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Topology, Rigid Cosymmetries and Linearization Instability in Higher Gauge Theories

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Abstract. We consider a class of non-linear PDE systems, whose equations possess Noether identities (the equations are redundant), including non-variational systems (not coming from Lagrangian field theories), where Noether identities and infinitesimal gauge transformations need not be in bijection. We also include theories with higher stage Noether identities, known as higher gauge theories (if they are variational). Some of these systems are known to exhibit linearization instabilities: there exist exact background solutions about which a linearized solution is extendable to a family of exact solutions only if some non-linear obstruction functionals vanish. We give a general, geometric classification of a class of these linearization obstructions, which includes as special cases all known ones for relativistic field theories (vacuum Einstein, Yang–Mills, classical $N = 1$ supergravity, etc.). Our classification shows that obstructions arise due to the simultaneous presence of rigid cosymmetries (generalized Killing condition) and non-trivial de Rham cohomology classes (spacetime topology). The classification relies on a careful analysis of the cohomologies of the on-shell Noether complex (consistent deformations), adjoint Noether complex (rigid cosymmetries) and variational bicomplex (conserved currents). An intermediate result also gives a criterion for identifying non-linearities that do not lead to linearization instabilities.

1. Introduction

It is well known that, on spatially compact manifolds, the solution space of Lagrangian gauge theories like general relativity (GR) and Yang–Mills (YM) theory is an infinite dimensional manifold with singularities [1,2]. The same phenomenon can occur in the solution space of other non-linear systems of partial differential equations (PDE systems), be they Lagrangian field theories or not. A background solution is said to be *linearization stable* if a solution space neighborhood of it can be modeled on the vector space of linearized solutions about that background. That is, for every linearized solution, there exists a 1-parameter family of exact solutions tangent to it. Otherwise, the background is said to be *linearization unstable*, in which case there exist linearized solutions not tangent to any 1-parameter family of exact solutions.

In the case of GR, the solutions space singularities at linearization unstable backgrounds are of conical type. That is, the corresponding solution space neighborhood can be modeled on the zero set of a quadratic constraint on the space of linearized solutions. The presence of such constraints is intimately linked with the spatial compactness of the underlying manifold. For spatially non-compact manifolds with asymptotically flat boundary conditions, conical singularities are absent [3,4].

Based on a remarkable observation of Moncrief [5,6], the necessary and sufficient conditions for linearization stability (resp. instability) have been identified with the absence (resp. presence) of Killing vectors on the background [1,2]. Similar results have been obtained for other gauge theories, including Yang–Mills [7], Einstein–Yang–Mills [8] and supergravity [9]. In each case, the notion of a Killing vector needed to be generalized to express the sufficient conditions for linearization instability.

In this note, we generalize the Killing condition to a general class of Lagrangian gauge theories, and even to non-Lagrangian theories, that satisfy non-trivial *Noether identities*,[1] which mean that the field equations are redundant). In Lagrangian gauge theories, the Noether identities are precisely dual to generators of gauge transformations, by Noether's theorem. In non-Lagrangian theories, the two need not be connected [11]. We then show how these theories acquire potential linearization instabilities, as in the above examples. In fact, this shows that it is the Noether identities that are responsible, rather than the gauge symmetries. Thus, since the classical Killing condition is associated with symmetries, we call the generalized condition responsible for linearization instabilities *co-Killing*. Since, as already mentioned, Noether's second theorem links gauge symmetries and Noether identities, this distinction may have been ambiguous in the past. Briefly, fields satisfying the co-Killing condition are those in the kernel of a adjoint linearized Noether operator and are called *cosymmetries* [17]. In specific cases, they have also been called *reducibility parameters* [12].

Starting with the original work of Fischer and Marsden on vacuum GR [18], the conditions for linearization instability have often been detected

[1] This terminology deserves a comment. Without question, the terminology *Noether identity* [10,11] is correct and standard for Lagrangian systems, where they have also been called *gauge identities* [12]. On the other hand, for non-Lagrangian systems, differential identities that annihilate a given linear differential operator have been traditionally called *compatibility operators* [13–15] or less commonly *Janet operators* [16]. So, what we will later call the *Noether complex* can also be referred to as the *compatibility complex* or *Janet sequence*. We choose to use the name Noether identity even for non-Lagrangian systems simply because of the strong historical link between linearization stability analysis and Lagrangian field theories.

through an analysis of the constraints imposed by the field equations on the initial data. In 4-dimensional GR, this requires the well-known 3+1 ADM decomposition. However, given that the resulting conditions (presence or absence of Killing vectors) are spacetime covariant it is desirable to have a covariant derivation of the standard and generalized co-Killing conditions as well. Such derivations are indeed available for specific examples as for instance for vacuum GR in [1] (see also [19], where similar ideas appear outside the specialized literature). In this paper, we give a covariant derivation of the generalized co-Killing condition, applicable to the same large class of theories mentioned above.

After having completed this work we learned of a little known paper of Arms and Anderson [20], which also gives a fully covariant derivation of generalized Killing conditions, in fact using ideas analogous to those presented below. However, their results are rather less general than ours and are presented in terms of explicit calculations on the example of Einstein–Yang–Mills theory (though with indications of how they are to be generalized). We give a brief comparison of their results with ours at the end of this section.

Since we analyze a large class of field theories instead of working on specific examples, we require an adequate abstract framework to state the necessary hypotheses and carry out the analysis. The abstract framework is provided by the jet-bundle formalism and various associated differential complexes and their cohomologies. One benefit of the abstraction is to attract attention to the fact that the necessary hypotheses are surprisingly liberal. In particular, the class of admissible PDE systems is larger than the determined elliptic and hyperbolic systems or gauge theories closely resembling those. Also, it becomes clear how the topology of the spacetime manifold gives rise to linearization instabilities and that any non-trivial de Rham cohomology class may be responsible (not only one dual to a compact Cauchy surface).

The jet-bundle formalism is introduced in Sect. 2; its contents are standard and serve mostly to fix notation. The main hypotheses imposed on the PDE systems under consideration are described in Sect. 2.2. Section 3 introduces three important concepts: consistent deformations (Sect. 3.1), cosymmetries and the co-Killing condition (Sect. 3.2), and null sources (Sect. 3.3). Section 4 contains the main results of the paper. Linearization instabilities and obstructions are defined in Sect. 4.1. Special conserved currents, deformation currents, are defined by pairing consistent deformations and null sources in Sect. 4.2. Our main result is obtained in Sect. 4.2: the deformation current defined by the non-linearity gives rise to a linearization obstruction valued in the de Rham cohomology of the spacetime manifold. The relation of this result to previous work is briefly discussed in Sect. 4.4, while Sect. 4.5 discusses obstructions generated by non-trivial asymptotic boundary conditions rather than non-trivial topology. Finally, the general result is specialized to several examples, some of which are novel, in Sects. 5 and 6 concludes with a discussion.

Since analogous ideas had previously appeared in [20], let us make a quick comparison. Arms and Anderson used the standard first and second Noether theorems for Einstein–Yang–Mills equations to link 'symmetries' and 'gauge symmetries' (rigid stage-0 and stage-1 symmetries, respectively, in our terminology) with conservation laws. They perturbed these conservation laws to derive conserved currents for the linearized equations. They called these currents 'Taub forms' and their charges (Cauchy surface integrals) 'Taub numbers'. The non-vanishing of some of the Taub numbers was then shown to be an obstruction to linearization stability. Thus, Taub forms are analogous to our deformation currents paired with the equation non-linearity and Taub numbers to our de Rham cohomology valued obstructions. In contrast, their work considered neither non-Lagrangian equations nor lower degree conservation laws. Moreover, the generation of conservation laws from the pairing of deformation currents and non-linear deformations was only implicit in their calculations. Finally, they showed that Taub numbers are gauge invariant. On the other hand, we do not consider this question below, since we make no hypothesis about the presence or absence of gauge symmetries in the class of equations that we consider.

2. Local Geometry of PDEs

2.1. Jets, Local Forms and the Variational Bicomplex

The natural setting for the local analysis of differential equations is that of *jet bundles*. The needed concepts and notation are briefly introduced below. More details can be found in the standard literature; see for instance [21–27].

Given a vector bundle $F \to M$ over a connected n-dimensional smooth manifold M, the *k-jet bundle* $J^k F \to M$ is a vector bundle whose defining characteristic is that for any (possibly non-linear) differential operator $f\colon \Gamma(F) \to \Gamma(F')$ of order k, there exists a canonical factorization $f[u] = f \circ j^k u$ for any section $u\colon M \to F$, where the *k-jet prolongation* $j^k\colon \Gamma(F) \to \Gamma(J^k F)$ is composed with a smooth bundle map $f\colon J^k F \to F'$, which by a slight abuse of notation we denote using the same symbol as the original differential operator. Composing the differential operator f with an l-jet prolongation canonically defines a new differential operator $p_l f\colon J^{l+k} F \to J^l F'$ called its *l-prolongation*, $j^l f[u] = p_l f \circ j^k u$. Given a trivializable restriction $F_U \to U$ of F to a chart $U \subset M$ with local coordinates (x^i) and fiber-adapted local coordinates (x^i, u^a), there is a corresponding adapted chart $J^k F_U \subset J^k F$ with adapted local coordinates (x^i, u^a_I), where $I = i_1 \cdots i_l$ runs through multi-indices of orders $|I| = l = 0, \ldots, k$. In these coordinates, the k-jet prolongation is given by $j^k u(x) = (x^i, \partial_I u^a(x))$, while the l-prolongation is given by $p_l f[u](x) = (x^i, \partial_I f^b[u](x))$, where $f[u](x) = (x^i, f^b[u](x))$ in fiber-adapted local coordinates (x^i, v^b) on F'. For any $l > k$, discarding the information about all derivatives of order $> k$ defines a natural projection $J^l F \to J^k F$. The projective limit $J^\infty F := \varprojlim_{k\to\infty} J^k F$ defines the *∞-jet bundle*. The ∞-jet prolongation j^∞ and ∞-prolongation p_∞ are defined in the obvious way. By composing with the natural projection $J^\infty F \to J^k F$, the differential operator f also canonically defines the smooth bundle map $f\colon J^\infty F \to J^k F \xrightarrow{f} F'$, which is again denoted by the same symbol f. Conversely, due to the projective limit construction, any smooth bundle map $f\colon J^\infty F \to F'$ can only depend on finitely many coordinates of its domain, which means that there exists a $k \geq 0$ such that this bundle map canonically factors as $f\colon J^\infty F \to J^k F \xrightarrow{f} F'$, with the smallest such k being the *order* of f.

Denote by TM and T^*M the tangent and cotangent bundles of M. Also, let $\Lambda^k M = \bigwedge^k T^*M$ be the bundles of k-forms. Denote by $\Omega^k = \Omega^k(M) = \Gamma(\Lambda^k M)$ the spaces of differential forms, with $\Omega^* = \bigoplus_k \Omega^k$. We call $\Omega^*(J^\infty F)$ the space of *local variational forms* on F. The de Rham differential on $\Omega^*(J^\infty F)$ canonically splits into the sum $\mathrm{d} = \mathrm{d}_\mathsf{h} + \mathrm{d}_\mathsf{v}$, where the respective *horizontal* and *vertical* differentials are individually nilpotent and anti-commutative, $\mathrm{d}_\mathsf{h}^2 = \mathrm{d}_\mathsf{v}^2 = 0$ and $\mathrm{d}_\mathsf{h}\mathrm{d}_\mathsf{v} + \mathrm{d}_\mathsf{v}\mathrm{d}_\mathsf{h} = 0$. The defining action of the horizontal differential on adapted local coordinates (x^i, u^a_I) is $\mathrm{d}_\mathsf{h} x^i = \mathrm{d}x^i$ and $\mathrm{d}_\mathsf{h} u^a_I = u^a_{Ii}\mathrm{d}x^i$; then simply $\mathrm{d}_\mathsf{v} = \mathrm{d} - \mathrm{d}_\mathsf{h}$. Denote by $\Omega^{h,0}(F), \Omega^{0,v}(F) \subset \Omega^*(J^\infty F)$ the subspaces of *local horizontal* and *local vertical forms*, which generate the entire space of local variational forms by wedge products. Hence, we have a natural bigrading $\Omega^*(J^\infty F) = \bigoplus_{h,v} \Omega^{h,v}(F)$, where $0 \leq h \leq n$ and $0 \leq v < \infty$. There is a natural inclusion $\Omega^k(M) \subset \Omega^{k,0}(F)$, via the pullback along the natural projection $\pi_\infty\colon J^\infty F \to M$, where the image of the inclusion is said to consist of *field-independent* forms.

The operators d_h and d_v together with the horizontal–vertical bigrading turns the space of local variational forms, augmented as shown below, into the *variational bicomplex of* F [22,23]:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \underline{\Omega}^{0,0} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \underline{\Omega}^{1,0} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \cdots \underline{\Omega}^{n-1,0} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \underline{\Omega}^{n,0} \arrow[d, "\mathrm{d}_\mathsf{v}"] \arrow[dr, bend left, "\delta_{\mathrm{EL}}"] & & \\
0 \arrow[r] & \Omega^{0,1} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \Omega^{1,1} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \cdots \Omega^{n-1,1} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \Omega^{n,1} \arrow[r, "I"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \mathcal{F}^1 \arrow[r] \arrow[d, "\delta_\mathsf{v}"] & 0 \\
0 \arrow[r] & \Omega^{0,2} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \Omega^{1,2} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \cdots \Omega^{n-1,2} \arrow[r, "\mathrm{d}_\mathsf{h}"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \Omega^{n,2} \arrow[r, "I"] \arrow[d, "\mathrm{d}_\mathsf{v}"] & \mathcal{F}^2 \arrow[r] \arrow[d, "\delta_\mathsf{v}"] & 0 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{tikzcd}
\tag{1}
$$

We have abbreviated $\Omega^k = \Omega^k(M)$ and $\Omega^{h,v} = \Omega^{h,v}(F)$. All arrows commute, which defines the *Euler–Lagrange derivative* $\delta_{\mathrm{EL}} = I \circ \mathrm{d}_\mathsf{v}$, where I is the so-called *interior Euler operator*. The space $\mathcal{F}^k \cong \bigoplus_a \mathrm{d}_\mathsf{v} u^a \wedge \Omega^{n,k-1}$ is called the space of *local functional k-forms* is defined by the image of I and the *variational differential* δ_v is defined as the unique map commuting with the rest of the diagram.

We can define the cohomologies, $H^{h,v}(\mathrm{d_h})$ and $H^{h,v}(\mathrm{d_v})$, of the horizontal and vertical differentials in the obvious way; they correspond to the cohomologies of the corresponding parts of the above rows and columns. All the columns and rows are exact, with the exception of the underlined nodes, with the proviso that the bent δ_{EL} arrow makes the following sequence exact also except at the underlined nodes:

$$0 \longrightarrow \underline{\Omega}^{0,0} \xrightarrow{\mathrm{d_h}} \cdots \underline{\Omega}^{n-1,0} \xrightarrow{\mathrm{d_h}} \underline{\Omega}^{n,0} \xrightarrow{\delta_{\mathrm{EL}}} \mathcal{F}^1 \xrightarrow{\delta_{\mathsf{v}}} \mathcal{F}^2 \xrightarrow{\delta_{\mathsf{v}}} \cdots . \quad (2)$$

At the underlined nodes, the horizontal cohomology is completely characterized by the de Rham cohomology of M, $H^{k,0}(\mathrm{d_h}) \cong H^k_{\mathrm{dR}}(M)$. At the same nodes, the vertical cohomology consists of all field-independent forms, $H^{k,0}(\mathrm{d_v}) \cong \Omega^k(M) \subset \Omega^{k,0}(F)$.

Note that any local horizontal form $\omega \in \Omega^{*,0}(F)$ naturally defines a smooth bundle morphism $\omega\colon J^\infty F \to \Lambda^* M$ as well as a differential operator $\omega\colon \Gamma(F) \to \Gamma(\Lambda^* M)$, $\omega[\psi] = \omega \circ j^\infty \psi$ for $\psi \in \Gamma(F)$, where we have slightly abused notation by denoting each of these naturally associated objects by the same symbol ω. It is convenient to generalize this construction by replacing the bundle $\Lambda^* M \to M$ with an arbitrary vector bundle $H \to M$. Any differential operator $f\colon J^\infty F \to H$ shall also be termed a *local section of* H (also *F-local* if such precision is necessary). Denote the space of all such local sections by $\Gamma_F(H)$. Evidently the space of local horizontal forms is the same as the space of local sections of $\Lambda^* M$, $\Omega^{*,0}(F) \cong \Gamma_F(\Lambda^* M)$. The pullback along the bundle projection $F \to M$ induces a natural inclusion $\Gamma(H) \to \Gamma_F(H)$ of *field-independent* local sections in the space of all local sections. Similarly, given two vector bundles $G \to M$ and $H \to M$, we call a differential operator $f\colon \Gamma_F(G) \to \Gamma_F(H)$ *local* (or *F-local*) when $f[\psi, \xi[\psi]] = f \circ (j^\infty \psi \times p_\infty \xi)$, where $\psi \in \Gamma(F)$, $\xi \in \Gamma_F(G)$ and on the right-hand side $f\colon J^\infty(F \times G) \to H$. If $f[\psi, \zeta]$ is linear in its second argument, we write it as $f[\psi; \zeta]$. Of course, any differential operator $f\colon \Gamma(G) \to \Gamma(H)$ naturally defines a *field-independent* local differential operator $f\colon \Gamma_F(G) \to \Gamma_F(H)$.

Local sections and local differential operators are discussed in [13,14] as $\mathcal{C}$-modules and $\mathcal{C}$-differential operators.

2.2. PDE Submanifold

In the jet bundle setting, a PDE system has a dual description [13,14,24,26], as a non-empty, smooth sub-bundle $\mathcal{E}^\infty \subset J^\infty(F)$ over M and as the zero-set of the ∞-prolongation $p_\infty e\colon J^\infty F \to J^\infty G$ of a smooth bundle morphism $e\colon J^\infty F \to G$, where the vector bundles $F \to M$ and $G \to M$ are, respectively, referred to as the *field bundle* and the *equation bundle*. The requirement of having both descriptions is non-vacuous. It is possible that the zero set of $p_\infty e$ is not a submanifold (say if the Jacobian of e is not of constant rank) and it is possible that $\mathcal{E}^\infty$ is not the zero set of any smooth bundle map (see [25, Section 7] for the necessary and sufficient topological condition on the normal bundle of the embedding $\mathcal{E}^\infty \subset J^\infty(F)$). A section $\phi\colon M \to F$ is a *solution* of the PDE system if its ∞-jet prolongation is contained in the PDE submanifold, $j^\infty \phi(M) \subseteq \mathcal{E}^\infty$, or equivalently if $e[\phi] = 0$.

Clearly the smooth PDE sub-bundle $J^\infty F \supset \mathcal{E}^\infty \to M$ provides an intrinsic description of the PDE system, while the choice of the *equation form*, the bundle $G \to M$ and the map e, are not unique. Suppose that the differential operator e is of order k and denote the natural projection $\pi^k\colon J^\infty F \to J^k F$. Without loss of generality, we can presume that the zero set $\mathcal{E}$ of $e\colon J^k F \to G$ is a smooth sub-bundle of $J^k F$ and that it agrees with the projection of $\mathcal{E}^\infty$, $\mathcal{E} = \pi^k \mathcal{E}^\infty$. Such an equation form is said to be *involutive*. In the sequel we freely use an equation form $e[\psi] = 0$ or the submanifold $\mathcal{E} \subset J^k F$ to define a PDE system.

Define the space of *on-shell local variational forms* $\Omega^*(\mathcal{E}^\infty)$ to be the pullback image of $\Omega^*(J^\infty F)$ along the inclusion $\mathcal{E}^\infty \subset J^\infty F$ and d the de Rham differential on it. The horizontal–vertical bigrading and the split of the de Rham differential, $\mathrm{d} = \mathrm{d}_{\mathsf{h}} + \mathrm{d}_{\mathsf{v}}$, commute with the pullback, which immediately defines the decomposition $\Omega^*(\mathcal{E}^\infty) \cong \sum_{h,v} \Omega^{h,v}(\mathcal{E})$ and the *on-shell* variational bicomplex with the projected operators d_{h}, d_{v}, δ_{EL} and δ_v. The cohomology $H^{*,0}_{\mathcal{E}}(\mathrm{d}_{\mathsf{h}})$ in the space of *on-shell local horizontal forms* is also called the *characteristic cohomology* of the PDE system and is denoted $H^*_{\mathrm{char}}(\mathcal{E})$ [21,24,27]. We still have a natural inclusion of the de Rham cohomology of M in the characteristic cohomology, $H^*_{\mathrm{dR}}(M) \subseteq H^*_{\mathrm{char}}(\mathcal{E})$, but it is no longer necessarily an isomorphism. The image of this inclusion is called the subspace of *field-independent* local horizontal forms. The quotient $H^p_{\mathrm{cur}}(\mathcal{E}) := H^{n-p}_{\mathrm{char}}(\mathcal{E})/H^{n-p}_{\mathrm{dR}}(M)$ is known as the space of *conserved (local) p-currents.*

We also define the space of *null local variational forms* $\hat{\Omega}^*(\mathcal{E}^\infty)$ as the kernel of the projection $\Omega^*(F) \to \Omega^*(\mathcal{E}^\infty)$, which consists of all local variational forms that vanish on the PDE submanifold $\mathcal{E}^\infty$. Therefore, we have the following exact sequence:

$$\hat{\Omega}^{*,*}(\mathcal{E}) \longrightarrow \Omega^{*,*}(F) \longrightarrow \Omega^{*,*}(\mathcal{E}). \tag{3}$$

We denote the cohomology of the complex $(\hat{\Omega}^{*,0}, \mathrm{d}_{\mathsf{h}})$ of null local horizontal forms by $H^*_{\mathrm{null}}(\mathcal{E})$. It should be noted that, provided the regularity conditions to be specified below are satisfied, null local forms are precisely those that are exact in terms of the Koszul–Tate part of the BV-BRST complex [27].

Given a vector bundle $H \to M$, we can analogously define the outer ends of the exact sequence

$$\hat{\Gamma}_{\mathcal{E}}(H) \to \Gamma_F(H) \to \Gamma_{\mathcal{E}}(H), \tag{4}$$

where the space $\hat{\Gamma}_{\mathcal{E}}(H)$ of *null local sections of* H consists of those that vanish on the PDE submanifold $\mathcal{E}^\infty$ and the space $\Gamma_{\mathcal{E}}(H)$ of *on-shell local sections of* H is the quotient.

Now, let us consider the equation form $e[\phi] = 0$ as defining a local differential operator $e\colon \Gamma_F(F) \to \Gamma_F(G)$ by the formula $e[\psi, \phi[\psi]] = e[\psi]$, which corresponds to the smooth bundle map $\mathrm{id} \times p_\infty e\colon J^\infty(F \times F) \to J^\infty G$. Consider the image $\mathcal{G}^\infty = (\mathrm{id} \times p_\infty e)(J^\infty(F \times F)) \subseteq J^\infty(F \times G)$. In this paper we are concerned with the case when $\mathcal{G}^\infty$ is not necessarily all of $J^\infty(F \times G)$. In particular, we are interested in the cases when $\mathcal{G}^\infty$ is itself a PDE in the sense given above. That is, there exists an involutive equation form $z^0\colon J^\infty(F\times G)\to Z^1$ such that $\mathcal{G}^\infty$ is the zero set of $\pi_F\times p_\infty z^0$ and a submanifold of $J^\infty(F\times G)$. The naturally associated local differential operator $z^0\colon \Gamma_F(G)\to\Gamma_F(Z^1)$ is variously known as a *(stage-0) Noether operator*, *compatibility operator* or *redundancy operator*, while we call $Z^1\to M$ the *(stage-1) Noether bundle*. When there are no topological obstructions, the same construction can be iterated. Let $Z^0=G$, $(\mathcal{Z}^0)^\infty=\mathcal{G}^\infty$ and define $(\mathcal{Z}^i)^\infty=(\pi_F\times p_\infty z^{i-1})(J^\infty(F\times Z^{i-1}))$ with involutive equation form $z^i\colon J^\infty(F\times Z^i)\to Z^{i+1}$, with z^i and Z^i, respectively, the *stage-i Noether operator* and *stage-i Noether bundle* (which also naturally define local differential operators). Provided there is no topological obstruction and the iteration terminates[2] at $i=r$ if $(\mathcal{Z}^r)^\infty=J^\infty Z^r$, the PDE system in question is said to be *stage-r irreducible*. When $r>0$, the PDE system is called a *gauge theory*, while when $r=1$ it is said to be an *irreducible gauge theory*. The end point of this construction is a *formally exact* complex of local differential operators

$$\Gamma_F(F)\xrightarrow{e}\Gamma_F(Z^0)\xrightarrow{z^0}\Gamma_F(Z^1)\xrightarrow{z^1}\cdots\Gamma_F(Z^r)\longrightarrow 0, \tag{5}$$

where *formal exactness* [13,14] means that the following is an exact sequence of smooth bundles over M:

$$\begin{aligned} J^\infty F^2 &\xrightarrow{\mathrm{id}\times p_\infty e} J^\infty(F\times Z^0)\xrightarrow{p_\infty(\pi_F\times z^0)} J^\infty(F\times Z^1)\\ &\xrightarrow{p_\infty(\pi_F\times z^1)}\cdots J^\infty(F\times Z^r)\longrightarrow M\times\{*\}. \end{aligned} \tag{6}$$

To even have a hope of proving some stability results for the space of solutions of a PDE system, the PDE system itself must satisfy some regularity properties. Moreover, in the next section, we state a characterization of the characteristic cohomology groups in terms of the Noether complex, which only holds when appropriate regularity properties are satisfied. These properties are discussed in detail in [27, Sections 5.1, 6.4.2–3] and are summarized in the following subsections.

2.2.1. Local Regularity. We say that a PDE system $\mathcal{E}^\infty\subset J^\infty F$ is *locally regular* if it is (i) *stage-r irreducible* and (ii) the corresponding differential operators $e\colon J^k\colon F\to G$, $z^i\colon J^{k_i}(F\times Z^i)\to Z^{i+1}$, where e and z^i are, respectively, of orders k and k_i, can be chosen to be smooth bundle maps with Jacobians of constant rank and such that the Noether operators $z^i[\psi,\zeta]$ are linear in their second arguments, which we denote as $z^i[\psi;\zeta]$.

This requirement is essentially equivalent to that of [27, Section 5.1]. In particular, it allows us to conclude that any smooth bundle map $f\colon J^\infty F\to F'$ that vanishes on the PDE manifold $\mathcal{E}^\infty$ (i.e., $e[\psi]=0$ implies $f[\psi]=0$) must

[2] Actually, at each stage, this procedure need only work on an open neighborhood of $(\mathcal{Z}^r)^\infty$, though then Z^{r+1} may need to be chosen as a non-linear smooth bundle. For simplicity, we ignore these technicalities and work only with vector bundles over M.

factor through the prolonged equation form:

$$f\colon J^\infty F \xrightarrow{p_\infty e} J^\infty(F\times G) \xrightarrow{f} F', \tag{7}$$

that is, $f[\psi] = f[\psi; e[\psi]]$, where $f[\psi;\xi]$ is linear in the last argument. Furthermore, we can also conclude that any smooth bundle map $g\colon J^\infty(F\times G)\to G'$ that vanishes on the stage-0 Noether manifold $\mathcal{G}^\infty = (\mathcal{Z}^0)^\infty$ (i.e., $g[\psi, e[\psi]] = 0$ for arbitrary ψ) must factor through the prolongued stage-0 Noether operator:

$$g\colon J^\infty(F\times G) \xrightarrow{p_\infty e} J^\infty(F\times Z^1) \xrightarrow{g} G', \tag{8}$$

that is, $g[\psi,\xi] = g[\psi; z^0[\psi;\xi]]$, where $g[\psi;\zeta]$ is linear in the last argument. Similar remarks can be made about the higher stage Noether operators. In total, they are equivalent to the acyclicity of the Koszul–Tate complex in positive anti-field degree, which is used extensively in the BV-BRST analysis of gauge theories [27].

2.2.2. Local Linearizability. In the question of linearization stability, we are interested in the relationship between the space of linearized solutions about a background exact solution $\varphi\colon M\to F$ and a neighborhood of φ in the space of exact solutions. As such, the linearization of the PDE system in question must be well defined.

Consider an arbitrary smooth 1-parameter family of sections $\phi_t\colon M\to F$, with $\phi_t = \varphi + t\psi + \Psi_t$, where $\Psi_t = O(t^2)$. For convenience, we denote $\psi_t = t\psi + \Psi_t$. We call a stage-r irreducible, locally regular PDE system *locally linearizable about* φ if (i) we can write, for ϕ_t as above and any $\zeta^i\in\Gamma(Z^i)$,

$$e[\phi_t] = e[\varphi] + e_\varphi[\psi_t] - f_\varphi[\psi_t], \tag{9}$$

$$z^i[\phi_t;\zeta^i] = z^i_\varphi[\zeta^i] - y^i_\varphi[\psi_t;\zeta^i], \tag{10}$$

where the local differential operators e_φ and z^i_φ ($z^i_\varphi[\zeta^i] = z^i[\varphi;\zeta^i]$) are linear, while $f_\varphi[\psi_t] = O(t^2)$ and $y_\varphi[\psi_t;\zeta] = O(t)$, and (ii) the linear PDE system defined by $e_\varphi[\psi]=0$ is stage-r irreducible and locally regular, with its *Noether complex*[3] given by

$$\Gamma_F(F) \xrightarrow{e_\varphi} \Gamma_F(Z^0) \xrightarrow{z^0_\varphi} \Gamma_F(Z^1) \xrightarrow{z^1_\varphi} \cdots \Gamma_F(Z^r) \longrightarrow 0. \tag{11}$$

We denote the PDE submanifolds defined by the linearized equation form $e_\varphi[\psi]=0$ by $\mathcal{E}^\infty_\varphi \subseteq J^\infty F$ and $\mathcal{E}_\varphi = \pi^{k_\varphi}\mathcal{E}^\infty_\varphi \subset J^{k_\varphi}F$, where k_φ is the order of e_φ. Incidentally, linearization has made the local differential operators e_φ and z^i_φ field-independent. We can then naturally define the *field-independent linearized Noether complex*

$$\Gamma(F) \xrightarrow{e_\varphi} \Gamma(Z^0) \xrightarrow{z^0_\varphi} \Gamma(Z^1) \xrightarrow{z^1_\varphi} \cdots \Gamma(Z^r) \longrightarrow 0, \tag{12}$$

which is a subcomplex of the field-dependent one above.

The requirement of local linearizability also imposes a restriction on the allowed background sections $\varphi\colon M\to F$. We call such an admissible section φ

[3] Also known as the *compatibility complex* or *Janet sequence*, cf. footnote 1.

a *linearizable background*. Denote by $\Gamma_{\rm lin}(F) \subseteq \Gamma(F)$ the subset of linearizable background sections. Also, denote by $\mathcal{S}(\mathcal{E}) \subset \Gamma(F)$ the subset of solutions, $\varphi \in \mathcal{S}(\mathcal{E})$ if $e[\varphi] = 0$. Finally, denote by $\mathcal{S}_{\rm lin}(\mathcal{E}) \subseteq \mathcal{S}(\mathcal{E})$ the subset of solutions that consists of all linearizable backgrounds. All of these spaces can be topologized as subsets of $\Gamma(F)$, with its natural smooth compact open (or Whitney) Fréchet topology [28,29]. We say that the PDE system in question is *locally linearizable* if the $\mathcal{S}_{\rm lin}(\mathcal{E})$ is an open subset of $\mathcal{S}(\mathcal{E})$.

2.2.3. Local Normality. Finally, there is almost no hope of identifying linearized solutions with the full set of exact solutions if the non-linear system is of a higher order than the linear one.

Let k, k_i, k^φ and k_i^φ denote the respective orders of the differential operators e, z^i, e_φ and z^i_φ. We say that a stage-r irreducible, locally regular, locally linearizable PDE system is *locally normal* if (i) the orders $k = k^\varphi$, $k_i = k_i^\varphi$ agree for all linearizable backgrounds $\varphi \in \mathcal{S}_\varphi(\mathcal{E})$ and (ii) the ranks of the linear bundle maps $e_\varphi \colon J^k F \to G$ and $z^i_\varphi \colon J^{k_i}(F \times Z^i) \to Z^{i+1}$ are, respectively, the same as those of the Jacobians of the smooth bundle maps $e \colon J^k F \to G$ and $z^i \colon J^{k_i}(F \times Z^i) \to Z^{i+1}$.

3. Deformations, Cosymmetries, Sources

3.1. Consistent Deformations

Consider the linearized Noether complex 11. Clearly, by linearity, the subspaces of null local sections, $\hat\Gamma_{\mathcal{E}_\varphi}(F) \subset \Gamma_F(F)$ and $\hat\Gamma_{\mathcal{E}_\varphi}(Z^i) \subset \Gamma_F(Z^i)$ are preserved by the action of e_φ and z^i_φ. Therefore, we can define the *null* and *on-shell* Noether complexes as the top and bottom rows of the following commuting bicomplex:

$$\begin{tikzcd}
\hat\Gamma_{\mathcal{E}_\varphi}(F) \arrow[r, "e_\varphi"] \arrow[d] & \hat\Gamma_{\mathcal{E}_\varphi}(Z^0) \arrow[r, "z^0_\varphi"] \arrow[d] & \hat\Gamma_{\mathcal{E}_\varphi}(Z^1) \arrow[r, "z^1_\varphi"] \arrow[d] & \cdots \Gamma_F(Z^r) \arrow[r] \arrow[d] & 0 \\
\Gamma_F(F) \arrow[r, "e_\varphi"] \arrow[d] & \Gamma_F(Z^0) \arrow[r, "z^0_\varphi"] \arrow[d] & \Gamma_F(Z^1) \arrow[r, "z^1_\varphi"] \arrow[d] & \cdots \Gamma_F(Z^r) \arrow[r] \arrow[d] & 0, \\
\Gamma_{\mathcal{E}_\varphi}(F) \arrow[r, "e_\varphi"] & \Gamma_{\mathcal{E}_\varphi}(Z^0) \arrow[r, "z^0_\varphi"] & \Gamma_{\mathcal{E}_\varphi}(Z^1) \arrow[r, "z^1_\varphi"] & \cdots \Gamma_F(Z^r) \arrow[r] & 0
\end{tikzcd} \tag{13}$$

where the columns are exact and the middle row is formally exact. We call the cohomologies of the rows of the above bicomplex, respectively, the *null*, *(off-shell)* and *on-shell (stage-i) consistent deformations*, denoted, respectively, by $\hat H^i_{\rm def}(e_\varphi)$, $H^i_{\rm def}(F, e_\varphi)$ and $H^i_{\rm def}(e_\varphi)$. Note that, since the Noether complex depends explicitly on the equation form e_φ, rather than just on the PDE submanifold $\mathcal{E}_\varphi$, we indicate the same dependence in the notation for consistent deformations.

From the definition, an on-shell consistent deformation class $[f] \in H^0_{\rm def}(e_\varphi)$ is represented by a local section $f \in \Gamma_F(Z_0) = \Gamma(G)$. The local section f is said to be *trivial* if $[f] = [0]$, which by local regularity means that it must be of the form

$$f[\psi] = e_\varphi[g[\psi]] + h[\psi; e_\varphi[\psi]], \tag{14}$$

for some local section $g \in \Gamma_F(F)$ and some local differential operator $h\colon \Gamma_F(G) \to \Gamma_F(G)$, with $h[\psi;\zeta]$ linear in its second argument.

Recall the linearization formulas 9 and 10. Let us define the leading order $m > 1$ and leading term $f_\varphi^{(m)}$ of f_φ as

$$f_\varphi[t\psi] = t^m f_\varphi^{(m)}[\psi] + O(t^{m+1}). \tag{15}$$

Similarly, let $m_i > 1$ and $y_\varphi^{i(m_i)}$ the leading order and leading term of y^i_φ. We also use the same notation for higher order terms in f and y^i. Clearly, $f_\varphi^{(m)}[\psi]$ and $y_\varphi^{i(m_i)}[\psi;\zeta]$ are homogeneous in ψ, of order m and m_i, respectively. Consider the following expansion of the exact Noether identity $z^0[\phi; e[\phi]] = 0$, which holds for arbitrary sections $\phi = \varphi + t\psi \in \Gamma(F)$:

$$\begin{aligned} t z^0_\varphi[e_\varphi[\psi]] - z^0_\varphi[f_\varphi[t\psi]] - t y^0_\varphi[t\psi; e_\varphi[\psi]] + y^0_\varphi[t\psi; f_\varphi[t\psi]] = 0 \\ \implies t z^0_\varphi[e_\varphi[\psi]] = t^m z^0_\varphi[f_\varphi^{(m)}[\psi]] + t y^0_\varphi[t\psi; e_\varphi[\psi]] + O(t^{m+1}). \end{aligned} \tag{16}$$

Completing the Taylor expansion and comparing the coefficients of powers of t, we find the condition $z^0_\varphi[e_\varphi[\psi]] = 0$ at order $O(t)$, which was already a requirement of local linearizability, and at order $O(t^m)$ that

$$z^0_\varphi[f_\varphi^{(m)}[\psi]] = -y_\varphi^{0(m-1)}[\psi; e_\varphi[\psi]]. \tag{17}$$

In other words, we find that $f_\varphi^{(m)}$ represents an equivalence class $[f_\varphi^{(m)}] \in H^0_{\mathrm{def}}(e_\varphi)$ of on-shell consistent deformations.

Of course, one could consider the cohomology spaces $H^i_{\mathrm{def}}(M, e_\varphi)$ of the field-independent linearized Noether complex (12). These *field-independent* consistent deformations are naturally included in those that are possibly field dependent, $H^i_{\mathrm{def}}(M, e_\varphi) \subseteq H^i_{\mathrm{def}}(F, e_\varphi)$. Of course, any non-trivial field-independent consistent deformation is still non-trivial on-shell, while any non-trivial null consistent deformation cannot be field-independent. Since in this paper we are chiefly concerned with non-linearities, all consistent deformations considered in the sequel will be field-dependent and of at least quadratic order.

3.2. Cosymmetries and Conserved Currents

For PDE systems that satisfy the regularity, linearizability and normality conditions given previously in Sect. 2.2, it is well known that there is a direct correspondence between the spaces $H^p_{\mathrm{cur}}(\mathcal{E}_\varphi)$ of conserved local p-currents (which is the same as the space $H^{(n-p)}_{\mathrm{char}}(\mathcal{E})/H^{(n-p)}_{\mathrm{dR}}(M)$ of on-shell closed local $(n-p)$-forms modulo field-independent ones) and the spaces of on-shell stage-$(p-1)$ cohomology of the complex formally adjoint to the linearized Noether complex 11. We refer to the latter objects as *stage*-$(p-1)$ *cosymmetries* and denote their space by $H^{(p-1)}_{\mathrm{cosym}}(e_\varphi)$; they are defined more precisely below. For simplicity, since we need only deal with the linearized Noether complex below, we restrict our discussion of this correspondence purely to the case of linear PDE systems.

If $h\colon J^\infty F \to H$ is a linear differential operator, we define its *formal adjoint* h^* as follows. For any vector bundle $H \to M$, denote by $H^* \to M$ its dual bundle and by $\tilde{H}^* = \Lambda^n M \otimes H^*$ the *densitized dual* bundle. There is a natural fiber-wise pairing between sections $\eta\colon M \to H$ and dual densities $\tilde{\alpha}^*\colon M \to \tilde{H}^*$, $\eta \cdot \tilde{\alpha}^*\colon M \to \mathbb{R}$. The *formal adjoint* $h^*\colon J^\infty \tilde{H}^* \to \tilde{F}^*$ is defined as the unique linear differential operator that satisfies

$$h[\psi] \cdot \tilde{\alpha}^* - \psi \cdot h^*[\tilde{\alpha}^*] = \mathrm{d}\mathcal{G}_h[\psi, \tilde{\alpha}^*] \tag{18}$$

for arbitrary sections $\psi\colon M \to F$ and $\tilde{\alpha}^*\colon M \to \tilde{H}^*$ and some bilinear bidifferential operator $\mathcal{G}_h[\psi, \tilde{\alpha}^*]\colon J^\infty(F \times \tilde{H}^*) \to \Lambda^{n-1}M$, where $\mathcal{G}_h$ is called a *Green form* associated with the adjoint pair h and h^*. Note that we can consider $\mathcal{G}_h$ as an $F \times \tilde{H}^*$-local horizontal form, $\mathcal{G}_h \in \Omega^{n-1,0}(F \times \tilde{H}^*)$. Also, $\mathcal{G}_h$ is only defined up to the addition of an exact local horizontal form $\mathcal{G}_h[\psi, \tilde{\alpha}^*] \sim \mathcal{G}_h[\psi, \tilde{\alpha}^*] + \mathrm{d}_h\mathcal{H}[\psi, \tilde{\alpha}^*]$, where $\mathcal{H}$ is itself a bilinear bi-differential operator, so only the corresponding equivalence class $[\mathcal{G}_h]$ is well defined, though we may restrict our attention only to representatives that are bilinear in their arguments. If $h[\psi; \xi]$ is a differential operator that is linear only in its second argument, we can still define its formal adjoint with respect to the second argument alone. It is again denoted by $h^*[\psi; \tilde{\alpha}^*]$ and is also linear in its second argument, while an associated Green form is denoted by $\mathcal{G}_h[\psi; \xi, \tilde{\alpha}^*]$ and is bilinear in its last two arguments.

It is straight forward to verify that for a stage-r irreducible, locally regular, locally linearizable, locally normal PDE system, the formal adjoint of its linearized Noether complex is also formally exact. Also, since all the operators involved are linear, the subspaces of null local sections, $\hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{F}^*) \subset \Gamma_F(\tilde{F}^*)$ and $\hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{Z}^{r*}) \subset \Gamma_F(\tilde{Z}^{r*})$, are preserved. That is, we can construct the following commuting bicomplex, where the columns are exact, while the middle row is formally exact:

$$\begin{tikzcd}
\hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{F}^*) \arrow[d] & \hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{Z}^{0*}) \arrow[l, "e^*_\varphi"'] \arrow[d] & \hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{Z}^{1*}) \arrow[l, "z^{0*}_\varphi"'] \arrow[d] & \cdots \arrow[l, "z^{1*}_\varphi"'] \hat{\Gamma}_{\mathcal{E}_\varphi}(\tilde{Z}^{r*}) \arrow[d] & 0 \arrow[l] \\
\Gamma_F(\tilde{F}^*) \arrow[d] & \Gamma_F(\tilde{Z}^{0*}) \arrow[l, "e^*_\varphi"'] \arrow[d] & \Gamma_F(\tilde{Z}^{1*}) \arrow[l, "z^{0*}_\varphi"'] \arrow[d] & \cdots \arrow[l, "z^{1*}_\varphi"'] \Gamma_F(\tilde{Z}^{r*}) \arrow[d] & 0. \arrow[l] \\
\Gamma_{\mathcal{E}_\varphi}(\tilde{F}^*) & \Gamma_{\mathcal{E}_\varphi}(\tilde{Z}^{0*}) \arrow[l, "e^*_\varphi"'] & \Gamma_{\mathcal{E}_\varphi}(\tilde{Z}^{1*}) \arrow[l, "z^{0*}_\varphi"'] & \cdots \arrow[l, "z^{1*}_\varphi"'] \Gamma_{\mathcal{E}_\varphi}(\tilde{Z}^{r*}) & 0 \arrow[l]
\end{tikzcd} \tag{19}$$

Finally, we define (for $r \geq 0$) *null*, *(off-shell)* and *on-shell local (stage-r) cosymmetries* to be elements of the kernel of the local differential operator z^{i*}_φ on, respectively, the top, middle and bottom rows of the above bi-complex; by convention we define $z^{-1}_\varphi = e_\varphi$ and $Z^{-1} = F$. That is, $\tilde{\xi}^{r*} \in \Gamma_F(\tilde{Z}^{r*})$ represents an on-shell local cosymmetry if $z^{(r-1)*}_\varphi[\psi; \tilde{\xi}^{r*}[\psi]] = 0$ whenever $e_\varphi[\psi] = 0$. The cohomologies of these rows define equivalence classes of cosymmetries. A cosymmetry $\tilde{\xi}^{r*} \in \Gamma_F(\tilde{Z}^{r*})$ is *trivial* if, respectively,

$$\text{(null)} \quad \tilde{\xi}^{r*}[\psi] = z_\varphi^r[\tilde{\xi}^{r+1*}[\psi; e_\varphi[\psi]]], \tag{20}$$

$$\text{(off-shell)} \quad \tilde{\xi}^{r*}[\psi] = z_\varphi^r[\tilde{\xi}^{r+1*}[\psi]], \tag{21}$$

$$\text{(on-shell)} \quad \tilde{\xi}^{r*}[\psi] = z_\varphi^r[\tilde{\xi}^{r+1*}[\psi]] + \tilde{\zeta}^{r*}[\psi; e_\varphi[\psi]]. \tag{22}$$

Two representatives are in the same equivalence class precisely when they differ by a trivial one. The spaces of equivalence classes of null, off-shell and on-shell stage-r local cosymmetries, respectively, by $\hat{H}^r_{\text{cosym}}(e_\varphi)$, $H^r_{\text{cosym}}(F, e_\varphi)$ and $H^r_{\text{cosym}}(e_\varphi)$. We call these equivalence classes *rigid cosymmetries*. The definition of a cosymmetry explicitly involves the equation form e_φ, rather than just the intrinsic PDE submanifold $\mathcal{E}_\varphi$, and the notation reflects that.

Since all operators involved in the definition of the adjoint linearized Noether complex are field-independent, we can also define the *field-independent* adjoint linearized Noether complex

$$\Gamma(\tilde{F}^*) \xleftarrow{e_\varphi^*} \Gamma(\tilde{Z}^{0*}) \xleftarrow{z_\varphi^{0*}} \Gamma(\tilde{Z}^{1*}) \xleftarrow{z_\varphi^{1*}} \cdots \Gamma(\tilde{Z}^{r*}) \longleftarrow 0, \tag{23}$$

whose cohomology spaces we denote by $H^r_{\text{cosym}}(M, e_\varphi)$. We have the natural inclusion $H^r_{\text{cosym}}(M, e_\varphi) \subseteq H^r_{\text{cosym}}(F, e_\varphi)$. Of course, any non-trivial field-independent local cosymmetry is still non-trivial on-shell, while any non-trivial null local cosymmetry cannot be field-independent.

We conclude this section recalling the following known bijection between equivalence classes of higher stage cosymmetries and higher conserved currents. We state the result only for the linearized equation, though a similar result works for non-linear equations as well.

Proposition 1 (Generalized Noether's first theorem). *For the linearized PDE system $\mathcal{E}_\varphi$, the space of classes of conserved local currents is isomorphic to the space of classes of null local forms, which in turn is isomorphic to the space of classes of rigid on-shell local cosymmetries. That is, for $0 < p \leq n$,*

$$H^{p+1}_{\text{cur}}(\mathcal{E}_\varphi) \cong H^{n-p}_{\text{null}}(\mathcal{E}_\varphi) \cong H^p_{\text{cosym}}(e_\varphi). \tag{24}$$

Though the proof of Proposition 1 can be found in several places in the literature [13,14,27].

The name *cosymmetry* is meant to be evocative. For a variational PDE system (one defined by the Euler–Lagrange equations of a local Lagrangian), the linearized equation form $e_\varphi \colon \Gamma_F(F) \to \Gamma_F(\tilde{F}^*)$ is formally self-adjoint, $e_\varphi^* = e_\varphi$. In that case, the adjoint linearized Noether complex is identical to the formally exact complex of generators of *(higher stage) gauge transformations* (generalized Noether's second theorem). It is well known that the cohomology of higher stage gauge generators is identified with higher stage rigid symmetries, those of stage-0 being the ordinary symmetries. Stage-1 rigid symmetries have also been called *reducibility parameters* [12]. For non-variational system, there need not be any correspondence between gauge generators and Noether operators or between symmetries and cosymmetries, hence the need for the distinct terminology.

In the case of stage-0 irreducible relativistic field theories and the Einstein or Yang–Mills gauge theories, rigid symmetries are known to correspond to so-called Killing vectors (or generalizations thereof [5,7]). By analogy, we can call the condition

$$z_\varphi^{(r-1)*}[\tilde{\xi}^{r*}[\psi]] = 0 \quad \text{and} \quad \tilde{\xi}^{r*}[\psi] \neq z_\varphi^{r*}[\tilde{\xi}^{(r+1)*}[\psi] \text{ (on-shell)}, \tag{25}$$

which defines a non-trivial cosymmetry, the *(generalized) co-Killing condition.*

3.3. Null Sources

A *null local p-source* ρ is a local horizontal $(n-p)$-form, $\rho \in \Omega^{n-p,0}(F)$, such that (i) it is horizontally exact, $\rho = \mathrm{d}_{\mathsf{h}} j$ for some local horizontal $(n-p-1)$-form $j \in \Omega^{n-p-1,0}(F)$, and (ii) it vanishes on linearized solutions, that is, it pulls back to 0 in $\Omega^{n-p,0}(\mathcal{E}_\varphi)$ along the inclusion $\mathcal{E}_\varphi^\infty \subset J^\infty F$, or simply $\rho \in \hat{\Omega}^{n-p,0}(\mathcal{E}_\varphi)$. A null local source ρ is said to be *trivial* if $\rho = \mathrm{d}_{\mathsf{h}} j$ where j itself vanishes on solutions. Two null local sources are said to be *equivalent* if they differ by a trivial one. We denote the space of null local p-source classes by $H^p_{\mathrm{src}}(\mathcal{E}_\varphi)$.

The term *source* is meant to be evocative. Consider an equation of the form

$$\mathrm{d}j = \rho; \tag{26}$$

if the left-hand side can be considered as the divergence of a current, then the right-hand side should be considered the source (or source density) for that current, whence the exactness requirement.

Null sources are clearly related to null local forms, as well as to characteristic cohomology.

Lemma 2. *We have $H^n_{\mathrm{src}}(\mathcal{E}_\varphi) = H^0_{\mathrm{null}}(\mathcal{E}_\varphi) = 0$ and for $0 \le p < n$*

$$H^p_{\mathrm{src}}(\mathcal{E}_\varphi) \cong H^{n-p}_{\mathrm{null}}(\mathcal{E}_\varphi) \cong H^{n-p-1}_{\mathrm{char}}(\mathcal{E}_\varphi)/H^{(n-p-1)}_{\mathrm{dR}}(M) \cong H^{p+1}_{\mathrm{cur}}(\mathcal{E}_\varphi). \tag{27}$$

Proof. If follows directly from the above definition that for each null local p-source ρ, we can find a local horizontal $(n-p-1)$-form j such that $\mathrm{d}_{\mathsf{h}} j = \rho$. Clearly j cannot be field-independent, unless it vanishes, and $\mathrm{d}_{\mathsf{h}} j$ vanishes on-shell. But this means precisely that j is a conserved $(p+1)$-current. Conversely, for each conserved $(p+1)$-current j, we can define $\rho = \mathrm{d}_{\mathsf{h}} j$ and easily check that ρ is a null p-source. Moreover, this correspondence respects equivalence classes. In other words, the classes of null p-sources are in bijection with the equivalence classes of conserved $(p+1)$-currents. The isomorphisms with $H^{n-p}_{\mathrm{null}}(\mathcal{E}_\varphi)$ and $H^{n-p-1}_{\mathrm{char}}(\mathcal{E}_\varphi)/H^{n-p-1}_{\mathrm{dR}}(M)$ follow directly from definitions.

The exactness of the variational bicomplex shows that any local horizontal 0-form is horizontally closed only if it is field-independent and constant. Therefore, any horizontally closed, null horizontal 0-form must be trivial. On the other hand, any null n-source must also be trivial, since the only exact local horizontal form in this degree is zero. In other words, $H^n_{\mathrm{src}}(\mathcal{E}_\varphi) = H^0_{\mathrm{null}}(\mathcal{E}_\varphi) = 0$. □

Here is another convenient way to represent null sources. A local horizontal $(n-p)$-form ρ naturally defines a local differential operator $\rho\colon \Gamma(M) \to$

$\Omega^{n-p}(M)$. If ρ is a null local source, local regularity of the linearized PDE system implies that it can be written as $\rho = \rho(e_\varphi)$, where on the right-hand side we denote the local section $e_\varphi \in \Gamma_F(G)$ is acted on by the linear local differential operator $\rho\colon \Gamma_F(G) \to \Omega^{n-p,0}(F)$. We write this as $\rho[\psi] = \rho[\psi; e_\varphi[\psi]]$. The local differential operator $\rho[\psi;\zeta]$ is not unique, since any other one of the form $\rho[\psi;\zeta] + \sigma[\psi; z^0[\zeta]] + \tau[\psi, e_\varphi[\psi], \zeta]$, where σ is linear in its second argument and τ is bilinear and anti-symmetric in its last two arguments, would represent the same null p-source $\rho[\psi]$. Local regularity implies that these possibilities exhaust the available ambiguity. We call a linear local differential operator $\rho\colon \Gamma_F(G) \to \Omega^{*,0}(F)$ *trivial* if it is of the form

$$\rho[\psi;\zeta] = \mathrm{d}j[\psi;\zeta] + \sigma[\psi; z^0_\varphi[\zeta]] + \tau[\psi; e_\varphi[\psi], \zeta]. \tag{28}$$

We have just observed that the space of equivalence classes of null p-sources $H^p_{\mathrm{src}}(\mathcal{E}_\varphi)$ is isomorphic to the space of equivalence classes $[\rho]$ of linear local differential operators $\rho\colon \Gamma_F(G) \to \Omega^{n-p,0}(F)$ as above, modulo trivial ones.

4. Linearization Instability

Consider a PDE system that is stage-r irreducible, locally regular, locally linearizable and locally normal, with the defining equation form and Noether complex (5). Similarly, its linearized equation form and Noether complex about a linearizable background solution $\varphi \in \mathcal{S}_{\mathrm{lin}}(\mathcal{E})$ are given by (11). In this section, we shall refer to the space $\mathcal{S}_{\mathrm{lin}}(\mathcal{E})$ of linearizable background solutions simply as the *(exact) solution space*. We shall refer to the space $\mathcal{S}(\mathcal{E}_\varphi)$ of solutions to the linearized PDE system at φ as the *linearized solution space* at φ. Both the exact and linearized solution spaces can be endowed with a topology as subspaces of the total space of sections $\Gamma(F)$. The choice of topology on $\Gamma(F)$ should be adapted to the problem at hand. The work on linearization stability that followed in the footsteps of [18] identifies the space of solutions with the space of Cauchy data satisfying initial value constraints, which is endowed with a norm topology. Thus, the accompanying functional analytical steps that require completeness introduce some weak solutions to the elliptic initial value constraints along with smooth classical ones. On the other hand, restricting our attention to a single Cauchy surface, rather than an open spacetime domain, does not take into account the possibility of singularity formation within the domain, which excludes some initial data from the solution space. The seminal result on the global non-linear stability of Minkowski space [30] can be seen through the prism of linearization stability analysis. In that work, another normed topology was used on $\Gamma(F)$. It would be interesting to also examine the same questions using a more natural Fréchet (Whitney) topology [28,29] in which $\Gamma(F)$ is complete, without the need to introduce weak solutions, beyond the minimal attention this question has already received in the existing literature.

We say that the background solution $\varphi \in \mathcal{S}_{\mathrm{lin}}(\mathcal{E})$ is *linearization stable* if (i) there is a neighborhood $U \subset \mathcal{S}_{\mathrm{lin}}(\mathcal{E})$ of φ that is homeomorphic to the linearized solution space $\mathcal{S}(\mathcal{E}_\varphi)$ and (ii) each 1-parameter family of linearized solutions of the form $t\psi$, $\psi \in \mathcal{S}(\mathcal{E}_\varphi)$ is mapped by this homeomorphism to a smooth 1-parameter family of exact solutions, $t\psi \mapsto \phi_t$, where $\phi_t \in \mathcal{S}_{\mathrm{lin}}(\mathcal{E})$ and $\phi_t = \varphi + t\psi + O(t^2)$.

Obviously proving linearization stability is a difficult problem that must involve a significant amount of functional analysis and at present it has only been treated in detail for a few selected equations. On the other hand, it is much easier to show that linearization stability is obstructed. We devote the following subsections to explicitly exhibiting global geometric conditions on the manifold M and a background solution φ on it, which we call *co-Killing conditions*, that imply the existence of obstructions, also called *linearization instabilities*.

4.1. Linearization Obstructions

Linearization stability is *obstructed* at a linearized solution $\psi \in \mathcal{S}(\mathcal{E}_\varphi)$ if it cannot be extended to a smooth 1-parameter family of exact solutions of the form $\phi_t = \varphi + t\psi + O(t^2)$; if such a 1-parameter family does exist, then ψ is called *extendable*. A *stability obstruction* is a function on the linearized solution space, $Q\colon \mathcal{S}(\mathcal{E}_\varphi) \to V$, where V is some vector space, such that $Q(\psi) = 0$ whenever ψ is extendable. A linearization obstruction is said to be *trivial* if $Q(\psi) = 0$ for every linearized solution. The obstruction is said to be of *order* m if it is homogeneous of the same order, $Q(t\psi) = t^m Q(\psi)$.

4.2. Deformation Currents from Null Sources

We are now ready to prove a theorem that relates null sources, consistent deformations and conserved currents. An easy consequence of it will be the existence of linearization obstructions, to be discussed in the next section.

Theorem 3 (Deformation currents). *Provided a linear PDE system $\mathcal{E}_\varphi$ with equation form $e_\varphi[\psi] = 0$ is locally regular, there exists a natural bilinear mapping pairing a null local p-source class with a stage-0 consistent deformation class giving a conserved local p-current.*

$$j\colon H^p_{\mathrm{src}}(\mathcal{E}_\varphi) \times H^0_{\mathrm{def}}(e_\varphi) \to H^p_{\mathrm{cur}}(\mathcal{E}_\varphi), \quad ([\rho],[f]) \mapsto [j_{\rho,f}]. \tag{29}$$

We call $j_{\rho,f}$ the deformation current *associated to ρ and f.*

Proof. As discussed in the last two sections, provided the regularity condition is satisfied, each null local p-source class $[\rho]$ can be represented by a linear local differential operator $\rho\colon \Gamma_F(G) \to \Omega^{n-p,0}(F)$, while a stage-0 on-shell consistent deformation class $[f]$ can be represented by a local section $f \in \Gamma_F(G)$. We define the local horizontal $(n-p)$-form $j_{\rho,f} \in \Omega^{n-p,0}(F)$ by the formula $j_{\rho,f} = \rho(f)$; in other words,

$$j_{\rho,f}[\psi] = \rho[\psi; f[\psi]], \tag{30}$$

for any section $\psi\colon M \to F$. It now remains to check that $j_{\rho,f}$ is in fact conserved and that the map j is well defined on equivalence classes.

We have already established that the horizontal differential of a null p-source representative ρ has the form

$$\mathrm{d}\rho[\psi;\zeta] = \sigma[\psi; z^0_\varphi[\zeta]] + \tau[\psi; e_\varphi[\psi], \zeta]. \tag{31}$$

The second term already vanishes on-shell because it is linear in the $e_\varphi[\psi]$ argument. If we set $\zeta = f[\psi]$, the first term on the right-hand also vanishes on-shell, since σ is linear in its second argument and $z^0_\varphi[f[\psi]]$ vanishes on-shell by the defining property of a stage-0 on-shell consistent deformation. Therefore, $j_{\rho,f}$ is in fact a conserved local p-current.

If ρ is a trivial null local p-source, then it must take the form given in Eq. (28), provided local regularity holds. We check that each of the three possible terms is a trivial conserved p-current after setting $\zeta = f[\psi]$. The term $\mathrm{d}j[\psi;\zeta]$ is trivial because it is exact. The term $\sigma[\psi; z^0_\varphi[f[\psi]]]$ is trivial because $z^0_\varphi[f[\psi]]$ vanishes on-shell by the definition of a consistent deformation. Finally, the term $\tau[\psi; e_\varphi[\psi], f[\psi]]$ vanishes on-shell because it is linear in the $e_\varphi[\psi]$ argument.

If f is a trivial on-shell consistent deformation, then it must take the form given in Eq. (14), provided local regularity holds. We check that each possible term substituted for ζ in $\rho[\psi;\zeta]$ gives a trivial conserved local p-current. The second term gives $\rho[\psi; h[\psi; e_\varphi[\psi]]]$, which is linear in $e_\varphi[\psi]$ and hence trivial. The remaining term $\rho[\psi; e_\varphi[g[\psi]]]$ is trivial for a slightly non-obvious reason: a deformation of the form $e_\varphi[g[\psi]]$ necessarily comes from a local field redefinition of the form $\psi \to \psi + tg[\psi] + O(t^2)$. Namely, recall that we can always write $\rho[\psi; e_\varphi[\psi]] = \mathrm{d}k[\psi]$ for some conserved local $(p+1)$-current k and consider the identity

$$\mathrm{d}k[\psi + tg[\psi]] = \rho[\psi + tg[\psi]; e_\varphi[\psi + tg[\psi]]] \tag{32}$$

$$= t\rho[\psi + tg[\psi]; e_\varphi[g[\psi]]] + \rho[\psi + tg[\psi]; e_\varphi[\psi]] \tag{33}$$

$$= t(\rho[\psi; e_\varphi[g[\psi]]] + \rho^{(1)}[\psi; e_\varphi[\psi]]) + O(t^2). \tag{34}$$

When expanded in powers of t, all coefficients on the left-hand side are exact. This shows that, up to the addition of the trivial conserved local p-current $\rho^{(1)}[\psi; e_\varphi[\psi]]$, the term $\rho[\psi; e_\varphi[g[\psi]]]$ is exact and hence trivial. This concludes the proof. □

Remark 1. It is interesting to note that the deformation current mapping

$$([\rho], [f]) \mapsto [j_{\rho,f}], \tag{35}$$

defined in Theorem 3, identifies a selection criterion on consistent deformations. We could say that a consistent deformation f is *conservative* or ρ-*conservative* when the deformation current $j_{\rho,f}$ is trivial (on-shell exact). The linearization obstruction generated by the deformation current $j_{\rho,f}$ for a ρ-conservative deformation f, as to be discussed in the next section, is necessarily trivial.

4.3. Obstructions from Deformation Currents

Consider the full non-linear PDE system $\mathcal{E}$ defined by the equation form $e[\phi] = 0$, which we take to be locally regular, linearizable and normal. Also, consider a linearizable background solution $\varphi \in \mathcal{S}_{\mathrm{lin}}(\mathcal{E})$. Recall that, a smooth 1-parameter family of solutions, $e[\phi_t] = 0$, of the form$\phi_t = \varphi + \psi_t$ satisfies

Eq. (9), that is,

$$e_\varphi[\psi_t] = f_\varphi[\psi_t]. \tag{36}$$

If $\psi_t = t\psi + O(t^2)$, then ψ is a linearized solution, $e_\varphi[\psi] = 0$, that can be extended to a smooth 1-parameter family of exact solutions, for instance ϕ_t. Recall also that the non-linearity f_φ defines a representative of a consistent deformation class $[f^{(m)}] \in H^0_{\mathrm{def}}(e_\varphi)$ given by the Taylor expansion

$$f_\varphi[t\psi] = t^m f^{(m)}[\psi] + O(t^{m+1}) \tag{37}$$

for some $m > 1$, where $f^{(m)}[\psi]$ is homogeneous of degree m in ψ. The main result of this paper, to be proven below, is that each deformation current defined by $f^{(m)}$ canonically defines a linearization obstruction.

As an intermediate step before formulating the main result, we need to introduce a grading by degree of polynomial dependence on field bundle sections on the spaces of local sections and local differential operators. Let ${}^{(l)}\Gamma_F(H) \subset \Gamma_F(H)$ denote the subspace of local sections of a vector bundle $H \to M$ of *homogeneous polynomial degree* l, that is, such that $h \in {}^{(l)}\Gamma_F(H)$ only if $h[\psi]$ depends polynomially on ψ and its derivatives, and is homogeneous in ψ of degree l. Note that the horizontal differential d_h, as well as all the differential operators in the linearized Noether complex 11, the adjoint linearized Noether complex 19 and the complex $\hat{\Gamma}_F(H) \to \Gamma_F(H) \to \Gamma_{\mathcal{E}}(H)$ all preserve subspaces of homogeneous polynomial degrees. This means that any of the cohomology spaces that we have defined also have subspaces of homogeneous polynomial degrees; the corresponding subspaces are denoted by ${}^{(l)}H^*_-(-) \subset H^*_-(-)$.

The representatives of each of these subspaces can always be chosen of the same homogeneous polynomial degree.

We are finally ready to formulate and prove the main theorem of this paper, which relates the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ and the space $H^*_{\mathrm{cosym}}(\mathcal{E}_\varphi)$ of rigid on-shell cosymmetries with linearization obstructions at φ, and thus with potential linearization instability.

Theorem 4. *If $\psi \in \mathcal{S}(\mathcal{E}_\varphi)$ is a linearized solution that can be extended to a smooth* 1*-parameter family of exact solutions, then it must necessarily satisfy the conditions $Q^l_p(\psi) = 0$, $0 \le p < n$, $0 \le l$, where*

$$Q^l_p \colon \mathcal{S}(\mathcal{E}_\varphi) \to {}^{(l)}H^p_{\mathrm{cosym}}(e_\varphi)^* \otimes H^{n-p}_{\mathrm{dR}}(M) \tag{38}$$

are linearization obstructions of order $l + m$ defined by the non-linearity f_φ, where * denotes the linear dual.

Proof. First, recall the isomorphisms $(0 \le p < n)$

$$H^p_{\mathrm{cosym}}(e_\varphi) \cong H^{p+1}_{\mathrm{cur}}(\mathcal{E}_\varphi) \cong H^p_{\mathrm{src}}(\mathcal{E}_\varphi) \tag{39}$$

and note that they preserve subspaces of homogeneous polynomial degree. That is, we can consider a null local p-source class $[\rho] \in {}^{(l)}H^p_{\mathrm{src}}(\mathcal{E}_\varphi)$, equally well, to be an element $[\rho] \in {}^{(l)}H^p_{\mathrm{cosym}}(e_\varphi)$. Using the on-shell consistent deformation class $[f^{(m)}_\varphi] \in H^0_{\mathrm{def}}(e_\varphi)$ and the map j defined in Theorem 3, we obtain a conserved local p-current class

$$j([\rho],[f_\varphi^{(m)}]) = [j_{\rho,f_\varphi^{(m)}}] \in {}^{(l+m)}H^p_{\mathrm{cur}}(\mathcal{E}_\varphi) \cong {}^{(l+m)}H^{n-p}_{\mathrm{char}}(\mathcal{E}_\varphi). \tag{40}$$

The representative $j_{\rho,f_\varphi^{(m)}} \in \Omega^{n-p,0}(F)$ is a local horizontal $(n-p)$-form and hence naturally defines a differential operator $j_{\rho,f_\varphi^{(m)}} : \Gamma(F) \to \Omega^{n-p}(M)$. By construction, we know that $j_{\rho,f_\varphi^{(m)}}[\psi]$ is of homogeneous polynomial degree $l+m$ and that $\mathrm{d}j_{\rho,f_\varphi^{(m)}}[\psi] = 0$ if ψ is a linearized solution. Hence it represents the de Rham cohomology class in $[\,j_{\rho,f_\varphi^{(m)}}[\psi]\,] \in H^{n-p}_{\mathrm{dR}}(M)$. Finally, we define the map Q^l_p by the formula

$$[\rho]\cdot Q^l_p(\psi) = [\,j_{\rho,f_\varphi^{(m)}}[\psi]\,], \tag{41}$$

where the dot on the left-hand side stands for the natural pairing between ${}^{(l)}H^p_{\mathrm{cosym}}(e_\varphi)$ and its dual ${}^{(l)}H^p_{\mathrm{cosym}}(e_\varphi)^*$.

Since $Q^l_p(\psi)$ is built out of a bilinear pairing between the differential operator ρ and $f_\varphi^{(m)}[\psi]$, the order $Q^l_p(\psi)$ must be exactly $l+m$.

To conclude the proof, it is sufficient to show that $[\rho]\cdot Q^l_p(\psi) = [0]$ for ψ that is extendable to a smooth 1-parameter family of exact solutions $\phi_t = \varphi + \psi_t$, with $\psi_t = t\psi + O(t^2)$. This family of exact solutions will satisfy the equation $e_\varphi[\psi_t] = f_\varphi[\psi_t]$. Recall also that we can find a conserved local $(p+1)$-current $k \in \Omega^{n-p-1,0}(F)$ such that $\rho[\Psi, e_\varphi[\Psi]] = \mathrm{d}k[\Psi]$ for arbitrary $\Psi \in \Gamma(F)$ and, in particular, this formula is still valid if we set $\Psi = \psi_t$. We then have the following equalities in terms of de Rham cohomology classes:

$$[\,\rho[\psi_t; f_\varphi[\psi_t]]\,] = t^{(l+m)}[\,\rho[\psi; f_\varphi^{(m)}[\psi]]\,] + O(t^{l+m+1}) \tag{42}$$

$$= [\,\rho[\psi_t; e_\varphi[\psi_t]]\,] = [\,\mathrm{d}k[\psi_t]\,] \tag{43}$$

$$= [0]. \tag{44}$$

Note that comparing the coefficient of t^{l+m} in these equations gives precisely the desired equality $[\rho]\cdot Q^l_p(\psi) = 0$. □

Remark 2. The above theorem only shows how to canonically construct potential linearization obstructions. It does not mean that the obstruction Q^l_p is necessarily non-trivial. For instance, if one can show that the conserved current $j_{\rho,f_\varphi^{(m)}}$ is trivial, $[j_{\rho,f_\varphi^{(m)}}] = [0] \in H^p_{\mathrm{cur}}(\mathcal{E}_\varphi)$ (equivalently, that the consistent deformation $f^{(m)}$ is ρ-conservative, as defined in the preceding section), then $[\rho]\cdot Q^l_p(\psi) = [0]$ is trivial and does not pose any obstruction to extending linearized solutions to exact ones. Further work must be done on a case by case basis to show that the obstruction Q^l_p is non-trivial. Alternatively, one can use the triviality of $[\rho]\cdot Q^l_p(\psi)$, checked in the way just given above, as a condition to select non-linear consistent deformations like $f_\varphi^{(m)}$ that will not create linearization obstructions (at least not of the kind constructed in Theorem 4).

Theorem 4 is a significant generalization of the results obtained in the literature that followed up the initial work of [18,31]. The known linearization obstructions have only been obtained in the case of either compact manifolds for equations in Riemannian geometry or in the case of manifolds with compact Cauchy surfaces for relativistic gauge theories. In light of our result, these situations are immediately recognizable as obstructions coming, respectively, from $H^n(M) \neq 0$ and from $H^{n-1}(M) \neq 0$, and also why only gauge theories are susceptible in the latter case. Since the equations considered in the literature have been stage-0 or stage-1 irreducible, our result also shows why similar obstructions are absent for manifolds with $H^{n-p}(M) \neq 0$ and $p > 1$. To see obstructions due to these lower cohomology groups, one needs to consider non-linear equations with higher stage reducibility and they are relatively infrequent. Such gauge theories do appear in some specialized physics literature, with non-linear p-form electromagnetism and some supergravities as examples. They have also recently attracted significant attention in the mathematics literature as *higher gauge theories* [32,33].

Finally, the calculations involved in identifying these linearization obstructions have been rather cumbersome. This is especially the case for relativistic gauge theories, where the necessary calculations lose much of their geometric character due to non-covariant restriction to some initial data surface. Our result, on the other hand, was obtained completely geometrically and explains the covariance of the final result. Famously, for Einstein equations, linearization obstructions are only present if the background solution possesses non-trivial Killing vectors (or satisfies the Killing condition). In particular, in the existing work on relativistic gauge theories, significant effort was needed in each case to obtain the analog of the Killing condition. Our result, on the other hand, shows that this condition is precisely Eq. 25 defining a non-trivial cosymmetry, which for obvious reasons we have also called the (generalized) co-Killing condition.

The relation of our result to previous work is discussed in more detail next.

4.4. Integrated Charges, Relation with Previous Work

Note that the linearization obstruction $[\rho] \cdot Q^l_p(\psi)$ constructed in the preceding section is not strictly of the kind that has been constructed in the existing literature on Einstein, Yang–Mills and related equations. While both formulations pass through conserved currents, we have given the obstructions as valued in the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ classes of these currents, while the usual formulation gives them in terms of corresponding integrated charges. An integrated charge is obtained by integrating a conserved current over a closed compact submanifold.

Our formulation can give rise to integrated charges as well. If $\Sigma \subset M$ is a closed compact submanifold, then

$$\langle [\Sigma], [\rho] \cdot Q^l_p(\psi) \rangle = \int_{\Sigma} j_{\rho, f^{(m)}_\varphi}[\psi] \tag{45}$$

is the corresponding integrated charge. The charges obtained in this way are not all independent. Each charge depends only on the (singular) homology class

$[\Sigma] \in H_*(M)$. Moreover, linear combinations of homology classes lead to linear combinations of charges. Therefore, to obtain a set of independent charges, we need to pick a basis $[\Sigma_i]$ for $H_*(M)$ and pair each basis element with $[\rho] \cdot Q^l_p(\psi)$. But this amounts to no more than composing Q^l_p with the isomorphism $H^*_{\mathrm{dR}}(M) \cong \bigoplus_i \mathbb{R}^{b_i}$ defined by the basis $[\Sigma_i]$, where the $b_i = \dim H_i(M)$ are the Betti numbers of M. The fact that this map is an isomorphism is simply a restatement of Poincaré duality [34]. So, an integrated charge over a non-trivial closed compact submanifold $\Sigma \subset M$ is simply a witness to the existence of a non-trivial cohomology class in $H^*(M)$. In other words, our formulation of linearization obstructions is equivalent to the usual one.

4.5. Asymptotic Boundary Conditions

On the other hand, our formulation is more convenient in the discussion of asymptotic boundary conditions on non-compact manifolds and corresponding non-standard de Rham cohomology. For instance, if M is non-compact, we may consider a suitable subspace of $\tilde{\Gamma}(F) \subset \Gamma(F)$ of the space sections of the field bundle $F \to M$, along with a corresponding refinement $\tilde{\Omega}^k(M) \subset \Omega^k(M)$ of the de Rham complex, with $\mathrm{d}\tilde{\Omega}^k(M) \subseteq \tilde{\Omega}^{k+1}(M)$. These subspaces could be selected by imposing some asymptotic boundary conditions at the open ends of M. An extreme example would be require all sections and forms to have compact support. The cohomology $\tilde{H}^*_{\mathrm{dR}}(M) = H(\tilde{\Omega}^*(M), \mathrm{d})$ could be different from the standard $H^*_{\mathrm{dR}}(M)$. Consider, with respect to the linearized PDE system, a conserved local current class $[k]$ and the corresponding null local source class $[\rho] = [\mathrm{d}_{\mathsf{h}} k]$. If the local horizontal forms ρ and k can be chosen such that the boundary conditions on $\psi \in \tilde{\Gamma}(F)$ imply that $\rho[\psi], k[\psi] \in \tilde{\Omega}^*(M)$. The construction of the linearization obstruction $[\rho] \cdot Q^l_p(\psi)$ still works, but the result is now valued in the non-standard de Rham cohomology $\tilde{H}^*_{\mathrm{dR}}(M)$ rather than $H^*_{\mathrm{dR}}(M)$. As before, it is still a non-trivial problem to check that the resulting potential linearization instability Q^l_p is non-trivial. However, if the boundary conditions are chosen such that $\tilde{H}^*(M) = 0$, then the linearization obstruction yielded by this construction is necessarily trivial.

The above logic appears to be the reason behind the absence of linearization obstructions for the common choice of asymptotically flat boundary conditions for the Einstein equations [3,4].

5. Examples

The PDE systems studied in the physics literature are mostly variational (coming from classical Lagrangian field theories). These are also the kind of systems analyzed in previous work on linearization instabilities. Our analysis is applicable to a more general class of PDE systems, including non-variational one. It would be nice to have explicit examples of theories from each of the corners missed by the existing literature. On the other hand, it is rather easy to provide examples of non-variational PDE systems by taking a variational system, $e[\psi] = 0$, and pre-composing with an arbitrary differential operator,

$e[g[\eta]] = 0$. The resulting system will generically no longer be variational, but will possess essentially the same Noether complex as the original one. Thus, the essentially new examples that we give below are restricted to 1-dimensional systems (ODEs) and higher gauge theories.

5.1. Stage-0 Irreducible Systems

5.1.1. ODEs. Consider an ordinary differential equation (ODE) that is defined on a circle, $M = S^1$, and scalar valued, $G = F = M \times \mathbb{R}$:

$$\ddot{\phi} = f[\phi], \tag{46}$$

with $\dot{\phi} = \mathrm{d}\phi/\mathrm{d}t$ and t a coordinate on M. This is essentially the 1-dimensional particle equation with cyclic time and force term f, which, say, is homogeneous of polynomial degree > 1. If we linearize about $\varphi = 0$, the linearized equation is just the free particle equation $\ddot{\phi} = 0$ with cyclic time. Its only solutions are constants, $\mathcal{S}(\mathcal{E}_\varphi) = \{\psi(t) \mid \psi(t) = \psi_0\} \cong \mathbb{R}^1$. It is straightforward to check that $\tilde{\xi}^* = \mathrm{d}t$ is a non-trivial cosymmetry with corresponding conserved 1-current $k[\psi] = \dot{\psi}$ (the momentum) and null source $\rho[\psi] = \ddot{\psi}\,\mathrm{d}t$.

If the non-linearity is $f_\varphi[\psi] = f[\psi] = \psi^2$, the corresponding deformation 0-current is

$$j_{\rho,\psi^2}[\psi] = \psi^2\,\mathrm{d}t, \tag{47}$$

which is easily seen to be non-trivial. In fact, the integrated charge

$$\langle [S^1], Q_\rho(\psi)\rangle = \int\limits_{S^1} j_{\rho,\psi^2}[\psi] = c\psi_0^2 \neq 0, \tag{48}$$

with c the circumference of S^1, for any linearized solution $\psi(t) = \psi_0 \neq 0$. Clearly, the zero set $Q_\rho(\psi) = 0$ consists of a single point, $\{\psi(t) \mid \psi(t) = 0\} \cong \mathbb{R}^0$.

On the other hand, if we consider $\tilde{\xi}^* = \dot{\psi}\,\mathrm{d}t$, $j[\psi] = \frac{1}{2}\dot{\psi}^2$ (the energy) and $\rho[\psi] = \dot{\psi}\ddot{\psi}\,\mathrm{d}t$, the corresponding deformation current is trivial:

$$j_{\rho,\psi^2}[\psi] = \dot{\psi}\psi^2\,\mathrm{d}t = \frac{1}{3}\mathrm{d}\psi^3. \tag{49}$$

In other words, the force term $f[\phi] = \phi^2$ is energy-conservative.

5.1.2. Semilinear Elliptic PDEs. A very similar situation occurs on higher dimensional compact manifolds, say $M = S^2$, with Laplace-type semilinear equations, $F = M \times \mathbb{R}$, $G = \tilde{F}^* = \Lambda^2 M$,

$$*(\Delta\phi + l(l+1)\phi) = f[\phi], \tag{50}$$

where we interpret S^2 as the standard, round unit sphere, $*$ is the Hodge star, $\Delta = *\mathrm{d}*\mathrm{d} + \mathrm{d}*\mathrm{d}*$ is the Laplacian on it and $l \geq 0$ is an integer. The linearized equation about $\varphi = 0$ is $*(\Delta\psi + l(l+1)\psi) = 0$ and its solution space is well known to be of dimension $2l+1$ and spanned by spherical harmonics, $\mathcal{S}(\mathcal{E}_\varphi) \cong \mathbb{R}^{2l+1}$. A non-trivial cosymmetry is given by a spherical harmonic $\tilde{\xi}^* = Y^l_m$. The corresponding conserved 1-current is $k_m[\phi] = *(Y^l_m \mathrm{d}\phi - \phi\mathrm{d}Y^l_m)$ and corresponding null 0-source is $\rho_m[\phi] = *Y^l_m(\Delta\phi + l(l+1)\phi)$.

If we set $f_\varphi[\psi] = f[\psi] = \psi^2$, we get the non-trivial deformation current

$$j_{\rho_m,\psi^2} = *Y^l_m \psi^2, \tag{51}$$

with integrated charge

$$\langle [S^2], Q_m(\psi)\rangle = \int\limits_{S^2} *Y^l_m \psi^2. \tag{52}$$

In other words, to satisfy all linearization obstructions of the form $Q_m(\psi) = 0$, $-l \leq m \leq l$, the decomposition of ψ^2 into spherical harmonics must have vanishing coefficients. These conditions are only satisfied by the zero solution; these obstructions drop the dimension of the linearized solution space from $2l+1$ to 0, $\{\psi \mid \psi = 0\} \cong \mathbb{R}^0$.

5.1.3. Other Examples. Other examples considered in the existing literature concern some problems from Riemannian geometry on compact manifolds, such as the equation describing metrics of constant scalar curvature [35]. The identified obstructions can also be computed using our main result.

5.2. Stage-1 Irreducible Systems

5.2.1. Einstein Equations. Einstein's equations describe the dynamics of the gravitational field, a Lorentzian metric, on a manifold M either in vacuum or, with appropriate modification, in the presence of matter. A Lorentzian metric $g \in \Gamma(S^2T^*M)$ is a section of the bundle $F = S^2T^*M \to M$ of rank-2 covariant symmetric tensors. Detailed formulas necessary to exhibit the structure of the equations, the non-linearity and the linearization obstructions are rather lengthy and, for our purposes, not particularly illuminating. The linearization instabilities of Einstein equations have been studied extensively, they in fact gave birth to this subject, and all the relevant details are summarized in the introductory sections of [1]. Below, we merely indicate how they fit into the framework of our main result.

The linearized Noether complex for Einstein equations, as well as its formal adjoint, about a background metric $\varphi = \bar{g}$, are given by

$$\Gamma_F(S^2T^*M) \xrightarrow{e_\varphi = L} \Gamma_F(S^2T^*M) \xrightarrow{B} \Gamma_F(T^*M) \longrightarrow 0, \tag{53}$$

$$\Gamma_F(S^2T^*M) \xleftarrow{e^*_\varphi = L} \Gamma_F(S^2T^*M) \xleftarrow{K} \Gamma_F(T^*M) \longleftarrow 0. \tag{54}$$

Note that we have identified $\tilde{F}^* \cong F$, and the same for the other tensor bundles, using $\bar{g}$ to raise and lower tensor indices as well as to construct a canonical volume density. Here, L is the differential operator of the linearized Einstein equations, also known as the *Lichnerowicz operator*. The operators B and K correspond, respectively, to the linearized *Bianchi identity* and *Killing equation*. In local coordinates, they are

$$(B[h])_i = \nabla^j h_{ij}, \tag{55}$$

$$(K[v])_{ij} = \frac{1}{2}(\nabla_i v_j + \nabla_j v_i), \tag{56}$$

where ∇ is the Levi-Civita connection with respect to $\bar{g}$, which is also used to raise and lower tensor indices. Solutions to the Killing equation are called *Killing vectors*; in our framework they are the rigid stage-1 cosymmetries of the linearized Einstein equations. The conserved 2-currents corresponding to these Killing vectors are known as Abbott–Deser fluxes [36] and the corresponding null 1-sources do not have a name in the literature. On the other hand, the corresponding deformation 1-currents, constructed using the quadratic term in Einstein's equations, are known as the Taub conserved currents [37]. The connection between the presence of non-trivial Killing vectors and the vanishing of their Taub charges as a linearization obstruction was first noticed by Moncrief [6].

5.2.2. Yang–Mills Equations. The notational background for this section is given in Appendix. In Yang–Mills theory [38], the basic dynamical field is a semi-simple Lie algebra $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$, so $F = \mathfrak{g} \otimes \Lambda^1 M$. Where M is a manifold of $\dim M = n$ and endowed with a (pseudo-) Riemannian metric. It is a stage-1 irreducible, non-linear deformation of the linear Maxwell theory. Its Lagrangian density with the leading order non-linearity, as a local variational form, is

$$\mathcal{L} = -\frac{1}{4}\langle \mathrm{d}_\mathsf{h}\alpha \wedge *\mathrm{d}_\mathsf{h}\alpha\rangle + \frac{1}{2}\langle \mathrm{d}_\mathsf{h}\alpha \wedge *[\alpha \wedge \alpha]\rangle + O(\alpha^4). \tag{57}$$

The linearized equations, about $\varphi = \alpha = 0$, and the leading order non-linear consistent deformation are obtained by a vertical variation of the Lagrangian density:

$$\begin{aligned} \mathrm{d}_\mathsf{v}\mathcal{L} &= -\frac{1}{2}\langle \mathrm{d}_\mathsf{v}\mathrm{d}_\mathsf{h}\alpha \wedge *\mathrm{d}_\mathsf{h}\alpha\rangle + \frac{1}{2}\langle \mathrm{d}_\mathsf{v}\mathrm{d}_\mathsf{h}\alpha \wedge *[\alpha \wedge \alpha]\rangle \\ &\quad + \langle \mathrm{d}_\mathsf{h}\alpha \wedge *[\mathrm{d}_\mathsf{v}\alpha \wedge \alpha]\rangle + O(\alpha^4) \qquad (58) \\ &= -\frac{1}{2}\langle \mathrm{d}_\mathsf{v}\alpha \wedge *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}\alpha\rangle + \frac{1}{2}\langle \mathrm{d}_\mathsf{v}\alpha \wedge *\delta_\mathsf{h}[\alpha \wedge \alpha]\rangle \\ &\quad + \langle \mathrm{d}_\mathsf{v}\alpha \wedge [\alpha \wedge *\mathrm{d}_\mathsf{h}\alpha]\rangle + \mathrm{d}_\mathsf{h}(\cdots) + O(\alpha^4). \qquad (59) \end{aligned}$$

We can read off the direct and adjoint linearized Noether complexes as

$$\Omega^{1,0}(F, \mathfrak{g}) \xrightarrow{e_\varphi = *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}} \Omega^{3,0}(F, \mathfrak{g}) \xrightarrow{\mathrm{d}_\mathsf{h}} \Omega^{4,0}(F, \mathfrak{g}) \longrightarrow 0, \tag{60}$$

$$\Omega^{3,0}(F, \mathfrak{g}) \xleftarrow{e^*_\varphi = *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}} \Omega^{1,0}(F, \mathfrak{g}) \xleftarrow{\mathrm{d}_\mathsf{h}} \Omega^{0,0}(F, \mathfrak{g}) \longleftarrow 0, \tag{61}$$

while the leading order consistent deformation is

$$f^{(2)}_\varphi[\alpha] = f[\alpha] = *\delta\frac{1}{2}[\alpha \wedge \alpha] + [\alpha \wedge *\mathrm{d}\alpha]. \tag{62}$$

A stage-1 rigid cosymmetry is any $\mathfrak{g}$-valued 0-form $\varepsilon \in \Omega^{0,0}(F, \mathfrak{g})$ such that $\mathrm{d}\varepsilon = 0$. The corresponding conserved 2-current and null 1-source are, respectively, $k = \langle \varepsilon \wedge *\mathrm{d}\alpha\rangle$ and $\rho = \langle \varepsilon \wedge *\delta\mathrm{d}\alpha\rangle$ (up to signs). The resulting deformation current, which we can check is not off-shell closed, is

$$j_{\rho,f}[\alpha] = \langle \varepsilon \wedge [\alpha \wedge *\mathrm{d}\alpha] \rangle, \tag{63}$$

$$\mathrm{d}j_{\rho,f}[\alpha] = -\langle \varepsilon \wedge [\alpha \wedge (\mathrm{d}*\mathrm{d}\alpha)] \rangle, \tag{64}$$

where the neglected term is trivial because of the identity $\mathrm{d}*\delta = 0$.

The above construction gives an obstruction Q_ε for each rigid cosymmetry $\varepsilon \in \Omega^{0,0}(F, \mathfrak{g})$ valued in $H^{n-1}_{\mathrm{dR}}(M)$. These are precisely the obstructions that were previously obtained by Moncrief [7], where he checked that they are non-trivial and also sufficient. Moncrief, like other existing literature, implicitly used the existence of a compact Cauchy surface as a witness to the non-triviality of $H^{n-1}_{\mathrm{dR}}(M)$. Obstructions appear also at non-vanishing backgrounds $\varphi = A$, where the rigid cosymmetries are geometrically identified with A-parallel $\mathfrak{g}$-valued scalars. If A is interpreted as a connection on a principal bundle, these parallel scalars are intimately connected with its the holonomy group.

5.2.3. Chern–Simons. The notational background for this section is given in Appendix. In Chern–Simons theory [39], where $\dim M = 3$, the basic dynamical field is also a semi-simple Lie algebra $\mathfrak{g}$-valued 1-form $\alpha \in \Omega^1(M, \mathfrak{g})$, so $F = \mathfrak{g} \otimes \Lambda^1 M$. It is a stage-1 irreducible, non-linear theory with Lagrangian density, as a local variational form,

$$\mathcal{L} = \frac{1}{2}\langle \alpha \wedge \mathrm{d}_{\mathsf{h}}\alpha \rangle - \frac{1}{3}\langle \alpha \wedge [\alpha \wedge \alpha] \rangle. \tag{65}$$

The linearized equations, about $\varphi = \alpha = 0$, and the non-linear consistent deformation are obtained by a vertical variation of the Lagrangian density:

$$\mathrm{d}_{\mathsf{v}}\mathcal{L} = \langle \mathrm{d}_{\mathsf{v}}\alpha \wedge \mathrm{d}_{\mathsf{h}}\alpha \rangle - \langle \mathrm{d}_{\mathsf{v}}\alpha \wedge [\alpha \wedge \alpha] \rangle + \mathrm{d}_{\mathsf{h}}(\cdots). \tag{66}$$

We can read off the direct and adjoint linearized Noether complexes as

$$\Omega^{1,0}(F, \mathfrak{g}) \xrightarrow{e_\varphi = \mathrm{d}_{\mathsf{h}}} \Omega^{2,0}(F, \mathfrak{g}) \xrightarrow{\mathrm{d}_{\mathsf{h}}} \Omega^{3,0}(F, \mathfrak{g}) \longrightarrow 0, \tag{67}$$

$$\Omega^{2,0}(F, \mathfrak{g}) \xleftarrow{e^*_\varphi = \mathrm{d}_{\mathsf{h}}} \Omega^{1,0}(F, \mathfrak{g}) \xleftarrow{-\mathrm{d}_{\mathsf{h}}} \Omega^{0,0}(F, \mathfrak{g}) \longleftarrow 0, \tag{68}$$

while the leading order consistent deformation is

$$f^{(2)}_\varphi[\alpha] = f[\alpha] = [\alpha \wedge \alpha]. \tag{69}$$

Just as in the case of Yang–Mills theory, a rigid stage-1 cosymmetry is any $\mathfrak{g}$-valued 0-form $\varepsilon \in \Omega^{0,0}(F, \mathfrak{g})$ such that $\mathrm{d}\varepsilon = 0$. The corresponding conserved 2-current and null 1-source are, respectively, $k = \langle \varepsilon \wedge \alpha \rangle$ and $\rho = \langle \varepsilon \wedge \mathrm{d}\alpha \rangle$. The resulting deformation current, which we can check is not off-shell closed, is

$$j_{\rho,f}[\alpha] = \langle \varepsilon \wedge [\alpha \wedge \alpha] \rangle, \tag{70}$$

$$\mathrm{d}j_{\rho,f}[\alpha] = -\langle \varepsilon \wedge [\alpha \wedge (\mathrm{d}\alpha)] \rangle. \tag{71}$$

The geometric interpretation of the rigid cosymmetries, and their connection with the holonomy group, remains the same as in Yang–Mills theory.

The solutions of the Chern–Simons equations constitute flat connections on the corresponding principal bundle over M (in the above simplified setting, the principal bundle is trivial). The moduli space of flat connections, that is, the solution space we have been considering modulo the gauge transformations $\alpha \to \alpha + \mathrm{d}\omega$, has been studied extensively. It is known that this moduli space is not a smooth manifold, because it may have quadratic algebraic singularities at connections with non-trivial holonomy groups [40]. That result is entirely consistent with the obstructions computed above.

5.3. Other Examples

Other irreducible gauge theories that have been considered in the existing literature include Einstein–Yang–Mills [8] and classical $N = 1$ supergravity [9]. The obstructions and (co-)Killing conditions identified in these theories are reproduced by our framework and have been checked to be non-trivial and sufficient. See [1,2] for a historical discussion and detailed references.

5.4. Stage > 1 Irreducible Systems

5.4.1. Non-Abelian Freedman-Townsend 2-Form. The notational background for this section is given in Appendix. Generalizations of Maxwell equations to p-forms (*abelian* p-forms) provide a standard source of examples of higher stage irreducible linear PDE systems. Their consistent deformations (*non-abelian* p-forms) provide examples of corresponding higher stage irreducible non-linear PDE systems.

On a manifold M of $\dim M = 4$ that is endowed with a (pseudo-) Riemannian metric, the only consistent, non-linear deformation of abelian 2-form field theory is the Freedman–Townsend model [41]. The basic dynamical field is a semi-simple Lie algebra $\mathfrak{g}$-valued 2-form $\beta \in \Omega^2(M, g)$, so $F = \mathfrak{g} \otimes \Lambda^2 M$. Its Lagrangian density, with the leading order non-linearity, as a local variational form, is

$$\mathcal{L}[\beta] = -\frac{1}{8}\langle \mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{h}\beta\rangle - \frac{1}{4}\langle \beta \wedge [*\mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{h}\beta]\rangle + O(\beta^4). \tag{72}$$

The linearized equations, about $\varphi = \beta = 0$, and the non-linear consistent deformation are obtained by a vertical variation of the Lagrangian density:

$$\begin{aligned}
\mathrm{d}_\mathsf{v}\mathcal{L}[\beta] = &-\frac{1}{4}\langle \mathrm{d}_\mathsf{v}\mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{h}\beta\rangle - \frac{1}{4}\langle \mathrm{d}_\mathsf{v}\beta \wedge [*\mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{h}\beta]\rangle \\
&+ \frac{1}{2}\langle \beta \wedge [*\mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{v}\mathrm{d}_\mathsf{h}\beta]\rangle + O(\beta^4) \qquad (73)\\
= &-\frac{1}{4}\langle \mathrm{d}_\mathsf{v}\beta \wedge *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}\beta\rangle - \frac{1}{4}\langle \mathrm{d}_\mathsf{v}\beta \wedge [*\mathrm{d}_\mathsf{h}\beta \wedge *\mathrm{d}_\mathsf{h}\beta]\rangle \\
&+ \frac{1}{2}\langle \mathrm{d}_\mathsf{v}\beta \wedge *\delta_\mathsf{h}[\beta \wedge *\mathrm{d}_\mathsf{h}\beta]\rangle + O(\beta^4). \qquad (74)
\end{aligned}$$

We can read off the direct and adjoint linearized Noether complexes as

$$\Omega^{2,0} \xrightarrow{e_\varphi = *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}} \Omega^{2,0} \xrightarrow{\mathrm{d}_\mathsf{h}} \Omega^{3,0} \xrightarrow{\mathrm{d}_\mathsf{h}} \Omega^{4,0} \longrightarrow 0\ , \tag{75}$$

$$\Omega^{2,0} \xleftarrow{e^*_\varphi = *\delta_\mathsf{h}\mathrm{d}_\mathsf{h}} \Omega^{2,0} \xleftarrow{-\mathrm{d}_\mathsf{h}} \Omega^{1,0} \xleftarrow{\mathrm{d}_\mathsf{h}} \Omega^{0,0} \longleftarrow 0\ , \tag{76}$$

while the leading order consistent deformation is

$$f_\varphi^{(2)}[\beta] = f[\beta] = [*\mathrm{d}\beta \wedge *\mathrm{d}\beta] - *\delta(2[*\mathrm{d}\beta \wedge \beta]). \tag{77}$$

A rigid stage-2 cosymmetry is any $\mathfrak{g}$-valued 0-form $\varepsilon \in \Omega^{0,0}(F, \mathfrak{g})$ such that $\mathrm{d}\varepsilon = 0$. The corresponding conserved 3-current and null 2-source are, respectively, $k = \langle \varepsilon \wedge *\mathrm{d}\beta \rangle$ (a 1-form) and $\rho = \langle \varepsilon \wedge *\delta\mathrm{d}\beta \rangle$ (2-form) (up to signs). The resulting deformation current, which we can check is not off-shell closed, is

$$j_{\rho,f}[\beta] = \langle \varepsilon \wedge [*\mathrm{d}\beta \wedge *\mathrm{d}\beta] \rangle, \tag{78}$$

$$\mathrm{d}j_{\rho,f}[\beta] = \langle \varepsilon \wedge [\mathrm{d}*\mathrm{d}\beta \wedge *\mathrm{d}\beta] \rangle, \tag{79}$$

where the neglected term is trivial because of the identity $\mathrm{d}*\delta = 0$.

Therefore, for each such rigid stage-1 cosymmetry ε, we have found a second order linearization obstruction for the background solution $\varphi = \beta = 0$ of the Freedman–Townsend model on any (pseudo-)Riemannian 4-manifold with non-vanishing $H^2_{\mathrm{dR}}(M)$:

$$[\varepsilon] \cdot Q_\rho \colon \mathcal{S}(\mathcal{E}_\varphi) \to H^2_{\mathrm{dR}}(M). \tag{80}$$

An example of a Lorentzian, globally hyperbolic spacetime is the Schwarzschild black hole spacetime, whose topology is $\mathbb{R}^2 \times S^2$. This observation appears to be new. We have not explicitly checked that these obstructions are non-trivial, but that is likely to be the case, by analogy with the Yang–Mills and Chern–Simons examples. It would be interesting to obtain an interpretation for these rigid cosymmetries in terms of the higher bundle interpretation of higher gauge theories [32].

5.4.2. Other Examples. Higher gauge theories have attracted a lot of attention recently in some mathematical literature (for instance, see [32,33,42]). Many of these theories involve Lie algebra $\mathfrak{g}$-valued p-forms as dynamical fields and exhibit non-linearities similar to those of Yang–Mills, Chern–Simons and Freedman–Townsend theories. In addition, $N > 1$ classical supergravity theories also involve dynamical p-forms, non-linearly coupled to other fields [43]. It appears that the question of linearization stability or instability of particular background solutions in these theories has yet to attract any attention.

6. Discussion

We have considered the question of linearization instability for a large class of non-linear PDE systems, which includes relativistic Lagrangian field theories, with both irreducible and reducible gauge theories among them. This class consists of all (not necessarily Lagrangian) PDE systems, for which a Noether complex can be defined.

The question of linearization stability has two aspects. One is to identify obstructions, which prevent an arbitrary linearized solution from being extended to a family of exact ones. The other, once sufficiently many obstructions have been identified, is to show that no further obstructions exist. We have concentrated only on the first aspect. Moreover, we have restricted our attention only those obstructions valued in the domain manifold's de Rham cohomology, with the cohomology representatives being constructed locally out of the linearized solutions. Though, within this class of obstructions, we likely give an exhaustive classification. (It is a matter of connecting the result of Theorem 3 with the consistent deformations coming from the non-linearity at leading, as was done in Theorem 4, or higher orders). Interestingly enough, it captures essentially all linearization obstructions that have been identified in the work following up the original articles Brill and Deser [31] and Fischer and Marsden [18]. In many cases these known obstructions have also been shown to be sufficient [1,2], at least when concentrating on initial data and ignoring blow-up singularities.

Our main result (Theorem 4) provides a streamlined, unified way of identifying such obstructions. Also, both the results and intermediate calculations are everywhere geometric and covariant with respect to the PDE domain. This is an improvement over previous calculations, which treated individually each PDE system of interest and often required some non-geometric intermediate steps.

The conditions identifying linearization unstable backgrounds in known examples have all been, in a sense, generalizations of the Killing condition (existence of non-trivial Killing vectors) in general relativity. Our main result puts this observation into a more general context. We term the most general form of this condition the *co-Killing condition* (with the prefix *co-* implying that it is defined by the Noether complex, rather than the complex of gauge generators, which are identical in Lagrangian theories). This condition corresponds to the presence of non-trivial rigid (higher stage) cosymmetries (Sect. 3.2). In Lagrangian theories (by a generalization of Noether's second theorem), these are dual to rigid (higher stage) symmetries, of which Killing vectors are in fact an example.

Our result also clarifies the role of the non-trivial topology of the PDE domain. The known linearization obstruction have so far appeared mostly on PDE domains (say of dimension n) that are compact ($H^n_{\mathrm{dR}} \neq 0$) or with a compact initial data surface ($H^{n-1}_{\mathrm{dR}} \neq 0$). No particularly clear statement has been made about non-compact domains. Now, from Theorem 4, we know that any non-trivial de Rham cohomology class in H^*_{dR} can generate a linearization obstruction. However, the lower degree cohomology classes come into play only in the presence of rigid higher stage cosymmetries. In particular, since reducible gauge theories have not been analyzed in the literature on linearization instabilities (Einstein, Yang–Mills and related theories are *irreducible* gauge theories), this explains why only the H^n_{dR} and H^{n-1}_{dR} cohomology classes have played a role. Furthermore, our result shows that linearization obstructions can appear even in the absence of non-trivial topology, as long as the de Rham complex develops non-trivial cohomology when restricted by some asymptotic boundary conditions on open manifolds (Sect. 4.5).

Further, the intermediate result of Theorem 3 may be of interest in its own right. It shows that consistent deformations of linear PDE systems relate on-shell conserved currents of different degrees. Namely, a consistent deformation and a conserved $(p+1)$-current bilinearly define a p-current, which we have called a *deformation current*. We have named *conservative* those consistent deformations that have trivial deformation currents. It may be interesting to analyze the subset of conservative deformations and its relation to the general problem of adding non-linear interactions to linear field theories [44]. In particular, it is interesting to investigate whether the rank (in either argument) of the deformation current mapping is expressible is some known invariant of the linear PDE system.

Finally, in the problem of deformation quantization of classical theories, the differential geometry of the phase space plays a crucial role. When the phase space is a smooth manifold, the Fedosov construction essentially solves the problem [45]. In classical field theories, the phase space can be identified with the space of solutions. Non-smooth, singular points (precisely those solutions that are linearization unstable) then pose an obstacle to quantization. The situation with deformation quantization in the presence of such singularities is much less clear, though some work has been done in that direction [46–48]. We hope that the results of this paper could serve as tools in this program.

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Appendix A: Dynamical Forms

Below, we collect useful formulas for explicit calculation with theories where the dynamical fields are differential k-forms on the manifold M of $\dim M = n$. We take the field bundle to be $F = \Lambda^k M$, often the equation bundle and the Noether bundles are bundles of forms over M as well, say $G \cong \Lambda^l M$ and $Z^i \cong \Lambda^{k_i} M$. The space of F-local section can then be identified with the space of local horizontal forms of the same degree, $\Gamma_F(F) \cong \Omega^{k,0}(F)$, $\Gamma_F(G) \cong \Omega^{l,0}(F)$ and $\Gamma_F(Z^i) \cong \Omega^{k_i,0}(F)$. Dual densities can be identified with forms of complementary degree, $\tilde{F}^* \cong \Lambda^{n-k}$, where the natural fiber-wise pairing is realized as

$$\lambda \cdot \mu = \lambda \wedge \mu, \tag{81}$$

with $\lambda \in \Gamma(F)$ and $\mu \in \Gamma(\tilde{F}^*)$. The de Rham differential d extended to act on local sections is none other than the horizontal differential d_h from the variational bicomplex. Using the algebra of differential forms, it is easy to find its formal adjoint:

$$\mathrm{d}\lambda \wedge \mu - \lambda \wedge (-)^{|\lambda|+1} \mathrm{d}\mu = \mathrm{d}(\lambda \wedge \mu), \tag{82}$$

which means that the formal adjoint of $\mathrm{d}_\mathsf{h}\colon \Omega^{l,0}(F) \to \Omega^{l+1,0}(F)$ is $\mathrm{d}_\mathsf{h}^* = (-)^{l+1}\mathrm{d}_\mathsf{h}\colon \Omega^{n-l-1,0}(F) \to \Omega^{n-k}$.

When the manifold M is endowed with a (pseudo-)Riemannian metric g, we can define the corresponding Hodge dual operator $*\colon \Omega^{l,0}(F) \to \Omega^{n-l,0}(F)$. Recall the following identities [49]:

$$\lambda_1 \wedge *\lambda_2 = \lambda_2 \wedge *\lambda_1, \tag{83}$$

$$**\lambda = \epsilon_{|\lambda|}\eta, \quad \text{with } \epsilon_k = (-)^{k(n-k)+(n-s)/2}, \tag{84}$$

where s is the signature of g. If $n = 4$ and the metric has Lorentzian signature $(-{+}{+}{+})$, $s = 2$, then $\epsilon_k = (-)^{k+1}$. With Hodge duality available, it is also possible to identify the dual densities of l-forms with l-forms themselves with the Hodge fiber-wise pairing

$$\lambda \cdot \mu = \lambda \wedge *\mu. \tag{85}$$

The formal adjoint of d_h with respect to the above pairing is the horizontal de Rham *codifferential* denoted by δ_h (and by δ when not acting on local horizontal forms):

$$\mathrm{d}_\mathsf{h}\lambda \wedge *\mu - \mu \wedge *\delta_\mathsf{h}\mu = \mathrm{d}_\mathsf{h}(\mu \wedge *\nu). \tag{86}$$

It satisfies the identity $\delta_\mathsf{h}\lambda = (-)^{|\lambda|}\epsilon_{|\lambda|}\mathrm{d}_\mathsf{h}$, so that $\delta_\mathsf{h}^2 = 0$ and $\mathrm{d}_\mathsf{h}*\delta_\mathsf{h} = \delta_\mathsf{h}*\mathrm{d}_\mathsf{h} = 0$.

For calculations that include vertical forms, it is convenient to extend the Hodge dual to all local variational forms so that

$$*(\nu \wedge \lambda) = \nu \wedge *\lambda, \tag{87}$$

where ν is any purely vertical form and λ is any purely horizontal one. This way, the Hodge dual commutes with the vertical differential, $\mathrm{d}_\mathsf{v}* = *\mathrm{d}_\mathsf{v}$. Moreover, the Hodge dual pairing satisfies the identity

$$\begin{aligned}(\omega \wedge \lambda) \wedge *(\pi \wedge \mu) &= (-)^{|\lambda||\pi|}\omega \wedge \pi \wedge (\lambda \wedge *\mu) &(88)\\ &= (-)^{(|\lambda|+|\omega|)|\pi|}\pi \wedge \omega \wedge (\mu \wedge *\lambda) &(89)\\ &= (-)^{|\lambda||\pi|+|\omega||\pi|+|\mu||\pi|}(\pi \wedge \mu) \wedge *(\omega \wedge \lambda), &(90)\end{aligned}$$

whenever ω and π are purely vertical, while λ and μ are purely horizontal.

Let $\mathfrak{g}$ be a Lie algebra. When the dynamical form fields admit the interpretation of being components of a (higher) connection on a (higher) principal bundle defined by $\mathfrak{g}$ over M, it becomes convenient to use $\mathfrak{g}$-valued forms as dynamical fields, that is, $F = \mathfrak{g} \otimes \Lambda^k M$. The Lie algebra naturally comes with the following bilinear operations:

$$\text{(commutator)} \quad [-]\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}, \tag{91}$$

$$\text{(Killing form)} \quad \langle - \rangle\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}, \tag{92}$$

where $[-]$ is antisymmetric, while $\langle - \rangle$ is symmetric and it is non-degenerate for semi-simple $\mathfrak{g}$. They satisfy the following compatibility identity:

$$\langle a \otimes [b \otimes c] \rangle = \langle [a \otimes b] \otimes c \rangle, \tag{93}$$

where either side defines a trilinear, totally antisymmetric functional. We extend slightly the notation for F-local horizontal forms and write $\Omega^{h,v}(F, \mathfrak{g})$ for the space of local $\mathfrak{g}$-valued variational (h, v)-forms; we also write $\Omega^k(M, \mathfrak{g}) \cong \mathfrak{g} \otimes \Omega^k(M)$. The operations d_v, d_h, $*$, $[-]$ and $\langle - \rangle$ extend to local $\mathfrak{g}$-valued variational forms simply by treating one or the other tensor factor trivially, while $\wedge$ is extended by acting as $\otimes$ on the Lie algebra factor. Then it is easy to check that the expressions

$$\langle \lambda_1 \wedge *\lambda_2 \rangle \quad \text{and} \quad \langle \mu_1 \wedge [\mu_2 \wedge \mu_3] \rangle \tag{94}$$

are totally symmetric in their arguments as long as μ_i are forms of odd degrees.

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