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Precise Arrhenius Law for p-forms: The Witten Laplacian and Morse–Barannikov Complex

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Abstract. Accurate asymptotic expressions are given for the exponentially small eigenvalues of Witten Laplacians acting on p-forms. The key ingredient, which replaces explicit formulas for global quasimodes in the case $p = 0$, is Barannikov's presentation of Morse theory in Barannikov (Adv Soviet Math 21:93–115, 1994).

1. Introduction and Main Statements

1.1. Presentation

The Brownian motion of a particle, at position $x(t)$ (in $\mathbb{R}^d$ for this rapid presentation), and experiencing a gradient field $-2\nabla f(x)$, can be modelled by the Smoluchowski stochastic differential equation (see a.e. [39]):

$$\mathrm{d}x = -2\nabla f(x)\mathrm{d}t + \sqrt{2h}\mathrm{d}W, \quad x(t) = x_0 \in \mathbb{R}^d. \tag{1}$$

Local minima of the energy profile, f, are stable steady states when $h = 0$ and become metastable states when h is positive and small. If $h > 0$ is thought as a temperature, the lifetime of such a metastable state U_0 is exponentially large in term of $1/h$. Its inverse $\varrho_{U_0}(h)$ follows an Arrhenius law $\varrho_{U_0}(h) \propto e^{-\frac{E_{\mathrm{act}}(U_0)}{h}}$, where the activation energy $E_{\mathrm{act}}(U_0)$ equals $2(f(U_1) - f(U_0))$ and U_1 is a proper saddle point, of the energy profile f, associated with U_0. Those inverse lifetime are actually the exponentially small eigenvalues of the Feller semigroup generator associated with (1),

$$-\left(-2\partial_x f\partial_x + h\Delta_x\right) = \frac{1}{h}(-h\partial_x + 2\partial_x f)(h\partial_x) \quad \text{on } \mathbb{R}^d, \tag{2}$$

defined on $L^2(\mathbb{R}^d, e^{-\frac{2f(x)}{h}}\mathrm{d}x)$, while $e^{-\frac{2f(x)}{h}}\mathrm{d}x$ is the invariant measure associated with (1).

The analysis of these activation energies, or exponentially small eigenvalues in terms of h, has motivated various mathematical studies within the probabilistic approach and simulated annealing techniques in the 80's (see for instance [14, 29]). More recently several works have been devoted to the accurate computation of the prefactors, $P_{U_0}(h)$ in $\varrho_{U_0}(h) = P_{U_0}(h)e^{-\frac{E_{\mathrm{act}}(U_0)}{h}}$, which is also known as the Eyring–Kramers law (see a.e. [2] for a recent review about this topic). It was carried out with a probabilistic and potential theory approach in [6, 7], or with PDE and spectral techniques in [19–21, 28, 33–35].

After conjugating with $e^{\frac{f}{h}}$ and multiplying by h, the operator (2) becomes a Witten Laplacian acting on functions (0-forms),

$$\begin{aligned}&(-h\partial_x + \partial_x f)(h\partial_x + \partial_x f)\\ &\quad = -h^2\Delta_x + |\nabla_x f|^2 - h(\Delta_x f) = d^*_{f,h} d_{f,h} = \Delta^{(0)}_{f,h},\end{aligned} \tag{3}$$

with $d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df\wedge$ and $d^*_{f,h} = hd^* + \mathbf{i}_{\nabla f}$.

On a general configuration space, that is a manifold, the Witten Laplacian acting on the space of all smooth differential forms,

$$\Delta_{f,h} = (d_{f,h} + d^*_{f,h})^2 = d^*_{f,h}d_{f,h} + d_{f,h}d^*_{f,h}, \tag{4}$$

is decomposed as the direct sum $\Delta_{f,h} = \oplus_{p=0}^{d}\Delta^{(p)}_{f,h}$ with $\Delta^{(p)}_{f,h}$ acting on p-forms. It provides the geometrically intrinsic writing, depending on the metric g and the Morse function f, and exhibits the relationship with other structures. In his celebrated article, Witten [45] showed that this deformation of Hodge theory allows one to recover analytically the Morse inequalities for the function f. The number of eigenvalues of $\Delta^{(p)}_{f,h}$ lying in $[0, h^{3/2})$ equals, for $h > 0$ small enough, the number m_p of critical points of f with index p, while conjugating the differential with $e^{\frac{f}{h}}$ provides an isomorphism in cohomology between the de Rham chain complex $(\Omega^*(M), d)$ and the chain complex $(\Omega^*_{f,h}, d_{f,h})$, where $\Omega^p_{f,h}$ is the space generated by the eigenmodes with eigenvalue less than $h^{3/2}$ for $\Delta^p_{f,h}$, and $d_{f,h}$ the Witten differential.

Shortly after Witten's article, it was proved in [24] that the $\mathcal{O}(h^{3/2})$ eigenvalues of $\Delta^{(p)}_{f,h}$ are actually exponentially small, $\lambda^{(p)}_k = \mathcal{O}(e^{-\frac{C^{(p)}_k}{h}})$ without specifying the $C^{(p)}_k$'s. The values of the activation energies $C^{(0)}_k$ for $p = 0$ were already known from [29, 14]. The accurate determination, in the case of functions ($p = 0$), of the prefactors, $P^{(0)}_k(h)$ in $\lambda^{(0)}_k = P^{(0)}_k(h)e^{-\frac{C^{(0)}_k}{h}}$, came later, motivated by probabilistic questions in [6, 7], or by the analysis of the Kramers–Fokker–Planck operators in [25].

The accurate computation of the small eigenvalues of $\Delta^{(p)}_{f,h}$ is made difficult by the interactions due to the tunneling effect between the m_p quantum wells, with a hierarchy of weakly resonant tunneling quantum wells, according to the terminology of [22, 23]. This hierarchy, which orders the exponentially small quantities, can be solved by considering the interaction via the deformed differential $d_{f,h}$ with the eigenmodes of $\Delta_{f,h}^{(p+1)}$ and via the deformed codifferential $d_{f,h}^*$ with the eigenmodes of $\Delta_{f,h}^{(p-1)}$. When $p = 0$, it is simply understood within the probabilistic approach by ordering the exit times, following in some sense the intuition of Arrhenius law. It actually amounts to elementary topological arguments by considering how the number of connected components of the sublevel set $f^\lambda = \{x, f(x) < \lambda\}$ varies as λ crosses a critical value. As this appeared in the case of surfaces treated in [36], such a simple presentation is no more relevant for higher degrees $p > 0$. Even without considering the prefactor problem, the equivalent of the Arrhenius law for p-forms, $p > 0$, already requires, as we shall see, the introduction of more sophisticated topological constructions. Moreover, the analysis carried out in [19–21,33–35], relied on the important remark that the eigenvalues of $\Delta_{f,h}^{(0)}$ are the squares of the singular values of the differential $d_{f,h}^{(0)}$. The Fan inequalities (see [41]) for singular values then allow to propagate relative errors (i.e., small errors relative to various exponentially small quantities).

For all these reasons, the study of exponentially small eigenvalues of $\Delta_{f,h}^{(p)}$ for a general p, is a natural question which is also encountered in geometry (see for instance [3,5,46] and references therein) or in statistical physics (see [44]). A first attempt was done in [36], extending the result for $p = 0$ to $p = 0, 1, 2$ on surfaces, with simple duality and chain complex arguments. For a general p, the global quasimodes of the form $\chi_{U_0}(x)e^{-\frac{f(x)-f(U_0)}{h}}$ used for the case $p = 0$ in [19–21,33–35] and which propagate the information through weakly resonant quantum wells, are missing.

The solution comes from the use of Barannikov's version of the Morse complex from [1], which fits exactly the handling of global quasimodes for Witten Laplacian. There are two reasons for this:

1. this new chain complex has nice restriction properties which are implemented by boundary Witten Laplacians;
2. a side result coming from this presentation of Morse theory allows to replace the analytical computations with $\chi_{U_0}(x)e^{-\frac{f(x)-f(U_0)}{h}}$ in the case $p = 0$, by a subtle repeated use of Stokes' formula.

We conclude this introduction by emphasizing that the accurate analysis of the tunnel effect required for the computation of exponentially small eigenvalues, goes far beyond the instantonic picture (see [9]), which sticks in some sense to the intuition of classical mechanics. However, it is remarkable that discriminating between so small quantities (exponentially small quantities $e^{-C_k/h}$ as $h \to 0$) is made possible by global topological arguments.

1.2. Assumptions and Result

Hypothesis 1. *We shall work on an oriented compact Riemannian manifold* (M, g) *and* f *will be an excellent Morse function:* f *is smooth, has non-degenerate critical points and these have distinct critical values. Moreover, homology and cohomology will always be with real coefficients.*

Barannikov's simple Morse complex, allows to partition the set of critical points, $\mathcal{U} = \{x \in M, \nabla f(x) = 0\}$ (resp. the set of critical points with index p, $\mathcal{U}^{(p)} = \{x \in M, \nabla f(x) = 0, \mathrm{sign}(\mathrm{Hess\ f})(x) = (d-p,p)\}$), into upper, lower and homological critical points:

$$Y\mathcal{U} = \mathcal{U}_{\mathrm{U}} \sqcup \mathcal{U}_{\mathrm{L}} \sqcup \mathcal{U}_{\mathrm{H}} \tag{5}$$

$$\text{resp.} \qquad \mathcal{U}^{(p)} = \mathcal{U}^{(p)}_{\mathrm{U}} \sqcup \mathcal{U}^{(p)}_{\mathrm{L}} \sqcup \mathcal{U}^{(p)}_{\mathrm{H}}. \tag{6}$$

Homological critical points in $\mathcal{U}^{(p)}_{\mathrm{H}}$ are associated with the kernel $\ker(\Delta^{(p)}_{f,h}) \sim \ker \Delta^{(p)}_{\mathrm{Hodge}}$ and their number is the p-th Betti number $\beta_p = \dim H^p(M,\mathbb{R})$. The boundary operator ∂_{B} of Barannikov's chain complex, defined on $\oplus_{U\in\mathcal{U}}\mathbb{R}U$, associates with any $U' \in \mathcal{U}^{(p)}_{\mathrm{U}}$ an element $U \in \mathcal{U}^{(p-1)}_{\mathrm{L}}$ such that $f(U) < f(U')$, and vanishes on all other critical points $U' \in \mathcal{U}_{\mathrm{H}} \cup \mathcal{U}_{\mathrm{L}}$. Details are given in Sect. 2. The second assumption avoids technical (nevertheless interesting) questions about multiplicities of non-zero exponentially small eigenvalues.

Hypothesis 2. *The values $f(U') - f(U)$ obtained for $U' \in \mathcal{U}_{\mathrm{U}}$ and $\partial_{\mathrm{B}} U' = U$ are all distinct.*

Here is our main result

Theorem 1.1. *Assume Hypotheses 1 and 2 hold. Let $\mathcal{U}_{\mathrm{H}}, \mathcal{U}_{\mathrm{L}}, \mathcal{U}_{\mathrm{U}}$, respectively, denote the sets of homological, lower and upper critical points.*

For h_0 small enough and $0 < h < h_0$, there exists a mapping j from $\mathcal{U} := \mathcal{U}_{\mathrm{H}} \cup \mathcal{U}_{\mathrm{L}} \cup \mathcal{U}_{U}$ onto $\sigma(\Delta_{f,h}) \cap [0, h^{\frac{3}{2}})$ and the restriction $j_p := j\big|_{\mathcal{U}^{(p)}}$ is onto $\sigma(\Delta^{(p)}_{f,h}) \cap [0, h^{\frac{3}{2}})$ and one to one provided the eigenvalues of $\sigma(\Delta^{(p)}_{f,h})$ are counted with multiplicities.

Moreover, the map j satisfies the following properties:

1. *For $U^{(p)}$ in $\mathcal{U}^{(p)}_{\mathrm{H}}$,*

$$j(U^{(p)}) = 0.$$

2. *For $U^{(p)}$ in $\mathcal{U}^{(p)}_{\mathrm{L}}$, let $U^{(p+1)}$ denote the element of $\mathcal{U}^{(p+1)}_{\mathrm{U}}$ s.t. $\partial_{\mathrm{B}}(U^{(p+1)}) = U^{(p)}$. Then, there exists a homological constant $\kappa(U^{(p+1)}) \in \mathbb{R}^*$ such that*

$$j(U^{(p)}) = \kappa^2(U^{(p+1)}) \frac{h}{\pi} \frac{|\lambda_1^{(p+1)} \cdots \lambda_{p+1}^{(p+1)}|}{|\lambda_1^{(p)} \cdots \lambda_p^{(p)}|} \frac{|Hess\, f(U^{(p)})|^{\frac{1}{2}}}{|Hess\, f(U^{(p+1)})|^{\frac{1}{2}}} \times e^{-2\frac{f(U^{(p+1)})-f(U^{(p)})}{h}}(1+\mathcal{O}(h)),$$

where $\lambda_1^{(\ell)}, \ldots, \lambda_\ell^{(\ell)}$ denote the negative eigenvalues of $Hess\, f(U^{(\ell)})$, for $\ell \in \{p, p+1\}$.

3. *Finally, for* $U^{(p)}$ *in* $\mathcal{U}_{\mathrm{U}}^{(p)}$, *the equality* $j(U^{(p)}) = j(\partial_{\mathrm{B}}(U^{(p)})) = j(U^{(p-1)})$ *holds with* $U^{(p-1)} := \partial_{\mathrm{B}}(U^{(p)}) \in \mathcal{U}_{\mathrm{L}}^{(p-1)}$, *i.e.*

$$j(U^{(p)}) = \kappa^2(U^{(p)})\frac{h}{\pi}\frac{|\lambda_1^{(p)}\cdots\lambda_p^{(p)}|}{|\lambda_1^{(p-1)}\cdots\lambda_{p-1}^{(p-1)}|}\frac{|Hess\, f(U^{(p-1)})|^{\frac{1}{2}}}{|Hess\, f(U^{(p)})|^{\frac{1}{2}}} \times e^{-2\frac{f(U^{(p)})-f(U^{(p-1)})}{h}}(1+\mathcal{O}(h)),$$

where $\lambda_1^{(\ell)},\ldots,\lambda_\ell^{(\ell)}$ *denote the negative eigenvalues of* $Hess\, f(U^{(\ell)})$, *for* $\ell \in \{p-1,p\}$.

A relative version of this result, implemented with boundary Witten Laplacians, is given in Sect. 4.5 at the end of this paper.

Remark 1.2. Although we are not able to prove it in general, there is a strong indication that the "homological" constant $\kappa(U^{(p)})$ equals ± 1. This is indeed the case for $p = 0$ as shown in [19,20,35]. By duality it is also true when $p = d$. Finally in the case of surfaces treated in [36], a combination of these results says that it is true for $p = 0, 1, 2$.

A general proof requires a better understanding of the topological aspects of Morse theory and of Barannikov's construction.

Surely, this constant is completely determined by the structure of the homology groups of the sublevel sets $H_*(\{f < \lambda\})$, $\lambda \in [-\infty, +\infty]$. It does not depend on h, on the Riemannian metric g or on the Morse function f (as long our generic assumptions are fulfilled), contrary to the other factors. This is a reason to use the attribute "homological" for this, up to now unknown, constant.

2. Barannikov's Simple Complex and Morse Theory

In this section, we adapt the approach of Barannikov, using notations and definitions better suited to the treatment of Witten Laplacians.

2.1. Sublevel Sets and Bases of Morse Theory

Remember that we work on a Riemannian compact oriented manifold (M, g), endowed with an excellent Morse function according to Hypothesis 1. With such an assumption we may identify a critical point U with the corresponding critical value $c = f(U)$. For any index p, $0 \le p \le d = \dim M$, the set of critical points of f with index p is $\mathcal{U}^{(p)} = \{U_k^{(p)}, 1 \le k \le m_p\}$. It is equivalently represented by a vertical line with a m_p points with heights $c_k^{(p)} = f(U_k^{(p)})$, $1 \le k \le m_p$. The vector space spanned by these points is denoted by $\mathcal{C}^{(p)}(f)$ and we set $\mathcal{C}(f) = \oplus_{p=0}^{d}\mathcal{C}^{(p)}(f)$. We shall construct explicitly a differential on $\mathcal{C}(f)$ quasi-isomorphic to the Morse chain complex associated with f.

For $\lambda \in [-\infty, +\infty]$, f^λ denotes the sublevel set $\{x \in M, f(x) < \lambda\}$ while f_λ denotes the upper level set $\{x \in M, f(x) > \lambda\}$ and more generally $f_\lambda^\mu = \{x \in M, \lambda < f(x) < \mu\}$. Let us recall a few elementary facts related with Morse theory known from [8,31,32,38]. We refer to [10,15,17,37] for basic material in homological algebra. When $\lambda \in (\min f, \max f)$ is not a critical value, f^λ and f_λ are boundary manifolds, while $f^\lambda = M$, $f_\lambda = \emptyset$ for $\lambda > \max f$ and $f^\lambda = \emptyset$, $f_\lambda = M$ for $\lambda < \min f$. When there is no critical value between λ_1 and λ_2, the natural inclusion of f^{λ_1} in f^{λ_2} (resp. f_{λ_2} in f_{λ_1}) induces a homotopy equivalence and, therefore, induces an isomorphism of their homology groups:

$$\forall p \in \{0,\ldots,d\}, \quad H_p(f^{\lambda_1}) = H_p(f^{\lambda_2}) \quad \text{and} \quad H_p(f_{\lambda_1}) = H_p(f_{\lambda_2}).$$

With the help of the five lemma, this holds also for the relative homology groups $H_*(f^\mu, f^{\lambda_1})$ and $H_*(f^\mu, f^{\lambda_2})$, and $H_*(f_{\lambda_1}, f_\mu)$, $H_*(f_{\lambda_2}, f_\mu)$ for $\mu > \lambda_1, \lambda_2$, as is easily seen using the long exact sequences

$$H_{*+1}(f^\lambda) \xrightarrow{i_*} H_{*+1}(f^\mu) \xrightarrow{j_*} H_{*+1}(f^\mu, f^\lambda) \xrightarrow{\partial} H_*(f^\lambda)\,,$$

$$H_{*+1}(f_\mu) \xrightarrow{i_*} H_{*+1}(f_\lambda) \xrightarrow{j_*} H_{*+1}(f_\lambda, f_\mu) \xrightarrow{\partial} H_*(f_\mu)\,,$$

when $\mu > \lambda$ are not critical values.

Passing a critical point with index p and critical value c, the pair $(f^{c+\varepsilon}, f^{c-\varepsilon})$ is homologous to the pair $(\mathbb{D}^p, \partial\mathbb{D}^p)$, associated with the p-cell $e^p = \mathbb{D}^p \backslash \partial\mathbb{D}^p$ (see [38]). This gives using excision

$$0 \to H_p(f^{c-\varepsilon}) \to H_p(f^{c+\varepsilon}) \to H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_{p-1}(f^{c-\varepsilon}) \to H_{p-1}(f^{c+\varepsilon}) \to 0\,, \tag{7}$$

with $\dim H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) = 1$ and ensures that for $k \neq p, p-1$ we have the equality of $H_k(f^{c\pm\varepsilon})$.

This yields two mutually exclusive cases

$$\begin{cases} H_p(f^{c-\varepsilon}) \xrightarrow{\sim} H_p(f^{c+\varepsilon}) \quad \text{and} \\ 0 \to H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_{p-1}(f^{c-\varepsilon}) \to H_{p-1}(f^{c+\varepsilon}) \to 0\,, \end{cases} \tag{8}$$

$$\text{or} \begin{cases} 0 \to H_p(f^{c-\varepsilon}) \to H_p(f^{c+\varepsilon}) \to H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \to 0 \\ \text{and} \quad H_{p-1}(f^{c-\varepsilon}) \xrightarrow{\sim} H_{p-1}(f^{c+\varepsilon})\,. \end{cases} \tag{9}$$

The Poincaré duality takes a nice form owing to Theorem 3.43 in [17] (M is oriented), with the excision argument $H_*(f^\mu, f^\lambda) = H_*(\overline{f^\mu_\lambda}, \{f = \lambda\})$: For two non-critical values $-\infty \leq \lambda < \mu \leq +\infty$, the cohomology group $H^k(f^\mu, f^\lambda)$ is isomorphic to $H_{d-k}(f_\lambda, f_\mu)$. In [42]-p. 296, this is called Alexander duality and proved, without excision, via coverings and Mayer–Vietoris techniques. With $f_\lambda = (-f)^{-\lambda}$, this is often summarized by changing f into $-f$ and inverting indexes p and $d-p$. Thus, the dual version of (7) for a critical value with index p is

$$\begin{array}{ccccccccc}
0 & \leftarrow & H_{d-p-1}(f_{c-\varepsilon}) & \leftarrow & H_{d-p-1}(f_{c+\varepsilon}) & \leftarrow & H_{d-p}(f_{c-\varepsilon}, f_{c+\varepsilon}) & \leftarrow & H_{d-p}(f_{c-\varepsilon})\,. \\
 & & & & & & & & \uparrow \\
 & & & & & & & & H_{d-p}(f_{c+\varepsilon}) \\
 & & & & & & & & \uparrow \\
 & & & & & & & & 0
\end{array}$$

For this, use the excision property

$$H_k(f^{c+\varepsilon}, f^{c-\varepsilon}) = H_k(\{c-\varepsilon \le f < c+\varepsilon\}, \{f = c-\varepsilon\})$$

while noticing that according to Poincaré duality $H_k(\{f = c-\varepsilon\})$ is isomorphic to $H^{d-1-k}(\{f = c-\varepsilon\})$.

Hence passing a critical value with upper level sets leads to the two exclusive cases

$$\begin{cases} H_{d-p}(f_{c+\varepsilon}) \xrightarrow{\sim} H_{d-p}(f_{c-\varepsilon}) \quad \text{and} \\ 0 \to H_{d-p}(f_{c-\varepsilon}, f_{c+\varepsilon}) \to H_{d-p-1}(f_{c+\varepsilon}) \to H_{d-p-1}(f_{c-\varepsilon}) \to 0\,, \end{cases} \tag{10}$$

$$\text{or} \begin{cases} 0 \to H_{d-p}(f_{c+\varepsilon}) \to H_{d-p}(f_{c-\varepsilon}) \to H_p(f_{c-\varepsilon}, f_{c+\varepsilon}) \to 0 \\ \text{and} \quad H_{d-p-1}(f_{c+\varepsilon}) \xrightarrow{\sim} H_{d-p-1}(f_{c+\varepsilon})\,. \end{cases} \tag{11}$$

2.2. Classification of Critical Points

2.2.1. Partition. The critical points are divided in three classes, and we prove that these classes make a partition of the set of critical points, satisfying a number of additional properties.

Definition 2.1.

1. A critical value (resp. point) c of f is called a lower critical value (resp. point), if the natural mapping

$$H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \longrightarrow H_*(M, f^{c-\varepsilon})$$

vanishes.

2. A critical value (resp. point) c of f is called an upper critical value (resp. point), if the natural mapping

$$H_*(f^{c+\varepsilon}) \longrightarrow H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$$

vanishes.

3. In all other cases the critical value (resp. point) c, is called a homological critical value (resp. point).

Remember the long exact sequence for the triple (X, A, B) where $B \subset A \subset X$:

$$\to H_*(A, B) \to H_*(X, B) \to H_*(X, A) \xrightarrow{\partial} H_{*-1}(A, B) \to$$

and the commutative diagram associated with a map $\varphi : X \to X'$ satisfying $\varphi(A) \subset A'$ and $\varphi(B) \subset B'$:

$$\begin{array}{ccccccccc} \to & H_*(A,B) & \to & H_*(X,B) & \to & H_*(X,A) & \xrightarrow{\partial} & H_{*-1}(A,B) & \to \\ & \downarrow \varphi_* & & \downarrow \varphi_* & & \downarrow \overline{\varphi_*} & & \downarrow \varphi_* & \\ \to & H_*(A',B') & \to & H_*(X',B') & \to & H_*(X',A') & \xrightarrow{\partial} & H_{*-1}(A',B') & \to \end{array} \tag{12}$$

Proposition 2.2. *The set of lower critical values (resp. points) and upper critical values (resp. points) are disjoint and the classification into lower, upper and homological critical values (resp. points) is a partition.*

Proof. Consider the long exact sequences corresponding to the triples

$$(X, A, B) = (f^{c+\varepsilon}, f^{c-\varepsilon}, \emptyset) \quad \text{and} \quad (X', B', A') = (M, f^{c-\varepsilon}, \emptyset)$$

with the mapping $i^{\infty,c+\varepsilon} : f^{c+\varepsilon} \to M = f^{+\infty}$:

$$\begin{array}{ccccccc} H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & \xrightarrow{\partial} & H_{*-1}(f^{c-\varepsilon}) . \\ \downarrow \mathrm{Id} & & \downarrow i_*^{\infty,c+\varepsilon} & & \downarrow \overline{i_*^{\infty,c+\varepsilon}} & & \downarrow \mathrm{Id} \\ H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(M) & \longrightarrow & H_*(M, f^{c-\varepsilon}) & \xrightarrow{\partial'} & H_{*-1}(f^{c-\varepsilon}) \end{array}$$

Now, assume that the mappings $H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_*(M, f^{c-\varepsilon})$ and $H_*(f^{c+\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ both vanish. This implies that ∂ is injective while $\overline{i^{\infty,c+\varepsilon}} = 0$. This contradicts the commutativity $\partial = \partial' \circ \overline{i^{\infty,c+\varepsilon}}$. □

2.2.2. Upper Critical Points. We first give other characterizations for upper critical points with index p, which will be used further. An additional property about the rank of $i_*^\lambda : H_*(f^\lambda) \to H_*(M)$ is given.

Proposition 2.3. *A critical value c with index p is an upper critical value, iff one of the following conditions is satisfied:*

1. *The critical value satisfies the condition (8), namely:*

$$\begin{cases} H_p(f^{c-\varepsilon}) \xrightarrow{\sim} H_p(f^{c+\varepsilon}) \quad \text{and} \\ 0 \to H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_{p-1}(f^{c-\varepsilon}) \to H_{p-1}(f^{c+\varepsilon}) \to 0 . \end{cases}$$

2. *The mapping $\partial : H_{*+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_*(f^{c-\varepsilon})$ is one to one.*
3. *There exists $\lambda \in [-\infty, c)$ such that the mapping $H_*(f^{c+\varepsilon}, f^\lambda) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ vanishes.*
4. *There exists $\lambda \in [-\infty, c)$ such that the mapping $\partial : H_{*+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_*(f^{c+\varepsilon}, f^\lambda)$ is one to one.*

Proof. The condition (1) is just the explicit form of the definition of an upper critical value, in view of the long exact sequence (7).

The condition (2) is obtained by considering the right-hand side of the long exact sequence

$$H_{*+1}(f^{c-\varepsilon}) \longrightarrow H_{*+1}(f^{c+\varepsilon}) \longrightarrow H_{*+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \xrightarrow{\partial} H_*(f^{c-\varepsilon}).$$

Similarly condition (4) is equivalent to condition (3).

Consider condition (3): It is necessary (take $\lambda = -\infty$). The middle square of the commutative diagram (12) with the embedding $\varphi = i^\lambda : (X, A, B) = (f^{c+\varepsilon}, f^{c-\varepsilon}, f^{-\infty} = \emptyset) \to (X', A', B') = (f^{c+\varepsilon}, f^{c-\varepsilon}, f^\lambda)$:

$$\begin{array}{ccccccc}
H_*(f^{c-\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}) & \longrightarrow & H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & \xrightarrow{\partial} & H_{*-1}(f^{c-\varepsilon}) \\
\downarrow{\scriptstyle i_*^\lambda} & & \downarrow{\scriptstyle i_*^\lambda} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle i_*^\lambda} \\
H_*(f^{c-\varepsilon}, f^\lambda) & \longrightarrow & H_*(f^{c+\varepsilon}, f^\lambda) & \xrightarrow{0} & H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & \xrightarrow{\partial} & H_{*-1}(f^{c-\varepsilon}, f^\lambda),
\end{array}$$

provides the sufficiency. □

Consider for $-\infty \leq \lambda < \mu \leq +\infty$ which are not critical values, the embeddings $i^\lambda : f^\lambda \to M$ and $i^\mu : f^\mu \to M$ and $i^{\mu,\lambda} : (f^\lambda, M) \to (f^\mu, M)$. Then the commutative diagram

$$\begin{array}{ccccccc}
H_*(f^\lambda) & \xrightarrow{i_*^\lambda} & H_*(M) & \longrightarrow & H_*(M, f^\lambda) & \xrightarrow{\partial} & H_{*-1}(f^\lambda) \\
\downarrow{\scriptstyle i_*^{\mu,\lambda}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \overline{i_*^{\mu,\lambda}}} & & \downarrow{\scriptstyle i_*^{\mu,\lambda}} \\
H_*(f^\mu) & \xrightarrow{i_*^\mu} & H_*(M) & \longrightarrow & H_*(M, f^\mu) & \xrightarrow{\partial} & H_{*-1}(f^\mu)
\end{array}$$

implies

$$i_*^\lambda = i_*^\mu \circ i_*^{\mu,\lambda} \quad (\lambda < \mu) \tag{13}$$

$$\mathrm{Im}\, i_*^\lambda \subset \mathrm{Im}\, i_*^\mu, \quad \mathrm{rank}\, i_*^\lambda \leq \mathrm{rank}\, i_*^\mu. \tag{14}$$

Proposition 2.4. *When c is an upper critical value, the ranges of $i_*^{c+\varepsilon} : H_*(f^{c+\varepsilon}) \to H_*(M)$ and $i_*^{c-\varepsilon} : H_*(f^{c-\varepsilon}) \to H_*(M)$ are the same.*

Proof. The condition (1) of Proposition 2.3 ensures that for any $k \in \{0, \ldots, d\}$, the mapping

$$i_*^{c+\varepsilon, c-\varepsilon} : H_k(f^{c-\varepsilon}) \to H_k(f^{c+\varepsilon})$$

is onto. Therefore, $i_*^{c+\varepsilon}$ and $i_*^{c-\varepsilon} = i^{c+\varepsilon} \circ i_*^{c+\varepsilon, c-\varepsilon}$ have the same range. □

2.2.3. Lower Critical Points. With $H_*(M, f^{c-\varepsilon}) = H_*(f^{+\infty}, f^{c+\varepsilon})$, the duality $H^*(f^\mu, f^\lambda) \simeq H_{d-*}(f_\lambda, f_\mu)$ says that the lower critical values are the ones for which the mapping

$$H_*(f_{c-\varepsilon}) = H_*(f_{c-\varepsilon}, f_{+\infty}) \longrightarrow H_*(f_{c-\varepsilon}, f_{c+\varepsilon})$$

vanishes. It is therefore the dual notion to the one of upper critical value and all the dual properties of the ones of the upper critical values will hold for lower critical points. We shall use the following characterization.

Proposition 2.5. *A critical value c with index p is a lower critical value, iff one of the following conditions is satisfied:*

1. *The critical value satisfies the condition* (10)*, namely:*

$$\begin{cases} H_{d-p}(f_{c+\varepsilon}) \stackrel{\sim}{\longrightarrow} H_{d-p}(f_{c-\varepsilon}) \quad \text{and} \\ 0 \longrightarrow H_{d-p}(f_{c-\varepsilon}, f_{c+\varepsilon}) \longrightarrow H_{d-p-1}(f_{c+\varepsilon}) \longrightarrow H_{d-p-1}(f_{c-\varepsilon}) \to 0\,. \end{cases}$$

2. *The mapping* $\partial : H_{*+1}(M, f^{c+\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ *is onto.*
3. *There exists* $\lambda \in (c, +\infty]$ *such that the mapping* $H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_*(f^\lambda, f^{c-\varepsilon})$ *vanishes.*
4. *There exists* $\lambda \in (c, +\infty]$ *such that the mapping* $\partial : H_{*+1}(f^\lambda, f^{c+\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ *is onto.*

Proof. The condition (1) is the dual statement of (8), which is equivalent to the dual notion of upper critical values.

The equivalence with the condition (2) is contained in the long exact sequence

$$H_{*+1}(M, f^{c-\varepsilon}) \longrightarrow H_{*+1}(M, f^{c+\varepsilon}) \stackrel{\partial}{\longrightarrow} H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \longrightarrow H_*(M, f^{c-\varepsilon})\,.$$

Similarly the conditions (3) and (4) are equivalent.

Consider the condition (3): It is necessary (take $\lambda = +\infty$). If there exists $\lambda \in (c, +\infty]$ such that the mapping $H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_*(f^\lambda, f^{c-\varepsilon})$ vanishes, the composition of the embeddings $i^{\lambda, c+\varepsilon} : (f^{c+\varepsilon}, f^{c-\varepsilon}) \to (f^\lambda, f^{c-\varepsilon})$ and $i^\lambda : (f^\lambda, f^{c-\varepsilon}) \to (M = f^{+\infty}, f^{c-\varepsilon})$ implies that the composed map

$$H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \xrightarrow{i_*^{\lambda, c+\varepsilon} = 0} H_*(f^\lambda, f^{c-\varepsilon}) \xrightarrow{i_*^\lambda} H_*(M, f^{c-\varepsilon})\,,$$

which equals $i_*^{c+\varepsilon}$, vanishes. □

We next prove that the lower critical values share the same property as the upper critical values, concerning the rank of $i_*^\lambda : H_*(f^\lambda) \to H_*(M)$.

Proposition 2.6. *When* c *is a lower critical value, the ranges of* $i_*^{c+\varepsilon} : H_*(f^{c+\varepsilon}) \to H_*(M)$ *and* $i_*^{c-\varepsilon} : H_*(f^{c-\varepsilon}) \to H_*(M)$ *are the same.*

Proof. Assume that c is a critical value with index p. Then for any $k \neq p$ the map $i_*^{c+\varepsilon, c-\varepsilon} : H_k(f^{c-\varepsilon}) \to H_k(f^{c+\varepsilon})$ is onto and the range of $i_*^{c-\varepsilon} = i_*^{c+\varepsilon} \circ i_*^{c+\varepsilon, c-\varepsilon}$ and $i_*^{c+\varepsilon}$, when restricted to H_k, are equal. For the case $k = p$, we start from the exact sequence

$$0 \longrightarrow H_p(f^{c-\varepsilon}) \xrightarrow{i_*^{c+\varepsilon, c-\varepsilon}} H_p(f^{c+\varepsilon}) \longrightarrow H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \stackrel{\partial}{\longrightarrow} H_{p-1}(f^{c-\varepsilon}) \longrightarrow 0\,,$$

where the ∂-arrow vanishes, because c cannot be an upper critical value. Hence the range of $i_*^{c+\varepsilon,c-\varepsilon}$ is an hyperplane of $H_p(f^{c+\varepsilon})$ and the equality $i_*^{c-\varepsilon} = i_*^{c+\varepsilon} \circ i_*^{c+\varepsilon,c-\varepsilon}$ implies that $i_*^{c+\varepsilon}$ and $i_*^{c-\varepsilon}$ have the same range iff

$$1 + \dim\ \ker(i_*^{c+\varepsilon}\big|_{\operatorname{Im} i_*^{c+\varepsilon,c-\varepsilon}}) - \dim\ \ker(i_*^{c+\varepsilon}) = 0.$$

We write for simplicity

$$i = i^{c+\varepsilon,c-\varepsilon}.$$

We have to prove that there exists $\alpha \in H_p(f^{c+\varepsilon})$ such that

$$i_*^{c+\varepsilon}(\alpha) = 0 \quad \text{and} \quad \alpha \notin \operatorname{Im} i_*.$$

The functoriality of the relative homology gives

$$\begin{array}{ccccccc} H_*(f^{c-\varepsilon}) & \xrightarrow{i_*^{c-\varepsilon}} & H_*(M) & \xrightarrow{j_*^{c-\varepsilon}} & H_*(M, f^{c-\varepsilon}) & \xrightarrow{\partial^{c-\varepsilon}} & H_{*-1}(f^{c-\varepsilon}) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \overline{i_*}} & & \downarrow{\scriptstyle i_*} \\ H_*(f^{c+\varepsilon}) & \xrightarrow{i_*^{c+\varepsilon}} & H_*(M) & \xrightarrow{j_*^{c+\varepsilon}} & H_*(M, f^{c+\varepsilon}) & \xrightarrow{\partial^{c+\varepsilon}} & H_{*-1}(f^{c+\varepsilon}) \end{array} \tag{15}$$

The condition (2) of Proposition 2.5 and the long exact sequence of relative homologies for the triple $(M, f^{c+\varepsilon}, f^{c-\varepsilon})$, provide the exact sequence

$$0 \longrightarrow H_{p+1}(M, f^{c-\varepsilon}) \xrightarrow{\overline{i_*}} H_{p+1}(M, f^{c+\varepsilon}) \xrightarrow{\partial} H_p(f^{c+\varepsilon}, f^{c-\varepsilon}) \to 0\,.$$

Let $\alpha_0 \in H_{p+1}(M, f^{c+\varepsilon})$ be such that $\partial\alpha_0 \neq 0$. By the second line of the above commutative diagram (15), $\partial^{c+\varepsilon}\alpha_0 \in H_p(f^{c+\varepsilon})$ belongs to $\ker i_*^{c+\varepsilon}$.

Assume $\partial^{c+\varepsilon}\alpha_0 \in \operatorname{Im} i_*$ and take $\beta \in H_p(f^{c-\varepsilon})$ such that

$$\partial^{c+\varepsilon}\alpha_0 = i_*\beta.$$

The commutative diagram (15) then implies

$$i_*^{c-\varepsilon}\beta = (\mathrm{Id} \circ i_*^{c-\varepsilon})\beta = (i_*^{c+\varepsilon} \circ i_*)\beta = i_*^{c+\varepsilon}(\partial^{c+\varepsilon}\alpha) = 0.$$

Hence $\beta \in \ker i_*^{c-\varepsilon} = \operatorname{Im}\partial^{c-\varepsilon}$. Hence there exists $\gamma \in H_{p+1}(M, f^{c-\varepsilon})$ such that

$$\partial^{c+\varepsilon}\alpha_0 = (i_* \circ \partial^{c-\varepsilon})\gamma = \partial^{c+\varepsilon}(\overline{i_*}\gamma).$$

The cycle $\alpha_0 - \overline{i_*}\gamma$ belongs to $\ker\partial^{c+\varepsilon} = \operatorname{Im} j_*^{c+\varepsilon}$. Hence there exists $\delta \in H_{p+1}(M)$ such that

$$\alpha_0 - \overline{i_*}\gamma = (j_*^{c+\varepsilon} \circ \mathrm{Id})\delta = (\overline{i_*} \circ j_*^{c-\varepsilon})\delta.$$

We finally get $\alpha_0 = \overline{i_*}(\gamma + j_*^{c-\varepsilon}\delta) \in \operatorname{Im}\overline{i_*} = \ker\partial$. But this contradicts the first assumption $\partial\alpha_0 \neq 0$. Thus we have found $\alpha = \partial^{c+\varepsilon}\alpha_0 \in H_p(f^{c+\varepsilon})$ such that

$$i^{c+\varepsilon}(\alpha) = (i^{c+\varepsilon} \circ \partial^{c+\varepsilon})\alpha_0 = 0 \quad \text{and} \quad \alpha \notin \operatorname{Im} i_*.$$

This ends the proof. ☐

2.2.4. Properties of Homological Critical Values. The attribute "homological" is justified by the following result. Let $C_H^{(p)}$ be the set of homological critical values with index p and set $C_H = \cup_{p=0}^{d} C_H^{(p)}$. Remember the mappings $i_*^\lambda : H_*(f^\lambda) \to H_*(M)$.

Theorem 2.7. *For every $p \in \{0, \ldots, d\}$, there is a one to one mapping $\alpha^{(p)} : C_H^{(p)} \to H_p(M)$ such that:*

- *The range of $\alpha^{(p)}$ is a basis of $H_p(M)$;*
- *For every $c \in C_H^{(p)}$, the quotient $Im\, i_*^{c+\varepsilon}/Im\, i_*^{c-\varepsilon}$ is the one-dimensional space spanned by the class of $\alpha^{(p)}(c)$.*

Finally the cardinal of $C_H^{(p)}$ is the p^{th} Betti number of M.

We need the following result, where homological critical values differ from the lower and upper critical values.

Proposition 2.8. *Assume that c is a homological critical value (resp. point) according to the Definition 2.1. Then the mapping*

$$H_*(f^{c+\varepsilon}) \to H_*(M, f^{c-\varepsilon})$$

is non-zero.

Moreover the mappings $i_*^{c\pm\varepsilon} : H_*(f^{c\pm\varepsilon}) \to H_*(M)$ satisfy

$$Im\, i_*^{c-\varepsilon} \subset Im\, i_*^{c+\varepsilon}, \quad rank\, i_*^{c-\varepsilon} = rank\, i_*^{c+\varepsilon} - 1.$$

Proof. By definition, homological critical value is neither a lower critical value nor an upper critical value. Therefore the mappings

$$H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H(M, f^{c-\varepsilon}) \quad \text{and} \quad H_*(f^{c+\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$$

are non-zero. Since $H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ is one-dimensional, the second one is onto and the composed map

$$H_*(f^{c+\varepsilon}) \xrightarrow{\sigma} H_*(M, f^{c-\varepsilon})$$

is non-zero. Consider now the second statement. We have already checked in (13)(14) the relations

$$i_*^{c-\varepsilon} = i_*^{c+\varepsilon} \circ i_*^{c+\varepsilon, c-\varepsilon} \quad \text{and} \quad \mathrm{Im}\, i_*^{c-\varepsilon} \subset \mathrm{Im}\, i_*^{c+\varepsilon}.$$

From the long exact sequence (when c is a critical value with index p)

$$0 \longrightarrow H_p(f^{c-\varepsilon}) \xrightarrow{i_*^{c+\varepsilon,c-\varepsilon}} H_p(f^{c+\varepsilon}) \longrightarrow H_{p-1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \longrightarrow 0,$$

we know that the codimension of $\mathrm{Im}\, i_*^{c+\varepsilon,c-\varepsilon}$ is at most one. Thus

$$\mathrm{rank}\, i_*^{c+\varepsilon} - 1 \leq \mathrm{rank}\, i_*^{c-\varepsilon} \leq \mathrm{rank}\, i_*^{c+\varepsilon},$$

and it suffices to find $\alpha \in \mathrm{Im}\, i_*^{c+\varepsilon}$ which does not belong to $\mathrm{Im}\, i_*^{c-\varepsilon}$. We use the diagram

$$\begin{array}{ccccc} & & H_*(f^{c+\varepsilon}) & & \\ & & \downarrow{\scriptstyle i_*^{c+\varepsilon}} & \searrow^{\sigma} & \\ H_*(f^{c-\varepsilon}) & \xrightarrow{i_*^{c-\varepsilon}} & H_*(M) & \xrightarrow{j_*^{c-\varepsilon}} & H_*(M, f^{c-\varepsilon}). \end{array}$$

We know that there exists $\alpha_0 \in H_*(f^{c+\varepsilon})$ such that $(j_*^{c-\varepsilon} \circ i_*^{c+\varepsilon})(\alpha_0) = \sigma(\alpha_0) \neq 0$. Take $\alpha = i_*^{c+\varepsilon}(\alpha_0)$. It belongs to $\mathrm{Im}\, i_*^{c+\varepsilon}$ and not in $\ker j_*^{c-\varepsilon} = \mathrm{Im}\, i_*^{c-\varepsilon}$. □

Proof of Theorem 2.7. Fix the degree p, $0 \leq p \leq d$, and consider $i_*^\lambda : H_p(f^\lambda) \to H_p(M)$. Start from $\lambda = \max f + \varepsilon$ for which $\mathrm{Im}\, i_*^\lambda = \mathrm{Im}\,\mathrm{Id} = H_p(M)$ and decrease λ down to $\min f - \varepsilon$ for which $\mathrm{Im}\, i_*^\lambda = \{0\}$. According to Morse theory, Proposition 2.4 and Proposition 2.6, the range of i_*^λ does not change except when λ passes a homological critical value with index p. For such a critical value, Proposition 2.8 says that the rank of i_*^λ is exactly decreased by 1. This yields the result. □

2.3. The Morse–Barannikov Chain Complex

2.3.1. Definition. Remember that $\mathcal{C}(f) = \oplus_{p=0}^d \mathcal{C}^{(p)}(f)$ is the vector space spanned by the critical points (identified with the critical values and the same notation c will be used for the two objects). The following definition will be proved to define a chain complex structure on $\mathcal{C}(f)$ of which the homology groups are isomorphic to the $H_p(M)$.

Definition 2.9. On $\mathcal{C}(f)$ consider the linear mapping ∂_B defined by:

- When c is a lower or a homological critical point, $\partial_{\mathrm{B}} c = 0$.
- When c is an upper critical point, take for c', according to the condition (3) of Proposition 2.3, the supremum of the λ's in $[-\infty, c)$ such that the mapping $H_*(f^{c+\varepsilon}, f^\lambda) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ vanishes and set

$$\partial_{\mathrm{B}} c = c'.$$

Theorem 2.10. *The mapping $\partial_{\mathrm{B}} : \mathcal{C}(f) \to \mathcal{C}(f)$ sends $\mathcal{C}^{(p)}(f)$ into $\mathcal{C}^{(p-1)}(f)$ and satisfies $\partial_{\mathrm{B}} \circ \partial_{\mathrm{B}} = 0$. Moreover the homology groups $H_*(\mathcal{C}(f))$ are isomorphic to $H_*(M)$ and a basis of $H_*(M)$ is indexed by the set $C_{\mathrm{H}}(f)$ of homological critical points (e.g. in Fig. 1).*

Proof. It suffices to prove that when c is an upper critical point with index p, the point c' is a lower critical point with index $p - 1$.

Assume that the mapping $H_*(f^{c+\varepsilon}, f^{c'-\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ vanishes while the mapping $\sigma : H_*(f^{c+\varepsilon}, f^{c'+\varepsilon}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ is non-zero. Consider the commutative diagram

$$\begin{array}{ccccc} H_*(f^{c+\varepsilon}, f^{c'-\varepsilon}) & \xrightarrow{\varphi^+} & H_*(f^{c+\varepsilon}, f^{c'+\varepsilon}) & \xrightarrow{\partial^+} & H_{*-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}) \\ \downarrow{\scriptstyle 0} & \swarrow{\scriptstyle \sigma} & & & \\ H_*(f^{c+\varepsilon}, f^{c-\varepsilon}) & & & & \end{array}$$

where the first line is the long exact sequence for the triple $f^{c'-\varepsilon} \subset f^{c'+\varepsilon} \subset f^{c+\varepsilon}$. Since σ is non-zero while $\sigma \circ \varphi^+ = 0$, φ^+ cannot be onto and ∂^+ is non-zero. We have found $\lambda = c + \varepsilon$ such that the mapping $\partial : H_*(f^{\lambda}, f^{c'+\varepsilon}) \to H_{*-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ is onto. By the characterization (4) of Proposition 2.5, c' is a lower critical point. Clearly it has the index $p-1$ when c has the index p.

Therefore $\partial_{\mathrm{B}} \circ \partial_{\mathrm{B}} = 0$.

Before we conclude, we check that if $\partial(c) = c'$, then c is the infimum of the λ's such that $\partial : H_*(f^{\lambda}, f^{c'+\varepsilon}) \to H_*(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ is onto.

We have to prove that the map $\partial^- : H_*(f^{c-\varepsilon}, f^{c'+\varepsilon}) \to H_*(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ vanishes. Consider the diagram

$$\begin{array}{ccccc} H_*(f^{c-\varepsilon}, f^{c'-\varepsilon}) & \xrightarrow{\varphi^-} & H_*(f^{c-\varepsilon}, f^{c'+\varepsilon}) & \xrightarrow{\partial^-} & H_{*-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}) \\ \downarrow{\scriptstyle i_*^{c+\varepsilon,c-\varepsilon}} & & \downarrow{\scriptstyle i_*^{c+\varepsilon,c-\varepsilon}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ H_*(f^{c+\varepsilon}, f^{c'-\varepsilon}) & \xrightarrow{\varphi^+} & H_*(f^{c+\varepsilon}, f^{c'+\varepsilon}) & \xrightarrow{\partial^+} & H_{*-1}(f^{c'+\varepsilon}, f^{c'-\varepsilon}) \\ \downarrow{\scriptstyle 0} & \swarrow{\scriptstyle \sigma} & & & \\ H_*(f^{c+\varepsilon}, f^{c-\varepsilon}). & & & & \end{array}$$

The maps σ and ∂^+ have one-dimensional ranges and their kernels have the codimension 1. Due to $\sigma \circ \varphi^+ = 0$, we know $\ker \partial^+ = \mathrm{Im}\, \varphi^+ \subset \ker \sigma$. With the same dimension, this yields $\ker \partial^+ = \ker \sigma$. If ∂^- does not vanish, there exists u such that $\partial^+(i_+^{c+\varepsilon,c-\varepsilon} u) = \partial^- u \neq 0$. Hence we get $(\sigma \circ i_*^{c+\varepsilon,c-\varepsilon})u \neq 0$, which contradicts the fact that $\sigma \circ i_*^{c+\varepsilon,c-\varepsilon} = 0$ as a part of the long exact sequence for the triple $f^{c'+\varepsilon} \subset f^{c-\varepsilon} \subset f^{c+\varepsilon}$.

Hence c is the infimum of the λ's such that $\partial : H_*(f^{\lambda}, f^{c'+\varepsilon}) \to H_*(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ is onto.

Now assume that c' is a lower critical point. By the characterization 4) of Proposition 2.5, the infimum of the λ's in $(c, +\infty]$, such that $\partial : H_*(f^{\lambda}, f^{c'+\varepsilon}) \to H_*(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ is onto, exists. Call it c. By the dual argument of the previous one, c is an upper critical point and c' is the supremum of the λ's such that the mapping $H_*(f^{c+\varepsilon}, f^{\lambda}) \to H_*(f^{c+\varepsilon}, f^{c-\varepsilon})$ vanishes. Hence c is an upper critical point such that $\partial_{\mathrm{B}}(c) = c'$.

We have finally proved that the range of $\partial_{\mathrm{B}} : \mathcal{C}(f) \to \mathcal{C}(f)$ contains all the lower critical points. The other statements are now straightforward consequences of Theorem 2.7. □

Remark 2.11. This result provides another proof of Morse inequalities, for excellent Morse functions, without making use of homotopy arguments to reduce the problem to self-indexed Morse functions (see [8,32,38]).

Proposition 2.12. *When c is an upper critical point with index $p+1$ such that $\partial_{\mathrm{B}} c = c'$, then the following commutative diagram holds:*

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon}) & & 0 & & \\
 & & & & \downarrow{\scriptstyle i_*^{c-\varepsilon,c'+\varepsilon}} & & \downarrow & & \\
0 & \longrightarrow & H_{p+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) & \xrightarrow{\partial} & H_p(f^{c-\varepsilon}, f^{c'-\varepsilon}) & \xrightarrow{i_*^{c+\varepsilon,c-\varepsilon}} & H_p(f^{c+\varepsilon}, f^{c'-\varepsilon}) & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle j^*} & & \uparrow\downarrow{\scriptstyle j*} & & \\
 & & 0 & \longrightarrow & H_p(f^{c-\varepsilon}, f^{c'+\varepsilon}) & \overset{i_*^{c+\varepsilon,c-\varepsilon}}{\rightleftarrows} & H_p(f^{c+\varepsilon}, f^{c'+\varepsilon}) & \longrightarrow & 0. \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

In particular, if $H_{p+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) = \mathbb{R}[e^{p+1}]$ and $H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon}) = \mathbb{R}[e^p]$, then there exists $\kappa \in \mathbb{R}^*$ such that ∂e^{p+1} and κe^p are homologous in $f^{c-\varepsilon}$ relatively to $f^{c'-\varepsilon}$: $[\partial e^{p+1}] = k[e^p]$ in $H_p(f^{c-\varepsilon}, f^{c'-\varepsilon})$ (e.g. in Fig. 2).

Proof. From the definition of $\partial_{\mathrm{B}}(c) = c'$, we deduce that the mapping $\partial^- : H_{p+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_p(f^{c-\varepsilon}, f^{c'-\varepsilon})$ is one to one while the mapping $\partial^+ : H_{p+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_p(f^{c-\varepsilon}, f^{c'+\varepsilon})$ vanishes. Put in the long exact sequences associated with the two triples $(f^{c'-\varepsilon}, f^{c-\varepsilon}, f^{c+\varepsilon})$ and $(f^{c'+\varepsilon}, f^{c-\varepsilon}, f^{c+\varepsilon})$, this provides the two lines of the diagram.

According to the end of the proof of Theorem 2.10, the relation $\partial_B(c) = c'$ implies as well that the mapping $\partial_- : H_{p+1}(f^{c+\varepsilon}, f^{c'+\varepsilon}) \to H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ is onto (or equivalently the mapping $H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon}) \to H_p(f^{c+\varepsilon}, f^{c'-\varepsilon})$ vanishes) and the mapping $\partial_- : H_{p+1}(f^{c-\varepsilon}, f^{c'+\varepsilon}) \to H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon})$ vanishes. With the long exact sequences associated with the triples $(f^{c'-\varepsilon}, f^{c'+\varepsilon}, f^{c+\varepsilon})$ and $(f^{c'-\varepsilon}, f^{c'+\varepsilon}, f^{c-\varepsilon})$, this provides the two columns of the diagram.

The diagram implies that the two mappings $i_*^{c-\varepsilon,c'+\varepsilon} : H_p(f^{c'+\varepsilon}, f^{c'-\varepsilon}) \to H_p(f^{c-\varepsilon}, f^{c'-\varepsilon})$ and $\partial : H_{p+1}(f^{c+\varepsilon}, f^{c-\varepsilon}) \to H_p(f^{c-\varepsilon}, f^{c'-\varepsilon})$ have the same one-dimensional range. This ends the proof. □

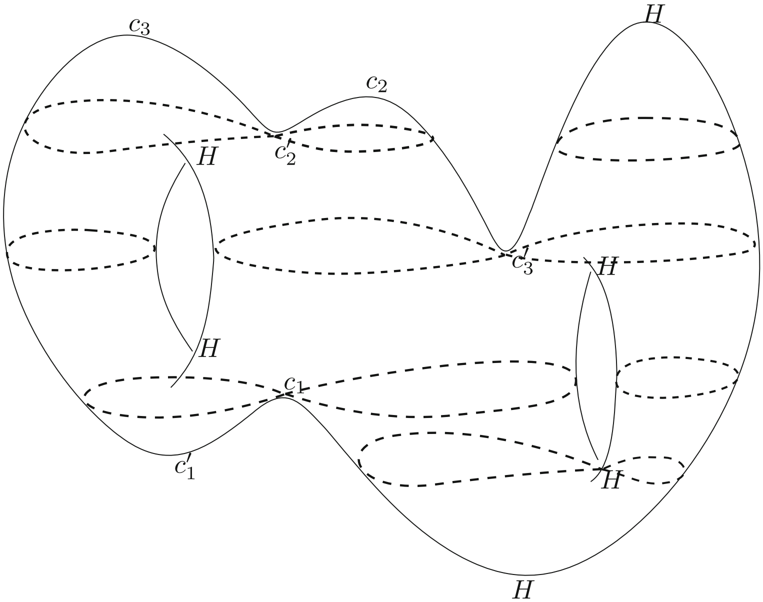

Figure 1. Example with a compact surface with genus 2 where f is the height function (*dotted lines* show some level curves). The homological critical points are labelled by H while the pairing of other critical points follows $\partial_B c_k = c'_k$

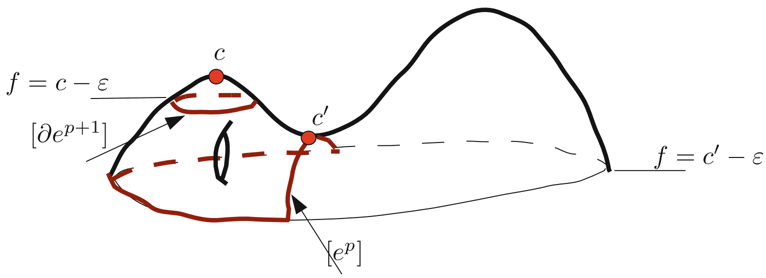

Figure 2. One example on a surface for which $[\partial e^{p+1}] = 1[e^p]$, $p = 1$

2.3.2. Restriction. In the previous construction the manifold M equals $f^{+\infty}$ while the homology group $H_*(f^\lambda)$ equals $H_*(f^\lambda, \emptyset) = H_*(f^\lambda, f^{-\infty})$. All the construction can be done with sublevel sets f^a and f^b with $-\infty \leq a < b \leq +\infty$ which are not critical values. For $\lambda \in [-\infty, +\infty]$ which is not a critical value, consider $\mathcal{C}(f, \lambda)$, the chain subcomplex of $\mathcal{C}(f)$ generated by critical values (points) below level λ. Since ∂_B preserves $\mathcal{C}(f, \lambda)$, we can introduce the quotient $\mathcal{C}(f, \lambda, \mu) = \mathcal{C}(f, \lambda)/\mathcal{C}(f, \mu)$ when $\mu < \lambda$. And there are relative homology groups $H_*(\mathcal{C}(f, \lambda), \mathcal{C}(f, \mu))$ for ∂_{B}, which will be denoted by $H_*(\mathcal{C}(f, \lambda), \mathcal{C}(f, \mu))$.

All the previous definitions and proofs can be translated to the restricted and relative homologies, after replacing $H_*(M, f^\lambda)$ by $H_*(f^b, f^\lambda)$ and $H_*(f^\lambda)$ by $H_*(f^\lambda, f^b)$, when $a < \lambda < b$ are not critical values. This observation gives at once:

Theorem 2.13. *For any a, b, $-\infty \leq a < b \leq +\infty$, which are not critical values, the relative homology groups $H_*(\mathcal{C}(f, b), \mathcal{C}(f, a))$ are isomorphic to $H_*(f^b, f^a)$ and the following diagram*

$$\begin{array}{ccccccc}
H_*(f^a) & \xrightarrow{i_*^{b,a}} & H_*(f^b) & \xrightarrow{j_*} & H_*(f^b, f^a) & \xrightarrow{\partial} & H_{*-1}(f^a) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
H_*(\mathcal{C}(f, a)) & \longrightarrow & H_*(\mathcal{C}(f, b)) & \longrightarrow & H_*(\mathcal{C}(f, b), \mathcal{C}(f, a)) & \xrightarrow{\partial_B} & H_{*-1}(\mathcal{C}(f, a))
\end{array}$$

is commutative.

Since we have a good basis of the chain complex $(\mathcal{C}(f), \partial_B)$, where the image by ∂_B of a generator is either 0 or another generator, we have a nice identification $H_*(\mathcal{C}(f, b), \mathcal{C}(f, a))$.

Proposition 2.14. *The relative homology group $H_*(f^b, f^a)$ has a basis made of critical values (resp. points) $c \in (a, b)$ satisfying one of the following conditions:*

1. *c is a homological critical value (resp. point) in M;*
2. *or c is an upper critical value (resp. point) such that $\partial_B c = c'$ is below a.*
3. *or c is a lower critical value (resp. point) in M, that is $\partial_B c' = c$ in $\mathcal{C}(f)$, but c' is above b;*

Remark 2.15. What the theorem says is that the homological critical points for $\mathcal{C}(f, b, a)$, that should be denoted $\mathcal{C}_{\mathrm{H}}(f, b, a)$ are **not** the points of $\mathcal{C}_{\mathrm{H}}(f)$ with critical value in $[a, b]$, but the union of those, together with the upper critical values in $[a, b]$ such that $\partial_{\mathrm{B}} c$ is below a, and the lower critical values c such that $\partial_{\mathrm{B}} c' = c$ with c' above b.

3. Relative Witten Chain Complex

The Witten Laplacian is a deformation of the Hodge Laplacian, related with de Rham cohomology, which allows to give within a semiclassical asymptotic framework, an analytic proof of the Morse inequalities (see [12,24,45]). The accurate computations of its exponentially small eigenvalues has connections with various topics such as stochastic analysis ([6,7,14]), kinetic theory ([21,25–28]), the computation of geometric invariants ([3,4]), or differential topology ([8,31,38]). The case of manifold with boundaries has been considered in [11,20,30,34,35] with a spectral approach and more recently in [32] with a pure topological point of view partly inspired by those previous works. We shall consider here directly the case with boundary, which is of interest here, and recall a few basic facts. We want to specify the realization of the Witten Laplacian on the manifold $\overline{f_a^b}$ with boundaries $\{f = a\}$ and $\{f = b\}$, when $-\infty \leq a < b \leq +\infty$ are not critical values, which is associated with the relative homology (after de Rham duality) $H_*(f^b, f^a)$.

3.1. Functional Analysis

We recall that $\bigwedge T_x^*M = \oplus_{p=0}^d \bigwedge^p T_x^*M$ is the exterior algebra on the cotangent fiber T_x^*M, $\bigwedge T^*M$ is the corresponding fiber bundle and $\mathcal{F}(M;\bigwedge T^*M)$ denotes the space of sections of class $\mathcal{F}$ on M ($\mathcal{F}$ stands for $\mathcal{C}^\infty$, L^p or $W^{m,r}$). The notation $\mathcal{F}(\overline{f}_a^b;\bigwedge T^*M)$ is the set of restrictions to $\overline{f_a^b}$ of elements in $\mathcal{F}(M;\bigwedge T^*M)$. The spaces $\bigwedge T_x^*M$ and $L^2(M;\bigwedge T^*M)$ are endowed with their natural scalar products inherited from the Riemannian metric g. A shorter notation for the Sobolev spaces will be

$$\Lambda W^{m,r} = W^{m,r}(M;\bigwedge T^*M), \quad \Lambda W^{m,r}(\overline{f_a^b}) = W^{m,r}(\overline{f_a^b};\bigwedge T^*M).$$

After the introduction of the Hodge-$\star$ operator, the scalar product of two p-forms equals

$$\langle \omega_1 \,|\, \omega_2 \rangle_{\Lambda^p L^2} = \int_\Omega \overline{\omega_1} \wedge \star\omega_2 .$$

Final calculations will be done with real valued p-forms and left or right antilinearity does not matter.

The notation $\mathbf{t}$ and $\mathbf{n}$ are specific to the case with boundary and useful for the analysis of boundary Hodge and Witten Laplacians (see [20,35,40]). Here is their specific meaning: On the boundary $\partial\Omega$ of a regular domain Ω, decompose the tangent vectors $X_i \in T_\sigma\Omega$, $\sigma \in \partial\Omega$, as $X_i = X_i^T + x_i^\perp n_\sigma$, where n_σ is the normalized outgoing normal vector to $\partial\Omega$ at σ, and set, for $\omega \in \mathcal{C}^\infty(\Omega;\bigwedge^p T^*\Omega)$,

$$\forall \sigma \in \partial\Omega, (\mathbf{t}\omega)_\sigma(X_1,\ldots,X_p) = \omega_\sigma(X_1^T,\ldots,X_p^T), \quad \mathbf{t}\omega \in \mathcal{C}^\infty(\partial\Omega;\bigwedge^p T^*\Omega),$$

$$\mathbf{n}\omega = \omega\Big|_{\partial\Omega} - \mathbf{t}\omega \quad \in \mathcal{C}^\infty(\partial\Omega;\bigwedge^p T^*\Omega).$$

Note that the $\mathbf{t}\omega$ and $\mathbf{n}\omega$ have a natural extension to a neighborhood of $\partial\Omega$ when the metric is fixed. After the right choice of coordinates, with x_d parametrizing normal curves to $\partial\Omega$, $\mathbf{t}\omega$ is the part with no dx_d while $\mathbf{n}\omega$ takes the form $dx_d \wedge \omega'$. On $\mathcal{C}^\infty$ differential forms, they satisfy various relations with the Hodge-$\star$ operator, the differential d and the codifferential d^* (see [20,35,40] for details),

$$\star d^{*,(p-1)} = (-1)^p d^{(d-p)}\star, \quad \star d^{(p)} = (-1)^{p+1} d^{*,(d-p-1)}\star, \tag{16}$$

$$\star\, \mathbf{n} = \mathbf{t}\,\star, \quad \star\,\mathbf{t} = \mathbf{n}\,\star, \tag{17}$$

$$\mathbf{t}\, d = d\,\mathbf{t}, \quad \mathbf{n}\, d^* = d^*\,\mathbf{n}, \tag{18}$$

and the Stokes' formula,

$$\forall \omega \in \mathcal{C}^\infty(\overline{\Omega};\bigwedge^{d-1} T^*\Omega), \quad \int_\Omega d\omega = \int_{\partial\Omega} \mathbf{t}\omega\ .$$

When there is no boundary, the Hodge-de Rham theory makes the relation between the spectral theory of the Hodge Laplacian and de Rham duality of homology and cohomology groups (see [15]). For boundary manifold relative and absolute (co-)homology groups can be considered. By excision, remember that $H_*(f^b, f^a) = H_*(\overline{f_a^b}, \{f = a\})$. We briefly recall why these relative homology groups are naturally associated with specific boundary conditions for the Hodge and Witten Laplacians. We refer the reader to [16] for a review and [43] for a complete but different presentation relying on the isometric doubling of the boundary manifold. When γ is a cycle in $\overline{f_a^b}$ relative to $\{f = a\}$, there is a natural (i.e. independent of the class representative of γ lying in $\{a \leq f\}$) integration $\int_\gamma \omega$ of forms $\omega \in \mathcal{C}^\infty(\overline{f_a^b}; \bigwedge T^*M)$ such that $\mathbf{t}\omega\big|_{\{f=a\}} = 0$. The dual condition along $\{f = b\}$, $\mathbf{n}\omega|_{\{f=b\}} = 0$, simply means $\mathbf{t}\omega|_{\{f=b\}} = \omega$ and ensures that such forms are determined by integration along chains lying in $\{f \leq b\}$. The Stokes' formula, when γ is a p-chain in $\overline{f_a^b}$, becomes

$$\forall \omega \in \mathcal{C}^\infty(\overline{f_a^b}; \bigwedge^{p-1} T^*\Omega) \text{ s.t. } \mathbf{n}\omega|_{\{f=b\}} = \mathbf{t}\omega|_{\{f=a\}} = 0, \quad \int_\gamma d\omega = \int_{\overline{\partial\gamma}} \omega,$$

where $\overline{\partial\gamma}$ is the cycle $\partial\gamma$ relative to $\{f = a\}$ (or $\{f \leq a\}$). In particular when $p = d$, it reads $\int_{f_a^b} d\omega = \int_{\{f=b\}} \omega$.

The class $\mathcal{F} = \mathcal{C}^\infty_{TN}$, $\mathcal{C}^\infty_T$ or $\mathcal{C}^\infty_N$ will denote the class of $\mathcal{C}^\infty$ forms fulfilling, respectively, both conditions, the first one or the second one, among

$$\mathbf{t}\omega|_{\{f=a\}} = 0, \quad \mathbf{n}\omega|_{\{f=b\}} = 0. \tag{19}$$

Note that the differential d preserves the class $\mathcal{C}^\infty_T$ while the codifferential preserves $\mathcal{C}^\infty_N$. Better commutations relations appear after considering the Hodge Laplacian (see further). The de Rham cohomology group $H^p(f^b, f^a)$ is then given by $\ker d^{(p)}/(\operatorname{Im} d^{p-1} \cap \mathcal{C}^\infty_{TN}(\overline{f_a^b}, \bigwedge^p T^*M))$, when d is defined on $\mathcal{C}^\infty_{TN}(\overline{f_a^b}, \bigwedge T^*M)$.

The Witten deformation consists in introducing a small parameter $h \to 0$ and to set

$$d_{f,h} = e^{-\frac{f}{h}}(hd)e^{\frac{f}{h}} = hd + df\wedge, \quad d^*_{f,h} = e^{\frac{f}{h}}(hd^*)e^{\frac{f}{h}} = hd^* + \mathbf{i}_{\nabla f}.$$

The Witten Laplacian is defined as a differential operator in $f_a^b = \{a < f < b\}$ by

$$\begin{aligned}\Delta_{f,h} = (d_{f,h} + d^*_{f,h})^2 &= d_{f,h}d^*_{f,h} + d^*_{f,h}d_{f,h} \\ &= h^2(d + d^*)^2 + |\nabla f|^2 + h(\mathcal{L}_{\nabla f} + \mathcal{L}^*_{\nabla f}).\end{aligned}$$

On the boundaries $\{f = a\}$ and $\{f = b\}$, the boundary conditions have to be completed with f-dependent additional boundary conditions in order to get a self-adjoint realization which is elliptic up to the boundary (see [40]). An additional property required here, is the commutation of the resolvent with $d_{f,h}$ and $d^*_{f,h}$. We follow the scheme of [11,20,35] where the "Dirichlet problem" and the "Neumann problem" have been considered separately. Here the "Dirichlet" boundary conditions occurs on $\{f = a\}$ while the "Neumann" boundary condition appears on $\{f = b\}$. Consider in $\Lambda W^{1,2}_{TN} = \overline{\mathcal{C}^\infty_{TN}(\overline{f_a^b}; \bigwedge T^*M)}^{W^{1,2}}$ the quadratic form given by

$$\mathcal{D}_{TN}(\omega,\eta) = \langle d_{f,h}\omega \,|\, d_{f,h}\eta\rangle + \langle d^*_{f,h}\omega \,|\, d^*_{f,h}\eta\rangle.$$
$$\mathcal{D}_{TN}(\omega) = \mathcal{D}_{TN}(\omega,\omega) = \|d_{f,h}\omega\|^2_{L^2} + \|d^*_{f,h}\omega\|^2_{L^2}.$$

Since $\{f=a\}$ and $\{f=b\}$ are disjoint, the main arguments are local (Sobolev trace theorem, Lopatinski–Schapiro conditions for the ellipticity up to the boundary) and finally playing with (16)(17)(18), we can combine without repeating the proofs the results of [20] and [35] in order to state the following result.

Note that due to the boundaries of the domain Ω, we avoid to consider the closure of the differential operators $d_{f,h}$ and $d^*_{f,h}$ in ΛL^2 which are not very explicit.

Proposition 3.1. *The non-negative quadratic form $\omega \to \mathcal{D}_{TN}(\omega)$ is closed on $\Lambda W^{1,2}_{TN}$. The associated (self-adjoint) Friedrichs extension is denoted by $\Delta^{TN}_{f,h}$. Its domain is*

$$D(\Delta^{TN}_{f,h}) = \left\{ \omega \in \Lambda W^{2,2}(\overline{f^b_a}); \quad \begin{matrix} \mathbf{t}\omega\big|_{f=a} = 0, & \mathbf{t}d^*_{f,h}\omega\big|_{f=a} = 0, \\ \mathbf{n}\omega\big|_{f=b} = 0, & \mathbf{n}d_{f,h}\omega\big|_{f=b} = 0 \end{matrix} \right\},$$

and acts as

$$\forall \omega \in D(\Delta^{TN}_{f,h}), \quad \Delta^{TN}_{f,h}\omega = \Delta_{f,h}\omega.$$

The operator $\Delta^{TN}_{f,h}$ has a compact resolvent and a discrete spectrum. Moreover, the commutations

$$\begin{aligned} (z-\Delta^{TN}_{f,h})^{-1} \circ d_{f,h}\omega &= d_{f,h} \circ (z-\Delta_{f,h})^{-1}\omega, \\ (z-\Delta^{TN}_{f,h})^{-1} \circ d^*_{f,h}\omega &= d^*_{f,h} \circ (z-\Delta_{f,h})^{-1}\omega, \\ 1_E(\Delta^{TN}_{f,h}) \circ d_{f,h}\omega &= d_{f,h} \circ 1_E(\Delta^{TN}_{f,h})\omega, \\ \text{and} \quad 1_E(\Delta^{TN}_{f,h}) \circ d^*_{f,h}\omega &= d^*_{f,h} \circ 1_E(\Delta^{TN}_{f,h})\omega, \end{aligned}$$

hold for all $z \in \mathbb{C}\backslash\mathbb{R}$, all Borel set E in $\mathbb{R}$ and all $\omega \in \Lambda W^{1,2}_{TN}$.

In the above result and in the sequel, we use the notation $\varphi(A)$ for a Borel function φ on $\mathbb{R}$ and a self-adjoint operator $(A, D(A))$. In particular when $\varphi = 1_E$ is the characteristic function of the Borel set $E \subset \mathbb{R}$, $1_E(A)$ simply denotes the spectral projection associated with the Borel set E and $(A, D(A))$.

Remark 3.2.
- The introduction of $\Delta^{TN}_{f,h}$, as a Friedrichs extension of a non-negative closed quadratic form defined on $\Lambda W^{1,2}_{TN}$, ensures that it is a non-negative self-adjoint operator. Requiring $\Delta^{TN}_{f,h}u \in \Lambda L^2$ for $u \in D(\Delta^{TN}_{f,h})$ forces the additional boundary conditions, after integration by part.
- The commutation relations do not result simply of the commutation of the differential operators $\Delta_{f,h} \circ d_{f,h} = d_{f,h} \circ \Delta_{f,h}$ valid in the interior f^b_a. Indeed $\Delta^{TN}_{f,h}$ can be applied only to elements of $D(\Delta^{TN}_{f,h})$ fulfilling the boundary conditions while $d_{f,h}$ do not preserve these boundary conditions even for $\mathcal{C}^\infty$-forms up to the boundaries. For details and complete proofs, we refer again the reader to [20,35] and [11].

- We can also define analogously the self-adjoint operator $\Delta_{f,h}^{NT}$, with domain $D(\Delta_{f,h}^{NT})$, by switching the above conditions on $\mathbf{n}$ and $\mathbf{t}$.

Here is the main result of this section.

Theorem 3.3. *There are two operators $L_{\pm} \in \mathcal{L}(\Lambda L^2; \Lambda W_{TN}^{1,2})$, commuting for all Borel sets E in $\mathbb{R}$ with $1_E(\Delta_{f,h}^{TN})$, such that every $u \in \Lambda L^2$ admits the orthogonal decomposition*

$$u = 1_{\{0\}}(\Delta_{f,h}^{TN})u + d_{f,h}L_- u + d_{f,h}^* L_+ u. \tag{20}$$

When F_M denotes the finite-dimensional space $Im\, 1_{[0,M]}(\Delta_{f,h}^{TN})$ and $\beta_M = d_{f,h}|_{F_M}$, its adjoint is $\beta_M^* = d_{f,h}^*|_{F_M}$ and F_M admits the orthogonal decomposition

$$F_M = \ker \Delta_{f,h}^{TN} \overset{\perp}{\oplus} Im\, \beta_M \overset{\perp}{\oplus} Im\, \beta_M^*.$$

After setting $F_M^{(p)} = Im\, 1_{[0,M]}(\Delta_{f,h}^{TN,(p)})$, the two finite-dimensional chain complexes

$$0 \rightleftarrows F_M^{(0)} \dots F_M^{(p-1)} \underset{\beta_M^{(p-1)*}}{\overset{\beta_M^{(p-1)}}{\rightleftarrows}} F_M^{(p)} \underset{\beta_M^{(p)*}}{\overset{\beta_M^{(p)}}{\rightleftarrows}} F_M^{(p+1)} \dots F_M^{(d)} \rightleftarrows 0 \tag{21}$$

are dual to each other and $\ker \beta_M^{(p)} / Im\, \beta_M^{(p-1)}$ is diffeomorphic to $H^p(f^b, f^a)$.

For any $p \in \{0, \dots, d\}$, the spectrum $\sigma(\Delta_{f,h}^{TN,(p)}) \cap (0, M]$, of $\Delta_{f,h}^{TN,(p)}$ lying in $(0, M]$, is made of the singular values squared of $\beta_M^{(p)}|_{Im\, \beta_M^{(p),*}}$ and $\beta_M^{(p-1)} |_{Im\, \beta_M^{(p-1),*}}$. Those values are counted with multiplicities.

We shall need the two following lemmas.

Lemma 3.4. *When ω belongs to $D(\Delta_{f,h}^{TN})$, $d_{f,h}\omega$ and $d_{f,h}^*\omega$ belong to $\Lambda W_{TN}^{1,2}$.*

Proof. The differential operators $d_{f,h}$ and $d_{f,h}^*$ are continuous from $D(\Delta_{f,h}^{TN}) \subset \Lambda W^{2,2}$ into $\Lambda W^{1,2}$. By the elliptic regularity up to the boundary of $\Delta_{f,h}^{TN}$, the set of $\mathcal{C}^\infty(\overline{f_a^b}; \bigwedge T^*M) \cap D(\Delta_{f,h}^{TN}))$ is dense in $D(\Delta_{f,h}^{TN})$ because $1 + \Delta_{f,h}^{TN} : D(\Delta_{f,h}^{TN}) \to \Lambda L^2$ is an isomorphism. For $\omega \in \mathcal{C}^\infty(\overline{f_a^b}; \bigwedge T^*M) \cap D(\Delta_{f,h}^{TN}))$, we have

$$\mathbf{n}(d_{f,h}\omega)|_{f=b} = 0 \quad \text{and} \quad \mathbf{t}(d_{f,h}\omega)|_{f=a} = 0$$

because $\omega \in D(\Delta_{f,h}^{TN})$. Moreover $\omega \in D(\Delta_{f,h}^{TN})$ also says

$$\mathbf{t}\omega|_{f=a} = 0 \quad \text{and} \quad \mathbf{n}\omega|_{f=b} = 0.$$

But since $\mathbf{t}e^{\pm\frac{f}{h}}\omega = e^{\pm\frac{f}{h}}\mathbf{t}\omega$ and $\mathbf{n}e^{\pm\frac{f}{h}}\omega = e^{\pm\frac{f}{h}}\mathbf{n}\omega$, the commutations (18) imply

$$\mathbf{t}(d_{f,h}\omega)|_{f=a} = 0 \quad \text{and} \quad \mathbf{n}(d_{f,h}^*\omega)|_{f=b} = 0.$$

This ends the proof. □

Lemma 3.5. *The relation*

$$\langle d_{f,h}\theta_1, \theta_2\rangle_{\Lambda L^2} = \langle \theta_1, d^*_{f,h}\theta_2\rangle_{\Lambda L^2}$$

holds for all $\theta_1, \theta_2 \in \Lambda W^{1,2}_{TN}$.

Proof. Since both quantities are continuous and $\mathcal{C}^\infty_{TN}(\overline{f^b_a}; \bigwedge T^*M)$ is dense in $\Lambda W^{1,2}_{TN}$, it suffices to consider $\theta_1, \theta_2 \in \mathcal{C}^\infty_{TN}(\overline{f^b_a}; \bigwedge T^*M)$. After writing $\omega_1 = e^{\frac{f}{h}}\overline{\theta_1}$ and $\omega_2 = e^{-\frac{f}{h}}\theta_2$ in $\mathcal{C}^\infty_{TN}(\overline{f^b_a}; \bigwedge T^*M)$, our identity amounts to

$$\langle d\overline{\omega_1}, \omega_2\rangle_{\Lambda L^2} = \langle \overline{\omega_1}, d^*\omega_2\rangle_{\Lambda L^2}.$$

But the Stokes' formula with the relations between d, $\star$ and $\wedge$ gives

$$\int_{f=a} \mathbf{t}(\omega_1 \wedge \star\omega_2) + \int_{f=b} \mathbf{t}(\omega_1 \wedge \star\omega_2) = \langle d\overline{\omega_1}\,|\,\omega_2\rangle - \langle \overline{\omega_1}\,|\,d^*\omega_2\rangle.$$

With the help of (17), write

$$\begin{aligned}\mathbf{t}(\omega_1 \wedge \star\omega_2) &= \mathbf{t}\left[(\mathbf{t}\omega_1)\wedge\star(\mathbf{t}\omega_2) + (\mathbf{t}\omega_1)\wedge\star(\mathbf{n}\omega_2) + (\mathbf{n}\omega_1)\wedge\star(\mathbf{t}\omega_2)\right.\\ &\qquad \left.+ (\mathbf{n}\omega_1)\wedge\star(\mathbf{n}\omega_2)\right]\\ &= \mathbf{t}\left[(\mathbf{t}\omega_1)\wedge\mathbf{n}(\star\omega_2) + (\mathbf{t}\omega_1)\wedge\mathbf{t}(\star\omega_2) + (\mathbf{n}\omega_1)\wedge\mathbf{t}(\star\omega_2)\right.\\ &\qquad \left.+ (\mathbf{n}\omega_1)\wedge\mathbf{n}(\star\omega_2)\right]\end{aligned}$$

and notice that $\mathbf{t}((\mathbf{n}u_1)\wedge(\mathbf{t}u_2)) = \mathbf{t}((\mathbf{t}u_1)\wedge(\mathbf{n}u_2)) = 0$. This leads to

$$\mathbf{t}(\omega_1 \wedge \star\omega_2) = \mathbf{t}\left[(\mathbf{t}\omega_1)\wedge\star(\mathbf{n}\omega_2) + (\mathbf{n}\omega_1)\wedge\star(\mathbf{t}\omega_2)\right],$$

where both terms vanishes on $\{f = a\} \cup \{f = b\}$ when $\omega_1, \omega_2 \in \mathcal{C}^\infty_{TN}(\overline{f^b_a}; \bigwedge T^*M)$. □

Proof of Theorem 3.3. The operator $\Delta^{TN}_{f,h}$ is a self-adjoint operator with a compact resolvent. Therefore, it is invertible when restricted to $\ker(\Delta^{TN}_{f,h})^\perp = \operatorname{Im} 1_{(0,+\infty)}(\Delta^*_{f,h})$. Take

$$L_+ = d_{f,h}(\Delta^{TN}_{f,h})^{-1}1_{(0,+\infty)}(\Delta^{TN}_{f,h}) \quad \text{and} \quad L_- = d^*_{f,h}(\Delta^{TN}_{f,h})^{-1}1_{(0,+\infty)}(\Delta^{TN}_{f,h}),$$

Since $(\Delta^{TN}_{f,h})^{-1}1_{(0,+\infty)}(\Delta^{TN}_{f,h}) \in \mathcal{L}(\Lambda L^2, D(\Delta^{TN}_{f,h}))$, $L_\pm \in \mathcal{L}(\Lambda L^2, \Lambda W^{1,2}_{TN}) \subset \mathcal{L}(\Lambda L^2, \Lambda L^2)$ according to Lemma 3.4. The commutation of $L_\pm$ with $1_E(\Delta_{f,h})$ is then a consequence of the commutation stated in Proposition 3.1. These commutations of Proposition 3.1 also imply

$$\begin{aligned} L_+\omega &= (\Delta^{TN}_{f,h})^{-1}1_{(0,+\infty)}(\Delta^{TN}_{f,h})d_{f,h}\omega\\ \text{and}\quad L_-\omega &= (\Delta^{TN}_{f,h})^{-1}1_{(0,+\infty)}(\Delta^{TN}_{f,h})d^*_{f,h}\omega,\end{aligned}$$

when $\omega \in \Lambda W^{1,2}_{TN}$ so that $L_\pm \in \mathcal{L}(\Lambda W^{1,2}_{TN}; D(\Delta^{TN}_{f,h}))$. By using again Lemma 3.4, $d_{f,h} \circ L_-$ and $d^*_{f,h} \circ L_+$ belong to $\mathcal{L}(\Lambda W^{1,2}_{TN})$. Consider now the decomposition

$$\omega = 1_{\{0\}}(\Delta^{TN}_{f,h})\omega + d_{f,h}L_-\omega + d^*_{f,h}L_+\omega$$

when $\omega \in \Lambda W^{1,2}_{TN}$. All the terms belong to $\Lambda W^{1,2}_{TN}$ while

$$d_{f,h}(d_{f,h}L_-\omega) = d_{f,h}(1_{\{0\}}(\Delta^{TN}_{f,h})\omega) = 0$$
$$\text{and} \qquad d^*_{f,h}(d^*_{f,h}L_+\omega) = d^*_{f,h}(1_{\{0\}}(\Delta^{TN}_{f,h})\omega) = 0.$$

Therefore Lemma 3.5 implies that the decomposition is orthogonal when $\omega \in \Lambda W^{1,2}_{TN}$, and this extends by continuity to $\omega \in \Lambda L^2$.

Proposition 3.1 ensures that $d_{f,h}$ and $d^*_{f,h}$ send F_M into itself and Lemma 3.5 with $F_M \subset D(\Delta^{TN}_{f,h}) \subset \Lambda W^{1,2}_{TN}$ implies

$$\beta^*_M = (1_{[0,M]}(\Delta^{TN}_{f,h})d_{f,h}1_{[0,M]}(\Delta^{TN}_{f,h}))^* = (1_{[0,M]}(\Delta^{TN}_{f,h})d^*_{f,h}1_{[0,M]}(\Delta^{TN}_{f,h})).$$

The orthogonal decomposition follows from the result for $\omega \in \Lambda L^2$. The chain complex structure comes from $d_{f,h} \circ d_{f,h} = d^*_{f,h} \circ d^*_{f,h} = 0$.

The space $\ker \beta^{(p)}_M / \operatorname{Im} \beta^{(p-1)}_M$ is isomorphic to $\ker(\Delta^{TN,(p)}_{f,h})$, which is contained like F_M in $\mathcal{C}^\infty_{TN}(\overline{f^b_a}; \bigwedge^p T^*M)$ by the elliptic regularity up to the boundary of $\Delta^{TN,(p)}_{f,h}$. Hence $\ker \beta^{(p)}_M / \operatorname{Im} \beta^{(p)}_M$ is isomorphic to $\ker d^{(p)}_{f,h} / \operatorname{Im} d^{(p-1)}_{f,h}$ after considering the differential operators $d_{f,h}$ restricted to $\mathcal{C}^\infty_{TN}(\overline{f}^a_b, \bigwedge T^*M)$. Since $\omega \mapsto e^{-\frac{f}{h}}\alpha$ is an isomorphism between the two spaces $\mathcal{C}^\infty_{TN}(\overline{f}^a_b, \bigwedge T^*M)$, respectively, defined for $\Delta^{TN}_{f,h}$ or $d_{f,h}$ and $\Delta^{TN}_{0,h} = h^2 \Delta^{TN}_{\text{Hodge}}$ or d, we obtain

$$\ker \beta^{(p)}_M / \operatorname{Im} \beta^{(p-1)}_M \sim \ker d^{(p)} / \operatorname{Im} d^{(p-1)} = H^p(f^a, f^b)$$

by Hodge-de Rham theory (see for instance [15] for the usual boundaryless case and [16,43] for the case with boundary).

The result concerned with the spectrum of $\Delta^{TN,(p)}_{f,h}$ is a direct consequence of the orthogonal decomposition of F_M with the chain complex structure (21). To be more specific, the decomposition of $\Delta^{TN,(p)}_{f,h}\big|_{F^{(p)}_M}$ according to $F^{(p)}_M = \ker \Delta^{TN,(p)}_{f,h} \overset{\perp}{\oplus} \operatorname{Im} \beta^{(p-1)}_M \overset{\perp}{\oplus} \operatorname{Im} \beta^{*,(p+1)}_M$ writes

$$\Delta^{TN,(p)}_{f,h}\big|_{F^{(p)}_M} = 0 \overset{\perp}{\oplus} \beta^{(p-1)}_M \beta^{(p-1),*}_M \overset{\perp}{\oplus} \beta^{(p),*}_M \beta^{(p)}_M$$

while $\beta^{(p-1),*}_M$ is an isomorphism from $\operatorname{Im} \beta^{(p-1)}_M$ onto $\operatorname{Im} \beta^{(p-1),*}$ with

$$\beta^{(p-1),*}_M(\beta^{(p-1)}_M \beta^{(p-1),*}_M) = (\beta^{(p-1),*}_M \beta^{(p-1)}_M)\beta^{(p-1),*}_M.$$

□

Remark 3.6. Note that the duality between the two chain complexes associated with β_M and β^*_M and their homology groups, is another version of the topological duality $f \to -f$. Actually changing f to $-f$ and p-forms with $(d-p)$-forms with the Hodge-$\star$ operator, interchanges $d_{f,h}$ and $d^*_{f,h}$.

3.2. Adapting Helffer-Sjöstrand Analysis

We still work in $\overline{f^b_a}$ and we introduce like in [24] the Agmon distance d_{Ag} associated with the degenerate metric $|\nabla f|^2 g$, where g is the initial Riemannian metric on M. This distance satisfies

$$d_{Ag}(x,y) \geq |f(x) - f(y)|$$

with equality when an integral curve of ∇f joins x and y.

Before stating the following crucial theorem, let us introduce two definitions which will be very useful in the sequel. The first one recalls Helffer-Sjöstrand notation $\tilde{\mathcal{O}}$, very convenient when handling exponentially small quantities.

Definition 3.7. For two quantities $A(h)$, estimated with a norm $|A(h)|$, and $B(h) \geq 0$, parametrized by $h \in (0, h_0)$, the notation $A(h) = \tilde{\mathcal{O}}(B(h))$ means:

$$\forall \varepsilon > 0, \exists C_\varepsilon > 0, \quad \forall h \in (0, h_0), |A(h)| \leq C_\varepsilon B(h) e^{\frac{\varepsilon}{h}}.$$

Definition 3.8. Let $U \in M$ be a critical point of f with index p and let $\Phi(x) := d_{Ag}(x, U)$. A local coordinate system $y_1, \ldots, y_d$ around U is said to be an adapted Morse coordinate system for f if $y_1, \ldots, y_d$ is centered at U, $dy_1, \ldots, dy_d$ is an orthonormal positively oriented basis of $T_U^* M$, and if, in these coordinates, the following Morse decompositions for f and Φ,

$$f(y) = f(U) + \frac{1}{2} \sum_{j=1}^{d} \lambda_j y_j^2, \quad \Phi(y) = \frac{1}{2} \sum_{j=1}^{d} |\lambda_j| y_j^2,$$

hold locally around U, with $\lambda_j < 0$ for $j \leq p$ and $\lambda_j > 0$ for $j > p$.

Let us notice that such a coordinate system always exists, according to [24] pp. 272–281. Moreover, in such a coordinate system, the stable and unstable manifolds of $-\nabla f$, respectively, denoted by $\mathcal{C}_{\mathrm{St}}$ and $\mathcal{C}_{\mathrm{Unst}}$ are locally parametrized by:

$$\mathcal{C}_{\mathrm{St}} = \{y\,;\; y_1 = \cdots = y_p = 0\} \quad \text{and} \quad \mathcal{C}_{\mathrm{Unst}} = \{y;\; y_{p+1} = \cdots = y_d = 0\}. \tag{22}$$

Theorem 3.9. *Let p belong to $\{0, \ldots, d\}$ and denote by $\mathcal{U}^{(p)} = \{U_1^{(p)}, \ldots, U_{m_p}^{(p)}\}$ the set of critical points with index p of f in f_a^b. There exists $h_0 > 0$ such that, for all $h \in (0, h_0]$, the spectral subspace $F^{(p)} = 1_{[0, Ch^{3/2}]}(\Delta_{f,h}^{TN(p)})$ is spanned by m_p normalized vector v_j, $1 \leq j \leq m_p$, which satisfy, for $x \in M$ and $\alpha \in \mathbb{N}^d$,*

$$|\partial_x^\alpha v_j| = \tilde{\mathcal{O}}\left(e^{-\frac{d_{\mathrm{Ag}}(x, U_j^{(p)})}{h}}\right), \tag{23}$$

$$|\partial_x^\alpha d_{f,h} v_j| = \tilde{\mathcal{O}}\left(e^{-\frac{\beta_j^+(x)}{h}}\right), \quad |\partial_x^\alpha d_{f,h}^* v_j| = \tilde{\mathcal{O}}\left(e^{-\frac{\beta_j^-(x)}{h}}\right) \tag{24}$$

$$\textit{with}\, \beta_j^+(x) = \min_{U \in \mathcal{U}^{(p+1)} \cup \mathcal{U}^{(p)} \setminus \left\{U_j^{(p)}\right\}} d_{\mathrm{Ag}}(U_j, U) + d_{\mathrm{Ag}}(U, x)$$

$$\textit{and}\, \beta_j^-(x) = \min_{U \in \mathcal{U}^{(p-1)} \cup \mathcal{U}^{(p)} \setminus \left\{U_j^{(p)}\right\}} d_{\mathrm{Ag}}(U_j, U) + d_{\mathrm{Ag}}(U, x).$$

The eigenvalues of $\Delta_{f,h}^{TN(p)}$ lying in $[0, h^{3/2}]$ are $\mathcal{O}(e^{-\frac{C}{h}})$.

When the metric g is Euclidean in some adapted Morse coordinates for f in $B(U_j^{(p)}, 2\eta)$, with $f(y) = f(U_j) + \frac{1}{2}\sum_{j=1}^{d} \lambda_j y_j^2$, then the form v_j satisfies

$$v_j = |\lambda_1 \ldots \lambda_d|^{1/4} (\pi h)^{-d/4} e^{-\frac{\sum_{j=1}^{d} |\lambda_j| y_j^2}{2h}} dy_1 \wedge \ldots \wedge dy_p + \mathcal{O}\left(e^{-\frac{C\eta}{h}}\right)$$

in $\mathcal{C}^\infty(B(U_j^{(p)}, \eta))$.

In the general case of a Riemannian metric, there exists, for η small enough, some adapted Morse coordinates for f in $B(U_j^{(p)}, 2\eta)$, with $f(y) = f(U_j) + \frac{1}{2}\sum_{j=1}^{d} \lambda_j y_j^2$, and such that the form v_j satisfies

$$e^{\frac{\sum_{j=1}^{d} |\lambda_j| y_j^2}{2h}} v_j = \omega_0(x) + \mathcal{O}(h^{1-d/4}) \quad \textit{in} \quad \mathcal{C}^\infty(B(U_j^{(p)}, \eta)),$$

with

$$\omega_0 = \frac{|\lambda_1 \ldots \lambda_d|^{1/4}}{(\pi h)^{d/4}} dy_1 \wedge \ldots \wedge dy_p \quad \textit{along} \quad \mathcal{C}_{Unst} \cap B(U_j^{(p)}, \eta),$$

and

$$\omega_0 = (-1)^{p(d-p)} \frac{|\lambda_1 \ldots \lambda_d|^{1/4}}{(\pi h)^{d/4}} \star (dy_{p+1} \wedge \ldots \wedge dy_d) \quad \textit{along} \quad \mathcal{C}_{St} \cap B(U_j^{(p)}, \eta).$$

We shall need an integration by part formula adapted from Lemma 4.3.3 in [20] and Lemma 4.3 in [35].

Lemma 3.10. *Let Ω be a regular domain of f_a^b with boundary made of three disjoint pieces $\partial\Omega = \{f = a\} \sqcup \{f = b\} \sqcup \Gamma$. Consider the self-adjoint realization $\Delta_{f,h}^{TND}$ of $\Delta_{f,h}$ given by the form*

$$\mathcal{D}(\omega, \omega') = \langle d_{f,h}\omega,\, d_{f,h}\omega' \rangle + \langle d_{f,h}^*\omega,\, d_{f,h}^*\omega' \rangle$$

with the form domain

$$\Lambda W_{TND}^{1,2} = \{\omega \in \Lambda W^{1,2}(\overline{\Omega}); \quad \mathbf{t}\omega|_{f=a} = 0, \mathbf{n}\omega\big|_{f=b} = 0, \omega\big|_{\Gamma} = 0\},$$

and the operator domain

$$D(\Delta_{f,h}^{TND}) = \left\{ \omega \in \Lambda W^{2,2}(\overline{\Omega}); \quad \begin{array}{ll} \mathbf{t}\omega\big|_{f=a} = 0, & \mathbf{t}d_{f,h}^*\omega|_{f=a} = 0, \\ \mathbf{n}\omega|_{f=b} = 0, & \mathbf{n}d_{f,h}\omega|_{f=b} = 0, \\ \omega|_\Gamma = 0 & \end{array} \right\}.$$

Let φ be any Lipschitz function. Then for all $\omega \in \Lambda W_{TND}^{1,2}$ we have the integration by part formula

$$\begin{aligned} \mathbb{R}\mathrm{e}\ \mathcal{D}(\omega, e^{\frac{2\varphi}{h}}\omega) = {} & h^2 \| d e^{\frac{\varphi}{h}} \omega \|^2 + h^2 \| d^* e^{\frac{\varphi}{h}} \omega \|^2 \\ & + \langle (|\nabla f|^2 - |\nabla \varphi|^2 + h\mathcal{L}_{\nabla f} + h\mathcal{L}_{\nabla f}^*) e^{\frac{\varphi}{h}} \omega,\, e^{\frac{\varphi}{h}}\omega \rangle \\ & + h \left(\int_{f=b} - \int_{f=a} \right) \langle \omega,\, \omega \rangle_{\Lambda T_\sigma^*\Omega} e^{\frac{2\varphi(\sigma)}{h}} \left(\frac{\partial f}{\partial n} \right)(\sigma) d\sigma, \end{aligned} \quad (25)$$

where $\frac{\partial f}{\partial n}$ is the exterior normal derivative. Moreover when $\omega \in D(\Delta_{f,h}^{TND})$ then $\mathcal{D}(\omega, e^{\frac{2\varphi}{h}}\omega) = \mathbb{R}\mathrm{e}\, \langle e^{\frac{2\varphi}{h}} \Delta_{f,h}^{TND} \omega, \omega \rangle$.

Proof of Theorem 3.9. First of all, applying the integration by part (25) with $\varphi = 0$ and the local harmonic approximation around critical points like in [12], one obtains that the number of eigenvalues in $[0, Ch^{3/2}]$ is m_p, with no other eigenvalues in $(Ch^{3/2}, h/C]$, when C is chosen large enough. The boundary term in (25) is non-negative because $\frac{\partial f}{\partial n}$ is non-positive (resp. non-negative) on $\{f = a\}$ (resp. $\{f = b\}$). The IMS localization formula $-h^2\Delta = -\sum_j \chi_j(h^2\Delta)\chi_j - h^2\sum_j |\nabla\chi_j|$ with one χ_{j_a} (resp. χ_{j_b}) localizing around $\{f = a\}$ (resp. $\{f = b\}$) shows that eigenfunctions associated with the $\mathcal{O}(h^{3/2})$ eigenvalues have asymptotically no mass around $\{f = a\} \cup \{f = b\}$. Contrary to [20] and [35], there are no generalized critical points at the boundary and the assumption that f restricted to the boundary is a Morse function is not necessary here.

We construct now a global quasimode ϕ_j^h associated with the critical point $U_j^{(p)}$. Following [22,24], consider, for a small constant $\gamma > 0$, the domain

$$\overline{\Omega_j} = \overline{f_a^b} \setminus \cup_{k \neq j} B(U_k^{(p)}, \gamma),$$

with $\partial\Omega_j = \{f = a\} \cup \{f = b\} \cup \Gamma$ and $\Gamma = \cup_{k\neq j}\partial B(U_k^{(p)}, \gamma)$, and take the self-adjoint realization $\Delta_{f,h}^{TND(p)}$ of Lemma 3.10 acting on p-forms. It admits a single eigenvalue μ_j^h which is $\mathcal{O}(h^{3/2})$ (with the rest of the spectrum in $[h/C, +\infty))$ and take ϕ_j^h a normalized eigenvector associated with μ_j^h:

$$\|\phi_j^h\|_{\Lambda L^2} = 1, \quad \Delta_{f,h}^{TND(p)}\phi_j^h = \mu_j^h \phi_j^h.$$

Applying Lemma 3.10, in the spirit of [13] pp. 49–55, with

$$\omega = \phi_j^h \;\; \varphi_\varepsilon(x) = (1-\varepsilon)d_{\mathrm{Ag}}(x, B(U_j^{(p)}, \varepsilon)) \;,\; |\nabla\varphi_\varepsilon| \leq (1-\varepsilon)|\nabla f|,$$

gives

$$\|e^{\frac{\varphi_\varepsilon}{h}}\phi_j^h\|_{\Lambda W^{1,2}} = \mathcal{O}_\varepsilon\left(\frac{1}{h}\right),$$

where the subscript $_\varepsilon$ recalls that the factor of $\frac{1}{h}$ depends on the parameter $\varepsilon > 0$. By elliptic regularity up to the boundary of $\Delta_{f,h}^{TND(p)}$, ϕ_j^h is $\mathcal{C}^\infty$ in $\overline{\Omega_j}$. The differential operator $e^{\frac{\varphi}{h}}\Delta_{f,h}e^{-\frac{\varphi}{h}}$ equals

$$h^2(dd^* + d^*d) + |\nabla f|^2 - |\nabla\varphi|^2 + h(\mathcal{L}_{\nabla\varphi} - \mathcal{L}_{\nabla\varphi}^* + \mathcal{L}_{\nabla f} + \mathcal{L}_{\nabla f}^*)$$

where the last part is a first order differential operator. With the boundary conditions, the form $u_j^h = e^{\frac{\varphi_\varepsilon}{h}}\phi_j^h$ satisfies the system

$$\begin{cases} (dd^* + d^*d)u_j^h = r_j \\ \mathbf{t}u_j^h|_{f=a} = 0 \quad \mathbf{t}d^*u_j^h|_{f=a} = \varrho_{j,a} \\ \mathbf{n}u_j^h|_{f=b} = 0 \quad \mathbf{n}du_j^h|_{f=b} = \varrho_{j,b} \\ u_j^h|_\Gamma = 0 \end{cases}$$

where $\|r_j\|_{\Lambda L^2}$, $\|\varrho_{j,*}\|_{\Lambda W^{1/2,2}}$ are $\mathcal{O}(\frac{1}{h^3})$. This provides a $\mathcal{O}(\frac{1}{h^3})$ estimate for $\|u_j^h\|_{\Lambda W^{2,2}}$ and bootstraping gives

$$\forall\alpha\in\mathbb{N}^d, \forall x\in\overline{\Omega_j},\quad |\partial_x^\alpha\phi_j^h(x)| = \tilde{\mathcal{O}}(e^{-\frac{\varphi_\varepsilon(x)}{h}}).$$

Since this holds for all $\varepsilon>0$, the definition of $\tilde{\mathcal{O}}$ provides the same result with $\varepsilon=0$. It means the following estimate holds, with $\varphi(x):=\varphi_0(x)=d_{Ag}(x,U_j^{(p)})$:

$$\forall\alpha\in\mathbb{N}^d, \forall x\in\overline{\Omega}_j,\quad |\partial_x^\alpha\phi_j^h(x)| = \tilde{\mathcal{O}}(e^{-\frac{\varphi(x)}{h}}).$$

The differential $d_{f,h}\phi_j^h$ solves in Ω_j the differential equation

$$\Delta_{f,h}(d_{f,h}\phi_j^h) = d_{f,h}(\Delta_{f,h}\phi_j^h) = \mu_j^h d_{f,h}\phi_j^h.$$

The same argument as the one for Lemma 3.4 leads to the fact that $d_{f,h}\phi_j^h$ satisfies the same boundary conditions as ϕ_j^h on $\{f=a\}\cup\{f=b\}$. Consider now the domain $\overline{\Omega'_j}=\overline{\Omega_j}\setminus\cup_{U\in\mathcal{U}^{(p+1)}}B(U,\gamma)$, note $\mathcal{V}_j=\mathcal{U}^{(p+1)}\cup\mathcal{U}^{(p)}\setminus\{U_j^{(p)}\}$, and work with the associated $\Delta_{f,h}^{TND(p+1)}$. The form $u'^h_j=\chi_\gamma d_{f,h}\phi_j^h$, where $\chi_\gamma\in\mathcal{C}^\infty(\overline{\Omega'_j})$ vanishes in $\cup_{U\in\mathcal{V}_j}B(U,2\gamma)$ and equals 1 outside $\cup_{U\in\mathcal{V}_j}B(U,3\gamma)$, belongs to $D(\Delta_{f,h}^{TND(p+1)})$ and solves

$$\Delta_{f,h}^{TDN(p+1)}(u'^h_j) = \mu_j^h u'^h_j + r'_j,$$

with $\operatorname{supp} r'_j\subset\cup_{U\in\mathcal{V}_j}B(U,3\gamma)$ and

$$|\partial_x^\alpha r'_j(x)| = \tilde{\mathcal{O}}\left(e^{\frac{-\min_{U\in\mathcal{V}_j} d_{\mathrm{Ag}}(U,U_j^{(p)})+c\gamma}{h}}\right)\qquad(\text{with } c>0).$$

With our choice of $\overline{\Omega'_j}$, $\Delta_{f,h}^{TND(p+1)}$ has no eigenvalue in $[0,h/C]$ and the same analysis as above leads to

$$\forall\alpha\in\mathbb{N}^d, \forall x\in\overline{\Omega'_j},\quad |\partial_x^\alpha d_{f,h}\phi_j^h(x)| = \tilde{\mathcal{O}}\left(e^{\frac{-\min_{U\in\mathcal{V}_j} d_{\mathrm{Ag}}(U_j^{(p)},U)-d_{Ag}(U,x)+c\gamma}{h}}\right),$$

where the previous estimates extend the result to all $\overline{\Omega_j}$. After changing $\mathcal{U}^{(p+1)}$ into $\mathcal{U}^{(p-1)}$, a similar result holds for $d_{f,h}^*\phi_j^h$. A simple computation then gives $\mu_j^h=\mathcal{D}(\phi_j^h,\phi_j^h)=\tilde{\mathcal{O}}(e^{2\frac{-\min_{U\in\mathcal{V}_j} d_{\mathrm{Ag}}(U,U_j)+c\gamma}{h}})$.

Let us now work with $\Delta_{f,h}^{TN(p)}$ on $\overline{f_a^b}$. Consider the cut-off $\theta_{j,\gamma}\in\mathcal{C}^\infty(\Omega_j)$ which vanishes in $\cup_{U\in\mathcal{U}^{(p)}\setminus\{U_j^{(p)}\}}B(U,2\gamma)$ and equals 1 in $\cup_{U\in\mathcal{U}^{(p)}\setminus\{U_j^{(p)}\}} B(U,3\gamma)$, and set

$$\psi_j^h=\theta_{j,\gamma}\phi_j^h.$$

These m_p vectors belong to $D(\Delta_{f,h}^{TN(p)})$ and satisfy

$$\begin{aligned}&\Delta_{f,h}^{TN(p)}\psi_j^h = \mu_j^h\psi_j^h + r_j\\ &\text{with}\quad \mu_j^h = \mathcal{O}(e^{-\frac{C}{h}}),\\ &\qquad |r_j(x)| = \mathcal{O}(e^{-\frac{d_{\mathrm{Ag}}(x,U_j)}{h}}),\quad \operatorname{supp} r_j \subset \cup_{U\in\mathcal{U}^{(p)}\setminus\{U_j^{(p)}\}} B(U,3\gamma),\end{aligned}$$

and $\quad (\langle\psi_j^h,\, \psi_k^h\rangle)_{1\le j,k\le m_p} = \mathrm{Id} + \mathcal{O}(e^{-\frac{C}{h}}),$

while $\Delta_{f,h}^{TN(p)}$ has only m_p eigenvalues in $[0, Ch^{3/2}]$. The Proposition 4.1 in [18] implies

$$\psi_j^h - 1_{[0,Ch^{3/2}]}(\Delta_{f,h}^{TN(p)})\psi_j^h = \mathcal{O}(e^{-\frac{C}{h}}),$$

and we set $u_j^h = 1_{[0,Ch^{3/2}]}(\Delta_{f,h}^{TN(p)})\psi_j^h$. The min-max principle applied with the ψ_j^h's also implies that the eigenvalues of $\Delta_{f,h}^{TN(p)}$ in $[0, Ch^{3/2}]$ are actually exponentially small. With the integration contour $\mathcal{C}_h = \left\{z\in\mathbb{C}, |z| = h^{3/2}\right\}$, write

$$u_j^h - \psi_j^h = \frac{1}{2i\pi}\int_{\mathcal{C}_h}(z-\mu_j^h)^{-1}(z-\Delta_{f,h}^{TN(p)})^{-1}r_j\ dz.$$

The resolvent estimates of Proposition 2.2.5 in [22] can be carried over to our boundary problem thanks to Lemma 3.10 and elliptic regularity up to the boundary. With the estimates and support condition on r_j, they lead to

$$\begin{aligned}\forall\alpha\in\mathbb{N}^d, \forall x\in\overline{f_a^b},\quad |\partial_x^\alpha\omega_z(x)| &= \tilde{\mathcal{O}}\left(e^{\frac{-\min_{U\in\mathcal{U}^{(p)}\setminus\left\{U_j^{(p)}\right\}} d_{\mathrm{Ag}}(U_j,U)+d_{\mathrm{Ag}}(U,x)+c\gamma}{h}}\right)\\ &= \tilde{\mathcal{O}}\left(e^{\frac{-d_{\mathrm{Ag}}(U_j,x)+c\gamma}{h}}\right)\end{aligned}$$

when $\omega_z = [(z-\Delta_{f,h}^{TN(p)})^{-1}r_j]$ and $z\in\mathcal{C}_h$.

With the estimates on $\psi_j = \theta_{j,h}\phi_j^h$, this leads to

$$\forall\alpha\in\mathbb{N}^d, \forall x\in\overline{f_a^b},\quad |\partial_x^\alpha u_j^h| = \tilde{\mathcal{O}}(e^{\frac{-d_{\mathrm{Ag}}(x,U_j)+c\gamma}{h}}),$$

and we take

$$v_j^h = \|u_j^h\|^{-1}u_j^h = (1+\mathcal{O}(e^{-C/h}))u_j^h.$$

The estimates for $d_{f,h}v_j^h$ (resp. $d_{f,h}^*v_j^h$) are obtained after writing the equation for $d_{f,h}(u_j^h-\psi_j^h)$ (resp. $d_{f,h}^*(u_j^h-\psi_j^h)$) and using the resolvent estimates for $\Delta_{f,h}^{TN(p+1)}$ (resp. $\Delta_{f,h}^{TN(p-1)}$).

Finally, since these estimates hold for any $\gamma>0$, the definition of $\tilde{\mathcal{O}}$ provides the same result with $\gamma = 0$.

The rest of the proof of Theorem 3.9 is a direct consequence of Theorem 2.5 of [24] and therein related WKB construction. □

3.3. An Important Remark

It is clear that the results stated for $\overline{f_a^b}$ hold when $a = -\infty$ or $b = +\infty$, that is when one boundary is empty.

Another variation on it consists in deforming homotopically $\{f = a\}$ (resp. $\{f = b\}$) while preserving the sign conditions $\frac{\partial f}{\partial n} < 0$ (resp. $\frac{\partial f}{\partial n} > 0$).

4. Barannikov–Morse Chain Complex and Construction of Accurate Global Quasimodes

4.1. Properties of Quasimodes Associated with Lower and Upper Critical Points

Consider the operator $\Delta_{f,h}^{TN}$ defined on $\overline{f_a^b}$ and set $F^{(p)} = \mathrm{Im}\,1_{[0,h^{3/2}]}(\Delta_{f,h}^{TN(p)})$ for $p \in \{0, \ldots, d\}$, $F = \oplus_p F^{(p)}$, $\beta = d_{f,h}|_F$ and $\beta^* = d_{f,h}^*|_F$. According to Sect. 3, dim $F^{(p)} = m_p$ and the chain complex associated with β

$$0 \rightleftarrows F^{(0)} \ldots F^{(p-1)} \underset{\beta^{(p-1)*}}{\overset{\beta^{(p-1)}}{\rightleftarrows}} F^{(p)} \underset{\beta^{(p)*}}{\overset{\beta^{(p)}}{\rightleftarrows}} F^{(p+1)} \ldots F^{(d)} \rightleftarrows 0$$

has the homology group $H^*(\overline{f_a^b}, \{f = a\})$ dual to $H_*(f^b, f^a) = H_*(\overline{f_a^b}, \{f = a\})$. Remember also that $F^{(p)}$ admits the orthogonal decompositions

$$\begin{aligned} F^{(p)} &= \ker \Delta_{f,h}^{TN(p)} \overset{\perp}{\oplus} \mathrm{Im}\,\beta^{(p-1)} \overset{\perp}{\oplus} \mathrm{Im}\,\beta^{(p)*} \\ &= \ker \beta^{(p)} \overset{\perp}{\oplus} \mathrm{Im}\,\beta^{(p)*} = \mathrm{Im}\,\beta^{(p-1)} \overset{\perp}{\oplus} \ker \beta^{(p-1)*} \end{aligned}$$

and that $F^{(p)}$ admits an almost orthonormal basis $\left\{v_U, U \in \mathcal{U}^{(p)}\right\}$ fulfilling the properties of Theorem 3.9. The orthogonal projection on any subspace G of the above orthogonal decomposition will be denoted by Π_G.

Proposition 4.1. *Assume that $U \in \mathcal{U}^{(p)}$ is not an upper critical points in f_a^b, then v_U is almost orthogonal to $Im\,\beta$:*

$$\|\Pi_{Im\,\beta^{(p-1)}} v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}}),$$

where the constant C_η depends on the small radius η fixed by the geometry in Theorem 3.9.

Proof. When $U \in \mathcal{U}^{(p)}$ with $f(U) = c_U$ is not an upper critical point, it means that the mapping $H_p(f^{c_U+\varepsilon}) \to H_p(f^{c_U+\varepsilon}, f^{c_U-\varepsilon})$ does not vanish. Since $H_p(f^{c_U+\varepsilon}, f^{c_U-\varepsilon})$ is one-dimensional and is generated by e_U^p, the unstable manifold for $-\nabla f$ leaving U and restricted to $f_{c_U-\varepsilon}$, there exists a cycle C_U^p in $f^{c_U+\varepsilon}$, or a cycle in M supported in $f^{c_U+\varepsilon}$, of which the restriction to $f_{c_U-\varepsilon}^{c_U+\varepsilon}$ is e_U^p. We choose $\varepsilon = \varepsilon_\eta > 0$ so that e_U^p is contained in the ball $B(U, \eta)$ of Theorem 3.9. In adapted Morse coordinates, e_U^p equals up to the orientation

$$\left\{ y_{p+1} = \cdots = y_d = 0, \quad \frac{1}{2}\sum_{j=1}^{p} |\lambda_j| y_j^2 < \varepsilon_\eta \right\}.$$

Meanwhile the p-form $e^{\frac{f-c_U}{h}}v_U$, where v_U is denoted by v_j in Theorem 3.9 when $U=U_j^{(p)}$, equals

$$|\lambda_1\dots\lambda_d|^{1/4}(\pi h)^{-d/4}e^{-\frac{\sum_{j=1}^p|\lambda_j|y_j^2}{h}}(\omega_0(y)+h\omega'(y,h))+\mathcal{O}(e^{-\frac{C_\eta}{h}})$$

with ω' bounded in $\mathcal{C}^\infty(B(U,\eta))$ and

$$\omega_0=dy_1\wedge\ldots\wedge dy_p\quad\text{along}\quad\{y_{p+1}=\cdots=y_d=0\}\cap B(U,\eta).$$

Decompose v_U according to

$$v_U=v_U'+v_U''=\Pi_{\mathrm{Im}\,\beta^{(p-1)}}v_U+\Pi_{\ker\beta^{(p-1)*}}v_U,$$
$$v_U'=\sum_{U'\in\mathcal{U}^{(p)}}t_{U'}v_{U'}\qquad v_U''=\sum_{U'\in\mathcal{U}^{(p)}}s_{U'}v_{U'}.$$

The decomposition $v_U=v_U'+v_U''$ is orthogonal

$$\|v_U'\|^2+\|v_U''\|^2=1.$$

Meanwhile, the exponential decay estimates of $(v_{U'})_{U'\in\mathcal{U}^{(p)}}$ stated in Theorem 3.9 provide the almost orthogonality

$$\sum_{U'}|t_{U'}|^2\le 1+\mathcal{O}(e^{-C/h}),\quad\sum_{U'}|s_{U'}|^2\le 1+\mathcal{O}(e^{-C/h}),$$
$$t_U+s_U=1+\mathcal{O}(e^{-C/h}),\quad\text{and}\quad t_{U'}+s_{U'}=\mathcal{O}(e^{-C/h})\quad\text{for } U'\neq U.$$

All the $v_{U'}$ have $\tilde{\mathcal{O}}(1)$ estimates in $\mathcal{C}^\infty(\overline{f_a^b};\bigwedge^p T^*M)$ and the support conditions on e_U^p and C_U^p give

$$\int_{e_U^p}e^{\frac{f-c_U}{h}}v_U=\int_{C_U^p}e^{\frac{f-c_U}{h}}v_U+\mathcal{O}(e^{-\frac{C_\eta}{h}}).$$

By using $v_U'=d_{f,h}\omega$ we get

$$\int_{e_U^p}e^{\frac{f-c_U}{h}}v_U=h\int_{C_U^p}d\left(e^{\frac{f-c_U}{h}}\omega\right)+\int_{C_U^p}e^{\frac{f-c_U}{h}}v_U'',$$

and finally with $\partial C_U^p=0$,

$$\int_{e_U^p}e^{\frac{f-c_U}{h}}v_U=\sum_{U'\in\mathcal{U}^{(p)}}s_{U'}\int_{C_U^p}e^{\frac{f-c_U}{h}}v_{U'}$$
$$=\sum_{U'\in\mathcal{U}^{(p)}}s_{U'}\left(\int_{e_U^p}+\int_{C_U^p\setminus e_U^p}\right)e^{\frac{f-c_U}{h}}v_{U'}.$$

With $s_{U'}=\mathcal{O}(1)$, $e^{\frac{f-c_U}{h}}=\mathcal{O}(e^{-\frac{C_\eta}{h}})$ on $C_U^p-e_U^p$ and $|v_{U'}|=\tilde{\mathcal{O}}(1)$ the second integral gives an exponentially small term. When $U'\neq U$ the exponential decay estimate of Theorem 3.9 imply that $s_{U'}\int_{e_U^p}e^{\frac{f-c_U}{h}}v_{U'}$ is $\mathcal{O}(e^{-C/h})$.

We have then proved

$$\int_{e_U^p} e^{\frac{f-c_U}{h}} v_U = s_U \int_{e_U^p} e^{\frac{f-c_U}{h}} v_U + \mathcal{O}(e^{-\frac{C_\eta}{h}}).$$

But a direct calculation in the Morse coordinates gives

$$\int_{e_U^p} e^{\frac{f-c_U}{h}} v_U = (1+\mathcal{O}(h)) \int_{\sum_{j=1}^p |\lambda_j| y_j^2 < 2\varepsilon_\eta} \frac{|\lambda_1 \ldots \lambda_d|^{1/4}}{(\pi h)^{d/4}} e^{-\frac{\sum_{j=1}^p |\lambda_j| y_j^2}{h}} \mathrm{d}y_1 \ldots \mathrm{d}y_p$$
$$= \frac{|\lambda_{p+1} \ldots \lambda_d|^{1/4}}{|\lambda_1 \ldots \lambda_p|^{1/4}} (\pi h)^{(2p-d)/4} (1+\mathcal{O}(h)).$$

This proves $s_U = 1+\mathcal{O}(e^{-\frac{C_\eta}{h}})$ while all the other coefficients are $\mathcal{O}(e^{-\frac{C_\eta}{h}})$. □

By duality $f \to -f$, other results can be deduced.

Proposition 4.2. *When $U \in \mathcal{U}^{(p)}$ is not a lower critical point in f_a^b, v_U is almost orthogonal to Im $\beta^{(p)*}$:*

$$\|\Pi_{Im\, \beta^{(p)*}} v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}}).$$

When $U \in \mathcal{U}^{(p)}$ is a homological critical point in f_a^b, v_U is exponentially close to $\ker \Delta_{f,h}^{(p)}$*:*

$$\|v_U - \Pi_{\ker \Delta_{f,h}^{TN(p)}} v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}}).$$

Finally, when $U \in \mathcal{U}^{(p)}$ is an upper (resp. a lower) critical point, v_U is exponentially close to Im $\beta^{(p-1)}$ (resp. Im $\beta^{(p)*}$):

$$\|v_U - \Pi_{Im\, \beta^{(p-1)}} v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}}) \quad (\textit{resp.} \quad \|v_U - \Pi_{Im\, \beta^{(p)*}} v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}})).$$

Proof. The first statement is dual to the one of Proposition 4.1. For the second one it suffices to notice that homological critical points are neither upper nor lower critical points. For the last one it suffices to notice that the number of homological critical points equals the dimension of $\ker \Delta_{f,h}^{TN(p)}$. Hence the set of $\Pi_{\ker \Delta_{f,h}^{TN(p)}} v_{U'}$ when U' ranges over the homological critical points, is an almost orthonormal basis of $\ker \Delta_{f,h}^{TN(p)}$. If U is an upper critical point, $\Pi_{\ker \beta^{(p)}} v_U = v_U + \mathcal{O}(e^{-\frac{C_\eta}{h}})$ is almost orthogonal to the $v_{U'}$ and therefore to $\ker \Delta_{f,h}^{TN(p)}$. We deduce

$$v_U = \Pi_{\mathrm{Im}\, \beta^{(p-1)}} v_U + \Pi_{\ker \Delta_{f,h}^{TN(p)}} v_U + \mathcal{O}(e^{-\frac{C_\eta}{h}}) = \Pi_{\mathrm{Im}\, \beta^{(p-1)}} v_U + \mathcal{O}(e^{-\frac{C_\eta}{h}}).$$

□

4.2. Construction of Accurate Global Quasimodes

We now define the global quasimodes for $\Delta_{f,h}$ on M which will be used in our computations.

- When U is a homological critical point, take simply
$$\omega_U = \Pi_{\ker \Delta_{f,h}} v_U,$$
where v_U is the form defined in Theorem 3.9 with $a = -\infty$ and $b = +\infty$.
- When U is an upper critical point, take
$$\omega_U = \Pi_{\mathrm{Im}\,\beta} v_U$$
where v_U is the form defined in Theorem 3.9 with $a = -\infty$ and $b = +\infty$.
- When $U \in \mathcal{U}^{(p)}$ is a lower critical point, there exists $U_1 \in \mathcal{U}^{(p+1)}$ with $f(U_1) = c_1$ such that $\partial_B U_1 = U$. In $f^{c_1-\varepsilon}$, U becomes a homological critical point. We take
$$\omega_U = 1_{[0,h^{3/2}]}(\Delta_{f,h})\chi_\varepsilon \tilde{v}_U$$
where $\tilde{v}_U$ is now the form defined in Theorem 3.9 with $a = -\infty$ and $b = c_1 - \varepsilon$ while $\Delta_{f,h}$ is the operator defined on all M. The function χ_ε vanishes in $f_{c_1-\frac{3}{2}\varepsilon}$ and equals 1 in $f^{c_1-2\varepsilon}$. The value of the parameter ε will be specified further according to η.

4.3. Computation of the Matrix of $d_{f,h}$

We work with the basis $(\omega_U)_{U\in\mathcal{U}}$ constructed before and we will denote by $\mathcal{U}_H, \mathcal{U}_L, \mathcal{U}_U$ the sets of homological, lower and upper critical points of f, and $\mathcal{U}_H^{(p)}, \mathcal{U}_L^{(p)}, \mathcal{U}_U^{(p)}$ their respective intersection with $\mathcal{U}^{(p)}$, the set of critical points of f with index p.

Proposition 4.3. *When U_0 belongs to $\mathcal{U}_U^{(p)} \cup \mathcal{U}_H^{(p)}$, then for any $U' \in \mathcal{U}^{(p+1)}$,*
$$\langle \omega_{U'} \,|\, d_{f,h}\omega_{U_0}\rangle = 0. \tag{26}$$

When U_0 belongs to $\mathcal{U}_L^{(p)}$, let U_1 denote the upper critical point with index $p+1$ s.t. $\partial_B(U_1) = U_0$. Then there exists a real constant $C > 0$ and a homological constant $\kappa = \kappa(U_1) \neq 0$ such that for $U' \in \mathcal{U}^{(p+1)}$:
$$\text{If } U' \neq U_1, \quad \langle \omega_{U'} \,|\, d_{f,h}\omega_{U_0}\rangle = \mathcal{O}(e^{-\frac{f(U_1)-f(U_0)+C}{h}}), \tag{27}$$
$$\text{If } U' = U_1, \quad \langle \omega_{U_1} \,|\, d_{f,h}\omega_{U_0}\rangle = (-1)^{pd}\kappa A(h) e^{-\frac{f(U_1)-f(U_0)}{h}}(1+\mathcal{O}(h)). \tag{28}$$

Moreover the prefactor $A(h)$ is given by the formula
$$A(h) = \left(\frac{h}{\pi}\right)^{\frac{1}{2}} \frac{|\lambda_1^1 \cdots \lambda_{p+1}^1|^{\frac{1}{4}}}{|\lambda_{p+2}^1 \cdots \lambda_d^1|^{\frac{1}{4}}} \frac{|\lambda_{p+1}^0 \cdots \lambda_d^0|^{\frac{1}{4}}}{|\lambda_1^0 \cdots \lambda_p^0|^{\frac{1}{4}}}, \tag{29}$$

where $\lambda_1^\ell < \cdots < \lambda_{p+\ell}^\ell < 0 < \lambda_{p+\ell+1}^\ell < \cdots < \lambda_d^\ell$ denote the eigenvalues of $\mathrm{Hess}\, f(U_\ell)$, for $\ell \in \{0,1\}$.

The rest of this section is devoted to the proof of this proposition. We are first going to prove the relations (26) and (27), then, in order to prove (28) and (29), we will in a first time work with a metric which is locally Euclidean around the critical points of f before showing that it remains valid for a general Riemannian metric.

Proof of equations (26) *and* (27). When U_0 is an upper critical point or a homological critical point, then the definition of ω_{U_0} says

$$d_{f,h}\omega_{U_0} = \beta\omega_{U_0} = 0,$$

which yields equation (26).

Let us now compute $\langle \omega_{U'},\, d_{f,h}\omega_{U_0}\rangle$ for $U' \in \mathcal{U}^{(p+1)}$ when $U_0 \in \mathcal{U}^{(p)}$ is a lower critical point with critical value c_0. Let $U_1 \in \mathcal{U}^{(p+1)}$ be the upper critical point with critical value c_1 such that $\partial_B U_1 = U_0$. The commutation $d_{f,h}1_{[0,h^2]}(\Delta_{f,h}) = 1_{[0,h^2]}(\Delta_{f,h})d_{f,h}$ gives

$$\begin{aligned}\langle \omega_{U'},\, d_{f,h}\omega_{U_0}\rangle &= \langle \omega_{U'},\, d_{f,h}1_{[0,h^2]}(\Delta^{(p)}_{f,h})\chi_\varepsilon\tilde{v}_{U_0}\rangle = \langle \omega_{U'},\, d_{f,h}\chi_\varepsilon\tilde{v}_{U_0}\rangle\\ &= \langle v_{U'},\, d_{f,h}\chi_\varepsilon\tilde{v}_{U_0}\rangle + \langle \omega_{U'} - v_{U'},\, d_{f,h}\chi_\varepsilon\tilde{v}_{U_0}\rangle.\end{aligned}$$

The relation $d_{f,h}\tilde{v}_{U_0} = 0$ in $\operatorname{supp}\nabla\chi_\varepsilon$ implies

$$d_{f,h}\chi_\varepsilon\tilde{v}_{U_0} = hd\chi_\varepsilon \wedge \tilde{v}_{U_0},$$

then, since $d\chi_\varepsilon$ is supported in $f^{c_1-\frac{3}{2}\varepsilon}_{c_1-2\varepsilon}$ and

$$|\tilde{v}_{U_0}(x)| = \tilde{\mathcal{O}}\left(e^{-\frac{d_{Ag}(x,U_0)}{h}}\right) = \mathcal{O}\left(e^{-\frac{c_1-c_0-C\varepsilon}{h}}\right) \quad \text{for } x \in \operatorname{supp}\nabla\chi_\varepsilon,$$

the remainder term $\langle \omega_{U'} - v_{U'},\, d_{f,h}\chi_\varepsilon\tilde{v}_{U_0}\rangle$ is bounded by

$$\|\omega_{U'} - v_{U'}\|\mathcal{O}\left(e^{-\frac{c_1-c_0-C\varepsilon}{h}}\right).$$

When U' is not a lower critical point, the relation $\|\omega_{U'} - v_{U'}\| = \mathcal{O}\left(e^{-\frac{C}{h}}\right)$ comes from Proposition 4.2. When U' is a lower critical point, simply note that both terms of the r.h.s. in

$$\|\omega_{U'} - v_{U'}\| \leq \|\omega_{U'} - \chi_\varepsilon\tilde{v}_{U'}\| + \|\chi_\varepsilon\tilde{v}_{U'} - v_{U'}\| \tag{30}$$

are $\mathcal{O}(e^{-\frac{C}{h}})$. Actually, the estimate for the second term is obtained after comparing $v_{U'}$ and $\tilde{v}_{U'}$ with the single eigenmode of a Dirichlet realization of $\Delta^{(p+1)}_{f,h}$ in $B(U', \eta_0)$, again by following [18].

Hence we have proved

$$\langle \omega_{U'},\, d_{f,h}\omega_{U_0}\rangle = \langle v_{U'},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle + \mathcal{O}\left(e^{-\frac{c_1-c_0+C}{h}}\right)$$

when $\varepsilon > 0$ is chosen small enough.

If $U' \neq U_1$, the exponential decay of $v_{U'}$,

$$|v_{U'}(x)| = \tilde{\mathcal{O}}\left(e^{-\frac{d_{Ag}(x,U')}{h}}\right) = \mathcal{O}(e^{-\frac{|c_1-f(U')|-C\varepsilon}{h}}) \quad \text{for } x \in \operatorname{supp}\nabla\chi_\varepsilon,$$

leads to

$$\langle \omega_{U'},\, d_{f,h}\omega_{U_0}\rangle = \mathcal{O}\left(e^{-\frac{c_1-c_0+C}{h}}\right),$$

and equation (27) is proved. □

4.3.1. Proof of Proposition 4.3 When the Metric is Euclidean in Some Adapted Morse Coordinates. Let us check Eqs. (28)-(29), when the metric is Euclidean in some local adapted Morse coordinates for f (around each critical point).

Note first that we have already proved, for a general metric, the following result,

$$\forall U' \in \mathcal{U}^{(p+1)}, \quad \langle \omega_{U'},\, d_{f,h}\omega_{U_0}\rangle = \langle v_{U'},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle + \mathcal{O}(e^{-\frac{c_1-c_0+C}{h}}),$$

where C is a positive constant. According to the choice of χ_ε, the first term of the right-hand side vanishes when $\partial_B U_1 \neq U_0$. Thus, we can focus on the term $\langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle$, when $U_1 \in \mathcal{U}^{(p+1)}$ satisfies $\partial_B(U_1) = U_0$.

In the ball $B(U_1,\eta)$, we use the above adapted Morse coordinates (y', y'') with $y' = (y_1,\dots,y_{p+1})$, $y'' = (y_{p+2},\dots,y_d)$, and $f(y) - c_1 = \frac{1}{2}\sum_{j=1}^{d} \lambda_j^1 y_j^2$. The parameter $\varepsilon > 0$ is chosen according to $\eta > 0$ so that

$$f_{c_1-2\varepsilon}^{c_1-\frac{3}{2}\varepsilon} \cap B(U_1,\eta) \neq \emptyset.$$

More precisely one takes C_1, $C_2 > 1$ and $\varepsilon = \varepsilon_\eta$ such that

$$f_{c_1-2\varepsilon}^{c_1-\frac{3}{2}\varepsilon} \cap \left\{|y''| < \frac{\eta}{C_1}\right\} \subset \left\{\frac{\eta}{C_2} < |y'| < \frac{2\eta}{C_2}, |y''| < \frac{\eta}{C_1}\right\} \subset B(U_1,\eta).$$

Lemma A.2.2 of [24] says that $d_{Ag}(x,y) = |f(x) - f(y)|$ if and only if there is a generalized integral curve of ∇f going from x to y. Hence, in $f_{c_1-2\varepsilon}^{c_1-\frac{3}{2}\varepsilon}$, the only points such that $d_{Ag}(U_1, y) = c_1 - f(y)$ are the points lying on the unstable manifold for $-\nabla f$. There exists consequently a constant $C_\eta > 0$ such that

$$\forall y \in f_{c_1-2\varepsilon}^{c_1-\frac{3}{2}\varepsilon} \setminus \left\{|y''| < \frac{\eta}{C_1}\right\}, \quad d_{\mathrm{Ag}}(U_1, y) \geq c_1 - f(y) + C_\eta.$$

By combining this with the exponential decay estimates for $\tilde{v}_{U_0}$, we deduce

$$\langle \omega_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle = \int_{|y''|\leq \frac{\eta}{C_1}} \langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle_{\Lambda T_y^* M} \;+\; \mathcal{O}\left(e^{-\frac{c_1-c_0+C_\eta}{h}}\right).$$

With the above inclusion and the approximation of v_{U_1} in $B(U_1,\eta)$ stated in Theorem 3.9 we get

$$\langle \omega_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle = K_{U_1}^h \int_{|y''|\leq \frac{\eta}{C_1}} \langle e^{-\frac{\sum_{j=1}^{d} |\lambda_j^1| y_j^2}{2h}} \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_{p+1},$$

$$hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle_{\Lambda T_y^* M} + \mathcal{O}\left(e^{-\frac{c_1-c_0+C_\eta}{h}}\right),$$

with $K_{U_1}^h = \frac{|\lambda_1^1 \ldots \lambda_d^1|^{1/4}}{(\pi h)^{d/4}}$. With an Euclidean metric, inserting $e^{-\frac{f-c_1}{h}} \times e^{\frac{f-c_0}{h}}$ in the bracket implies that $\frac{e^{\frac{c_1-c_0}{h}}}{K_{U_1}^h} \langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0} \rangle$ equals

$$\int_{|y''| \leq \frac{\eta}{C_1}} \langle e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \mathrm{d}y_1 \wedge \ldots \wedge \mathrm{d}y_{p+1}, hd\chi_\varepsilon \wedge e^{\frac{f-c_0}{h}} \tilde{v}_{U_0} \rangle_{\Lambda T_y^* M} + \mathcal{O}(e^{-\frac{C\eta}{h}})$$

$$= h \int_{|y''| \leq \frac{\eta}{C_1}} d\chi_\varepsilon \wedge e^{\frac{f-c_0}{h}} \tilde{v}_{U_0} \wedge (e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \mathrm{d}y_{p+2} \wedge \ldots \wedge \mathrm{d}y_d) + \mathcal{O}(e^{-\frac{C\eta}{h}}).$$

But our assumption says that $d\left(e^{\frac{f}{h}} \tilde{v}_{U_0}\right) = 0$ in $\operatorname{supp} \nabla \chi_\varepsilon$. Moreover, one has clearly $d\left(e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \mathrm{d}y_{p+2} \wedge \ldots \wedge \mathrm{d}y_d\right) = 0$. Hence the integrand is nothing but

$$(-1)^{pd} \mathrm{d}\left(\chi_\varepsilon e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \mathrm{d}y_{p+2} \wedge \ldots \wedge \mathrm{d}y_d \wedge e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}\right).$$

By Stokes' formula, the quantity $\frac{e^{\frac{c_1-c_0}{h}}}{K_{U_1}^h} \langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0} \rangle$ equals

$$(-1)^{pd} h \int_{|y''| \leq \frac{\eta}{C_1}} \int_{|y'| = \frac{2\eta}{C_2}} e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \mathrm{d}y_{p+2} \wedge \ldots \wedge \mathrm{d}y_d \wedge e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}$$
$$+ \mathcal{O}(e^{-\frac{C\eta}{h}}),$$

and by introducing, for every fixed y'' such that $|y''| \leq \frac{\eta}{C_1}$, the cycle $C_{y''}$ supported by $\left\{(y', y''), |y'| = \frac{2\eta}{C_2}\right\}$ and homotopic to $\partial e_{U_1}^{p+1}$, we get

$$\frac{e^{\frac{c_1-c_0}{h}}}{hK_{U_1}^h} \langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0} \rangle$$
$$= (-1)^{pd} \int_{|y''| \leq \frac{\eta}{C_1}} e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \int_{C_{y''}} e^{\frac{f-c_0}{h}} \tilde{v}_{U_0} + \mathcal{O}(e^{-\frac{C\eta}{h}}).$$

For any y'', the cycle $C_{y''}$ is homologous to $\partial e_{U_1}^{p+1}$ and, according to Proposition 2.12, to $\kappa[e_{U_0}^p]$ in $f^{c_1-\varepsilon}$ relatively to $f^{c_0-\gamma_\eta}$, with $\gamma_\eta > 0$ small enough. Owing to $d\left(e^{\frac{f}{h}} \tilde{v}_{U_0} = 0\right)$ in $f^{c_1-\varepsilon}$ and to the exponential decay estimate of $\tilde{v}_{U_0}$ stated in Theorem 3.9, we obtain

$$\frac{e^{\frac{c_1-c_0}{h}}}{hK_{U_1}^h} \langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0} \rangle$$
$$= (-1)^{pd} \kappa \int_{|y''| \leq \frac{\eta}{C_1}} e^{-\frac{\sum_{j=p+2}^d |\lambda_j^1| y_j^2}{h}} \int_{e_{U_0}^p} e^{\frac{f-c_0}{h}} \tilde{v}_{U_0} + \mathcal{O}(e^{-\frac{C\eta}{h}}).$$

Using again Theorem 3.9 with $\tilde{v}_{U_0}$, $f(U_0) = c_0$, and the decomposition of f around U_0, $f(z) - c_0 = \frac{1}{2}\sum_{j=1}^{d} \lambda_j^0 z_j^2$ in some (adapted) Morse coordinates $(z', z'') = (z_1, \ldots, z_p, z_{p+1}, \ldots, z_d)$, we get,

$$\begin{aligned}
&\frac{e^{\frac{c_1-c_0}{h}}}{hK_{U_1}^h K_{U_0}^h}\langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle \\
&\quad = (-1)^{pd}\kappa \int\limits_{|y''|\leq \frac{\eta}{C_1}} e^{-\frac{\sum_{j=p+2}^{d}|\lambda_j^1| y_j^2}{h}} \int\limits_{e_{U_0}^p} e^{-\frac{\sum_{j=1}^{p}|\lambda_j^0| z_j^2}{h}} + \mathcal{O}(e^{-\frac{C\eta}{h}}),
\end{aligned}$$

with $K_{U_0}^h = \frac{|\lambda_1^0 \ldots \lambda_d^0|^{1/4}}{(\pi h)^{d/4}}$.

Now, writing successively two Laplace methods, we obtain

$$\frac{e^{\frac{c_1-c_0}{h}}}{hK_{U_1}^h K_{U_0}^h}\langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle = (-1)^{pd}\kappa \frac{(\pi h)^{\frac{d-1}{2}}}{|\lambda_{p+2}^1 \cdots \lambda_d^1|^{\frac{1}{2}}\,|\lambda_1^0 \cdots \lambda_p^0|^{\frac{1}{2}}}(1 + \mathcal{O}(h)),$$

which leads, finally, to the following formula:

$$\begin{aligned}
&\langle \omega_{U_1}, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle \\
&\quad = (-1)^{pd}\kappa\Big(\frac{h}{\pi}\Big)^{\frac{1}{2}} \frac{|\lambda_1^1 \cdots \lambda_{p+1}^1|^{\frac{1}{4}}}{|\lambda_{p+2}^1 \cdots \lambda_d^1|^{\frac{1}{4}}} \frac{|\lambda_{p+1}^0 \cdots \lambda_d^0|^{\frac{1}{4}}}{|\lambda_1^0 \cdots \lambda_p^0|^{\frac{1}{4}}} e^{-\frac{c_1-c_0}{h}}(1 + \mathcal{O}(h)).
\end{aligned}$$

The picture below (Fig. 3) summarizes the scheme of the calculation and the use of Stokes' formula, for $d = 3$ and $p = 1$.

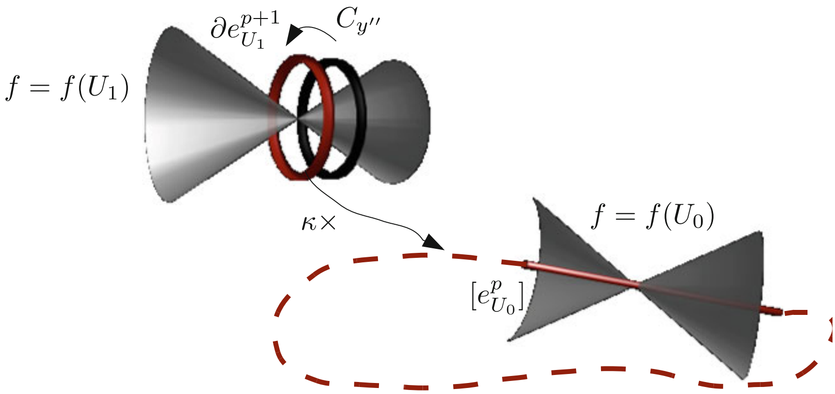

Figure 3. The arrows show the use of Skokes' formula. The *dotted part* of $[e_{U_0}^p]$ shows the part of $[e_{U_0}^p]$ lying below $f(U_0) - \gamma_\eta$

4.3.2. Proof of Proposition 4.3 for a General Riemannian Metric. As in the previous subsection, we look at the term $\langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle$, where $U_1 \in \mathcal{U}^{(p+1)}$ satisfies $\partial_B(U_1) = U_0$, and we use some adapted Morse coordinates $(y', y'') = (y_1, \ldots, y_{p+1}, y_{p+2}, \ldots, y_d)$ in the ball $B(U_1, \eta)$. Let us recall that the function f has the following decomposition in these coordinates:

$$f(y) - c_1 = \frac{1}{2}\sum_{j=1}^{d} \lambda_j^1 y_j^2.$$

Again, one takes C_1, $C_2 > 1$ and $\varepsilon = \varepsilon_\eta$ such that

$$f_{c_1-2\varepsilon}^{c_1-\frac{3}{2}\varepsilon} \cap \left\{|y''| < \frac{\eta}{C_1}\right\} \subset \left\{\frac{\eta}{C_2} < |y'| < \frac{2\eta}{C_2}, |y''| < \frac{\eta}{C_1}\right\} \subset B(U_1, \eta),$$

and we have the existence of $C_\eta > 0$ s.t.

$$\langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle = \int_{|y''| \le \frac{\eta}{C_1}} \langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle_{\Lambda T_y^* M} + \mathcal{O}(e^{-\frac{c_1 - c_0 + C_\eta}{h}}).$$

Choose $\eta > 0$ small enough such that U_1 is the only critical point of f in $f^{-1}([c_1 - 2\eta, c_1 + 2\eta])$.

Now, let us introduce the metric g_1,

$$g_1(y) = \chi(y) g_e(y) + (1 - \chi(y)) g(y),$$

where $0 \le \chi \le 1$ is a smooth cut-off function such that $\chi = 1$ in $B(U_1, \eta)$, $\chi = 0$ outside $B(U_1, \frac{3}{2}\eta)$, g is the usual metric on M, and g_e is the Euclidean metric $g_e = \sum_{i=1}^d (dy_i)^2$.

For g and g_1, let $\tilde{v}_{U_1}$ and $\tilde{v}_{U_1}^1$ denote, respectively, the forms defined in Theorem 3.9 with $a = c_1 - 2\eta$ and $b = c_1 + 2\eta$. Since U_1 is the only critical point of f in $f^{-1}([c_1 - 2\eta, c_1 + 2\eta])$, this means that $\tilde{v}_{U_1}^i$ is a normalized form in the one-dimensional kernel of $\Delta_{g,f,h}^{TN,(p+1)}$ (resp. $\Delta_{g_1,f,h}^{TN,(p+1)}$), the Witten Laplacian corresponding to the metric g (resp. g_1).

Note also that the boundary conditions are strictly the same for both $\tilde{v}_{U_1}$ and $\tilde{v}_{U_1}^1$, since the metrics g and g_1 coincide near the boundary. In particular, $\tilde{v}_{U_1}$ and $\tilde{v}_{U_1}^1$ belong to the same domain $D(\Delta_{f,h}^{TN,(p+1)}) := D(\Delta_{g,f,h}^{TN,(p+1)}) = D(\Delta_{g_1,f,h}^{TN,(p+1)})$. Analogously, $\star\tilde{v}_{U_1}$ and $\star_1\tilde{v}_{U_1}^1$ belong to the same domain $D(\Delta_{-f,h}^{NT,(d-p-1)}) := D(\Delta_{g,-f,h}^{NT,(d-p-1)}) = D(\Delta_{g_1,-f,h}^{NT,(d-p-1)})$ (we refer to Remark 3.2 for the meaning of $\Delta_{-f,h}^{NT}$)).

Since

$$v_{U_1} = \tilde{v}_{U_1} + \mathcal{O}(e^{-\frac{C_\eta}{h}})$$

holds in $f^{-1}([c_1 - 2\eta, c_1 + 2\eta]) \cap \operatorname{supp} d\chi_\varepsilon$, it suffices to estimate

$$\langle v_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle = \int_{|y''| \le \frac{\eta}{C_1}} \langle \tilde{v}_{U_1},\, hd\chi_\varepsilon \wedge \tilde{v}_{U_0}\rangle_{\Lambda T_y^* M} \;+\; \mathcal{O}(e^{-\frac{c_1 - c_0 + C_\eta}{h}}).$$

The following lemma gives some useful relations between $\tilde{v}_{U_1}$ and $\tilde{v}^1_{U_1}$, especially the second one which will be crucial in the sequel.

Lemma 4.4. *There exist ω in $D(\Delta^{TN,(p)}_{f,h})$ and ω' in $D(\Delta^{NT,(d-p-2)}_{-f,h})$ s.t.*

$$e^{\frac{f-c_1}{h}}\,\tilde{v}_{U_1} = h\,d\left(e^{\frac{f-c_1}{h}}\omega\right) + (1+\mathcal{O}(h))e^{\frac{f-c_1}{h}}\,\tilde{v}^1_{U_1}, \tag{31}$$

$$\star\left(e^{-\frac{f-c_1}{h}}\,\tilde{v}_{U_1}\right) = h\,d\left(e^{-\frac{f-c_1}{h}}\omega'\right) + (1+\mathcal{O}(h))\star_1\left(e^{-\frac{f-c_1}{h}}\,\tilde{v}^1_{U_1}\right). \tag{32}$$

Proof. The form $\tilde{v}_{U_1}$ (resp. $\tilde{v}^1_{U_1}$) is in the one-dimensional kernel of $\Delta^{TN,(p+1)}_{g,f,h}$ (resp. $\Delta^{TN,(p+1)}_{g_1,f,h}$) and the isomorphisms

$$\ker\Delta^{TN,(p+1)}_{g,f,h} \sim \ker d^{TN}_{f,h}\,/\,\mathrm{Ran}\,d^{TN}_{f,h} \sim \ker\Delta^{TN,(p+1)}_{g_1,f,h},$$

with the middle set independent of the metrics, implies the existence of a constant $\alpha_1 \neq 0$ s.t.

$$\tilde{v}_{U_1} - \alpha_1\,\tilde{v}^1_{U_1} \in \mathrm{Ran}\,d^{TN}_{f,h}.$$

This means the existence of ω in $D(\Delta^{TN,(p+1)}_{f,h})$ s.t.

$$e^{\frac{f-c_1}{h}}\,\tilde{v}_{U_1} - \alpha_1\,e^{\frac{f-c_1}{h}}\,\tilde{v}^1_{U_1} = h\,d\left(e^{\frac{f-c_1}{h}}\omega\right). \tag{33}$$

Moreover, the form $\star\tilde{v}_{U_1}$ (resp. $\star_1\tilde{v}^1_{U_1}$) belongs to $\ker(\Delta^{NT,(d-p-1)}_{g,-f,h})$ (resp. $\ker(\Delta^{NT,(d-p-1)}_{g_1,-f,h})$), and there exists another constant $\alpha'_1 \neq 0$ s.t.

$$\star\tilde{v}_{U_1} - \alpha'_1\star_1\tilde{v}^1_{U_1} \in \mathrm{Ran}\,d^{NT}_{-f,h}.$$

According to the definition of $d_{-f,h}$, it means that there exists ω' in $D(\Delta^{NT,(d-p-2)}_{-f,h})$ s.t.

$$\star\left(e^{-\frac{f-c_1}{h}}\,\tilde{v}_{U_1}\right) - \alpha'_1\star_1\left(e^{-\frac{f-c_1}{h}}\,\tilde{v}^1_{U_1}\right) = h\,d\left(e^{-\frac{f-c_1}{h}}\omega'\right). \tag{34}$$

In order to show that $\alpha_1 = 1+\mathcal{O}(h)$, let us integrate Eq. (33) along the unstable manifold $\mathcal{C}_{\mathrm{Unst}}$, which is a $(p+1)$-cycle in $\overline{f^{c_1+2\eta}_{c_1-2\eta}}$ relatively to $\{f = c_1 - 2\eta\}$. Using Stokes' formula, we obtain

$$\int_{\mathcal{C}_{\mathrm{Unst}}} e^{\frac{f-c_1}{h}}\left(\tilde{v}_{U_1} - \alpha_1\tilde{v}^1_{U_1}\right) = h\int_{\mathcal{C}_{\mathrm{Unst}}} d\left(e^{\frac{f-c_1}{h}}\omega\right) = 0.$$

Consequently, the constant α_1 is given by

$$\alpha_1 = \frac{\int_{\mathcal{C}_{\mathrm{Unst}}} e^{\frac{f-c_1}{h}}\,\tilde{v}_{U_1}}{\int_{\mathcal{C}_{\mathrm{Unst}}} e^{\frac{f-c_1}{h}}\,\tilde{v}^1_{U_1}} = 1+\mathcal{O}(h),$$

where the last equality comes from a Laplace method applied to each integral, after introducing the first order WKB approximations of $\tilde{v}_{U_1}$ and $\tilde{v}^1_{U_1}$ recalled from [24] in the last statements of Theorem 3.9.

In order to obtain the same estimate for α'_1, we make the same computation with equation (34) along the stable manifold $\mathcal{C}_{\mathrm{St}}$, which is a $(d-p-1)$-cycle in $\overline{f_{c_1-2\eta}^{c_1+2\eta}}$ relatively to $\{f = c_1 + 2\eta\}$. This gives, according again to the last statements of Theorem 3.9,

$$\alpha_1' = \frac{\int_{\mathcal{C}_{\mathrm{St}}} e^{-\frac{f-c_1}{h}} \star \tilde{v}_{U_1}}{\int_{\mathcal{C}_{\mathrm{St}}} e^{-\frac{f-c_1}{h}} \star_1 \tilde{v}_{U_1}^1} = 1 + \mathcal{O}(h).$$

□

Consider now the quantity

$$\begin{aligned} A &:= \int_{|y''| \le \frac{\eta}{C_1}} \langle \tilde{v}_{U_1}, h d\chi_\varepsilon \wedge \tilde{v}_{U_0} \rangle_{\Lambda T_y^* M} \\ &= e^{-\frac{c_1-c_0}{h}} h \int_{|y''| \le \frac{\eta}{C_1}} d\chi_\varepsilon \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) \wedge \star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}), \end{aligned}$$

by recalling that $\tilde{v}_{U_1}$, like $\tilde{v}_{U_1}^1$ and therefore α_1, α_1' ω and ω', can be chosen real. By our assumption, $\tilde{v}_{U_1}$ is in $f^{-1}([c_1 - 2\eta, c_1 + 2\eta])$ solution to $\Delta_{g,f,h}^{TN,(p+1)} \tilde{v}_{U_1} = 0$, then we have on this domain the following equality:

$$d\left(\star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1})\right) = (-1)^{p+1} \star \left(e^{-\frac{f-c_1}{h}} d_{g,f,h}^* \tilde{v}_{U_1}\right) = 0.$$

Keeping in mind the relation $d(e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) = 0$ in $\operatorname{supp} d\chi_\varepsilon$, this implies

$$d\chi_\varepsilon \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) \wedge \star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}) = (-1)^{pd} d \left(\chi_\varepsilon \star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}) \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0})\right).$$

Then we have by Stokes' formula,

$$\begin{aligned} e^{\frac{c_1-c_0}{h}} A &= (-1)^{pd} h \int_{\partial(\{|y''| \le \frac{\eta}{C_1}\})} \chi_\varepsilon \star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}) \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) \\ &= (-1)^{pd} h \int_{|y''| \le \frac{\eta}{C_1}} \int_{|y'| = \frac{2\eta}{C_2}} \star (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}) \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) + \mathcal{O}(e^{-\frac{C}{h}}). \end{aligned}$$

Using now the second equation of Lemma 4.4, let us write

$$\begin{aligned} (-1)^{pd} e^{\frac{c_1-c_0}{h}} A = h(1 + \mathcal{O}(h)) \int_{|y''| \le \frac{\eta}{C_1}} \int_{|y'| = \frac{2\eta}{C_2}} \star_1 (e^{-\frac{f-c_1}{h}} \tilde{v}_{U_1}^1) \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) \\ + h^2 \int_{|y''| \le \frac{\eta}{C_1}} \int_{|y'| = \frac{2\eta}{C_2}} d\left(e^{-\frac{f-c_1}{h}} \omega'\right) \wedge (e^{\frac{f-c_0}{h}} \tilde{v}_{U_0}) + \mathcal{O}(e^{-\frac{C}{h}}). \end{aligned} \tag{35}$$

But, looking at the second equation of Lemma 4.4 and using our exponential decay estimates, the second term of the r.h.s. is also (up to an exponentially error term),

$$h^2 \int_{\partial(\{|y''|\leq \frac{\eta}{C_1}\})} \mathrm{d}\left(e^{-\frac{f-c_1}{h}}\omega'\right) \wedge (e^{\frac{f-c_0}{h}}\tilde{v}_{U_0}) + \mathcal{O}(e^{-\frac{C}{h}}),$$

where the integral term is 0 owing to Stokes' formula.

Equation (35) can then be rewritten

$$e^{\frac{c_1-c_0}{h}}A = (-1)^{pd}h(1+\mathcal{O}(h)) \int_{|y''|\leq \frac{\eta}{C_1}} \int_{|y'|=\frac{2\eta}{C_2}} \star_1(e^{-\frac{f-c_1}{h}}\tilde{v}^1_{U_1}) \wedge (e^{\frac{f-c_0}{h}}\tilde{v}_{U_0})$$
$$+\mathcal{O}(e^{-\frac{C}{h}}),$$

and we can focus on the integral part of the r.h.s.,

$$B := \int_{|y''|\leq \frac{\eta}{C_1}} \int_{|y'|=\frac{2\eta}{C_2}} \star_1(e^{-\frac{f-c_1}{h}}\tilde{v}^1_{U_1}) \wedge (e^{\frac{f-c_0}{h}}\tilde{v}_{U_0}).$$

Since the metric g_1 is Euclidean in the ball $B(U_1,\eta)$, the Morse decomposition of f combined with Theorem 3.9 gives

$$B = K^h_{U_1} \int_{|y''|\leq \frac{\eta}{C_1}} \int_{|y'|=\frac{2\eta}{C_2}} \times e^{-\frac{\sum_{j=p+2}^{d}|\lambda^1_j|y_j^2}{h}} \mathrm{d}y_{p+2}\wedge\ldots\wedge \mathrm{d}y_d \wedge (e^{\frac{f-c_0}{h}}\tilde{v}_{U_0})$$
$$+\mathcal{O}(e^{-\frac{C_\eta}{h}}),$$

with $K^h_{U_1} = \frac{|\lambda^1_1\ldots\lambda^1_d|^{1/4}}{(\pi h)^{d/4}}$.

We can then follow the same proof as the one used in the locally Euclidean case in order to obtain the wanted result. The only difference arises in the fact that the WKB expansion of $\tilde{v}_{U_0}$, given in the last statement of Theorem 3.9, contains higher order correcting terms, because the full WKB expansion of $\tilde{v}_{U_0}$ depends on the metric, and this produces a relative $\mathcal{O}(h)$ error term.

4.4. End of the Proof of Theorem 1.1

We recall firstly some notations of Sect. 4.1. The operator $\Delta_{f,h}$ is defined on M (i.e. on $\overline{f^{+\infty}_{-\infty}}$) and

$$F = \oplus_{p=0}^{d} F^{(p)} \quad \text{with} \quad F^{(p)} = \operatorname{Im} 1_{[0,h^2]}(\Delta^{(p)}_{f,h}).$$

According to Sect. 3, $F^{(p)}$ admits an almost orthonormal basis, $\{v_U, U\in\mathcal{U}^{(p)}\}$, fulfilling the properties of Theorem 3.9.

Consider now the family of quasimodes, $\{\omega_U, U\in\bigcup_{p=0}^{d}\mathcal{U}^{(p)}\}$, constructed in Sect. 4.2. For any p in $\{0,\ldots,d\}$ and $U\in\mathcal{U}^{(p)}$, the p-form ω_U belongs to $F^{(p)}$ and, as already mentioned, ω_U satisfies the relation $\|\omega_U - v_U\| = \mathcal{O}(e^{-\frac{C_\eta}{h}})$ (see Proposition 4.2 for upper or homological critical points and (30) for lower critical points).

The family $\left\{\omega_U, U\in\mathcal{U}^{(p)}\right\}$ is then an almost orthonormal basis of $F^{(p)}$ and, thanks to Proposition 4.3, Theorem 2.3 of [33] applies. For h_0 small enough and $h\in(0,h_0]$, we obtain an accurate writing of the non-zero eigenvalues of $d^*_{f,h}d_{f,h} : F\to F$.

More precisely, when restricted to $F^{(p)}$, the non-zero eigenvalues of $d^*_{f,h}$ $d_{f,h}$ are the quantities

$$\kappa^2 B(h) e^{-2\frac{f(U_U^{(p+1)})-f(\partial_B(U_U^{(p+1)}))}{h}}(1+\mathcal{O}(h)), \quad U_U^{(p+1)} \in \mathcal{U}_U^{(p+1)},$$

with

$$B(h) = \frac{h}{\pi} \frac{|\lambda_1^1 \cdots \lambda_{p+1}^1|^{\frac{1}{2}}}{|\lambda_{p+2}^1 \cdots \lambda_d^1|^{\frac{1}{2}}} \frac{|\lambda_{p+1}^0 \cdots \lambda_d^0|^{\frac{1}{2}}}{|\lambda_1^0 \cdots \lambda_p^0|^{\frac{1}{2}}},$$

where $\lambda_1^\ell < \cdots < \lambda_{p+\ell}^\ell < 0 < \lambda_{p+\ell+1}^\ell < \cdots < \lambda_d^\ell$ denote the eigenvalues of Hess $f(U_\ell)$; for $\ell \in \{0,1\}$, $U_1 := U_{\mathrm{U}}^{(p+1)}$, and $U_0 := \partial_{\mathrm{B}}(U_{\mathrm{U}}^{(p+1)})$.

In particular, $d^*_{f,h} d_{f,h} : F \to F$ has exactly card$\,\mathcal{U}_{\mathrm{U}}$ = card$\,\mathcal{U}_{\mathrm{L}}$ non-zero eigenvalues and these eigenvalues are distinct.

This provides all the exponentially small non-zero eigenvalues of $\Delta_{f,h}$ according to the last statement of Theorem 3.3.

This ends the proof of Theorem 1.1.

4.5. A Relative Version of Theorem 1.1

The analysis for Theorem 1.1 is done on $M = f_{-\infty}^{+\infty}$. All the constructions and the good restriction properties of Morse–Barannikov chain complex, when considering $H_*(f^b, f^a)$, $-\infty \leq a < b \leq +\infty$, have their counterpart with the Witten Laplacians $\Delta_{f,h}^{TN}$ defined on $\overline{f_a^b}$ in Section 3. Hence all the proof of Theorem 1.1 is still valid for $\Delta_{f,h}^{TN}$ on $\overline{f_a^b}$ except that some end points of the relation $\partial_B U^{(p+1)} = U^{(p)}$ disappear when they lie in f^a or f_b. We state without more detail the spectral result for $\Delta_{f,h}^{TN}$ on $\overline{f_a^b}$.

Theorem 4.5. *With the same assumptions as in Theorem 1.1 and when a, b are not critical values of f, $-\infty \leq a < b \leq +\infty$, the exponentially small eigenvalues of $\Delta_{f,h}^{TN}$, defined on $\overline{f_a^b}$ according to Proposition 3.1, are given by a mapping $j_a^b : \mathcal{U} \cap f_a^b \to \sigma(\Delta_{f,h}^{TN})$ derived from the mapping j of Theorem 1.1 by:*

- $j_a^b(U) = j(U)(1+\mathcal{O}(h)) \neq 0$ *if* $U \in \mathcal{U}_L \cap f_a^b$ *and* $U = \partial_B U'$ *with* $U' \in f_a^b$ *;*
- $j_a^b(U) = j(U)(1+\mathcal{O}(h)) \neq 0$ *if* $U \in \mathcal{U}_U \cap f_a^b$ *and* $\partial_B U = U'$ *with* $U' \in f_a^b$ *;*
- $j_a^b(U) = 0$ *else.*

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References

[1] Barannikov, S.A.: The framed Morse complex and its invariants. Singularities and bifurcations. Adv. Soviet Math. **21**, 93–115 (1994)

[2] Berglund, N.: Kramers' law: Validity, derivation and generalisations. (Preprint). arXiv:1106.5799v1 (2011)

[3] Bismut, J.M., Lebeau, G.: The hypoelliptic Laplacian and Ray-Singer metrics. Annals of Mathematics Studies, vol. 167. Princeton University Press, New Jersy (2008)

[4] Bismut, J.M.: The Witten complex and the degenerate Morse inequalities. J. Differ. Geom. **23**, 207–240 (1986)

[5] Bismut, J.M.: Hypoelliptic Laplacian and Bott-Chern cohomology. Comptes Rendus Mathematique **349**(2), 75–80 (2011)

[6] Bovier, A., Eckhoff, M., Gayrard, V., Klein, M.: Metastability in reversible diffusion processes I: sharp asymptotics for capacities and exit times. JEMS **6**(4), 399–424 (2004)

[7] Bovier, A., Gayrard, V., Klein, M.: Metastability in reversible diffusion processes II: precise asymptotics for small eigenvalues. JEMS **7**(1), 69–99 (2004)

[8] Bott, R.: Lectures on Morse theory, old and new. Bull. Am. Math. Soc. (N.S.) **7**(2), 331–358 (1982)

[9] Bott, R.: Morse theory indomitable, vol. 78, pp. 99–114. Publications IHES, Tome (1988)

[10] Bott, R., Tu, L.: Differential forms in algebraic topology. Graduate Texts in Mathematics, vol. 82. Springer, Berlin (1982)

[11] Chang, K.C., Liu, J.: A cohomology complex for manifolds with boundary. Topol. Methods NonLinear Anal. **5**, 325–340 (1995)

[12] Cycon, H.L., Froese, R.G., Kirsch, W., Simon, B.: Schrödinger operators with application to quantum mechanics and global geometry. Text and Monographs in Physics. Springer, Berlin (1987)

[13] Dimassi, M., Sjöstrand, J.: Spectral Asymptotics in the semi-classical limit. London Mathematical Society. Lecture Note Series, Vol. 268. Cambridge University Press, Cambridge (1999)

[14] Freidlin, M.I., Wentzell, A.D.: Random perturbations of dynamical systems. Transl. from the Russian by Joseph Szuecs, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 260. Springer, Berlin (1998).

[15] Fulton, W.: 1995 Algebraic topology. A first course. Graduate Texts in Mathematics, vol. 153. Springer (1995)

[16] Guérini, P.: Prescription du spectre du laplacien de Hodge-de Rham. Ann. Sci. École Norm. Sup. (4) **37**(2), 270–303 (2004)

[17] Hatcher, A.: Algebraic topology. Cambridge University Press, Cambridge (2002)

[18] Helffer, B.: Introduction to the semi-classical Analysis for the Schrödinger operator and applications. Lecture Notes in Mathematics, vol. 1336, Springer, Berlin (1988)

[19] Helffer, B., Klein, M., Nier, F.: Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach. Matematica Contemporanea **26**, 41–85 (2004)

[20] Helffer, B., Nier, F.: Quantitative analysis of metastability in reversible diffusion processes via a Witten complex approach: the case with boundary. Mémoire 105, Société Mathématique de France, (2006)

[21] Helffer, B., Nier, F.: Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians. Lecture Notes in Mathematics, vol. 1862. Springer, Berlin (2005)

[22] Helffer, B., Sjöstrand, J.: Puits multiples en limite semi-classique II—Interaction moléculaire—Symétries - Perturbations. Ann. Inst. H. Poincaré Phys. Théor. **42**(2), 127–212 (1985)

[23] Helffer, B., Sjöstrand, J.: Multiple wells in the semi-classical limit III. Math. Nachrichte **124**, 263–313 (1985)

[24] Helffer, B., Sjöstrand, J.: Puits multiples en limite semi-classique IV - Étude du complexe de Witten -. Commun. Partial Differ. Equ. **10**(3), 245–340 (1985)

[25] Hérau, F., Nier, F.: Isotropic hypoellipticity and trend to the equilibrium for the Fokker–Planck equation with high degree potential. Arch. Ration. Mech. Anal. **171**(2), 151–218 (2004)

[26] Hérau, F., Sjöstrand, J., Stolk, C.: Semiclassical analysis for the Kramers–Fokker–Planck equation. Commun. Partial Differ. Equ. **30**(4-6), 689–760 (2005)

[27] Hérau, F., Hitrik, M., Sjöstrand, J.: Tunnel effect for Kramers–Fokker*-Planck type operators. Ann. Henri Poincaré **9**(2), 209–274 (2008)

[28] Hérau, F., Hitrik, M., Sjöstrand, J.: Tunnel effect and symmetries for Kramers–Fokker–Planck type operators. J. Inst. Math. Jussieu **10**(3), 567–634 (2011)

[29] Holley, R., Kusuoka, R., Stroock, D.: Asymptotic of the spectral gap with applications to the theory of simulated annealing. J. Funct. Anal. **83**(2), 333–347 (1989)

[30] Koldan, N., Prokhorenkov, I., Shubin, M.: Semiclassical asymptotics on manifolds with Boundary. Spectral analysis in geometry and number theory, Contemprory Mathematics, vol. 484, pp. 239–266. American Mathematical Society, USA (2009)

[31] Laudenbach, F.: On the Thom-Smale complex. Astérisque **205**, 219–233 (1992)

[32] Laudenbach, F.: A Morse complex on manifolds with boundary. Geometrica Dedicata **153**(1), 47–57 (2011)

[33] Le Peutrec, D.: Small singular values of an extracted matrix of a Witten complex. Cubo A Math. J. **11**(4), 49–57 (2009)

[34] Le Peutrec, D.: Local WKB construction for Witten Laplacians on manifolds with boundary. Analysis PDE **3**(3), 227–260 (2010)

[35] Le Peutrec, D.: Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian. Ann. Fac. Sci. Toulouse Math (6) **19**(3–4), 735–809 (2010)

[36] Le Peutrec, D.: Small eigenvalues of the Witten Laplacian acting on p-forms on a surface. Asymptot. Anal. **73**(4), 187–201 (2011)

[37] Massey, W.S.:A basic course in algebraic topology. Graduate Texts in Mathematics, vol. 127. Springer, Berlin (1991)

[38] Milnor, J.: Morse theory. Annals of Mathematics Studies No. 51. Princeton University Press, Princeton (1963)

[39] Nelson, E.: Dynamical Theories of Brownian Motion, 2nd edn. Princeton University Press, Princeton (2002)

[40] Schwarz, G.: Hodge Decomposition. A method for solving boundary value problems. Lecture Notes in Mathematics, vol. 1607. Springer, Besrlin (1995)

[41] Simon, B.: Trace Ideals and Their Applications. London Mathematical Society. Lecture Note Series, vol. 35. Cambridge University Press, Cambridge (1979)

[42] Spanier, E.H.: Algebraic Topology. McGraw-Hill Series in Higher Mathematics, USA (1966)

[43] Taylor, M.E.: Partial Differential Equations 1, basic theory. Applied Mathematical Sciences, vol. 115. Springer, Berlin (1997)

[44] Tailleur, J., Tanase-Nicola, S., Kurchan, J.: Kramers equation and supersymmetry. J. Stat. Phys. **122**(4), 557–595 (2006)

[45] Witten, E.: Supersymmetry and morse inequalities. J. Diff. Geom. **17**(4), 661–692 (1982)

[46] Zhang, W.: Lectures on Chern-Weil theory and Witten deformations. Nankai Tracts in Mathematics, vol. 4. World Scientific Publishing Co., Hackensack (2001)

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