Book Review

Nonlinear Dynamical Systems of Mathematical Physics by Denis Blackmore, Anatoliy K. Prykarpatsky and Valeriy Hr Samolyenko, Spectral and Symplectic Integrability Analysis, World Scientific, 2011; ISBN: 13 978-981-4327-15-2

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Physics concerns itself with the most fundamental principles and laws of the nature. But what is mathematical physics? The concise answer may be following: mathematical physics deals with the formulation and analysis of physical problems by means of mathematical constructions and apparatus. These include the application of existing mathematical techniques to physics problems and the development of new mathematical ideas for their study.

Nonlinear Dynamical Systems of Mathematical Physics by D. Blackmore et al. is an excellent text devoted to nonlinear dynamical systems in context of the spectral and symplectic integrability analysis, addressed to advanced undergraduate and graduate students of exact and natural sciences.

The book is organized into fourteen chapters.

Chapter 1, *General Properties of Nonlinear Dynamical Systems*, is the terse introduction to the theory of dynamical systems on finite-dimensional symplectic manifolds. Basic mathematical concepts are presented and described concisely: the notion of the phase space of the dynamical system; the Liouville condition; the famous Poincaré theorem; the Birkhoff–Khinchin theorem on ergodicity and an analogous theorem for a discrete dynamical systems; the Poissonian formalism; the Hamilton–Jacobi technique and the procedure of Dirac reduction of Poisson operators on submanifolds.

Chapter 2 focuses on the *Geometric and Algebraic Properties of Nonlinear Dynamical Systems with Symmetry*. Some remarks on the Poisson structures and Lie group actions on Poisson manifolds together with the research programme relating to dynamical systems possessing an intrinsic symmetry structure, are presented first. Next, the canonical reduction method applied to geometric forms on symplectic manifolds with symmetry and associated canonical connections are discussed. The special fiber-bundle structure used to the study of the motion of a charged particle under an abelian Yang–Mills type gauge field equations, is analysed also. This approach is known as the principle of minimal interaction and is very useful for studying various interacting physical systems. The last section introduces the reader to the world of classical and quantum integrable systems. Starting with the technical definition of an integrable classical system provided by the well-known Liouville theorem, the basic aspects of integrability are studied including elements of the Hopf and quantum algebras theory. On the basis of the structure of Casimir elements associated with Hopf algebras, an integrable flows connected with the naturally induced Poisson structures on an arbitrary co-algebra and their deformations, are described. The special cases, including co-algebra structures related to the oscillator co-algebra (Heisenberg–Weil co-algebra) are constructed also.

Integrability by Quadratures of Hamiltonian and Picard Fuchs Equations is the theme of Chapter 3. First, a terse overview of the basic results on integrability by quadratures in the historical context is presented. This includes: the Bour–Liouville theorem and its

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generalizations—Liouville–Arnold, and Mischenko–Fomenko theorems. Subsequently, a symplectic theory is developed for solving the problem of algebraic-analytical construction of the corresponding integral submanifold embeddings for integrable Hamiltonian dynamical systems on canonically symplectic phase spaces. First, the construction via the abelian Liouville–Arnold theorem is considered for which the related Picard–Fuchs–type equations are obtained. The derivation is presented in a detailed manner, and attention is given to some subtleties. Next, the appropriate construction via the nonabelian Liouville–Arnold theorem is described. As a result, a wide class of exact solutions represented by quadratures can be produced. Thereafter, two examples of nonabelian Liouville–Arnold integrability by quadratures are discussed, including point vortices in the plane and the material point motion in a central potential field. The remainder of Chapter 3 is devoted to the analysis of the existence problem for a global set of invariants that shows, in particular, that nonabelian Liouville–Arnold integrability by quadratures does not in general imply integrability via the abelian Liouville–Arnold theorem. Finally, two applications of the integral submanifold embedding for abelian Liouville–Arnold integrable dynamical systems are presented. The famous Hénon–Heiles system and the truncated four-dimensional Fokker–Planck flow on two-dimensional tori are described and analysed.

In Chapter 4, an introduction to the theory of *Infinite-dimensional Dynamical Systems* is presented. This theory is exceptionally complicated and very far from the end, especially in the ergodicity and the complete integrability. Some aspects of the Hamiltonian formalism for infinite-dimensional dynamical systems in the frames of the Poisson description are discussed first. Then, the symmetry properties of systems written in the form $du/dt = K[u]$ (flows), where M is the manifold, $u \in K : M \rightarrow T(M)$ is the locally Fréchet smooth on M , $T(M)$ is the tangent space, are considered. The Bäcklund transformations for such flows are defined and their properties are described. The problem of integrability of a Hamiltonian infinite-dimensional dynamical system under the condition that it has an infinite set of nontrivial conservation laws is also sketched. Two interesting examples of dynamical systems, in which the evolution is governed by integro-differential equations,

namely, the dynamics of a closed vortex filament in an perfect fluid, and the continuum approximation model of granular flows, finish the chapter.

Chapter 5 deals with the *Integrability Criteria for Dynamical Systems*. The gradient-holonomic method for the Lax-type integrability of certain dynamical systems in the form $du/dt = K[u]$, where u belongs to a subspace of smooth 2π -periodic functions, is presented and thoroughly analysed. The asymptotic construction applied to the problem of finding the analytical solutions for recursive and implectic structures for Lax integrable dynamical systems is described in some detail. The usefulness of this approach is shown by applying it to spectacular, well-known equations, including the Korteweg–de Vries (KdV) equation, the modified Korteweg–de Vries equation, and the nonlinear Schrödinger model (NLS) equation. Some structural properties of the symmetry Lie algebra for a compatibly bi-Hamiltonian dynamical system is also studied. The KdV equation on a Schwartz manifold is considered, as an illustrative example. Some remarks on an algebraic proof of complete integrability of Fréchet smooth dynamical system on an infinite-dimensional functional manifold are presented also.

Chapter 6 concentrates on the *Algebraic, Analytic and Differential Geometric Aspects of Integrability*. This is the longest chapter, with nine sections encompassing 142 pages of the book's 553 total. The first section contains the elements of the theory of non-isospectrally Lax integrable dynamical systems through two examples: KdV and NLS equations. A deeper discussion concerning the algebraic structure of the gradient-holonomic algorithm for Lax integrability is performed in the next section. The illustrative examples include the generalized KdV equation, the Benney–Kaup (BK) dynamical system, the inverse KdV equation (inv KdV) and the inverse BK system. The third section is devoted to the analysis of a Whitham type nonlocal dynamical system for a relaxing medium with spatial memory. The well-posedness for the Whitham-type nonlinear and nonlocal dynamical system with a suitable regularization scheme is discussed in the further section. The fifth section describes the richness of conservation laws, both dispersive non-polynomial and dispersionless polynomial, for the generalized Riemann-type hydrodynamic equations.

The next section presents a new and very effective differential-algebraic approach to examine the Lax integrability of the generalized Riemann type hydrodynamic equations and the classical KdV equation. In the subsequent section, an interesting analysis of the Maxwell field equations in the context of the well-known Dirac–Fock–Podolsky hamiltonian formulation is achieved. Symplectic approach to selected magnetohydrodynamic superfluid problems is presented in the succeeding section. In particular, the Peradzyński helicity theorem is revisited and generalized for the case of an incompressible superfluid flow. The last section concerns the algebraic-analytic structure of integrability by quadratures of Abel–Riccati (AR) equations. Some progress is observed in a special case, namely the basic Riccati equation, however general problems concerning the integrability of the AR equation still remain open.

Chapter 7 refers to a *Versal Deformations of a Dirac Operator on a Sphere*. The construction of a general expression for transversal deformations of the Dirac-type differential operator is realized. Then it is clarified by using the Lie-algebraic theory of induced $\text{Diff}(S^1)$ -actions on a suitable Poisson manifold. [$\text{Diff}(S^1)$ denotes the group of all diffeomorphisms of S^1 , where S^1 is the one-dimensional torus]. A broad family of versally deformed Dirac-type differential operators depending on complex parameters is constructed by means of the Marsden–Weinstein reduction with respect to specific Casimir-generated distributions.

Chapter 8 discusses *Integrable Spatially Three-dimensional Coupled Dynamical Systems*. A new approach is developed for constructing Lax-type integrable multidimensional nonlinear dynamical systems which allows to introduce one more variable into Lax integrable $(2 + 1)$ -dimensional dynamical systems originating as flows on dual spaces to a centrally extended matrix Lie-algebra of integro-differential operators. Such systems admit the infinite set of conservation laws and related triple Lax-type linearizations. So, their solutions can be easily obtained by the standard transformations.

The short chapter 9 concerns with *Hamiltonian Analysis and Integrability of Tensor Poisson Structures and Factorized Operator Dynamical Systems*. The problem of factorization of operator dynamical

systems formulated by Dickey (see Dickey L. A., *Lett. Math. Phys.*, 34: 379–384; 35: 229–236, 1995) written in the form of Lax flows, is analysed and solved. The method employs exclusively the standard properties of tensor-multiplied Poisson structures and suitably constructed Bäcklund transformations.

Chapter 10 considers *A Multi-dimensional Generalization of Delsarte–Lions Transmutation Operator Theory*. The spectral and analytical properties of Delsarte–Lions transformed operators are studied using the differential-geometric and topological machineries. Next, their connections with generalized de Rham–Hodge theory of special differential complexes are affirmed. As the result, the effective analytical expressions for the corresponding Delsarte transmutation bounded invertible Volterra-type operators in a prescribed Hilbert space are obtained. A special case of the application of presented theory to the Lax systems is briefly discussed. The remaining sections deal with the geometric and spectral properties of Delsarte–Darboux binary transformations and the spectral structure of Delsarte–Darboux transmutation operators in multi-dimensions with three spectacular examples.

A very concise chapter 11 treats *Characteristic Classes of Chern Type and Integrability of Systems on Riemannian Manifolds*. The geometric objects of Chern-type characteristic classes and characters are introduced and analysed. The special differential invariants of the Chern type are constructed and the integrability of multidimensional nonlinear differential systems of Gromov type on Riemannian manifolds is discussed.

Chapter 12 is the authors' personal look at the concept of the so-called *Quantum Mathematics*. A brief history of the relations between mathematics and theoretical (quantum) physics is presented first. After some preliminaries, it is shown that a reasonable class of nonlinear dynamical systems in functional spaces (described by partial differential equations in Hilbert spaces) can be perceived as generic objects in suitably constructed Fock spaces in which the corresponding evolution flows are fully linearized. The methods used here include the Gelfand–Vilenkin representation theory and the Goldin–Menikoff–Sharp theory generating Bogolubov type (or characteristic) functionals. Next, the notion of computability from the point of

view of quantum computation is investigated. The quantum algorithms for the holonomic quantum computation are described, based on geometric Lie-algebraic objects on Grassmann manifolds using the various differential techniques. Several examples that have been considered using quantum computing algorithms, e.g., (RSA)-cryptosystem, two-mode quantum-optical models and the Lax flow model are analysed in some detail.

Chapter 13 deals with the *Lagrangian and Hamiltonian Analysis of Relativistic Electrodynamics and String Models*. It offers a concise overview of the classical electrodynamics problems and demonstrates a key principles characterizing the vacuum potential field dynamical equations in the frames of the Lagrangian and Hamiltonian formalisms. Utilizing the procedures developed here, the application of canonical Dirac quantization and the corresponding derivation of the Schrödinger-type evolution equations are presented. The following section studies the classical relativistic least action principle from the vacuum field theory point of view. The new expressions for the relativistic Lorentz-type forces are derived, and the new relativistic hadronic string model is formulated and analyzed. Finally, the basic

principles of the vacuum field theory are recalled and used to derive the Maxwell electromagnetic field equations.

The book ends with the supplement pertaining to the *Basics of Differential Geometry for Dynamical Systems*. This chapter familiarizes the reader with the fundamentals of linear algebra, differential geometry and topology that are of the primary importance in dynamical systems theory.

In summary, the book is clearly and interestingly written and contains only a few minor misprints. It should be very useful for both students and active researchers in the area of contemporary integrable nonlinear dynamical systems theory. The particular part of the material should be also valuable to mathematicians and physicists working on quantum and classical field theories. I think that theoretically oriented geophysicists may also benefit greatly from this book.

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