \mathfrak{K}$ and $(o, e) \in \binom{E}{2}$ be fixed. Let $(E, \Omega) := \Pi_E(\Sigma)$ be the germ of Σ (by Theorem 1.4, (E, Ω) is a sharply 2-transitive symmetric permutation set) and let $(E, +, \cdot)$ be the double-loop-derivation of (E, Ω) in the points (o, e) (cf. Sect. 2.2). By applying the germ operation Π_E onto Σ we obtain from Theorems 3.3.(3) and Corollary 2.5.(2):

Theorem 3.4. *The germ $(E, \Omega) := \Pi_E(\Sigma)$, has the following properties:*

- (1) (E, Ω) satisfies besides (S) also (S'') hence $(E, +, \cdot)$ is a quasi-domain.
- (2) (E, Ω) is of class III.
- (3) Any two points $a, b \in E$ have exactly one midpoint $m \in E$, i.e. there is exactly one involution $\mu \in \mathfrak{J}_E$ with $\text{Fix } \mu = m$ and $\mu(a) = b$.

Now we state our main result on point symmetric 2-structures:

Theorem 3.5. *If Σ is a point symmetric 2-structure and $(E, \Omega) = \Pi_E(\Sigma)$ the germ then $(E, +, \cdot)$ is a quasi-domain of $\text{char } E \neq 2$ and:*

- (1) $\forall a, b, c, x \in E : c(d_{a,b}x) = (cd_{a,b})x$.
- (2) $(E, +)$ is a uniquely 2-divisible K-loop where o is the neutral element.
- (3) $(E^*, \cdot)$ is a Bol loop where e is the neutral element.
- (4) $\Omega = \text{Aff}(E, +, \cdot)$.

Conversely: Let $(E, +, \cdot)$ be a quasi-domain of $\text{char } E \neq 2$ satisfying the conditions (1) and (3) and let $\Omega = \text{Aff}(E, +, \cdot)$ then the chain derivation $\kappa((E, \Omega))$ is a point symmetric 2-structure.

Proof. “ $\Rightarrow$ ” By Theorem 1.4.(4), (E, Ω) is a sharply 2-transitive symmetric permutation set and so, by Theorem 2.2., $(E, +)$ is a uniquely 2-divisible K-loop. By Theorem 3.3.(3), $\mathfrak{K} \cdot \mathfrak{J}_{\mathfrak{K}} = \mathfrak{K}$ hence $\Omega \circ \mathfrak{J}_E = \Omega$, i.e. (E, Ω) satisfies (S'') , hence the other statements follow from Theorem 2.3 (4).

“ $\Leftarrow$ ” The condition (1) implies by Theorem 1.5(6) that (E, Ω) is a sharply 2-transitive permutation set. The condition (3) implies by Theorem 1.6(2)(b) that (E, Ω) is symmetric. Therefore by Theorem 1.4(4) $\kappa((E, \Omega))$ is a symmetric 2-structure. Moreover by Corollary 1.7, (1) and (3) imply that $(E, +, \cdot)$ satisfies the condition (7)(a) of Theorem 1.5, hence also the condition (8)(b) of Theorem 1.5. Now if $a^+ \circ b^\bullet \in \Omega_o \cap \mathfrak{J}_E$ then $o = a^+ \circ b^\bullet(o) = a + o = a$ hence $a^+ \circ b^\bullet = b' \in \mathfrak{J}_E$ implies $b(bx) = x$ in particular $b \cdot b = e$ and $b \neq e$. Therefore $\Omega_o \cap \mathfrak{J}_E = \{(-e)^\bullet\}$, by [2, (1.4)(a)] and by Theorem 3.1(A)(3), $\kappa((E, \Omega))$ is of class III. $\square$

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