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Neutrino Radiation Showing a Christodoulou Memory Effect in General Relativity

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Abstract. We describe neutrino radiation in general relativity by introducing the energy–momentum tensor of a null fluid into the Einstein equations. Investigating the geometry and analysis at null infinity, we prove that a component of the null fluid enlarges the Christodoulou memory effect of gravitational-waves. The description of neutrinos in general relativity as a null fluid can be regarded as a limiting case of a more general description using the massless limit of the Einstein–Vlasov system. Gigantic neutrino bursts occur in our universe in core-collapse supernovae and in the mergers of neutron star binaries.

1. Introduction and Main Results

In this paper, we prove that there is a nonlinear memory effect for neutrino radiation. We describe the neutrinos from a typical source like binary neutron star merger or core-collapse supernova as a null fluid in the Einstein equations. We compute the radiated energy, derive limits at null infinity and compare them with the Einstein vacuum (EV) case and the Einstein–Maxwell (EM) case.

We consider the Einstein equations

$$G_{ij} := R_{ij} - \frac{1}{2} g_{ij} \ R = 8\pi T_{ij} \tag{1}$$

(setting $G = c = 1$), $i, j = 0, 1, 2, 3$. g_{ij} denotes the metric tensor, R_{ij} is the Ricci curvature tensor, R the scalar curvature tensor, G_{ij} denotes the Einstein tensor and T_{ij} is the energy–momentum tensor.

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We describe the burst of neutrinos in a typical source such as core-collapse supernovae and binary neutron star mergers as a null fluid. This means that the energy–momentum tensor will have the form

$$T^{ij} = \mathcal{N} K^i K^j \tag{2}$$

with K denoting a null vector to be specified below and $\mathcal{N}$ a positive scalar function. Later on, we make use of $\sqrt{\mathcal{N}} K^i = k^i$ and

$$T^{ij} = k^i k^j. \tag{3}$$

Let L as well as $\underline{L}$ denote null vectors, properly defined below. Note that we refer to L as the null vectorfield generating the corresponding outgoing null hypersurfaces and to $\underline{L}$ as the null vectorfield generating corresponding incoming null hypersurfaces. L and $\underline{L}$ are complemented to a null frame by the S-tangent orthonormal frame $(e_A, A = 1, 2)$. Thus, the corresponding components of the energy–momentum tensor are

$$T^{LL}, T^{\underline{L}\underline{L}}, T^{L\underline{L}}, T^{AL}, T^{A\underline{L}}, T^{AB}. \tag{4}$$

Initially, the burst will take off in all directions, but eventually the null part T^{LL} will dominate, as we prove below.

Recall that the contravariant tensor T^{ij} turns into the covariant tensor T_{ij} by contracting with the metric. Thus we have

$$T^{LL} = \frac{1}{4} T_{\underline{L}\underline{L}}.$$

We show that the components of the energy–momentum tensor have the following decay behavior:

$$\begin{aligned}
T^{LL} &= O(r^{-2}\tau_-^{-4})\\
T^{AL} &= O(r^{-3}\tau_-^{-3})\\
T^{AB} &= O(r^{-4}\tau_-^{-2})\\
T^{L\underline{L}} &= O(r^{-4}\tau_-^{-2})\\
T^{A\underline{L}} &= O(r^{-5}\tau_-^{-1})\\
T^{\underline{L}\underline{L}} &= O(r^{-6}).
\end{aligned}$$

This means

$$\begin{aligned}
T_{\underline{L}\underline{L}} &= O(r^{-2}\tau_-^{-4})\\
T_{A\underline{L}} &= O(r^{-3}\tau_-^{-3})\\
T_{AB} &= O(r^{-4}\tau_-^{-2})\\
T_{L\underline{L}} &= O(r^{-4}\tau_-^{-2})\\
T_{AL} &= O(r^{-5}\tau_-^{-1})\\
T_{LL} &= O(r^{-6}).
\end{aligned}$$

The quantity τ_- is defined as $\tau_- = \sqrt{1+u^2}$. We use foliations of our spacetime by a time function t and an optical function u as explained below.

We state our equations with all the components of the energy–momentum tensor. Their decay behavior emerges from the physical model and the corresponding mathematical consequences.

In this paper, the covariant differentiation on the spacetime M we denote by D or ∇, whereas the one on a spacelike hypersurface H is $\overline{\nabla}$ or ∇. It is clear from the context what ∇ refers to.

The twice contracted Bianchi identities imply that

$$D^j G_{ij} = 0. \tag{5}$$

This is equivalent to the following equation, namely, that the divergence of the energy–momentum tensor of the null fluid vanishes:

$$D^j T_{ij} = 0. \tag{6}$$

Since T_{ij} is traceless, the Einstein equations (1) for a null fluid reduce to

$$R_{ij} = 8\pi T_{ij}. \tag{7}$$

We prove that there is a contribution from neutrino radiation to the nonlinear Christodoulou memory effect of gravitational-waves. When describing neutrino radiation by a null fluid and coupling the energy–momentum tensor of a null fluid to the Einstein vacuum equations, we find the energy radiated away per unit angle in a given direction to be $F/8\pi$ with

$$F(\cdot) = \int_{-\infty}^{+\infty} \left(\mid \Xi(u,\cdot) \mid^2 + 2\pi T^*_{\underline{LL}}(u,\cdot) \right) \mathrm{d}u. \tag{8}$$

The limit $T^*_{\underline{LL}}$ of $T_{\underline{LL}}$ is positive. See Eq. (2) and the subsequent paragraph. Here and in what follows Ξ and Σ denote limits for fixed u as $t \to \infty$ of the shears $\hat{\underline{\chi}}$ and $\hat{\chi}$, respectively. The latter are the traceless parts of the second fundamental forms $\underline{\chi}$ and χ, which are introduced at the beginning of Sect. 2.1. The null limits are derived in Theorems 2 and 3, respectively.

Considering gravitational radiation in the presence of an electromagnetic field as well as neutrino radiation, as present typically in binary neutron star mergers, we investigate the Einstein–Maxwell null-fluid equations and find similarly the energy radiated away per unit angle in a given direction to be $F/8\pi$ with

$$F(\cdot) = \int_{-\infty}^{+\infty} \left(\mid \Xi(u,\cdot) \mid^2 + \frac{1}{2} \mid A_F(u,\cdot) \mid^2 + 2\pi T^*_{\underline{LL}}(u,\cdot) \right) \mathrm{d}u. \tag{9}$$

The term A_F denotes the limit of the corresponding component of the electromagnetic field as given in [5,6]. The permanent displacement formula of the nonlinear memory effect involves formula (8), respectively (9). In both cases, we have a contribution from neutrino radiation given by the term $2\pi T^*_{\underline{LL}}(u,\cdot)$.

For the corresponding result in Einstein–Maxwell theory, where F turns out to be like in (9) but without the null-fluid term, see the articles [5,6] by the first present author with co-authors P. Chen and S.-T. Yau. In the pioneering paper by Christodoulou [10], the formula reduces to the contribution from only Ξ.

The present authors with co-authors P. Chen and S.-T. Yau have work in progress to generalize the results of this paper using the more general description of the massless limit of the Einstein–Vlasov system to derive a complete kinetic theory for neutrinos in general relativity.

The Einstein null-fluid equations (1), respectively (7) investigated in this article are proven to have geodesically complete solutions for physical initial data by the first present author in [4]. What could happen in general, is that the null geodesics may intersect. To prevent that, one has to specify corresponding initial data for the null fluid. We work with initial data which is asymptotically flat. Whereby the geometric part, that is the induced metric at time $t = 0$ and the corresponding second fundamental form of the initial spacelike hypersurface may behave as in [11]. Necessary conditions on the null fluid for this spacetime to be non-singular are given in [4]. In particular, considering (2), in the region of compact support we choose the density to be 1 and the vectorfield in the outside to decrease appropriately. Moreover, outside the region of compact support, the vectorfield is directed outwards. In [4] the Cauchy problem is solved to prove that the corresponding solution is non-singular. In particular, the local result is implied by the implicit function theorem, whereas the global result is achieved by a bootstrap argument.

The main theorem of the present paper, proven in Sect. 4.3, is stated as follows.

Main Theorem. (Theorem 6) *Denote by $\Sigma^+(\cdot)$ the limit $\Sigma^+(\cdot) = \lim_{u\to\infty} \Sigma(u,\cdot)$ and by $\Sigma^-(\cdot)$ the limit $\Sigma^-(\cdot) = \lim_{u\to-\infty} \Sigma(u,\cdot)$. Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \big(\mid \Xi(u,\cdot) \mid^2 + 2\pi T^*_{\underline{L}\underline{L}}(u,\cdot)\big)\, \mathrm{d}u.$$

Also, let Φ be the solution with $\bar{\Phi} = 0$ on S^2 of the equation

$$\overset{\circ}{\not{\triangle}}\ \Phi = F - \bar{F}.$$

Then $\Sigma^+ - \Sigma^-$ is determined by the following equation on S^2.

$$\overset{\circ}{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\ (\Sigma^+ - \Sigma^-) = \overset{\circ}{\not{\nabla}}\ \Phi.$$

Σ is a function of u at null infinity. This theorem relates the difference $\Sigma^+ - \Sigma^-$ to the energy radiated away as given in formula (8). The proof reveals the interesting nonlinear analytic as well as geometric structure captured by the quantities that go into the energy. In the last section, $\Sigma^+ - \Sigma^-$ is shown to be related to a permanent displacement of test masses in a gravitational-wave detector.

Strategy of Proof and Known Results. In this article, we show that the geometric outcomes of [11] for the Einstein vacuum equations carry over to the Einstein null-fluid situation in an appropriate sense. Important results for a non-vanishing energy–momentum tensor are obtained in [21,22] solving the nonlinear stability problem for Einstein–Maxwell equations, and more recently in [5,6] by the first present author and co-authors solving the radiation problem for Einstein–Maxwell equations. In these works including electromagnetic fields it is shown in detail how the corresponding energy–momentum tensor affects the emerging spacetime geometry. The main results and strategies carry over directly to our new investigations for the energy–momentum tensor of a null fluid. In fact, the most important changes from Einstein vacuum are related with the main theorem. More precisely, under physical assumptions as in these papers, it is shown that in particular the curvature and its derivatives continue to be controlled in corresponding norms even though the energy–momentum tensor is non-trivial. Moreover, it is proven that the geometric quantities are not affected in any other sense than the one mentioned above. In the present paper, we show carefully where the energy–momentum tensor of a null fluid plays a crucial role and where its effects are negligible. The important steps of the proof are carried out in detail. Whenever the structure of the equations and extra terms are of the same nature as in the Einstein–Maxwell case and the current problems can be solved in the same way, we refer to the works cited above.

For our purposes, the main property of the stress–energy is that the component T^{LL} goes like r^{-2} in the approach to null infinity and all other components decay faster. (In physical terms, $r^2 T^{LL}$ at large r is the energy radiated in neutrinos per unit solid angle per unit time.) This is the same behavior found in [5,6,21,22] for the stress–energy of the electromagnetic field. This fall off of the stress–energy ensures that certain components of the Weyl tensor have the same fall off at null infinity as they do in the vacuum case, because any contribution of the stress–energy becomes negligible at large r. It also ensures the existence of the Bondi mass as the limit at null infinity of the Hawking mass. Furthermore it ensures a formula for the time rate of change of the Bondi mass in which T^{LL} plays the same role as the corresponding quantity in the Einstein–Maxwell case and indeed the same role as the corresponding energy flux in gravitational radiation. In the vacuum case, the gravitational-wave energy radiated per unit solid angle gives rise to the Christodoulou nonlinear memory effect. We show that the neutrino energy radiated per unit solid angle also contributes to this memory effect.

Outline of Proof. The proof of the main theorem relies on the geometric-analytic properties of the Einstein null-fluid equations. In Sect. 2, we introduce the setup and derive properties of the energy–momentum tensor. We also prove that the neutrino flow converges to null geodesics generated by L for later times. In Sect. 3, we give the equations controlling the shears, which play a fundamental role for the memory effect. The first parts of Sect. 4 derive important limits at null infinity, such as the Bondi mass, and investigate the quantities that go into the energy F as in formula (8). Then the proof of the main theorem is carried out in Sect. 4.3. First, we investigate the null Codazzi equation for the shear $\hat{\chi}$. It is then shown that the main questions reduce to solving a Hodge system on $S_{t,u} = H_t \cap C_u$, the intersection of a spacelike hypersurface H_t with a null hypersurface C_u, for a specific component ϵ of the second fundamental form k. In these equations, the energy–momentum components $T_{\underline{LL}}$ and T_{LL} occur. Whereas T_{LL} has enough decay in order not to interfere with the leading order 'purely geometric' terms, the component $T_{\underline{LL}}$ plays the crucial role and is proven to contribute to the energy F and therefore to the difference of the limits $\Sigma^+ - \Sigma^-$. The specific structures of these equations as well as of their null limits are used to derive the main result. The remaining part of this paper derives further limits and relates the result of the main theorem to the permanent displacement in a gravitational-wave detector using the Jacobi equation and the generic structures of the Riemannian curvature tensor of the Einstein null-fluid spacetime.

1.1. Nonlinear Christodoulou Memory Effect of Gravitational Waves

The general theory of relativity predicts gravitational-waves. There has been vast literature about this topic. Ongoing and future experiments like LIGO or LISA want to measure these waves directly. Other experiments based on radio astronomy aim at measuring the Christodoulou effect of these waves, which means that they would be detected through this nonlinear effect.

A gravitational-wave train will have two different effects on the test masses. A wave train traveling from its source to us, will pass the experiment. During the passage of such a wave train the test masses will experience 'instantaneous displacements'. Afterwards, the test masses will show 'permanent displacements'. The latter is the nonlinear memory effect (Christodoulou effect) of gravitational-waves. Thus, the spacetime has been changed permanently. Such an effect was known in a linear theory [20], but its contribution was very small and people believed it negligible. See also [8,9]. Christodoulou [10] shows that this is a truly nonlinear effect and as such its contribution is much larger than expected. He computes and investigates exact solutions of the Einstein equations; no approximation is used.

Since the pioneering days, when Christodoulou [10] established his nonlinear result for the EV equations, showing that gravitational-waves displace test masses permanently, it has been an open problem if electromagnetic fields in the EM equations contribute to the nonlinear effect. In the work of the first author with Chen and Yau [5,6], we solve this problem and apply the new findings to astrophysical data. We show [5] how the electromagnetic field in the EM equations contributes to the nonlinear memory effect of gravitational-waves. And we investigate the effect on gravitational-wave detectors. Precise formulas derived with geometric-analytic methods from the EM equations are related to experiment. And predictions for measurements are stipulated. We apply the new results to astrophysical data [6].

We would like to emphasize that neither in the works [5,6] nor in this article, any approximation is used. The first author with Chen and Yau [5,6] as well as the present authors in this paper derive and investigate exact solutions of the EM, respectively, Einstein null-fluid equations. All the results mentioned in this subsection hold for large data. Examples of the latter include supernovae, mergers of black holes or neutron stars.

Typical sources of neutrino radiation are big events in the universe such as core-collapse supernovae or binary neutron star mergers. In a core-collapse supernova, the 1.4 solar mass iron core of a massive star is converted into neutrons and neutrinos. In the process, about 3×10^{53} ergs of gravitational binding energy is released, almost all of it in the form of neutrino radiation [15]. In addition to the gravitational-wave effects of the neutrino radiation, the neutrinos themselves can be detected thus enhancing the possibility of detection through a joint gravitational-wave and neutrino search [13]. Since neutrinos are the dominant form of energy loss, one might think that the gravitational-waves generated by neutrinos (and the associated gravitational-wave memory) would be the most easily detected gravitational-wave signature from core-collapse supernovae. Unfortunately, this turns out not to be the case [14,15]. To begin with, the spherically symmetric part of the energy emission produces no gravitational-waves, so it is only the fraction (about 2 % or so [14]) of the neutrino energy emission that is anisotropic that produces gravitational-waves. These gravitational-waves are higher amplitude than those produced by the matter in the supernova core; however, their frequency is also smaller. This is because though the neutrinos are produced promptly in the collapse of the core, it takes them a time of the order of a second to escape from the extremely high density region of the collapsing core. Thus neutrino emission and the associated gravitational-waves have a time scale of about a second, or equivalently a frequency scale of about 1 Hz. For LIGO and the other ground-based gravity wave detectors, 1 Hz is the low frequency range where seismic noise in the detectors is large [18]. This makes detecting gravitational-wave memory in core-collapse supernova very challenging. However, current improvements being made to gravity wave detectors include an improvement in seismic isolation and thus a lessening in detector noise at low frequency. Thus in addition to the overall improvement in the possibility of detecting gravitational-waves, the detectors should also have an improved possibility of detecting gravitational-wave memory.

A binary neutron star system consists of two neutron stars in orbit around each other. Such a system loses energy in gravitational radiation, causing the neutron stars to orbit at ever smaller distances (referred to as "inspiral") until they merge [19]. The merged object is a supermassive neutron star with large thermal energy as well as a flattened shape caused by its high rotation speed. It is estimated [12] that the supermassive neutron star radiates neutrinos with a power of about 5×10^{52} erg/s over a time of about 1 s. Eventually the supermassive neutron star is expected to collapse to form a black hole and perhaps generate a gamma ray burst. The gravitational radiation generated by the inspiral of the neutron stars is considered to be the most promising candidate for detection of gravitational-waves [16]. The gravitational-wave memory from the burst of neutrinos will be more challenging to detect because of the longer time scale and the seismic noise in the detectors.

2. Setting

It will be useful to split the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ into its Weyl tensor $W_{\alpha\beta\gamma\delta}$ being traceless, and a part including the spacetime Ricci curvature $R_{\alpha\beta}$ and spacetime scalar curvature R:

$$R_{\alpha\beta\gamma\delta} = W_{\alpha\beta\gamma\delta} + \frac{1}{2}(g_{\alpha\gamma}R_{\beta\delta} + g_{\beta\delta}R_{\alpha\gamma} - g_{\beta\gamma}R_{\alpha\delta} - g_{\alpha\delta}R_{\beta\gamma}) - \frac{1}{6}\left(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}\right)R. \tag{10}$$

We work with two different foliations of the spacetime (M, g). First, we can choose an appropriate time function t and obtain a foliation given by the level sets H_t. We denote the time vector field by T, i.e. the future-directed normal to the foliation. Thus it is $T^i = -\Phi^2 g^{ij}\partial_j t$ and $Tt = T^i\partial_j t = 1$. The resulting spacetime foliation is diffeomorphic to the product $\mathbb{R} \times \bar{M}$ with $\bar{M}$ being a 3-manifold and each level hypersurface H_t of t is diffeomorphic to $\bar{M}$. The metric then reads

$$g = -\Phi^2 \mathrm{d}t^2 + \bar{g} \tag{11}$$

where $\bar{g} = \bar{g}(t)$ is the induced metric on H_t. We denote the components of the inverse metric as $g^{ij} = (g^{-1})^{ij}$. The lapse function Φ is given as $\Phi := (-g^{ij}\partial_i t\partial_j t)^{-\frac{1}{2}}$. Choosing a frame field $\{e_i\}$ for $i = 1, 2, 3$ on H_t, we Lie-transport it along the integral curves of T. Thus, it is

$$[T, e_i] = 0.$$

Write $\bar{g}_{ij} = \bar{g}(e_i, e_j) = g(e_i, e_j)$. Then the second fundamental form k is given by

$$k_{ij} = k(e_i, e_j) \tag{12}$$

$$= \frac{1}{2}\Phi^{-1}\frac{\partial \bar{g}_{ij}}{\partial t}. \tag{13}$$

We choose to work with a maximal time function, that is

$$\mathrm{tr}k = 0. \tag{14}$$

Moreover, denote by N the spacelike unit normal vector of $S_{t,u}$ in H_t and from now on let T be the timelike unit normal vector of H_t in the spacetime. Often we shall use the frame (T, N, e_2, e_1).

Second, we work with a null foliation of the spacetime. For this purpose, we foliate the spacetime by an optical function u and denote its lapse function by a. Now, the level sets C_u of u are outgoing null hypersurfaces. Along C_u we pick a suitable pair of normal vectors. Denote by e_4 and e_3 the null pair, i.e. $g(e_3, e_4) = -2$, where $e_4 = T + N$ and $e_3 = T - N$. We consider the intersection $S_{t,u} = H_t \cap C_u$. Let $\{e_A\}, A = 1, 2$ be an orthonormal frame on $S_{t,u}$. This yields a null frame (e_4, e_3, e_2, e_1) in the spacetime. Often we write $e_4 = L$ and $e_3 = \underline{L}$.

Thus, $\underline{L}$ is transversal to C, the latter being generated by L. This vectorfield L corresponds through the spacetime metric g to the 1-form $\mathrm{d}u$. In an arbitrary frame, we write

$$L^{\mu} = g^{\mu\nu}\,\partial_{\nu}\,u. \tag{15}$$

We also need the second fundamental forms with respect to e_4 and e_3, respectively. Let X, Y be arbitrary tangent vectors to S at a point in S. Then the second fundamental forms are defined to be

$$\chi(X,Y) = g(\nabla_X e_4, Y), \quad \underline{\chi}(X,Y) = g(\nabla_X e_3, Y).$$

Further, denote their traceless parts by $\hat{\chi}$ and $\underline{\hat{\chi}}$, respectively. These are in fact the shears.

More details about these foliations can be found in [11] as well as in [21,22] and [1,2].

Given this null pair, e_3 and e_4, we can define the tensor of projection from the tangent space of M to that of $S_{t,u}$.

$$\Pi^{\mu\nu} = g^{\mu\nu} + \frac{1}{2}\left(e_4^{\nu}e_3^{\mu} + e_3^{\nu}e_4^{\mu}\right).$$

We shall decompose the Einstein equations as well as the curvature and all the geometric quantities with respect to these two foliations. These decompositions were first introduced in [11] and then applied and further investigated in [21,22] and in [1–3]. We refer to these works for the detailed procedures.

Let $\underline{u} = u + 2r$ and $\tau_-^2 = 1 + u^2$ as well as $\tau_+^2 = 1 + \underline{u}^2$.

The following vectorfields are expressed in terms of L and $\underline{L}$. The time vectorfield T reads

$$T = \frac{1}{2}\left(L + \underline{L}\right). \tag{16}$$

The generator S of scalings is defined to be

$$S = \frac{1}{2}\,\left(\underline{u}\,L + u\,\underline{L}\right). \tag{17}$$

The generator K of inverted time translations is defined as

$$K = \frac{1}{2}\,\left(\underline{u}^2\,L + u^2\,\underline{L}\right). \tag{18}$$

Then the vectorfield $\bar{K} = K + T$ is

$$\bar{K} = \frac{1}{2}\left(\tau_+^2\,L + \tau_-^2\,\underline{L}\right). \tag{19}$$

We decompose the second fundamental form k_{ij} of H_t according to

$$k_{NN} = \delta \tag{20}$$

$$k_{AN} = \epsilon_A \tag{21}$$

$$k_{AB} = \eta_{AB}. \tag{22}$$

Then, define

$$\theta_{AB} = \langle \nabla_A N, e_B \rangle. \tag{23}$$

More generally, the second fundamental form θ_{ab} with $a, b = \{1, 2, 3\}$ of the u-foliation within H is obtained by projecting $\nabla_s N_t$ from H to S. Thus, the resulting tensor is tangent to S. Choosing on H the orthonormal frame $\{N, \{e_A\}_{A=1,2}\}$ where $\{e_1, e_2\}$ is an orthonormal frame on S, we find formula (23). It is then easy to see that relative to arbitrary coordinates on S, the second fundamental form reads

$$\theta_{AB} = \frac{1}{2a} \frac{\partial}{\partial u} \gamma_{AB} \tag{24}$$

where γ_{AB} denotes the induced metric on S.

The Ricci coefficients of the null standard frame $T - N, T + N, e_2, e_1$ are

$$\chi'_{AB} = \theta_{AB} - \eta_{AB} \tag{25}$$

$$\underline{\chi}'_{AB} = -\theta_{AB} - \eta_{AB} \tag{26}$$

$$\underline{\xi}'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi - a^{-1} \nabla\!\!\!\!/_A a \tag{27}$$

$$\underline{\zeta}'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi - \epsilon_A \tag{28}$$

$$\zeta'_A = \phi^{-1} \nabla\!\!\!\!/_A \phi + \epsilon_A \tag{29}$$

$$\nu' = -\phi^{-1} \nabla\!\!\!\!/_N \phi + \delta \tag{30}$$

$$\underline{\nu}' = \phi^{-1} \nabla\!\!\!\!/_N \phi + \delta \tag{31}$$

$$\omega' = \delta - a^{-1} \nabla\!\!\!\!/_N a. \tag{32}$$

The Ricci coefficients of the null frame $a^{-1}(T - N), a(T + N), e_2, e_1$ are denoted by χ, $\underline{\chi}$, etc. In what follows, we drop the primes, but point out, which frame is used.

Definition 1. Define the null components of the Weyl curvature tensor W to be

$$\underline{\alpha}_{\mu\nu}\ (W) = \Pi_\mu^{\ \rho}\ \Pi_\nu^{\ \sigma}\ W_{\rho\gamma\sigma\delta}\ e_3^\gamma\ e_3^\delta \tag{33}$$

$$\underline{\beta}_\mu\ (W) = \frac{1}{2}\ \Pi_\mu^{\ \rho}\ W_{\rho\sigma\gamma\delta}\ e_3^\sigma\ e_3^\gamma\ e_4^\delta \tag{34}$$

$$\rho\ (W) = \frac{1}{4}\ W_{\alpha\beta\gamma\delta}\ e_3^\alpha\ e_4^\beta\ e_3^\gamma\ e_4^\delta \tag{35}$$

$$\sigma\ (W) = \frac{1}{4}\ {}^*W_{\alpha\beta\gamma\delta}\ e_3^\alpha\ e_4^\beta\ e_3^\gamma\ e_4^\delta \tag{36}$$

$$\beta_\mu\ (W) = \frac{1}{2}\ \Pi_\mu^{\ \rho}\ W_{\rho\sigma\gamma\delta}\ e_4^\sigma\ e_3^\gamma\ e_4^\delta \tag{37}$$

$$\alpha_{\mu\nu}\ (W) = \Pi_\mu^{\ \rho}\ \Pi_\nu^{\ \sigma}\ W_{\rho\gamma\sigma\delta}\ e_4^\gamma\ e_4^\delta. \tag{38}$$

In [11] for Einstein vacuum and in [22] for Einstein–Maxwell the following behavior is shown.

$$\begin{aligned}
\underline{\alpha}(W) &= O\left(r^{-1}\tau_-^{-\frac{5}{2}}\right) \\
\underline{\beta}(W) &= O\left(r^{-2}\tau_-^{-\frac{3}{2}}\right) \\
\rho(W) &= O(r^{-3}) \\
\sigma(W) &= O\left(r^{-3}\tau_-^{-\frac{1}{2}}\right) \\
\alpha(W), \beta(W) &= o\left(r^{-\frac{7}{2}}\right).
\end{aligned}$$

Remark. In the present Einstein null-fluid situation, the energy–momentum components behave in an analogous way as the ones for an electromagnetic field in [22]. Chapter 7 of [22] gives the main equations for the curvature and its derivatives and shows how the energy–momentum tensor enters the picture. Bounds for the curvature are then proven. Chapter 8 of [22] takes care of the error terms and concludes the control of the curvature components. It is easily checked that the leading order energy–momentum components for the Einstein null fluid and for the electromagnetic field are of the same order with respect to decay in r. Moreover, the structures of the Einstein null-fluid situation are such that they do not change the behavior of the curvature. Thus, the above formulas hold in our situation as well.

2.1. Ricci Rotation Coefficients

The Ricci rotation coefficients of the null frame are:

$$\begin{aligned}
\chi_{AB} &= g(D_A e_4, e_B) \\
\underline{\chi}_{AB} &= g(D_A e_3, e_B) \\
\underline{\xi}_A &= \frac{1}{2} g(D_3 e_3, e_A) \\
\zeta_A &= \frac{1}{2} g(D_3 e_4, e_A) \\
\underline{\zeta}_A &= \frac{1}{2} g(D_4 e_3, e_A) \\
\nu &= \frac{1}{2} g(D_4 e_4, e_3) \\
\underline{\nu} &= \frac{1}{2} g(D_3 e_3, e_4) \\
\epsilon_A &= \frac{1}{2} g(D_A e_4, e_3).
\end{aligned}$$

In [11,22] the authors compute fundamental derivatives, of which here we use:

$$\begin{aligned}
D_4 e_A &= \not{D}_4 e_A + \underline{\zeta}_A e_4 \\
D_4 e_3 &= 2\underline{\zeta}_A e_A + \nu e_3.
\end{aligned}$$

The notation for ζ and $\underline{\zeta}$ above as introduced in [11] and used in [22] is used slightly in a different way in this paper. We explain the underlying structures next. To do so, we introduce the *torsion* 1*-form* ζ on S as

$$\zeta(X) = \frac{1}{2} g(\nabla_X L, \underline{\mathrm{L}}) \quad \forall X \in TS. \tag{39}$$

One can then show that

$$\nabla_L \underline{L} = -\ 2\ Z \tag{40}$$

is the S_s-tangent vectorfield corresponding to the 1-form ζ.

We recall $g(L, \underline{L}) = -2$ and $\nabla_L L = 0$, $\nabla_{\underline{L}} \underline{L} = 0$. Then it is

$$Lg(L, \underline{L}) = g(\underbrace{\nabla_L L}_{=0}, \underline{L}) + g(L, \underbrace{\nabla_L \underline{L}}_{=-2Z}) = 0.$$

Thus we have

$$g(L, \nabla_L \underline{L}) = 0$$

telling us that $\nabla_L \underline{L}$ is tangential to C. Moreover, compute

$$Lg(\underline{L}, \underline{L}) = 2g(\nabla_L \underline{L}, \underline{L}) = 0$$

yielding

$$g(\underline{L}, \nabla_L \underline{L}) = 0.$$

Therefore, $\nabla_L \underline{L}$ does not have any component along L either but is indeed tangential to S. This fact is used in (49).

Let us further explore the torsion. Take any $X \in T_x S$ and extend X to a Jacobi field along the generator through x. We compute

$$\begin{aligned} g(\nabla_L \underline{L}, X) &= L(\underbrace{g(\underline{L}, X)}_{=0}) - g(\underline{L}, \nabla_L X) \\ &= -g(\underline{L}, \nabla_X L) \\ &= -2\zeta(X). \end{aligned}$$

The second line holds because $[L, X] = 0$. Then consider any vector $X \in T_x S$ together with $g(\nabla_{\underline{L}} L, X)$. To calculate the following, we use formula (15) in an arbitrary frame:

$$L^\mu = g^{\mu\nu} \partial_\nu u.$$

We make use of the fact that the Hessian of a function is symmetric and we compute

$$\begin{aligned} g(\nabla_{\underline{L}} L, X) &= \nabla^2 u \cdot (\underline{L}, X) \\ &= \nabla^2 u \cdot (X, \underline{L}), \\ &= g(\nabla_X L, \underline{L}) \\ &= 2\zeta(X). \end{aligned}$$

Summarizing, we find that $\zeta_A = -\frac{1}{2} g(\nabla_L \underline{L}, e_A) = \frac{1}{2} g(\nabla_{\underline{L}} L, e_A)$. Thus, in the notation from above, this means $\zeta_A = -\underline{\zeta}_A$.

Then the corresponding derivatives take the form:

$$D_4 e_A = \not{D}_4 e_A - \zeta_A e_4 \tag{41}$$

$$D_4 e_3 = -2\zeta_A e_A + \nu e_3. \tag{42}$$

Further, we calculate

$$g(\nabla_L e_A, \underline{L}) = \underbrace{L g(e_A, \underline{L})}_{=0} \underbrace{- g(e_A, \nabla_L \underline{L})}_{=2\zeta_A} = 2\zeta_A\,.$$

In a similar way, it is shown that $g(\nabla_L e_A, L) = 0$ and $g(\nabla_L e_A, e_B) = 0$.

2.2. Behavior of the Energy–Momentum Tensor

Behavior and Decay of Vector Fields at Null Infinity. For the current purpose, let us use the notation as in Eq. (2), where we read

$$T^{ij} = \mathcal{N} K^i K^j.$$

The vector k initially will be of the form

$$k = aL + b\underline{L} + V \tag{43}$$

with V denoting a vector tangent to S.

In the following, we show that a long time after the burst the L direction will dominate, that is that $\underline{L}$ and V will decay along L. Moreover, we show that b decays and that a converges to 1 in the corresponding limits.

Let us first consider the vectorfield L and write

$$T^{ij} = \mathcal{N} L^i L^j.$$

We have seen that the twice contracted Bianchi identities imply (5), and therefore the Einstein equations enforce (6), that is

$$\nabla_i T^{ij} = 0.$$

Thus we have

$$(\nabla_i \mathcal{N}) L^i L^j = -\mathcal{N}(\nabla_i L^i) L^j - \mathcal{N} L^i (\nabla_i L^j)$$

$$(\nabla_L \mathcal{N}) L^j = -\mathcal{N} div L L^j - \mathcal{N} \nabla_L L^j.$$

As L is a geodesic vectorfield, the last term is zero and we have

$$(\nabla_L \mathcal{N}) = -\mathcal{N} \mathrm{div} L. \tag{44}$$

From the studies in [11] we know how the geodesics behave. Some of this geometric structure worked out in [11] is used in the following. In particular, it is

$$\lim_{C_u, t\to\infty} r \mathrm{tr}\chi = 2, \qquad \lim_{C_u, t\to\infty} r \mathrm{tr}\underline{\chi} = -2.$$

We compute

$$\mathrm{div} L = \mathrm{tr}\chi + \text{l.o.t.} = \frac{2}{r} + \text{l.o.t.} \tag{45}$$

$$\mathrm{div} \underline{L} = \mathrm{tr}\underline{\chi} + \text{l.o.t.} = -\frac{2}{r} + \text{l.o.t.} \tag{46}$$

Then we have in Eq. (44)

$$(\nabla_L \mathcal{N}) = -\mathcal{N} \mathrm{div} L = -2\mathcal{N} r^{-1} + \text{l.o.t.}$$

Therefore, we conclude that

$$\mathcal{N} = O(r^{-2}).$$

Thus we have

$$T^{LL} = O(r^{-2}).$$

We can absorb $\mathcal{N}$ into L and denote $\sqrt{\mathcal{N}} L^i = L'^i$. However, we continue by dropping the prime and will point out which vectorfield is used. We use the following two settings: (1) we take the vectorfield L to generate an affinely parametrized geodesic, and (2) L stands for L'.

Working with the geodesic vectorfield L we compute decay rates for e_A and $\underline{L}$ along L. Next, we focus on the vectorfield e_A. We find from (41) that (47) holds and by direct computation (48) holds for a vector V tangential to S:

$$\nabla_L e_A = -\zeta_A L \tag{47}$$

$$\nabla_L V = \chi^A{}_B V^B e_A - V^A \zeta_A L. \tag{48}$$

To find the behavior of e_A along L, focus on Eq. (47) and use the fact that $\zeta = O(r^{-2})$ to find

$$\nabla_L e_A = -\zeta_A L = O(r^{-2}).$$

Thus, along L the vectorfield e_A decays like r^{-1}.

Finally, consider vectorfield $\underline{L}$. Equation (42) exhibits a component along $\underline{L}$. But we show above that this component vanishes and $\nabla_L \underline{L}$ is tangential to S. Now, with $\zeta = O(r^{-2})$ we find

$$\nabla_L \underline{L} = -2\zeta_A e_A = O(r^{-2}) O(r^{-1}) = O(r^{-3}). \tag{49}$$

And along L the vectorfield $\underline{L}$ decays like r^{-2}.

Switching to the vectorfield $L = L'$, we point out: consequently, we find that along $L = L'$ the vectorfield $\underline{L}$ decays like r^{-3} and for $A = \{1, 2\}$ the vectorfield e_A decays like r^{-2}.

It then directly follows that the energy–momentum tensor has the following behavior in r:

$$\begin{aligned}
T^{LL} &= O(r^{-2}) \\
T^{AL} &= O(r^{-3}) \\
T^{AB} &= O(r^{-4}) \\
T^{L\underline{L}} &= O(r^{-4}) \\
T^{A\underline{L}} &= O(r^{-5}) \\
T^{\underline{L}\underline{L}} &= O(r^{-6}).
\end{aligned}$$

Remark. When absorbing $\mathcal{N}$ into L denoting $L' = \sqrt{\mathcal{N}} L$, then the components of the energy–momentum tensor take the previous form. In the discussion to follow we take the vectorfield L to generate an affinely parametrized geodesic.

Thus, the following holds: along L the vectorfield $\underline{L}$ decays like r^{-2} and for $A = \{1, 2\}$ the vectorfield e_A decays like r^{-1}.

The null vector k satisfies

$$k_a k^a = 0.$$

Thus, we have

$$0 = (aL_a + b\underline{L}_a + V_a)(aL^a + b\underline{L}^a + V^a) = -4ab + V_a V^a.$$

That is

$$4ab = V_a V^a. \tag{50}$$

The goal is to prove that k tends to L along C_u as $t \to \infty$.

For that purpose, we first investigate the deformation tensor for T. Let Y, Z denote any vectorfields in M. The deformation tensor of T then is

$$(\mathcal{L}_T g)(Y, Z) = g(\nabla_Y T, Z) + g(Y, \nabla_Z T). \tag{51}$$

We denote the components of the deformation tensor for T by

$$^{(T)}\pi_{\alpha\beta} = (\mathcal{L}_T g)_{\alpha\beta}. \tag{52}$$

With (16) we write

$$(\mathcal{L}_T g)(Y, Z) = \frac{1}{2}(\mathcal{L}_L g)(Y, Z) + \frac{1}{2}(\mathcal{L}_{\underline{L}} g)(Y, Z) \tag{53}$$

$$= \frac{1}{2}g(\nabla_Y L, Z) + \frac{1}{2}g(Y, \nabla_Z L) + \frac{1}{2}g(\nabla_Y \underline{L}, Z) + \frac{1}{2}g(Y, \nabla_Z \underline{L}). \tag{54}$$

In a null frame, the vectorfields Y and Z read $Y = Y^L L + Y^{\underline{L}}\underline{L} + Y^A e_A$ and $Z = Z^L L + Z^{\underline{L}}\underline{L} + Z^A e_A$ with $A = \{1, 2\}$. Then, with respect to a general null frame the deformation tensor of T decomposes into the following components:

$$\begin{aligned}
&(\mathcal{L}_T g)(Y, Z) \\
&\quad = Y^L Z^L\ {}^{(T)}\pi_{44} + (Y^L Z^{\underline{L}} + Y^{\underline{L}} Z^L)\ {}^{(T)}\pi_{34} + (Y^A Z^L + Y^L Z^A)\ {}^{(T)}\pi_{A4} \\
&\qquad + Y^{\underline{L}} Z^{\underline{L}}\ {}^{(T)}\pi_{33} + (Y^A Z^{\underline{L}} + Y^{\underline{L}} Z^A)\ {}^{(T)}\pi_{3A} + Y^A Y^B\ {}^{(T)}\pi_{AB}.
\end{aligned} \tag{55}$$

Direct computations yield

$$\begin{aligned}
{}^{(T)}\pi_{44} &= -2\nu \\
{}^{(T)}\pi_{34} &= 2\delta \\
{}^{(T)}\pi_{A4} &= -2\epsilon_A + \phi^{-1}\nabla\!\!\!\!/\,_A \phi \\
{}^{(T)}\pi_{33} &= -2\underline{\nu} \\
{}^{(T)}\pi_{3A} &= 2\epsilon_A + \phi^{-1}\nabla\!\!\!\!/\,_A \phi \\
{}^{(T)}\pi_{AB} &= -2\eta_{AB}
\end{aligned}$$

with the right-hand sides being the connection coefficients introduced earlier. Note that for geodesic vectorfields some of these terms vanish.

In the case where Y and Z are tangential to S, Eq. (54) reduces to

$$(\mathcal{L}_T g)(Y,Z) = \chi(Y,Z) + \underline{\chi}(Y,Z) = -2\eta(Y,Z) \tag{56}$$
$$= \mathrm{tr}\chi + \hat{\chi} + \mathrm{tr}\underline{\chi} + \hat{\underline{\chi}} \tag{57}$$
$$= \hat{\chi} + \hat{\underline{\chi}} + O(r^{-2}). \tag{58}$$

The last equation holds because of

$$\mathrm{tr}\chi = \frac{2}{r} + O(r^{-2}) \quad \mathrm{tr}\underline{\chi} = -\frac{2}{r} + O(r^{-2}).$$

Further, as $\hat{\underline{\chi}}_{AB} = O(r^{-1})$ is the lowest order term in (54), respectively, in (56), we find that the deformation tensor for T behaves like

$$(\mathcal{L}_T g)(Y,Z)\gamma_{AB} = Y^A Z_A \hat{\underline{\chi}}_{AB} + \mathrm{l.o.t.}$$

Moreover, we recall the facts that for any $Y, Z \in T_x C$ it is $\chi(Y,Z) = \chi(\Pi^C Y, \Pi^C Z)$ and for any $Y, Z \in T_x \underline{C}$ it is $\underline{\chi}(Y,Z) = \underline{\chi}(\Pi^{\underline{C}} Y, \Pi^{\underline{C}} Z)$ with Π^C denoting the projection along L onto the tangent space of S and $\Pi^{\underline{C}}$ the projection along $\underline{L}$ onto the tangent space of S.

Next, we need to get bounds on the coefficients V^A of the above vectorfield $V = V^A e_A$. For this purpose, we consider the deformation tensor of the vectorfield K introduced in (18). With V being tangential to S we find

$$\begin{aligned}(\mathcal{L}_K g)(V,V) &= \frac{1}{2}\underline{u}^2(\mathcal{L}_L g)(V,V) + \frac{1}{2}u^2(\mathcal{L}_{\underline{L}} g)(V,V) \\ &= V^A V^B \underline{u}^2 \chi_{AB} + V^A V^B u^2 \underline{\chi}_{AB} \qquad (59) \\ &= V^A V^B \left\{ \underline{u}^2 \hat{\chi}_{AB} + u^2 \hat{\underline{\chi}}_{AB} + \frac{1}{2}\underline{u}^2 tr\chi\gamma_{AB} + \frac{1}{2}u^2 tr\underline{\chi}\gamma_{AB} \right\}. \qquad (60)\end{aligned}$$

We split $(\mathcal{L}_K g)$ into its trace and traceless part, the latter being denoted as $\widehat{(\mathcal{L}_K g)}$. This gives

$$\widehat{(\mathcal{L}_K g)}(V,V) = V^A V^B \underline{u}^2 \hat{\chi}_{AB} + V^A V^B u^2 \hat{\underline{\chi}}_{AB}. \tag{61}$$

Observing the orders of these terms, we find

$$\underline{u}^2 \hat{\chi}_{AB} = C_2 + O(r^{-1}) \quad \text{and} \quad u^2 \hat{\underline{\chi}}_{AB} = O(u^{\frac{1}{2}} r^{-1}).$$

As $\widehat{(\mathcal{L}_K g)}(V,V)$ has to go to zero as $t \to \infty$, we obtain bounds on the components V^A. We conclude for fixed u and as $r \to \infty$

$$(\mathcal{L}_K g)(V,V)\gamma_{AB} = V^A V_A \cdot C_2 + \mathrm{l.o.t.} = C_2|V|^2 + \mathrm{l.o.t.} \tag{62}$$

As $t \to \infty$ the quantity $\widehat{(\mathcal{L}_K g)}(V,V)$ goes to zero. Therefore, the coefficients V^A have to decay like $r^{-\epsilon}$. Equation (62) yields the bounds for the coefficients

$$V^A = O(r^{-\epsilon}).$$

Proof of Convergence of k to L as $t \to \infty$. Let k be a null geodesic, that is,

$$k^a \nabla_a k = \nabla_k k = 0$$
$$k^a k_a = 0.$$

If our manifold were the Minkowski space, then there would exist conformal Killing fields X, that is,

$$\nabla_{(a}X_{b)} = \phi g_{ab}$$

for some scalar ϕ, that is

$$(\mathcal{L}_X g) = \phi g.$$

Then it follows that

$$k^a \nabla_a (X_a k^a) = 0$$

and consequently that for each geodesic there exists a constant c such that $k^a X_a = c$.

As our manifold is a Lorentzian manifold with a lot of curvature structure, there are no 'pure' conformal Killing fields, and the afore-mentioned equations in Minkowski space do not hold in a general Lorentzian manifold. However, another property of our Lorentzian manifold, namely its asymptotic flatness, guarantees the existence of almost- and quasi-conformal Killing fields. This means that the corresponding deformation tensors are suitably small and tend to zero as $t \to \infty$ in a suitable way. In that case, the afore-mentioned equations 'hold in an asymptotic sense'. This is what we have to prove now.

The null geodesic vectorfield k takes the form as in (43)

$$k = aL + b\underline{L} + V.$$

Consider $(T_b k^b)$ and write in view of the above equations

$$\nabla_k (T_b k^b) = (\nabla_k T_b) k^b = (\nabla_k T) \cdot k = (\nabla_k T) \cdot (aL + b\underline{L} + V). \tag{63}$$

First, we investigate $\nabla_k T$.

$$\nabla_k T = \frac{1}{2} a \nabla_L L + \frac{1}{2} b \nabla_{\underline{L}} L + \frac{1}{2} V^A \nabla_A L + \frac{1}{2} a \nabla_L \underline{L} + \frac{1}{2} b \nabla_{\underline{L}} \underline{L} + \frac{1}{2} V^A \nabla_A \underline{L}.$$

To compute the subsequent orders of the terms, we use the information from [11]. First, we derive

$$\begin{aligned} \nabla_k T = 0 & \\ & + b\zeta_A e_A + \frac{1}{2} b\underline{\nu} L \\ & + \frac{1}{2} V^A \chi_{AB} e_B - \frac{1}{2} V^A \epsilon_A L \\ & - a\zeta_A e_A + \frac{1}{2} a\nu \underline{L} \\ & + b\underline{\xi} e_A - \frac{1}{2} b\underline{\nu L} \\ & + \frac{1}{2} V^A \underline{\chi}_{AB} e_B + \frac{1}{2} V^A \epsilon_A \underline{L}. \end{aligned}$$

Some of these expressions are zero. Above we show that $\nabla_L \underline{L}$ does not have any component along $\underline{L}$. In a similar manner it follows that $\nabla_{\underline{L}} L$ does not have any component along L. Straightforward computations along the lines as we do them above yield

$$\nabla_k T = b\zeta_A e_A + \frac{1}{2}V^A \chi_{AB} e_B - \frac{1}{2}V^A \epsilon_A L - a\zeta_A e_A + \frac{1}{2}V^A \underline{\chi}_{AB} e_B + \frac{1}{2}V^A \epsilon_A \underline{L}. \tag{64}$$

All the connection coefficients except $\hat{\underline{\chi}}$ are $O(r^{-2})$, only $\hat{\underline{\chi}} = O(r^{-1})$, moreover the highest order terms of the traces of χ and $\underline{\chi}$ cancel. We take this into account as well as the decays for the vectorfields established above. Then we compute in the order of appearance for the terms on the right-hand side of the previous equation

$$\begin{aligned}(\nabla_k T)\cdot(aL + b\underline{L} + V) = {} & bO(r^{-4-\epsilon}) + O(r^{-4-\epsilon}) + bO(r^{-4-\epsilon}) + aO(r^{-4-\epsilon}) \\ & + O(r^{-3-\epsilon}) + aO(r^{-4-\epsilon}).\end{aligned}$$

Then we obtain

$$(\nabla_k T)\cdot(aL + b\underline{L} + V) = O(r^{-3-\epsilon}) + aO(r^{-4-\epsilon}) + bO(r^{-4-\epsilon}).$$

We derive

$$a + b = T_b k^b = c_1 + O(r^{-2-\epsilon}) + aO(r^{-3-\epsilon}) + bO(r^{-3-\epsilon}). \tag{65}$$

The lower order terms multiplied by a, respectively, b can be absorbed into a, respectively, b.

Next, we do the corresponding computations with the vectorfield K given in (18), thus $K = \frac{1}{2}(\underline{u}^2 L + u^2 \underline{L})$. It is

$$\nabla_k (K_b k^b) = (\nabla_k K_b) k^b = (\nabla_k K)\cdot k = (\nabla_k K)\cdot(aL + b\underline{L} + V). \tag{66}$$

First, we consider $\nabla_k K$.

$$\nabla_k K = \underbrace{\frac{1}{2}\underline{u}^2 \nabla_k L + \frac{1}{2}u^2 \nabla_k \underline{L}}_{=:A} + \underbrace{\underline{u}(\nabla_k \underline{u})L + u(\nabla_k u)\underline{L}}_{=:B}.$$

We investigate A, then $A\cdot k$. Then, we investigate B, then $B\cdot k$. The terms in A emerge from the results above for T, but multiplied with the corresponding weights $\underline{u}^2$ and u^2, respectively. With (64) we find

$$\begin{aligned}A = \frac{1}{2}\underline{u}^2 \nabla_k L + \frac{1}{2}u^2 \nabla_k \underline{L} = {} & b\underline{u}^2 \zeta_A e_A + \frac{1}{2}\underline{u}^2 V^A \chi_{AB} e_B - \frac{1}{2}\underline{u}^2 V^A \epsilon_A L \\ & - au^2 \zeta_A e_A + \frac{1}{2}u^2 V^A \underline{\chi}_{AB} e_B + \frac{1}{2}u^2 V^A \epsilon_A \underline{L}.\end{aligned}$$

It then follows directly for $A\cdot k$ (in order of appearance of the terms on the right-hand side in the previous formula):

$$\begin{aligned}A\cdot k = {} & bO(r^{-2}) + O(r^{-1-\epsilon}) + bO(r^{-2-\epsilon}) + aO(u^2 r^{-4}) \\ & + O(u^2 r^{-3-\epsilon}) + aO(u^2 r^{-4-\epsilon}).\end{aligned}$$

Thus it is

$$A\cdot k = O(r^{-1-\epsilon}) + O(u^2 r^{-3-\epsilon}) + aO(u^2 r^{-4}) + bO(r^{-2}).$$

Let us focus on B. We have

$$B = \underbrace{\underline{u}(\nabla_k \underline{u})L}_{=:B_1} + \underbrace{u(\nabla_k u)\underline{L}}_{=:B_2}.$$

The main part of the first term writes

$$\nabla_k \underline{u} = a\nabla_L \underline{u} + b\nabla_{\underline{L}} \underline{u} + V^A \nabla_A \underline{u}$$

whereas the main part of the second term is

$$\nabla_k u = a\nabla_L u + b\nabla_{\underline{L}} u + V^A \nabla_A u.$$

Then it is

$$\begin{aligned} B_1 &= \underline{u}(\nabla_k \underline{u})L = a\underline{u}(\nabla_L \underline{u})L + b\underline{u}(\nabla_{\underline{L}} \underline{u})L + \underline{u}V^A(\nabla_A \underline{u})L \\ B_2 &= u(\nabla_k u)L = au(\nabla_L u)\underline{L} + bu(\nabla_{\underline{L}} u)\underline{L} + uV^A(\nabla_A u)\underline{L}. \end{aligned}$$

Straightforward computations yield

$$\begin{aligned} B \cdot k &= ab\underline{u}(\nabla_L \underline{u})L\underline{L} + b^2\underline{u}(\nabla_{\underline{L}} \underline{u})L\underline{L} + b\underline{u}V^A(\nabla_A \underline{u})L\underline{L} \\ &\quad + a^2u(\nabla_L u)\underline{L}L + abu(\nabla_{\underline{L}} u)\underline{L}L + auV^A(\nabla_A u)\underline{L}L \\ &= ab\underline{u}(\nabla_L \underline{u})L\underline{L} + abu(\nabla_{\underline{L}} u)\underline{L}L \\ &= ab\underline{u}O(r^{-2}) + abuO(r^{-2}). \end{aligned}$$

The above holds because $0 = \nabla_L u = \nabla_{\underline{L}} \underline{u} = \nabla_A \underline{u} = \nabla_{\underline{L}} u = \nabla_A u$ and $\nabla_L \underline{u}$ as well as $\nabla_{\underline{L}} u$ are constant. We recall from above that $ab = \frac{1}{4}V^a V_a = O(r^{-2-\epsilon})$. This gives

$$B \cdot k = O(r^{-3-\epsilon}) + O(ur^{-4-\epsilon}).$$

Putting the pieces together we find

$$\begin{aligned} (\nabla_k K) \cdot k &= A \cdot k + B \cdot k \\ &= O(r^{-1-\epsilon}) + O(u^2 r^{-3-\epsilon}) + aO(u^2 r^{-4}) + bO(r^{-2}). \end{aligned}$$

Further we compute

$$u^2 a + \underline{u}^2 b = k_a K^a = c_2 + O(r^{-\epsilon}) + O(u^2 r^{-2-\epsilon}) + aO(u^2 r^{-3}) + bO(r^{-1}). \tag{67}$$

Again we absorb the lower order terms in a, respectively, b into a, respectively b.

Combining the results for T and K,

$$T_a k^a = a + b = c_1 + O(r^{-2-\epsilon}) \tag{68}$$

$$K_a k^a = u^2 a + \underline{u}^2 b = c_2 + O(r^{-\epsilon}). \tag{69}$$

We are free to choose $c_1 = 1$. Then it is with (68)

$$a = 1 - b + O(r^{-2-\epsilon}).$$

Inserting in (69) yields

$$
\begin{aligned}
u^2(1-b+O(r^{-2-\epsilon}))+\underline{u}^2 b = c_2 + O(r^{-\epsilon})\\
b(\underline{u}^2-u^2) = c_2 - u^2 - u^2 O(r^{-2-\epsilon}) + O(r^{-\epsilon})\\
b = \frac{c_2 - u^2 + u^2 O(r^{-2-\epsilon}) + O(r^{-\epsilon})}{\underline{u}^2 - u^2}\\
b = \frac{c_2 - u^2 + u^2 O(r^{-2-\epsilon}) + O(r^{-\epsilon})}{4r(u+r)}.
\end{aligned}
$$

The latter equation holds because of $\underline{u} = u + 2r$.

Thus we find that $b = O(r^{-2})$. As $r \to \infty$ it follows that $b \to 0$ and $a \to 1$.

Then it follows that k tends to L along C_u for $t \to \infty$. This ends the proof of convergence.

Remark. In the above arguments, there could in principle be terms involving a^2 and b^2. However, it can be easily shown that they vanish. The remaining terms involving ab are estimated by identity (50). The quantities with a and b emerge with factors of lower order in r and therefore are absorbed into a and b in the computations.

We recall that performing our experiment, we are at null infinity of our spacetime and receive gravitational-wave signals traveling from the source along outward null hypersurfaces.

The above shows that while a short time after the gravitational-wave burst, the neutrinos following null curves may still fly into various directions, after some time the amount escaping towards the $\underline{L}$- and V-directions decay and the neutrino flow for later times will approach the null geodesic generated by L.

3. Spacetime Structure

Decomposing the Einstein null-fluid equations with respect to the two main foliations, we derive the following equations.

3.1. Equations for *t*-Foliation

In this chapter, we give the most important equations to be used later in the paper.

Given the Laplace operator $\triangle$ in H, its radial decomposition reads

$$\triangle = \nabla_N^2 + \not{\triangle} + \mathrm{tr}\theta \nabla_N + a^{-1} \not{\nabla} a \cdot \not{\nabla}. \tag{70}$$

The second fundamental form k obeys the equations

$$\mathrm{tr} k = 0 \tag{71}$$

$$(\mathrm{curl}\ k)_{ij} = H(W)_{ij} + \frac{1}{2}\epsilon_{ij}{}^{l} R_{0l} \tag{72}$$

$$(\mathrm{div}\ k)_i = R_{0i}. \tag{73}$$

We have to take into account that the Ricci curvature $\bar{R}_{ij}$ in the spacelike hypersurfaces H_t is composed as in the following formula, with $R_{\alpha\beta}$ denoting the spacetime Ricci curvature of M.

$$\bar{R}_{ij} \;=\; k_{ia}k_j^a + E(W)_{ij} + \frac{1}{2}g_{ij}R_{00} + \frac{1}{2}R_{ij}. \tag{74}$$

In particular, the components δ, ϵ, η of k satisfy:

$$\begin{aligned} \mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \epsilon &= -\nabla_N\delta - \frac{3}{2}\mathrm{tr}\theta\delta + \hat{\eta}\cdot\hat{\theta} - 2(a^{-1}\not{\nabla} a)\cdot\epsilon \\ &\quad - \pi T_{\underline{L}\underline{L}} + \pi T_{LL} && (75) \\ \mathrm{c}\not{\mathrm{u}}\mathrm{rl}\, \epsilon &= \sigma(W) + \hat{\theta}\wedge\hat{\eta} && (76) \\ \not{\nabla}_N\epsilon + \mathrm{tr}\theta\epsilon &= -\not{\nabla}\delta - \hat{\theta}\cdot\epsilon + \frac{3}{2}(a^{-1}\not{\nabla} a)\cdot\delta - \hat{\eta}\cdot(a^{-1}\not{\nabla} a) \\ &\quad + \frac{1}{2}(\beta - \underline{\beta}) + \pi(T_{A\underline{L}} - T_{AL}) && (77) \\ \mathrm{d}\not{\mathrm{i}}\mathrm{v}\, \hat{\eta} &= -\frac{1}{2}\not{\nabla}\delta + \hat{\theta}\cdot\epsilon - \frac{1}{2}\mathrm{tr}\theta\cdot\epsilon \\ &\quad + \frac{1}{2}(\beta - \underline{\beta}) + 4\pi(T_{A\underline{L}} - T_{AL}) && (78) \\ \not{\nabla}_N\hat{\eta} + \frac{1}{2}\mathrm{tr}\theta\hat{\eta} &= \frac{3}{2}\delta\cdot\hat{\theta} + \frac{1}{2}\not{\nabla}\hat{\otimes}\epsilon + (a^{-1}\not{\nabla} a)\hat{\otimes}\epsilon \\ &\quad + \frac{1}{4}(\alpha - \underline{\alpha}). && (79) \end{aligned}$$

3.2. Null Structure Equations

The main quantities to derive nonlinear memory are the shears $\hat{\chi}$ and $\hat{\underline{\chi}}$. Equations for the latter on two surfaces are coupled to evolution equations of the corresponding traces. Propagation equations of $\mathrm{tr}\underline{\chi}$ and $\mathrm{tr}\chi$ with respect to l-pair:

$$\frac{\mathrm{dtr}\underline{\chi}}{\mathrm{d}s} = -\frac{1}{2}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} - 2\underline{\mu} + 2\,|\zeta|^2 \tag{80}$$

$$\frac{\mathrm{dtr}\chi}{\mathrm{d}s} = -\frac{1}{2}\,(\mathrm{tr}\chi)^2 - |\widehat{\chi}|^2 - 4\pi T_{LL}. \tag{81}$$

The Gauss equation reads

$$K = -\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi} + \frac{1}{2}\widehat{\chi}\cdot\underline{\widehat{\chi}} - \rho\,(W) - 2\pi T_{L\underline{L}}.$$

Define the function $\underline{\mu}$ as

$$\underline{\mu} = -\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\underline{\zeta} + \frac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - \rho(W) - 2\pi T_{L\underline{L}}.$$

The latter, with the help of the Gauss curvature K, can be written as

$$\underline{\mu} = -\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\underline{\zeta} + K + \frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}. \tag{82}$$

The null Codazzi and conjugate null Codazzi equations read

$$\mathrm{d}\!\!/\!\mathrm{iv}\, \hat{\chi} = -\hat{\chi} \cdot \zeta + \frac{1}{2}(\nabla\!\!\!/ \,\mathrm{tr}\chi + \zeta \mathrm{tr}\chi) - \beta - 4\pi T_{AL} \tag{83}$$

$$\mathrm{d}\!\!/\!\mathrm{iv}\, \underline{\hat{\chi}} = \underline{\hat{\chi}} \cdot \zeta + \frac{1}{2}(\nabla\!\!\!/ \,\mathrm{tr}\underline{\chi} - \zeta \mathrm{tr}\underline{\chi}) + \underline{\beta} + 4\pi T_{A\underline{L}}. \tag{84}$$

Useful identities

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{2} r \, \overline{a\phi \mathrm{tr}\chi} \tag{85}$$

$$\frac{\mathrm{d}r}{\mathrm{d}u} = \frac{1}{2} r \, \overline{a \mathrm{tr}\theta}. \tag{86}$$

4. Null Infinity

4.1. Null Asymptotic Limits

Theorem 1. *Let C_u denote any null hypersurface. Then the normalized curvature components $r\underline{\alpha}(W)$, $r^2\underline{\beta}(W)$, $r^3\rho(W)$, $r^3\sigma(W)$, and normalized energy–momentum components $r^2 T_{\underline{L}\underline{L}}, r^4 T_{L\underline{L}}, r^4 T_{AB}, r^3 T_{A\underline{L}}, r^5 T_{AL}$ have limits as $t \to \infty$. That is*

$$\lim_{C_u, t\to\infty} r\underline{\alpha}(W) = A_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^2\underline{\beta}(W) = B_W(u,\cdot)$$

$$\lim_{C_u, t\to\infty} r^3\rho(W) = P_W(u,\cdot), \qquad \lim_{C_u, t\to\infty} r^3\sigma(W) = Q_W(u,\cdot)$$

$$\lim_{C_u, t\to\infty} r^2 T_{\underline{L}\underline{L}} = T^*_{\underline{L}\underline{L}}(u,\cdot),$$

$$\lim_{C_u, t\to\infty} r^3 T_{A\underline{L}} = T^*_{A\underline{L}}(u,\cdot),$$

$$\lim_{C_u, t\to\infty} r^4 T_{L\underline{L}} = T^*_{L\underline{L}}(u,\cdot),$$

$$\lim_{C_u, t\to\infty} r^4 T_{AB} = T^*_{AB}(u,\cdot),$$

$$\lim_{C_u, t\to\infty} r^5 T_{AL} = T^*_{AL}(u,\cdot),$$

where the limits are on S^2 and depend on u. Moreover, these limits satisfy

$$|A_W(u,\cdot)| \le C(1+|u|)^{-5/2} \qquad |B_W(u,\cdot)| \le C(1+|u|)^{-3/2}$$

$$\left|P_W(u,\cdot) - \overline{P}_W(u)\right| \le (1+|u|)^{-1/2} \qquad \left|Q_W(u,\cdot) - \overline{Q}_W(u)\right| \le (1+|u|)^{-1/2}$$

$$T^*_{\underline{L}\underline{L}}(u,\cdot) \le C(1+|u|)^{-4}$$

$$T^*_{A\underline{L}}(u,\cdot) \le C(1+|u|)^{-3}$$

$$T^*_{L\underline{L}}(u,\cdot) \le (1+|u|)^{-2}$$

$$T^*_{AB}(u,\cdot) \le (1+|u|)^{-2}$$

$$T^*_{AL}(u,\cdot) \le (1+|u|)^{-1}$$

and

$$\lim_{u\to-\infty} \overline{P}_W(u) = 0, \qquad \lim_{u\to-\infty} \overline{Q}_W(u) = 0.$$

Proof. Whereas the proof of the properties of the Weyl tensor components is along the lines of [11,22], respectively, we establish the results for the null fluid above. For the influence of the energy–momentum tensor for a null fluid on the Weyl curvature, see remark after Definition 1. The estimates of this theorem follow directly.

The following theorem shows behavior of the shears and the fundamental relation between the shears and the curvature, which is in accordance with the picture found by Christodoulou and Klainerman [11] and by Zipser [21,22]. In our setting here, we prove that the null-fluid terms do not change these equations.

Theorem 2. *Consider the null hypersurface C_u. The normalized shear $r^2\widehat{\chi}'$ tends to the following limit as $t \to \infty$:*

$$\Sigma\,(u,\cdot) = \lim_{C_u, t\to\infty} r^2\widehat{\chi}'.$$

The limit Σ is a symmetric traceless covariant 2-tensor on S^2 that depends on u.

The proof is the same as in [11,22], respectively, because the propagation equation is not affected by the extra terms from the energy–momentum tensor of the null fluid. This propagation equation reads

$$\frac{\mathrm{d}\widehat{\chi}_{AB}}{\mathrm{d}s} = -\mathrm{tr}\chi\widehat{\chi}_{AB} - \alpha(W)_{AB}.$$

Theorem 3. *Consider any null hypersurface C_u. The limit of $r\widehat{\eta}$ exists as $t \to \infty$, in particular*

$$\Xi\,(u,\cdot) = \lim_{C_u, t\to\infty} r\widehat{\eta}.$$

The limit Ξ is a symmetric traceless 2-covariant tensor on S^2 that depends on u and obeys

$$|\Xi\,(u,\cdot)|_{\overset{\circ}{\gamma}} \leq C\,(1+|u|)^{-3/2}.$$

In addition, the following holds:

$$\lim_{C_u, t\to\infty} r\widehat{\theta} = -\frac{1}{2}\lim_{C_u, t\to\infty} r\underline{\widehat{\chi}}' = \Xi$$

and

$$\frac{\partial \Xi}{\partial u} = -\frac{1}{4}A_W. \tag{87}$$

$$\frac{\partial \Sigma}{\partial u} = -\Xi \tag{88}$$

Proof. To prove this, we take into account the decay behavior of the energy–momentum tensor components for the null fluid. The argument is along the lines of the proof of conclusion 17.0.3 in [11]. To verify that the components of the energy–momentum tensor for the null fluid do not interfere with the limits, we have a look at the transport equations of $\hat{\eta}$ along L and for $\hat{\chi}$ and $\hat{\eta}$ along N.

First, we point out that the definition of $\hat{\eta}_4$ in the Einstein null-fluid case in [4] is

$$\hat{\eta}_4 := \not{D}_4 \hat{\eta} + \frac{1}{2}\mathrm{tr}\chi\hat{\eta} - \pi T_{\underline{LL}}\delta_{AB}. \tag{89}$$

This differs from the Einstein vacuum case in [11] by the term $\pi T_{\underline{LL}}\delta_{AB}$. As a comparison to the Einstein–Maxwell case, where the extra term is given by a quadratic of the leading order component of the electromagnetic field, see also chapter 9 in [22]. The extra term in (89) has the 'right' decay behavior in r and is bounded. In particular, we have

$$\begin{aligned}\hat{\eta}_4 := \not{D}_4 \hat{\eta} + \frac{1}{2}\mathrm{tr}\chi\hat{\eta} - \pi T_{\underline{LL}}\delta_{AB} = {} & \frac{1}{2}\alpha(W) + \pi T_{LL}\delta_{AB} \\ & + \frac{1}{2}\not{\nabla}\hat{\otimes}\epsilon - \phi^{-1}\not{\nabla}^2\phi + (a^{-1}\not{\nabla} a)\hat{\otimes}\epsilon + \epsilon\hat{\otimes}\epsilon \\ & + \frac{3}{2}\delta\hat{\theta} - \delta\hat{\eta} - (\zeta - \phi^{-1}\not{\nabla}\phi)\hat{\otimes}\epsilon.\end{aligned}$$

The right-hand side of this equation is of order $O(r^{-3})$. It then follows that $\frac{\mathrm{d}}{\mathrm{d}s}(r\hat{\eta}) = O(r^{-2})$ is integrable and therefore $r\hat{\eta}$ has a limit. For more details see [4].

The second limit holds because of the relations of the Ricci coefficients in (25) and (26).

To prove (87) and (88), consider the transport equations along N for $\hat{\chi}$ and $\hat{\eta}$. They are

$$\begin{aligned}\not{\nabla}_N \hat{\chi} &= -\frac{1}{2}\mathrm{tr}\chi\hat{\eta} - \left(\frac{1}{2}\delta + \mathrm{tr}\chi\right)\hat{\chi} - \zeta\hat{\otimes}\zeta - \frac{1}{2}\not{\nabla}\hat{\otimes}\zeta - \frac{1}{2}\alpha(W) - \pi\gamma_{AB}T_{LL} \\ \not{\nabla}_N \hat{\eta} &= -\frac{1}{2}\mathrm{tr}\theta\hat{\eta} + \left(2\delta - \frac{1}{2}\mathrm{tr}\chi\right)\hat{\eta} + \frac{3}{2}\delta\hat{\chi} + \epsilon\hat{\otimes}\zeta - \epsilon\hat{\otimes}\epsilon + \frac{1}{2}\not{\nabla}\hat{\otimes}\epsilon \\ &\quad + \frac{1}{4}\alpha(W) - \frac{1}{4}\underline{\alpha}(W).\end{aligned}$$

We note that the energy–momentum tensor only shows up in the first equation, namely T_{LL}. This component is of much lower order than the other terms and does not contribute to the limit to be taken. Next, multiply the first equation by r^2 and the second by r. This yields

$$\begin{aligned}\not{\nabla}_N(r^2\hat{\chi}) &= -r\hat{\eta} + \mathrm{l.o.t.} \\ \not{\nabla}_N(r\hat{\eta}) &= -\frac{1}{4}\underline{\alpha}(W) + \mathrm{l.o.t.}\end{aligned}$$

Taking the limits, we conclude (87) and (88). This ends the proof.

In the proof of Theorem 6 below we use a fact on the limit of $\mathrm{tr}\chi'$ which we want to establish now. For that purpose, we define

$$H = \lim_{C_u, t\to\infty}\left(r^2\left(\mathrm{tr}\chi' - \frac{2}{r}\right)\right).$$

Lemma 1. *The following holds for the function H:*

$$\frac{\partial H}{\partial u} = 0 \tag{90}$$

$$\bar{H} = 0. \tag{91}$$

Proof. At the beginning, we want to remind the reader that the equivalent statement for the EV equations is shown in conclusion 17.0.5 and in lemma 17.0.1 of [11]. There the authors use

$$\nabla_N \mathrm{tr}\chi' + \frac{1}{2}\chi' = O(r^{-3}).$$

In our new setting for the Einstein null fluid, it follows in a straightforward manner that the additional terms due to the null fluid are of order $O(r^{-3})$. Then by the same argument as in [11], Eq. (90) above follows in the presence of a null fluid.

To prove Eq. (91) in the null-fluid case, we recall the EV situation from lemma 17.0.1 in [11]. One has to show that $r^2\bar{\delta}$ converges to $2M(u)$. Now, Proposition 4.4.4 in [11] says that

$$4\pi r^3\bar{\delta} = \int\limits_{u_0}^{u} \mathrm{d}u' \left(\int\limits_{S_{t,u}} ar\hat{\theta}\cdot\hat{\eta} - \frac{1}{2}\kappa(\delta - \bar{\delta}) - ra^{-1}\nabla\!\!\!/\, a\cdot\epsilon + r(\mathrm{div}k)_N \right).$$

Along the lines of the proof of lemma 17.0.1 in [11], it follows that

$$\int\limits_{S_{t,u}} ar\hat{\theta}\cdot\hat{\eta} - \frac{1}{2}\kappa(\delta - \bar{\delta}) - ra^{-1}\nabla\!\!\!/\, a\cdot\epsilon = r\int\limits_{S^2} |\Xi|^2 d\mu_{\overset{\circ}{\gamma}} + O(1).$$

The constraints of the Einstein null-fluid equations give

$$(\mathrm{div}k)_N = R_{0N} = 8\pi T_{0N} = 2\pi(T_{\underline{LL}} - T_{LL}). \tag{92}$$

From Eq. (92), we deduce

$$\int\limits_{S_{t,u}} r(\mathrm{div}k)_N = 2\pi r\int\limits_{S^2} T_{\underline{LL}}\mathrm{d}\mu_{\overset{\circ}{\gamma}} + O(1).$$

As a consequence, we infer that

$$r\bar{\delta} = \frac{2}{r^2}\int\limits_{u_0}^{u} r\frac{\partial}{\partial u}m(t,u) + O(r^{-1}).$$

This concludes the main part where the null-fluid components enter. The remaining steps follow easily.

4.2. Bondi Mass

Next, we study the Bondi mass in our setting. First, we introduce the Hawking mass m enclosed by a two surface $S_{t,u}$ as in [10] to be

$$m(t,u) = \frac{r}{2}\left(1 + \frac{1}{16\pi}\int\limits_{S_{t,u}} \mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right). \tag{93}$$

We first investigate $\frac{\partial}{\partial t}m(t,u)$ and then $\frac{\partial}{\partial u}m(t,u)$. From the limiting behavior of the former we conclude a convergence result of $m(t,u)$ to the Bondi mass, and from the limiting behavior of the latter we compute the Bondi mass-loss formula.

Consider the null structure Eqs. [(80), (81)]. To compute $\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi})$ we add $\mathrm{tr}\underline{\chi}\cdot$(81) and $\mathrm{tr}\chi\cdot$(80), which yields

$$\frac{\mathrm{d}}{\mathrm{d}s}(\mathrm{tr}\chi\mathrm{tr}\underline{\chi}) = -(\mathrm{tr}\chi)^2\,\mathrm{tr}\underline{\chi} - \mathrm{tr}\underline{\chi}\,|\widehat{\chi}|^2 - 2\underline{\mu}\mathrm{tr}\chi + 2\mathrm{tr}\chi\,|\zeta|^2 - 4\pi\mathrm{tr}\underline{\chi}T_{LL}.$$

Then we derive

$$\begin{aligned}\frac{\partial}{\partial t}\int\limits_{S_{t,u}} \mathrm{tr}\chi\mathrm{tr}\underline{\chi} &= \int\limits_{S_{t,u}} a\phi\left(-\mathrm{tr}\underline{\chi}|\widehat{\chi}|^2 + 2\mathrm{tr}\chi\,|\zeta|^2 - 4\pi\mathrm{tr}\underline{\chi}T_{LL}\right)\\ &\quad - 2\int\limits_{S_{t,u}} a\phi\underline{\mu}\mathrm{tr}\chi.\end{aligned} \tag{94}$$

Next, we use (82) to integrate $\underline{\mu}$ on $S_{t,u}$. Applying Gauss–Bonnet yields

$$\int\limits_{S_{t,u}} \underline{\mu} = \int\limits_{S_{t,u}} 4\pi\left(1 + \frac{1}{16\pi}\int\limits_{S_{t,u}} tr\mathrm{tr}\chi\mathrm{tr}\underline{\chi}\right) = \frac{8\pi}{r}m. \tag{95}$$

Finally, from (94) and (95), using identity (85), we conclude

$$\begin{aligned}\frac{\partial}{\partial t}m(t,u) &= \frac{r}{8\pi}\int\limits_{S_{t,u}} a\phi\left(-\frac{1}{4}\mathrm{tr}\underline{\chi}|\widehat{\chi}|^2 + \frac{1}{2}\mathrm{tr}r\chi\,|\zeta|^2 - \pi\mathrm{tr}\underline{\chi}T_{LL}\right)\\ &\quad - \frac{r}{16\pi}\int\limits_{S_{t,u}} (a\phi\mathrm{tr}\chi - \overline{a\phi\mathrm{tr}\chi})\underline{\mu}.\end{aligned} \tag{96}$$

Let us have a look at the terms on the right-hand side. From the fact that $\underline{\mu} = O(r^{-3})$ it follows that the integrand on the second line of (96) is $O(r^{-5})$. Moreover, all the integrands on the first line of (96) are also $O(r^{-5})$ or higher order. This leads to the conclusion

$$\frac{\partial}{\partial t}m(t,u) = O(r^{-2}).$$

Then we keep u fixed and observe $m(t,u)$ to reach its limit $M(u)$ as $t \to \infty$. This limit $M(u)$ is called the Bondi mass, and it is defined for each null hypersurface C_u. Thus, in each C_u as $t \to \infty$ the Hawking mass $m(t,u)$ equals the Bondi mass $M(u)$ plus terms decaying like $O(r^{-1})$. Note that the null-fluid term on the right-hand side of (96) decays fast enough not to interfere with the 'purely geometrical' parts. Thus, we have proven the following theorem.

Theorem 4. *On any null hypersurface C_u the Hawking mass $m(t,u)$ tends to the Bondi mass $M(u)$ as $t \to \infty$, in particular it is: $m(t,u) = M(u) + O(r^{-1})$.*

Having understood how the Hawking mass tends to the Bondi mass, the next question is how the mass changes going from one null hypersurface to another. From above we have $m = \frac{r}{8\pi}\int_{S_{t,u}} \underline{\mu}$. Our goal is to derive the Bondi mass-loss formula. To do so, we turn to $\frac{\partial}{\partial u}m(t,u)$ and write

$$\frac{\partial}{\partial u}m(t,u) = \underbrace{\frac{1}{16\pi}\left(\int_{S_{t,u}} \underline{\mu}\right) \cdot r}_{=\frac{1}{2}m} \cdot \overline{a\mathrm{tr}\theta} + \frac{r}{8\pi} \underbrace{\frac{\partial}{\partial u}\int_{S_{t,u}} \underline{\mu}}_{=\int_{S_{t,u}} a(\nabla_N \underline{\mu} + \mathrm{tr}\theta\underline{\mu})}$$

$$= \frac{1}{2}m \cdot \overline{a\mathrm{tr}\theta} + \frac{r}{8\pi}\int_{S_{t,u}} a(\nabla_N \underline{\mu} + \mathrm{tr}\theta\underline{\mu}).$$

The last integrand can be written as follows using $e_4 = a^{-1}(T+N)$ and $e_3 = a(T-N)$:

$$a(\nabla_N \underline{\mu} + \mathrm{tr}\theta\underline{\mu}) = \frac{1}{2}a^2(D_4\underline{\mu} + \mathrm{tr}\chi\underline{\mu}) - \frac{1}{2}(D_3\underline{\mu} + \mathrm{tr}\underline{\chi}\underline{\mu}).$$

We compute

$$D_4\underline{\mu} + \mathrm{tr}\chi\underline{\mu} = O(r^{-4})$$

$$D_3\underline{\mu} + \mathrm{tr}\underline{\chi}\underline{\mu} = -\frac{1}{4}\mathrm{tr}\chi|\hat{\underline{\chi}}|^2 - 2\pi\mathrm{tr}\chi T_{\underline{LL}} + O(r^{-4}).$$

We derive

$$\frac{\partial}{\partial u}m(t,u) = \frac{r}{64\pi}\int_{S_{t,u}} \mathrm{tr}\chi\left(|\hat{\underline{\chi}}|^2 + 8\pi T_{\underline{LL}}\right) + O(r^{-4}).$$

To derive the limit yielding the Bondi mass-loss formula, we need to check the limits of each term in the expression $\frac{\partial m(t,u)}{\partial u}$ for the Hawking mass. First, we recall that for each u, $\phi^*_{t,u}$ denotes a diffeomorphism from the unit sphere S^2 to $S_{t,u}$. Then by arguments along the lines as in [11] it follows that for each u as $t \to \infty$ the metric $\tilde{\gamma} = \phi^*_{t,u}(r^{-2}\gamma)$ converges to the standard metric $\overset{\circ}{\gamma}$ on S^2. It follows in a straightforward manner that for each u as $t \to \infty$, $r\mathrm{tr}\chi$ converges to 2, and $r\hat{\underline{\chi}}$ converges to -2Ξ. Recall that $T^*_{\underline{LL}}$ is positive. Taking the limits we obtain the Bondi mass-loss formula: - See [7] for the first appearance of Bondi mass-loss.

$$\frac{\partial}{\partial u}M(u) = \frac{1}{8\pi}\int_{S^2}\left(|\Xi|^2 + 2\pi T^*_{\underline{LL}}\right)\mathrm{d}\mu_{\overset{\circ}{\gamma}}.$$

This term is positive and integrable in u, from which it follows that the Bondi mass $M(u)$ is a non-decreasing function of u. Moreover, it has finite limits

$M(-\infty)$ and $M(\infty)$ as u tends to $-\infty$ and $+\infty$, respectively. From (95) it follows that $M(-\infty) = 0$ and that $M(\infty)$ is the total mass. We have therefore proven the next theorem:

Theorem 5. *The Bondi mass $M(u)$ obeys the following Bondi mass-loss formula:*

$$\frac{\partial}{\partial u} M(u) = \frac{1}{8\pi} \int_{S^2} \left(|\Xi|^2 + 2\pi T^*_{\underline{L}\underline{L}} \right) \mathrm{d}\mu_{\overset{\circ}{\gamma}}$$

where $\mathrm{d}\mu_{\overset{\circ}{\gamma}}$ denotes the area element of the standard unit sphere S^2.

We compare this result with the corresponding formulas in the purely gravitational case and in the electromagnetic case. See [5,10,11,22]. Thus, we find that the energy–momentum tensor of the null fluid describing neutrino radiation contributes to the change of the Bondi mass through the term $\frac{1}{4} \int_{S^2} T^*_{\underline{L}\underline{L}} d\mu_{\overset{\circ}{\gamma}}$.

The behavior of Ξ and $T^*_{\underline{L}\underline{L}}$ in u are consequences of Theorems 1 and 3. Now, we define the function

$$F = \int_{-\infty}^{+\infty} \left(|\Xi|^2 + 2\pi T^*_{\underline{L}\underline{L}} \right) \mathrm{d}u. \tag{97}$$

We then find the total energy radiated to infinity in a given direction per unit solid angle to be $\frac{F}{8\pi}$. We note that neutrino radiation contributes through its corresponding null component limit $T^*_{\underline{L}\underline{L}}$.

4.3. Permanent Displacement Formula

The difference $\Sigma^+ - \Sigma^-$ governs the permanent displacement formula of test particles in a gravitational-wave detector. In this section, we prove a theorem for $\Sigma^+ - \Sigma^-$ in the case of neutrino radiation described by a null fluid in the Einstein equations.

The theorem we prove in this chapter employs the full and rich geometric-analytic structure of our spacetime. The emerging result is then related to experiment in the last part of the present article.

At this point, we emphasize that $T^*_{\underline{L}\underline{L}}$ is positive.

Theorem 6. *Denote by $\Sigma^+(\cdot)$ the limit $\Sigma^+(\cdot) = \lim_{u\to\infty} \Sigma(u,\cdot)$ and by $\Sigma^-(\cdot)$ the limit $\Sigma^-(\cdot) = \lim_{u\to-\infty} \Sigma(u,\cdot)$. Let*

$$F(\cdot) = \int_{-\infty}^{\infty} \left(\mid \Xi(u,\cdot) \mid^2 + 2\pi T^*_{\underline{L}\underline{L}}(u,\cdot) \right) \mathrm{d}u. \tag{98}$$

Also, let Φ be the solution with $\bar{\Phi} = 0$ on S^2 of the equation

$$\overset{\circ}{\triangle}\, \Phi = F - \bar{F}.$$

Then $\Sigma^+ - \Sigma^-$ is determined by the following equation on S^2.

$$\overset{\circ}{\operatorname{div}}\, (\Sigma^+ - \Sigma^-) = \overset{\circ}{\nabla}\, \Phi. \tag{99}$$

Proof. First, one has to check on the limits of Σ. Theorem 3, Eq. (88) ensures that Σ tends to limits Σ^+ as $u \to \infty$ and Σ^- as $u \to -\infty$. Moreover, one has

$$\Sigma(u) - \Sigma^- = -\int_{-\infty}^{u} \Xi(u')\mathrm{d}u'$$

as well as

$$\Sigma^+ - \Sigma^- = -\int_{-\infty}^{\infty} \Xi(u')\mathrm{d}u'.$$

Let us now explore how to get the limiting equation at null infinity for Σ. For this purpose, we focus on the normalized null Codazzi equation

$$(\mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\hat{\chi})_A - \frac{1}{2}\nabla\!\!\!/_A \mathrm{tr}\chi + \epsilon_B \hat{\chi}_{AB} - \frac{1}{2}\epsilon_A \mathrm{tr}\chi = -\beta(W)_A - 4\pi T_{AL}. \tag{100}$$

Then we multiply Eq. (100) by r^3 and take the limit as $t \to \infty$ on C_u. We also introduce

$$E = \lim_{C_u, t\to\infty} \left(r^2 \epsilon\right).$$

We derive thereby the limiting equation on S^2:

$$\overset{\circ}{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\ \Sigma = \overset{\circ}{\nabla\!\!\!/}\ H + E, \tag{101}$$

This structure is the same as in the EV case, which is proven in [11] p. 510, conclusion 17.0.8.

Next, from our result on H in Lemma 1 Eq. (90) we obtain

$$\overset{\circ}{\mathrm{d}\not{\mathrm{i}}\mathrm{v}}\ (\Sigma) = E. \tag{102}$$

Thus, the next task is to investigate E at null infinity through its limiting Hodge system on S^2. We therefore study the Hodge system for ϵ:

$$\begin{aligned} \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\epsilon &= -\nabla_N \delta - \frac{3}{2}\mathrm{tr}\theta\delta + \hat{\eta}\cdot\hat{\theta} \\ &\quad - 2(a^{-1}\nabla\!\!\!/\, a)\cdot\epsilon - \pi T_{\underline{LL}} + \pi T_{LL} \end{aligned} \tag{103}$$

$$\mathrm{c}\not{\mathrm{u}}\mathrm{rl}\,\epsilon = \sigma(W) + \hat{\theta}\wedge\hat{\eta}. \tag{104}$$

To derive Eq. (103), we consider (73) and write for the normal component.

$$\begin{aligned} R_{0N} = (\mathrm{div}k)_N &= \nabla_N k_{NN} + \gamma^{AB}\nabla_B k_{NA} \\ &= \nabla_N \delta + 2(a^{-1}\nabla\!\!\!/\, a)\cdot\epsilon + \mathrm{d}\not{\mathrm{i}}\mathrm{v}\,\epsilon + \frac{3}{2}\delta\cdot\mathrm{tr}\theta - \hat{\eta}\cdot\hat{\theta}. \end{aligned} \tag{105}$$

From here with (7) we compute directly and obtain (103). Further, we use (72) to find the c$\not{\mathrm{u}}$rl Eq. (104). The latter in fact coincides with the one obtained by Christodoulou and Klainerman [11], whereas the d$\not{\mathrm{i}}$v Eq. (103) contains the extra terms T_{LL} and $T_{\underline{LL}}$ from the null fluid.

In the $\not{d}\mathrm{iv}/\not{c}\mathrm{url}$ system [(103), (104)] we make use of underlying structures when taking the limit on C_u as $t \to \infty$. To extract these structures, we introduce $\Psi,\ \Psi'$ as follows:

$$\triangle\Psi = r \mid \hat{\eta} \mid^2 - \pi r T_{\underline{LL}} \tag{106}$$

$$\triangle\Psi' = -ra^{-1}\lambda\big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\big) + \pi r^2 a^{-1}\big(a\not{D}_4 T_{\underline{LL}} - \overline{a\not{D}_4 T_{\underline{LL}}}\big). \tag{107}$$

The reader may want to compare this with the formulas in the EV case by Christodoulou and Klainerman [11], chapter 11.2, (11.2.2b) and (11.2.7b) which read

$$\triangle\Psi = r \mid \hat{\eta} \mid^2 \tag{108}$$

$$\triangle\Psi' = -ra^{-1}\lambda\big(\mid \hat{\eta} \mid^2 - \overline{\mid \hat{\eta} \mid^2}\big). \tag{109}$$

In our new situation of neutrino radiation given by the null fluid, we compute the limits as

$$\lim_{C_u, t\to\infty} \Psi = \mathbf{\Psi} \qquad\qquad \lim_{C_u, t\to\infty} \Psi' = \mathbf{\Psi'}$$
$$\lim_{C_u, t\to\infty} r\nabla_N \Psi = \Omega(u,\cdot) \qquad\qquad \lim_{C_u, t\to\infty} r\nabla_N \Psi' = \Omega'(u,\cdot). \tag{110}$$

We proceed by investigating $\nabla_N \delta$ in Eq. (103). Writing the following equation for $\nabla_N \delta$ and comparing it to [11], chapter 17, (17.0.12c), we note that our formula (111) differs from that by the extra term from the null fluid.

$$\begin{aligned}
&\nabla_N \delta - \hat{\theta}\cdot\hat{\eta} + \pi T_{\underline{LL}} \\
&\quad = -2r^{-3}(\nabla_N r)p + r^{-2}\nabla_N p - r^{-2}(\nabla_N r)\nabla_N \Psi + r^{-1}\nabla_N^2 \Psi \\
&\quad = -\hat{\chi}\cdot\hat{\eta} - r^{-1}\not{\triangle}\,\Psi - r^{-2}\big(r\mathrm{tr}\theta + a^{-1}\lambda\big)\nabla_N \Psi \\
&\qquad - r^{-1}a^{-1}\not{\nabla}\, a \cdot \not{\nabla}\,\Psi + r^{-2}\nabla_N p - 2r^{-3}a^{-1}\lambda p
\end{aligned} \tag{111}$$

with

$$p = r\nabla_N q + q' + \Psi' \qquad \text{and} \qquad p = r(r\delta - \nabla_N \Psi).$$

In a straightforward manner, along the lines of the argument in [11] and also used in [22], we show that

$$\triangle q = r(\mu - \overline{\mu}) + I \tag{112}$$

with

$$\begin{aligned}
I &= \frac{1}{2}\, {}^{(rN)}\hat{\pi}^{ij} k_{ij} - \triangle\Psi - \pi r T_{\underline{LL}} + \pi r T_{LL} \\
&= r\hat{\chi}\cdot\hat{\eta} - \kappa\delta - 2ra^{-1}\not{\nabla}\, a\cdot\epsilon + \pi r T_{LL}.
\end{aligned}$$

Recall the mass aspect function μ to be

$$\mu = -\widehat{\chi}\cdot\hat{\eta} - \rho(W).$$

In the next step, we make use of the radial decomposition of $\triangle$ given in formula (70) above. Direct conclusions from the last equations yield

$$\begin{aligned}
\triangle q &= \nabla_N^2 q + \mathrm{tr}\theta\nabla_N q + \not{\triangle}\, q + a^{-1}\not{\nabla}\, a\cdot\not{\nabla}\, q \\
&= -r(\rho - \bar{\rho}) - r\overline{\hat{\chi}\cdot\hat{\eta}} - \kappa\delta - 2ra^{-1}\not{\nabla}\, a\cdot\epsilon + \pi r T_{LL}.
\end{aligned} \tag{113}$$

Now, first we substitute for $\nabla_N p$ from (113) in (111) and then the resulting terms from (111) in (103) to obtain

$$\begin{aligned} \mathrm{d}\!\!/\mathrm{iv}\, \epsilon &= \rho - \bar{\rho} + \hat{\chi} \cdot \hat{\eta} - \overline{\hat{\chi} \cdot \hat{\eta}} \\ &\quad + r^{-1} \not\Delta \Psi - r^{-2} \nabla_N \Psi' - r^{-3} a^{-1} \lambda \Psi' + \text{l.o.t.} \end{aligned} \tag{114}$$

$$\mathrm{c}\!\!/\mathrm{url}\, \epsilon = \sigma(W) + \hat{\theta} \wedge \hat{\eta}. \tag{115}$$

Then we multiply equations (114) and (115) by r^3 and take the limits on C_u as $t \to \infty$. We thereby derive the following limiting equations for E on S^2. That is the Hodge system for E at null infinity.

$$\overset{\circ}{\mathrm{c}\!\!/\mathrm{url}}\ E = Q + \Sigma \wedge \Xi \tag{116}$$

$$\begin{aligned} \overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}}\ E &= P - \bar{P} + \Sigma \cdot \Xi - \overline{\Sigma \cdot \Xi} \\ &\quad + \overset{\circ}{\not\Delta}\ \Psi - \Psi' - \Omega'. \end{aligned} \tag{117}$$

We investigate the limits as $u \to +\infty$ and $u \to -\infty$. Taking into account the above equations for ϵ and E, applying Theorems 3 and 1 it follows that E tends to a limit E^+ as $u \to +\infty$ and to E^- as $u \to -\infty$.

As the $\overset{\circ}{\mathrm{c}\!\!/\mathrm{url}}$ equation does not include energy–momentum terms, similar arguments as in [11], chapter 17, yield

$$\overset{\circ}{\mathrm{c}\!\!/\mathrm{url}}\ (E^+ - E^-) = 0.$$

The situation for $\overset{\circ}{\mathrm{d}\!\!/\mathrm{iv}}\ (E^+ - E^-)$ is more subtle and requires detailed investigations. The energy–momentum tensor plays a crucial role. We point out that the computation of the limits involving $\boldsymbol{\Psi}$ and $\boldsymbol{\Psi}'$ and Ω' are crucial.

In the EV case, Christodoulou and Klainerman prove the corresponding result in lemma 17.0.2, on page 504 of [11]. Here, we establish the new results where the null-fluid term $T_{\underline{LL}}$ and its limit $T^*_{\underline{LL}}$ change the picture. Whereby it is used that

$$\not{D}_4 T_{\underline{LL}} = \not{D}_4 \mathcal{N} = -\mathrm{tr}\chi \mathcal{N} + \text{l.o.t.} \tag{118}$$

This is a direct consequence from (44) and results thereafter.

Calculating the limits (110), applying (118), (106) and (107), we derive formulas for $\boldsymbol{\Psi}$, $\boldsymbol{\Psi}'$, Ω, Ω'. We find

$$\begin{aligned} \Omega' = -\frac{1}{2^{\frac{3}{2}} 4\pi} \int_{-\infty}^{+\infty} \Bigg\{ \int_{S^2} \frac{|\,\Xi\,|^2\ (u', \omega') - \overline{|\,\Xi\,|^2\ (u')}}{(1 - \omega\omega')^{\frac{1}{2}}} \mathrm{d}\omega' \\ + 2\pi \int_{S^2} \frac{T^*_{\underline{LL}}(u', \omega') - \overline{T^*_{\underline{LL}}(u')}}{(1 - \omega\omega')^{\frac{1}{2}}} \mathrm{d}\omega' \Bigg\} \mathrm{d}u' \\ - \frac{1}{2} \int_{-\infty}^{+\infty} \Big\{ \mathrm{sgn}(u - u') \Big(\Big(|\,\Xi\,|^2\ (u', \omega') - \overline{|\,\Xi\,|^2\ (u')} \Big) \end{aligned}$$

$$+ 2\pi\Big(T^*_{\underline{LL}}(u',\omega') - \overline{T^*_{\underline{LL}}(u')}\Big)\Big)\Big\}\mathrm{d}u'$$

$$\Omega = \frac{1}{2^{\frac{3}{2}}4\pi}\int_{-\infty}^{+\infty}\left\{\int_{S^2}\frac{|\,\Xi\,|^2\ (u',\omega')}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega' + 2\pi\int_{S^2}\frac{T^*_{\underline{LL}}(u',\omega')}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega'\right\}du'$$

$$+ \frac{1}{2}\int_{-\infty}^{+\infty}\Big\{\mathrm{sgn}(u-u')\Big(\,|\,\Xi\,|^2\ (u',\omega') + 2\pi T^*_{\underline{LL}}(u',\omega')\Big)\Big\}\,\mathrm{d}u'$$

$$\mathbf{\Psi}' = \frac{1}{2^{\frac{1}{2}}4\pi}\int_{-\infty}^{+\infty}\left\{\int_{S^2}\frac{|\,\Xi\,|^2\ (u',\omega') - \overline{|\,\Xi\,|^2\ (u')}}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega'\right.$$

$$\left. +2\pi\int_{S^2}\frac{T^*_{\underline{LL}}(u',\omega') - \overline{T^*_{\underline{LL}}(u')}}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega'\right\}\mathrm{d}u'$$

$$\mathbf{\Psi} = -\frac{1}{2^{\frac{1}{2}}4\pi}\int_{-\infty}^{+\infty}\left\{\int_{S^2}\frac{|\,\Xi\,|^2\ (u',\omega')}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega' + 2\pi\int_{S^2}\frac{T^*_{\underline{LL}}(u',\omega')}{(1-\omega\omega')^{\frac{1}{2}}}\mathrm{d}\omega'\right\}\mathrm{d}u'.$$

By straightforward computations we find the following: Investigating the difference of the limits as $u \to +\infty$ and $u \to -\infty$ in (117), there are no contributions from $\overset{\circ}{\not\Delta}\ \Psi$, Ψ'. Only terms in Ω' contribute. It follows directly that Ω' tends to limits $\Omega'^+(\cdot)$ and $\Omega'^-(\cdot)$ as $t \to \infty$ and $t \to -\infty$, respectively. From this we derive

$$\begin{aligned}&\Omega'^+(\cdot) - \Omega'^-(\cdot)\\ &\quad = \int_{-\infty}^{+\infty}\Big(|\,\Xi(u,\cdot)\,|^2 - \overline{|\,\Xi(u,\cdot)\,|^2} + 2\pi T^*_{\underline{LL}}(u,\cdot) - 2\pi\overline{T^*_{\underline{LL}}(u,\cdot)}\Big)\,\mathrm{d}u.\end{aligned} \tag{119}$$

We conclude

$$\begin{aligned}\overset{\circ}{\not{\mathrm{div}}}\ (E^+ - E^-) &= -\Omega'^+ + \Omega'^-\\ &= \int_{-\infty}^{+\infty}\Big(-\,|\,\Xi(u,\cdot)\,|^2 + \overline{|\,\Xi(u,\cdot)\,|^2} - 2\pi T^*_{\underline{LL}}(u,\cdot)\\ &\quad +2\pi\overline{T^*_{\underline{LL}}(u,\cdot)}\Big)\,\mathrm{d}u.\end{aligned} \tag{120}$$

This yields

$$(E^+ - E^-) = \overset{\circ}{\not\nabla}\ \Phi \tag{121}$$

where Φ is the solution of

$$\overset{\circ}{\not\Delta}\ \Phi = -\Omega'^+ + \Omega'^- \quad \text{on } S^2,$$

with $\bar{\Phi} = 0$ on S^2.

Now, we need Eq. (102) from above

$$\overset{\circ}{\text{div}}\!\!\!/\ \Sigma = E.$$

Relating result (121) to this equation, where in the latter we first take the limits as $u \to \infty$ and $u \to -\infty$, we conclude

$$\overset{\circ}{\text{div}}\!\!\!/\ (\Sigma^+ - \Sigma^-) = E^+ - E^- = \overset{\circ}{\nabla}\!\!\!/\ \Phi, \tag{122}$$

which is Eq. (99).

This concludes the proof of the theorem.

4.4. Limit for r as $t \to \infty$ on Null Hypersurface C_u

We shall use the fact that the constraint on the spacelike scalar curvature, which is given by

$$R = |k|^2 + R_{00},$$

differs from the constraint in the vacuum case only by the term R_{00}.

We can now prove the following results.

Theorem 7. *As $t \to \infty$ we obtain on any null hypersurface C_u*

$$r = t - 2M(\infty)\log t + O(1).$$

Proof. We recall from [11], p. 503, with $\phi' = \phi - 1$,

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}t} &= \frac{r}{2}\overline{\phi \mathrm{tr}\chi'}\\ &= \frac{r}{2}\overline{(1+\phi')\left(\frac{2}{r} + \left(\mathrm{tr}\chi' - \frac{2}{r}\right)\right)}\\ &= 1 + \overline{\phi'} + O(r^{-2}).\end{aligned}$$

In the last equality, we use Eq. (91). In fact, this holds in our case as well. See [22] for the Einstein–Maxwell situation for which this computation is analogous to ours.

Now, in the Einstein null-fluid case, we have for R_{00} the following expression in terms of the components T_{LL} and $T_{\underline{LL}}$ of the null fluid:

$$R_{00} = 8\pi T_{00} = 2\pi(T_{\underline{LL}} - T_{LL}). \tag{123}$$

Moreover, the lapse equation in our situation is given by

$$\triangle\phi = (\mid k \mid^2 + R_{00})\phi. \tag{124}$$

We integrate the lapse Eq. (124) on H_t in the interior of $S_{t,u'}$ to obtain

$$\int_{S_{t,u}} \nabla_N \phi' = \int_{u_0}^{u} \mathrm{d}u' \int_{S_{t,u'}} a\phi(\mid k \mid^2 + R_{00}).$$

In view of (123) and the fact that all the terms on the right-hand side of (123) except $T_{\underline{LL}}$ are of lower order, we estimate

$$\int_{S_{t,u}} \nabla_N \phi' = \int_{u_0}^{u} \mathrm{d}u' \int_{S_{t,u'}} a\phi(\mid k \mid^2 + 2\pi T_{\underline{LL}}) + \text{l.o.t.}$$

We see that

$$\int_{S_{t,u'}} a\phi(\mid k \mid^2 + 2\pi T_{\underline{LL}}) \to \int_{S^2} \mid \Xi \mid^2 + 2\pi T^*_{\underline{LL}}.$$

Consider the Bondi mass-loss formula in Theorem 5. Then, as $t \to \infty$ we conclude

$$\int_{S_{t,u}} \nabla_N \phi' - 8\pi M(u) = O(r^{-1}) \tag{125}$$

on each C_u. In view of ϕ' we compute:

$$\begin{aligned}
\overline{\phi'} &= \frac{1}{4\pi r^2} \int_{S_{t,u}} \phi' = -\frac{1}{4\pi} \int_{B} \operatorname{div}(r^{-2}\phi' N) \\
&= \frac{1}{4\pi} \int_{B} \left(-\frac{1}{a(r(t,u'))^2} \overline{a\mathrm{tr}\theta} N\phi' + \frac{1}{(r(t,u'))^2} \phi' \underbrace{(\operatorname{div} N)}_{=\mathrm{tr}\theta} + \frac{1}{(r(t,u'))^2} \nabla_N \phi' \right) \\
&= -\frac{1}{4\pi} \int_{u}^{\infty} \frac{1}{(r(t,u'))^2} \mathrm{d}u' \left(\int_{S_{t,u'}} a\nabla_N \phi' + (a\mathrm{tr}\theta - \overline{a\mathrm{tr}\theta})\phi' \right) \\
&= -\frac{1}{4\pi} \int_{u}^{\infty} \frac{1}{(r(t,u'))^2} \mathrm{d}u' \left(\int_{S_{t,u'}} \nabla_N \phi' \right) + O(r^{-2}),
\end{aligned}$$

where B denotes the exterior of $S_{t,u}$. Therefore, from (125) it follows on C_u as $t \to \infty$,

$$\overline{\phi'}(t,u) = -2 \int_{u}^{\infty} \frac{1}{(r(t,u'))^2} M(u') \mathrm{d}u' + O(r^{-2}) = -\frac{2}{r} M(\infty) + O(r^{-2}).$$

Thus, we obtain on any cone C_u for $t \to \infty$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = 1 - \frac{2}{r} M(\infty) + O(r^{-2}). \tag{126}$$

Thus, the statement of our theorem follows, which closes the proof.

5. Gravitational Wave Experiments

In the previous chapters, we derived a contribution from neutrino radiation to the nonlinear Christodoulou memory effect of gravitational-waves. This effect will be shown as a permanent displacement of test masses in a laser interferometer gravitational-wave detector. In this section, we show how the mathematical results relate to experiment. In a typical source of a neutrino burst such as core-collapse supernovae or binary neutron star mergers, over a timescale of tens of seconds a huge amount of energy is radiated away in the form of neutrinos. In particular, in the case of a supernova, it is expected that approximately 99 % of the gravitational binding energy of the remnant in the process is converted into neutrinos. See Scholberg's [17] article for a recent review. In such a process, gravitational-waves are emitted and the wave packet is traveling at the speed of light along the null hypersurfaces of our spacetime. We may think of doing the experiment at null infinity of the spacetime.

Our findings are twofold: first, we discuss the instantaneous displacements of test masses occurring while the packet is moving through the experiment. Second, we investigate the permanent displacements of test masses after the gravitational-wave train has passed, namely the nonlinear Christodoulou memory effect. We prove that the contribution from neutrino radiation described as a null fluid has only lower order[1] contribution to the first effect, but contributes at the same highest order as the 'purely geometrical' term to the nonlinear Christodoulou effect. The information about the null-fluid part is 'encoded' in $\Sigma^+ - \Sigma^-$ and described in Theorem 6. Precisely, this latter term governs the permanent displacement as we show at the end of the present chapter.

Now, we briefly review the setup of such a detector with three test masses. A detailed explanation is given in Christodoulou's [10] pioneering paper, and a derivation in the Einstein–Maxwell case is given in the article by the first present author with Chen and Yau [5].

Let us think of the experiment having a reference test mass m_0 at the location of the beam splitter. Initially, masses m_1 and m_2 are at equal distances d from m_0 forming a right angle there. In an Earth-based detector such as LIGO, the masses are suspended by pendulums and thus are free. If the observation is performed in space as in LISA, then the masses are free by nature. In fact, in the first case, for time scales much shorter than the period of the pendulums the motion of the masses in the horizontal plane can be considered free. By laser interferometry, the distance of m_1 and m_2 from the reference mass m_0 is measured. Whenever the light travel times between the masses differ, then we see a difference of phase of the laser light at m_0.

The three masses move along geodesics in spacetime. We denote the geodesic for m_0 by Γ_0. Let T be the future-directed tangent vector field of Γ_0 of unit magnitude. Moreover, let t denote the arc length along Γ_0. At $\Gamma_0(0)$ we choose an orthonormal frame (e_1, e_2, e_3) for the spacelike, geodesic hyperplane

[1] Here and in what follows, the word 'order' refers to decay behavior of the exact solution, not to any approximations. That is, 'higher order' means 'less decay'. For details, see [2,5,22].

H_0. For each t denote by H_t the spacelike, geodesic hyperplane through $\Gamma_0(t)$ orthogonal to T. We obtain the orthonormal frame field (T, e_1, e_2, e_3) along Γ_0 by parallel propagation of (e_1, e_2, e_3). The latter being an orthonormal frame for every H_t at $\Gamma_0(t)$. To a point in spacetime close to Γ_0 and lying in H_t we can now assign cylindrical normal coordinates (t, x^1, x^2, x^3).

Assume that the source of the waves is in the e_3-direction, and that the light travel time corresponding to the distance d is significantly shorter than the time scale of large variations of the spacetime curvature. Then the geodesic equation for the trajectories of m_1 and m_2 is replaced by the Jacobi equation (127) measuring the geodesic deviation from Γ_0. With $R_{kTlT} = R(e_k, T, e_l, T)$ it is

$$\frac{\mathrm{d}^2 x^k}{\mathrm{d}t^2} = - \ R_{kTlT} \ x^l. \tag{127}$$

The acceleration in (127) is controlled by the curvature R_{kTlT}. To reveal the roles played by the null fluid and by the gravitational part, we have to investigate the structure of R_{kTlT}. Thus, we decompose the latter into its Weyl and Ricci parts:

$$R_{kTlT} = W_{kTlT} + \frac{1}{2}(g_{TT} R_{kl} + g_{kl} R_{TT} - g_{Tl} R_{kT} - g_{Tk} R_{lT}). \tag{128}$$

The Einstein null-fluid equations (7) tell us that

$$R_{TT} = 8\pi T_{TT},$$

ensuring the following identity:

$$R_{TT} = 8\pi T_{TT} = 2\pi (T_{\underline{L}\underline{L}} - T_{LL}). \tag{129}$$

The worst decay behavior on the right-hand side of (128) occurs in R_{TT}, namely we find it in the null-fluid component $T_{\underline{L}\underline{L}}$.

To take limits at null infinity, we change to the null frame with $L = T - e_3$ and $\underline{L} = T + e_3$. Then the leading components of the curvature can be expressed as

$$\underline{\alpha}_{AB} = R(e_A, \underline{L}, e_B, \underline{L})$$
$$\underline{\alpha}_{AB} = \frac{A_{AB}}{r} + o(r^{-2}),$$

and the leading component of the null fluid as

$$T_{\underline{L}\underline{L}} = \frac{T^*_{\underline{L}\underline{L}}}{r^2} + \text{l.o.t.} \tag{130}$$

We now observe that the null fluid enters the right-hand side of the Jacobi equation at order (r^{-2}). As a consequence the null fluid does not contribute at leading order to the deviation measured. This brings us back to the situation for the Einstein vacuum equations investigated by Christodoulou [10]. At leading order, our result coincides with his

$$\frac{\mathrm{d}^2 x^k{}_{(A)}}{\mathrm{d}t^2} = -\frac{1}{4} r^{-1} A_{AB} x^l_{(B)} + O(r^{-2}). \tag{131}$$

Similarly to [5,10] we find that in the Einstein null-fluid case there is no acceleration in the vertical direction to leading order (r^{-1}). Before the wave packet travels through the experiment, the masses m_1 and m_2 are at rest at equal distance d and at right angles from m_0. That is, we have the initial conditions as $t \to -\infty$: ${x^3}_{(A)} = 0$, ${\dot{x}^3}_{(A)} = 0$, ${x^B}_{(A)} = \mathrm{d}\delta^B_A$, ${\dot{x}^B}_{(A)} = 0$. As the right-hand side is very small, one can substitute the initial values on the right-hand side. Then the motion is confined to the horizontal plane. To leading order it is:

$$\ddot{x}^{A}{}_{(B)} = -\frac{1}{4} r^{-1} d A_{AB}. \tag{132}$$

Integrating yields

$$\dot{x}^A_{(B)}(t) = -\frac{1}{4} d r^{-1} \int_{-\infty}^{t} A_{AB}(u) \mathrm{d}u. \tag{133}$$

At this point, the identities (87) and (88) will be applied. First, using Eq. (87), namely $\frac{\partial \Xi}{\partial u} = -\frac{1}{4}\ A$ with $\lim_{|u|\to\infty} \Xi = 0$ we have

$$\Xi(t) = -\int_{-\infty}^{t} A_{AB}(u) \mathrm{d}u \tag{134}$$

and thus

$$\dot{x}^A{}_{(B)}\ (t) = \frac{d}{r}\ \Xi_{AB}\ (t). \tag{135}$$

As $\Xi \to 0$ for $u \to \infty$, the test masses return to rest after the passage of the gravitational-wave train. Next, using (88), that is $\frac{\partial \Sigma}{\partial u} = -\Xi$, another integration yields

$$x^A{}_{(B)}\ (t) = -\left(\frac{d}{r}\right)(\Sigma_{AB}(t) - \Sigma^-). \tag{136}$$

Now, we take the limit $t \to \infty$ to obtain

$$\triangle x^A{}_{(B)} = -\left(\frac{d}{r}\right)(\Sigma^+_{AB} - \Sigma^-_{AB}). \tag{137}$$

Thus, we find that the test masses are permanently displaced. In particular, $\Sigma^+ - \Sigma^-$ is equivalent to an overall displacement of the test masses. Precisely this term, in Theorem 6, is proven to exhibit a contribution from the null fluid besides the purely gravitational part. This 'purely' geometrical part is treated in [10].

Thus, we find that the instantaneous displacements of test masses are not affected at highest order by the null fluid. However, the null fluid does contribute at highest order to the permanent displacement of the masses and therefore enlarges the nonlinear Christodoulou memory effect of gravitational-waves.

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