Erratum to “Resonance Free Domains for Non Globally Analytic Potentials” Ann. Henri Poincaré 3(4) (2002), 739–756

André Martinez*

In this erratum we correct the proof of Lemma 6.1 of [4]. Lemma 6.1 (here rewritten as Lemma 1, in a slightly improved form). This lemma was used in [4] for the proof of the main result (another proof of which can be found in DSZ), and later in [2].

At first, we recall some notations (we refer to [4] for more details about the definitions).

For $G \in C_0^\infty(\mathbb{R}^{2n}; \mathbb{R})$, we set,

$$H(x, \xi) = G(x, \xi) - \xi \cdot \partial_\xi G(x, \xi)$$

and, for $t \in \mathbb{R}$ small enough,

$$\Lambda_{tG} = \left\{ (x + it\partial_\xi G(x, \xi), \xi - it\partial_x G(x, \xi)) ; (x, \xi) \in \mathbb{R}^{2n} \right\}.$$

We also introduce the F.B.I. transform T , defined by,

$$Tu(x, \xi) = c \int e^{i(x-y)\xi/h - (x-y)^2/2h} u(y) dy$$

where $c = 2^{-\frac{n}{2}}(\pi h)^{-\frac{3n}{4}}$, and, for $g = g(\alpha, y, \eta)$ defined and smooth on $\Lambda_{tG} \times \mathbb{R}^{2n}$, and admitting some polynomial behavior as $|(y, \eta)| \rightarrow \infty$, we set,

$$T_g u(\alpha) := c \int e^{i(\alpha_x - y)\alpha_\xi/h - (\alpha_x - y)^2/2h} \text{Op}_h^W(g(\alpha, \cdot)) u(y) dy,$$

where $\text{Op}_h^W(g(\alpha, \cdot))$ stands for the semiclassical Weyl-quantization of the α -dependent symbol: $(y, \eta) \mapsto g(\alpha, y, \eta)$. In the sequel, g will be one of the two

* Investigation supported by University of Bologna. Funds for selected research topics.

functions $f = f(\alpha, y, \eta)$ and $mr = m(\alpha)r(\alpha, y, \eta)$, where f, m, r verify,

$$\begin{aligned}\partial^\gamma f(\alpha, y, \eta) &= \mathcal{O}(\langle \alpha_\xi, \eta \rangle^d); \\ \partial^\gamma r(\alpha, y, \eta) &= \mathcal{O}(\langle \alpha_\xi, \eta \rangle^d \mu(t)); \\ \partial^{\gamma'} m(\alpha) &= \mathcal{O}(\langle \alpha_\xi \rangle^\ell),\end{aligned}\tag{1}$$

for some $d, \ell \geq 0$, all $\gamma \in \mathbb{N}^{4n}$, $\gamma' \in \mathbb{N}^{2n}$, and where $\mu(t)$ is just a notation, and stands either for $|t|^\infty$ or for $e^{-\varepsilon/|t|^{1/(s-1)}}$ for some $\varepsilon > 0$, depending whether f and r are assumed to be C^∞ or s -Gevrey with respect to the variables (y, η) .

Finally, for $\sigma \in \mathbb{R}$ fixed arbitrarily and $g \in \{f, mr\}$, we set,

$$K_g = \langle \alpha_\xi \rangle^\sigma e^{-tH/h} T_g T_g^* e^{tH/h} \langle \alpha_\xi \rangle^{-(d+\ell+\sigma)},$$

considered as an operator on $L^2(\Lambda_{tG})$. Then, we have,

Lemma 1. *The operators K_f and K_{mr} are bounded on $L^2(\Lambda_{tG})$, and their norms verify,*

$$\begin{aligned}\|K_f\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(1 + h^{n/2} \mu(t) e^{J_0|t|/h}\right) \\ \|K_{mr}\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(h^{n/2} \mu(t) e^{J_0|t|/h} + h^\infty\right),\end{aligned}$$

uniformly as $h \rightarrow 0_+$ and $t \rightarrow 0$, and with $J_0 := 2 \sup_{\mathbb{R}^{2n}} (|G| + |\nabla G|^2)$.

Remark. In [4], the incorrect proof was mainly inspired from [1], where the positive parameter t is fixed (independent of h), and this fact was indeed used several times to obtain the estimates. However, the main result of [4] requires $t \sim |h \ln h|$.

Proof. The distribution kernel of K_g is given by the oscillatory integral:

$$K_g(\alpha, \beta) = \frac{c^2 \langle \alpha_\xi \rangle^\sigma}{(2\pi h)^n \langle \beta_\xi \rangle^{d+\ell+\sigma}} \int e^{\Phi_1/h} g\left(\alpha, \frac{y+x'}{2}, \eta\right) d(x', y, \eta) \tag{2}$$

with,

$$\Phi_1 = t[H(\operatorname{Re} \beta) - H(\operatorname{Re} \alpha)] + \Phi_2;$$

$$\Phi_2 = i\alpha_x \alpha_\xi - i\beta_x \beta_\xi + i(\eta - \alpha_\xi)y + i(\beta_\xi - \eta)x' - \frac{(\alpha_x - y)^2 + (\beta_x - x')^2}{2}$$

($\alpha, \beta \in \Lambda_{tG}$). At first, we perform repeated integrations by parts in (2) by using the operator:

$$\begin{aligned}L_1 &= \frac{1}{1 + h^{-1}|\mathbf{q}|^2} \left[1 + (\eta - \bar{\alpha}_\xi + i(\bar{\alpha}_x - y)) \cdot D_y + (\bar{\beta}_\xi - \eta + i(\bar{\beta}_x - x')) \cdot D_{x'} \right. \\ &\quad \left. + (y - x') \cdot D_\eta \right]\end{aligned}$$

where we have set,

$$\begin{aligned}\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_5) &:= (\eta - \operatorname{Re} \alpha_\xi + \operatorname{Im} \alpha_x, y - \operatorname{Re} \alpha_x - \operatorname{Im} \alpha_\xi, \eta - \operatorname{Re} \beta_\xi - \operatorname{Im} \beta_x, \\ &\quad x' - \operatorname{Re} \beta_x + \operatorname{Im} \beta_\xi, y - x') \in \mathbb{R}^{5n}.\end{aligned}$$

One has $L_1(e^{\Phi_2/h}) = e^{\Phi_2/h}$, and this permits to obtain for any $N \geq 1$,

$$K_g(\alpha, \beta) = \frac{c^2 \langle \alpha_\xi \rangle^\sigma}{(2\pi h)^n \langle \beta_\xi \rangle^{d+\ell+\sigma}} \int e^{\Phi_1/h} g_N(\alpha, \beta, x', y, \eta) d(x', y, \eta) \quad (3)$$

with,

$$g_N(\alpha, \beta, x', y, \eta) = ({}^t L_1)^N \left(g \left(\alpha, \frac{y+x'}{2}, \eta \right) \right). \quad (4)$$

In particular, one easily sees that g_N satisfies to the estimate,

$$g_N = \mathcal{O} \left(\frac{\langle \mathbf{q} \rangle^N}{\langle h^{-1} |\mathbf{q}|^2 \rangle^N} \right) \sum_{|\gamma| \leq N} \left| \partial_{(y, x', \eta)}^\gamma g \left(\alpha, \frac{y+x'}{2}, \eta \right) \right|. \quad (5)$$

We denote by $\tilde{g}_N$ an almost analytic extension of g_N constructed from an almost analytic extension $\tilde{g}$ of g via (4) (so that it satisfies (5) too), and, for $C > 0$ large enough (to be specified later on) we consider the following family of contours in $\mathbb{C}^{3n}$:

$$\mathcal{C}_s : \mathbb{R}^{3n} \ni \begin{pmatrix} y \\ x' \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} y \\ x' \\ \eta \end{pmatrix} + iC \text{diag}(s_1, \dots, s_{3n}) \begin{pmatrix} (\eta - \overline{\alpha_\xi}) / \langle \eta - \alpha_\xi \rangle \\ (\beta_\xi - \eta) / \langle \beta_\xi - \eta \rangle \\ (y - x') / \langle y - x' \rangle \end{pmatrix}$$

for $s = (s_1, \dots, s_{3n}) \in [0, |t|]^{3n}$. Then, along $\mathcal{C}_s$ we have:

$$\begin{aligned} \text{Re} \Phi_1 &\leq |t| F_t(\alpha, \beta) - \left(\min_j s_j \right) \left[\frac{|y-x'|^2}{\langle y-x' \rangle} + \frac{|\eta - \alpha_\xi|^2}{\langle \eta - \alpha_\xi \rangle} + \frac{|\eta - \beta_\xi|^2}{\langle \eta - \beta_\xi \rangle} \right] \\ &\quad - \frac{1}{4} [|\text{Re} \alpha_x - y|^2 + |\text{Re} \beta_x - x'|^2] \end{aligned}$$

with

$$F_t(\alpha, \beta) = |G(\text{Re} \beta) - G(\text{Re} \alpha)| + 4|t| (|\nabla G(\text{Re} \alpha)|^2 + 4|\nabla G(\text{Re} \beta)|^2).$$

(Here we have used estimates such as: $|(x' - \text{Re} \beta_x) \text{Im} \beta_\xi| \leq \frac{1}{8} |x' - \text{Re} \beta_x|^2 + 2t^2 |\nabla G(\text{Re} \beta)|^2$.) Therefore, since $\bar{\partial} \tilde{g}_N = ({}^t L_1)^N \bar{\partial} \tilde{g}$, and using Stokes' formula and the fact that, for any $u, v \in \mathfrak{C}^n$,

$$\frac{|u|^2}{\langle u \rangle} + \frac{|v|^2}{\langle v \rangle} \geq \frac{1}{2} \frac{|u-v|^2}{\langle u-v \rangle},$$

(just separate the two cases $|u-v| \leq \max(|u|, |v|)$ and $|u-v| \geq \max(|u|, |v|)$), we obtain from (3),

$$\begin{aligned} K_g(\alpha, \beta) &= \mathcal{O} \left(h^{-5n/2} e^{\frac{|t|}{h} F_t(\alpha, \beta) - \frac{C|t|}{2h} \frac{|\alpha_\xi - \beta_\xi|^2}{\langle \alpha_\xi - \beta_\xi \rangle} - \frac{C|t|}{2h} \frac{|\alpha_x - \beta_x|^2}{\langle \alpha_x - \beta_x \rangle}} \right) \\ &\quad \times \int_{\mathcal{C}_{|t|}} \frac{\langle \alpha_\xi \rangle^\sigma}{\langle \beta_\xi \rangle^{d+\ell+\sigma}} \tilde{g}_N d(x', y, \eta) \end{aligned}$$

$$\begin{aligned}
& + \mathcal{O}\left(h^{-5n/2}e^{|t|F_t(\alpha,\beta)/h}\right) \sup_{\Omega_t} \left(\sum_{|\gamma| \leq N} |\langle(\alpha_\xi, \eta)\rangle^{-(d+\ell)} \partial^\gamma \bar{\partial} \tilde{g}| \right) \\
& \times \int_{\Omega_t} \frac{\langle(\alpha_\xi, \eta)\rangle^{d+\ell+\sigma}}{\langle\beta_\xi\rangle^{d+\ell+\sigma}} \frac{\langle\mathbf{q}\rangle^N}{\langle h^{-1}\mathbf{q}^2 \rangle^N} \mathbf{L}(d(x', y, \eta))
\end{aligned} \quad (6)$$

where $\mathcal{C}_{|t|}$ stands for $\mathcal{C}_{(|t|, \dots, |t|)}$, $\Omega_t := \cup_{s \in [0, |t|]} \mathcal{C}_s$, and $\mathbf{L}(d(x', y, \eta))$ is the Lebesgue measure on $\mathfrak{C}^{3n}$.

Then, for N large enough, we use the estimate,

$$\int_{\mathcal{C}_{|t|}} \frac{\langle\mathbf{q}\rangle^N}{\langle h^{-1}|\mathbf{q}|^2 \rangle^N} d(x', y, \eta) \leq \frac{C_N h^{3n/2}}{(1 + h^{-1}|\mathbf{q}_0|^2)^{N/2}} + \frac{C_N h^N}{(1 + |\mathbf{q}_0|)^{N/2}}, \quad (7)$$

for some constant $C_N > 0$, and where we have set,

$$\begin{aligned}
\mathbf{q}_0 &= \mathbf{q}_0(\alpha, \beta) := (\mathbf{q}_3 - \mathbf{q}_1, \mathbf{q}_4 + \mathbf{q}_5 - \mathbf{q}_2) \\
&= (\operatorname{Re}(\alpha_\xi - \beta_\xi) - \operatorname{Im}(\alpha_x + \beta_x), \operatorname{Re}(\alpha_x - \beta_x) + \operatorname{Im}(\alpha_\xi + \beta_\xi)) \in \mathbb{R}^{2n}.
\end{aligned}$$

This estimate can be proved, e.g., by splitting the set of integration into the two subsets,

$$\begin{aligned}
\mathcal{C}_{|t|}^1 &:= \mathcal{C}_{|t|} \cap \{|\mathbf{q}| \leq 1\} \\
\mathcal{C}_{|t|}^2 &:= \mathcal{C}_{|t|} \cap \{|\mathbf{q}| \geq 1\}.
\end{aligned}$$

On $\mathcal{C}_{|t|}^1$, we use the obvious fact that $|\mathbf{q}|^2 \geq |\mathbf{q}_0|^2/4$, and,

$$\int_{\mathcal{C}_{|t|}^1} \langle h^{-1}|\mathbf{q}|^2 \rangle^{-N/2} d(x', y, \eta) = \mathcal{O}(h^{3n/2}),$$

while, on $\mathcal{C}_{|t|}^2$, we just observe that $\langle\mathbf{q}\rangle \langle h^{-1}|\mathbf{q}|^2 \rangle^{-1} \leq 2h|\mathbf{q}|^{-1}$, and $|\mathbf{q}| \geq |\mathbf{q}_0|/2$. Then, (7) easily follows.

In a similar (but easier) way, using the fact that $\langle\mathbf{q}\rangle^{-1} \langle\beta_\xi\rangle^{-1} \langle(\alpha_\xi, \eta)\rangle$ remains uniformly bounded for $\alpha, \beta \in \Lambda_{tG}$ and $(x', y, \eta) \in \mathbb{R}^{3n}$, by taking N large enough, one also finds,

$$\int_{\Omega_t} \frac{\langle(\alpha_\xi, \eta)\rangle^{d+\ell+\sigma}}{\langle\beta_\xi\rangle^{d+\ell+\sigma}} \frac{\langle\mathbf{q}\rangle^N}{\langle h^{-1}\mathbf{q}^2 \rangle^N} \mathbf{L}(d(x', y, \eta)) = \mathcal{O}\left(\frac{h^{3n}}{(1 + |\mathbf{q}_0|^2)^{N/2}}\right). \quad (8)$$

Therefore, we deduce from (6)–(8),

$$\begin{aligned} K_g(\alpha, \beta) = & \mathcal{O} \left(\frac{h^{-n} e^{\frac{|t|}{h} F_t(\alpha, \beta) - \frac{C|t|}{2h} \frac{|\alpha_\xi - \beta_\xi|^2}{\langle \alpha_\xi - \beta_\xi \rangle} - \frac{C|t|}{2h} \frac{|\alpha_x - \beta_x|^2}{\langle \alpha_x - \beta_x \rangle}}{(1 + h^{-1} |\mathbf{q}_0|^2)^{N/2}} B_N(g) \right) \\ & + \mathcal{O} \left(\frac{h^{N/2} e^{\frac{|t|}{h} F_t(\alpha, \beta) - \frac{C|t|}{2h} \frac{|\alpha_\xi - \beta_\xi|^2}{\langle \alpha_\xi - \beta_\xi \rangle} - \frac{C|t|}{2h} \frac{|\alpha_x - \beta_x|^2}{\langle \alpha_x - \beta_x \rangle}}{(1 + |\mathbf{q}_0|)^{N/2}} B_N(g) \right) \\ & + \mathcal{O} \left(\frac{h^{n/2} e^{|t| F_t(\alpha, \beta)/h}}{(1 + |\mathbf{q}_0|)^{N/2}} B_N(\bar{\partial} \tilde{g}) \right), \end{aligned} \quad (9)$$

where we have used the notation,

$$B_N(g) := \sup_{\Omega_t} \left(\sum_{|\gamma| \leq N} |\langle (\alpha_\xi, \eta) \rangle^{-(d+\ell)} \partial^\gamma g| \right).$$

Let $C' > 0$ be such that $\text{Supp } G \subset \{|(x, \xi)| \leq C'\}$. Then, as in [1], we split $\Lambda_{tG} \times \Lambda_{tG}$ into 4 regions:

1st region: $|\text{Re } \alpha| \geq C'$ and $|\text{Re } \beta| \geq C'$

In this region $F_t(\alpha, \beta) = 0$ and thus, we immediately obtain from (9),

$$\begin{aligned} K_g(\alpha, \beta) = & \mathcal{O} \left(\frac{h^{-n} B_N(g)}{(1 + h^{-1} |\mathbf{q}_0|^2)^{N/2}} \right) + \mathcal{O} \left(\frac{h^{N/2} B_N(g)}{(1 + |\mathbf{q}_0|)^{N/2}} \right) \\ & + \mathcal{O} \left(\frac{h^{n/2} B_N(\bar{\partial} \tilde{g})}{(1 + |\mathbf{q}_0|)^{N/2}} \right), \end{aligned} \quad (10)$$

for any fixed $N \geq 1$ large enough. In particular, if we use (1) and the properties of the almost analytic extensions, we obtain,

$$K_f(\alpha, \beta) = \mathcal{O} \left(\frac{h^{-n}}{(1 + h^{-1} |\mathbf{q}_0|^2)^{2n+1}} + \frac{1}{(1 + |\mathbf{q}_0|)^{2n+1}} \right), \quad (11)$$

and,

$$K_{mr}(\alpha, \beta) = \mathcal{O} \left(\frac{h^{-n} \mu(t)}{(1 + h^{-1} |\mathbf{q}_0|^2)^{N/2}} \right) + \mathcal{O} \left(\frac{h^{n/2} \mu(t)}{(1 + |\mathbf{q}_0|)^{2n+1}} \right). \quad (12)$$

2nd region: $|\text{Re } \alpha| \geq 2C'$ and $|\text{Re } \beta| \leq C'$

In this region we have

$$|\alpha - \beta| \geq \frac{1}{2} |\text{Re } \alpha| \geq \frac{C'}{2} + \frac{1}{4} |\text{Re } \alpha|. \quad (13)$$

Therefore, setting $J_0 := 2 \sup_{\mathbb{R}^{2n}} (|G| + |\nabla G|^2)$, we see on (9) that if C and N have been chosen sufficiently large, we have in this region,

$$K_g(\alpha, \beta) = \mathcal{O} \left(\frac{h^{-n}}{(1 + h^{-1}|\mathbf{q}_0|^2)^{2n+1}} + \frac{h^{n/2}}{(1 + |\mathbf{q}_0|)^{2n+1}} \right) B_N(g) \\ + \mathcal{O} \left(h^{n/2} e^{J_0|t|/h} \right) \frac{1}{(1 + |\mathbf{q}_0|)^{2n+1}} B_N(\bar{\partial} \tilde{g}).$$

In particular,

$$K_f(\alpha, \beta) = \mathcal{O} \left(\frac{h^{-n}}{(1 + h^{-1}|\mathbf{q}_0|^2)^{2n+1}} + \frac{1}{(1 + |\mathbf{q}_0|)^{2n+1}} \right) \\ + \mathcal{O} \left(h^{n/2} \mu(t) e^{J_0|t|/h} \right) \frac{1}{(1 + |\mathbf{q}_0|)^{2n+1}}, \quad (14)$$

and,

$$K_{mr}(\alpha, \beta) = \mathcal{O} \left(h^{n/2} \mu(t) e^{J_0|t|/h} \right) \left(\frac{h^{-n}}{(1 + h^{-1}|\mathbf{q}_0|^2)^{2n+1}} + \frac{1}{(1 + |\mathbf{q}_0|)^{2n+1}} \right). \quad (15)$$

3rd region: $|\operatorname{Re} \alpha| \leq C'$ and $|\operatorname{Re} \beta| \geq 2C'$

In the same way, we still obtain (14) and (15).

4th region: $|\operatorname{Re} \alpha| \leq 2C'$ and $|\operatorname{Re} \beta| \leq 2C'$

In this last region we try to apply the stationary phase theorem to (2). The unique stationary point (given by $\partial_y \Phi_1 = \partial_{x'} \Phi_1 = \partial_\eta \Phi_1 = 0$) is:

$$\begin{cases} y = x' = \frac{\alpha_x + \beta_x}{2} - \frac{i}{2}(\alpha_\xi - \beta_\xi) \\ \eta = \frac{\alpha_\xi + \beta_\xi}{2} + \frac{i}{2}(\alpha_x - \beta_x). \end{cases}$$

In order to be able to apply the stationary phase theorem with complex-valued phase function (see [5, p. 145]), we need that the critical point remains close enough to the real (it already remains bounded because so are α and β in this region). That is, we need that $|\alpha - \beta|$ is small enough. But if it is not the case (that is if $|\alpha - \beta| \geq \delta'$ for some $\delta' > 0$), then, as for the 2nd and 3rd regions (and possibly by increasing the value of the constant C again), (9) still gives (14) and (15).

Otherwise, if $|\alpha - \beta| \leq \delta'$ with δ' small enough, then Theorem 2.3 of [5] can be applied and gives, for any $N \geq 1$ (recalling that $c^2 \sim h^{-3n/2}$),

$$K_g(\alpha, \beta) = h^{-n} \tilde{b}_N(\alpha, \beta, h) e^{\varphi(\alpha, \beta)/h} + \mathcal{O}(h^N) \quad (16)$$

where $\varphi(\alpha, \beta)$ is the critical value of Φ_1 , that is:

$$\varphi(\alpha, \beta) = t(H(\beta) - H(\alpha)) + \frac{i}{2}(\alpha_x - \beta_x)(\alpha_\xi + \beta_\xi) - \frac{1}{4}(\alpha_x - \beta_x)^2 - \frac{1}{4}(\alpha_\xi - \beta_\xi)^2$$

and $\tilde{b}_N$ is a bounded symbol which can be estimated by the derivatives of g at the critical point, up to order $2(N + n)$. Moreover, using the definition of H we see that,

$$\operatorname{Re} \varphi(\alpha, \beta) = -\frac{1}{4} \operatorname{Re}(\alpha_x - \beta_x)^2 - \frac{1}{4} \operatorname{Re}(\alpha_\xi - \beta_\xi)^2 + t\psi(\alpha, \beta) \quad (17)$$

with

$$\begin{aligned}\psi(\alpha, \beta) &= G(\beta) - G(\alpha) - \frac{1}{2} \operatorname{Re}(\beta_x - \alpha_x)(\partial_x G(\alpha) + \partial_x G(\beta)) \\ &\quad - \frac{1}{2} \operatorname{Re}(\beta_\xi - \alpha_\xi)(\partial_\xi G(\alpha) + \partial_\xi G(\beta)) \\ &= \mathcal{O}(|\alpha - \beta|^2).\end{aligned}$$

As a consequence, if we take t small enough we deduce from (1), (16) and (17):

$$\begin{aligned}K_f(\alpha, \beta) &= \mathcal{O}\left(h^{-n}e^{-|\alpha-\beta|^2/8h} + h^N\right) \\ K_{mr}(\alpha, \beta) &= \mathcal{O}\left(h^{-n}(\mu(t) + \mu(|\alpha - \beta|))(e^{-\nu|\alpha-\beta|^2/h} + h^N)\right)\end{aligned}\quad (18)$$

with $N \geq 1$ arbitrary. (Here we have also used that $\mu(t+s) = \mathcal{O}(\mu(t/2) + \mu(s/2))$ for $s, t \geq 0$.)

Now, gathering the estimates (11), (12), (14), (15), and (18), we obtain uniform bounds for both $\int_{\Lambda_{tG}} |K_g(\alpha, \beta)| d\operatorname{Re} \alpha$ and $\int_{\Lambda_{tG}} |K_g(\alpha, \beta)| d\operatorname{Re} \beta$ ($g = f, mr$), and thus, by the Schur lemma, this gives the following estimates on the operators K_f K_{mr} :

$$\begin{aligned}\|K_f\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(1 + h^{n/2}\mu(t)e^{J_0|t|/h}\right) \\ \|K_{mr}\|_{\mathcal{L}(L^2(\Lambda_{tG}))} &= \mathcal{O}\left(h^{n/2}\mu(t)e^{J_0|t|/h} + h^\infty\right).\end{aligned}$$

This completes the proof of Lemma 1. $\square$

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André Martinez
Dipartimento di Matematica
Università di Bologna
Piazza di porta San Donato, 5
I-40127 Bologna
Italy
e-mail: martinez@dm.unibo.it