Stochastic Phase Field Equations: Existence and Uniqueness

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Abstract. We consider a conservative system of stochastic PDE's, namely a one dimensional *phase field* model perturbed by an additive space–time white noise. We prove a global existence and uniqueness result in a space of continuous functions on $\mathbb{R}_+ \times \mathbb{R}$. This result is obtained by extending previous results of Doering [3] on the stochastic Allen–Cahn equation.

1 Introduction and results

We consider a *phase field* system with additive stochastic noise, which is formally written as

$$\begin{aligned}\partial_t m(t) &= \frac{1}{2} \Delta m(t) - V'(m(t)) + \lambda h(t) + a \eta(t) \\ \partial_t [h(t) + m(t)] &= \frac{1}{2} \Delta h(t)\end{aligned}\tag{1.1}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, $m(t) = m(t, x)$, $h(t) = h(t, x)$ are two scalar random fields (we omit to write explicitly the dependence on the randomness), λ is a positive parameter, Δ is the Laplacian on $\mathbb{R}$, $V(m) = m^4/4 - m^2/2$ is a double well potential, $a = a(x)$ is a bounded and continuous function, and $\eta(t) = \eta(t, x)$ is a space–time white noise, i.e. $\mathbb{E}(\eta(t, x) \eta(s, y)) = \delta(t - s) \delta(x - y)$. In particular a translationally covariant noise is obtained for $a = 1$.

The deterministic system obtained by setting $a = 0$ in (1.1), usually referred to as *phase field* equations, describes the kinetic of phase segregation when the presence of the *latent heat* is taken into account. The first equation describes in fact the evolution of the order parameter m which is coupled to the external field h (which can be thought as the excess temperature measured from the melting temperature); h is however itself a dynamic variable which diffuses and, via the coupling λ , introduces a feedback into m whose effect is to slow down the phase segregation process. We stress that $q = m + h$ is locally conserved as apparent from the second equation in (1.1). Scaling limits of the deterministic phase field equations as $\lambda \rightarrow 0$ have been considered in [2] and [6].

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Due to the phenomenological character of the phase field model, the introduction of a random forcing term appears natural and makes possible to discuss statistical properties of solutions. We also mention that the stochastic system (1.1) is very similar to the so-called, in the physical literature on critical phenomena, model C of Hohenberg and Halperin [5]. The specific choice of an additive white noise only in the first equation has been done to keep the model as simple as possible: the random forcing term is non-conservative, whereas the conservation law is still linear and not perturbed. Moreover, for λ small, m and h are weakly coupled so that we may refer to (1.1) as a *weakly conservative system*. This simplifying feature might help in developing a mathematical theory for phase segregation in conservative models. Indeed, in the companion paper [1], front fluctuations for (1.1) are analyzed in a suitable scaling limit as $\lambda \rightarrow 0$ and $a = O(\lambda)$. The need of an existence and uniqueness result for the stochastic system (1.1) in [1] is the main motivation for the present paper.

Referring to [1] for a more exhaustive discussion on the stochastic phase field equations, we next state precisely our results. For $\alpha > 0$ and $\gamma \in (0, 1]$, let us define the following norms on $C(\mathbb{R})$:

$$\begin{aligned}\|\varphi\|_{C_\alpha(\mathbb{R})} &:= \sup_{x \in \mathbb{R}} e^{-\alpha|x|} |\varphi(x)| \\ \|\varphi\|_{C_\alpha^\gamma(\mathbb{R})} &:= \|\varphi\|_{C_\alpha(\mathbb{R})} + \sup_{x \neq y} e^{-\alpha(|x|+|y|)} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\gamma}\end{aligned}$$

and also the following ones on $C(\mathbb{R}_+ \times \mathbb{R})$

$$\begin{aligned}\|f\|_{C_\alpha(\mathbb{R}_+ \times \mathbb{R})} &:= \sup_{t \in \mathbb{R}_+} e^{-\alpha^2 t/2} \|f(t)\|_{C_\alpha(\mathbb{R})} \\ \|f\|_{C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})} &:= \sup_{t \in \mathbb{R}_+} e^{-\alpha^2 t/2} \|f(t)\|_{C_\alpha^\gamma(\mathbb{R})}\end{aligned}$$

we shall denote by $C_\alpha(\mathbb{R}), \dots$, the corresponding Banach spaces.

Let $p_t = e^{t\Delta/2}$ be the heat semigroup, namely the integral operator with kernel $p_t(x, y) = (2\pi t)^{-1/2} \exp\{-(x - y)^2/2t\}$. We introduce the process $Z(t) = Z(t, x)$ given by $Z(t) = \int_0^t ds p_{t-s}[a\eta(s)]$. Then Z is the mean zero Gaussian process with covariance

$$\mathbb{E}(Z(t, x)Z(s, y)) = \Gamma(t, s; x, y) := \int_0^{t \wedge s} du \int dz p_{t-u}(x - z) p_{s-u}(y - z) a(z)^2 \quad (1.2)$$

where $t \wedge s := \min\{t, s\}$. In the next lemma we state some properties of the paths of Z , which follow, by standard Gaussian arguments, from the properties of the covariance Γ , see e.g. [3] and references therein.

Lemma 1.1 *For each $\alpha > 0$ and $\gamma \in (0, 1/2)$ we have that $Z \in C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})$, $\mathbb{P}$ -a.s.*

We shall denote by $\mathcal{F}_t$ the filtration given by $\mathcal{F}_t := \sigma\{Z(s, x); (s, x) \in [0, t] \times \mathbb{R}\}$. In the following we consider a fixed realization of $Z \in C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})$.

Let $q := m + h$ and $U'(m) := V'(m) - \lambda m$; we formulate the problem (1.1) in the integral form

$$\begin{aligned} m(t) &= p_t m(0) + \int_0^t ds p_{t-s} [-U'(m(s)) + \lambda q(s)] + Z(t) \\ q(t) &= p_t q(0) - \frac{1}{2} \int_0^t ds \Delta p_{t-s} m(s) \end{aligned} \quad (1.3)$$

note that the integral on the r.h.s. of the second equation is well defined (the result is a continuous function) provided $x \mapsto m(s, x)$ is Hölder continuous.

Our main result is the following existence and uniqueness result for the system (1.3) on the space of Hölder continuous functions.

Theorem 1.2 *Let $m(0), q(0) \in C_\alpha^\gamma(\mathbb{R})$ for any $\alpha > 0$ and $\gamma \in (0, 1/2)$. Then there exists a unique $\mathcal{F}_t$ -adapted process $(m, q) \in C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R}) \times C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$ and $\gamma \in (0, 1/2)$ which solves (1.3).*

We introduce the linear operators $G, \tilde{G}$ on $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ given by

$$\begin{aligned} G[F](t) &:= \int_0^t ds p_{t-s} F(s) \\ \tilde{G}[F](t) &:= \int_0^t ds (t-s) \frac{1}{2} \Delta p_{t-s} F(s) \end{aligned} \quad (1.4)$$

and set $R(t) := p_t m(0) + \lambda t p_t q(0) + Z(t)$. By plugging the second equation in (1.3) into the first one we get that m solves the problem

$$m = R - G[U'(m)] - \lambda \tilde{G}[m]. \quad (1.5)$$

By standard estimates on the heat kernel, it is easy to verify that for each $\alpha > 0$ and $\gamma \in (0, 1]$ there exist an $\alpha' > 0$ and a constant $C = C(\alpha, \gamma)$ such that

$$\begin{aligned} \|Gf\|_{C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})} &\leq C \|f\|_{C_{\alpha'}^\gamma(\mathbb{R}_+ \times \mathbb{R})}, \\ \|\tilde{G}f\|_{C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})} &\leq C \|f\|_{C_{\alpha'}^\gamma(\mathbb{R}_+ \times \mathbb{R})}. \end{aligned}$$

Furthermore, for each $\alpha > 0$ and $\gamma \in (0, 1)$, there exists an $\alpha' > 0$, $\gamma' \in (0, 1)$ and a constant $C = C(\alpha, \gamma)$ such that

$$\left\| \int_0^t ds \Delta p_{t-s} f(s) \right\|_{C_\alpha^\gamma(\mathbb{R}_+ \times \mathbb{R})} \leq C \|f\|_{C_{\alpha'}^{\gamma'}(\mathbb{R}_+ \times \mathbb{R})}.$$

Therefore Theorem 1.2 can be easily deduced from the following existence and uniqueness result for the problem (1.5) on the space of continuous functions.

Theorem 1.3 *Let $m(0), q(0) \in C_\alpha(\mathbb{R})$ for any $\alpha > 0$. Then there exists a unique $\mathcal{F}_t$ -adapted process $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$ which solves (1.5).*

In the rest of the paper we prove Theorem 1.3. A uniqueness and existence result for equation 1.5 with $\lambda = 0$ is given in [3]. See also [4] for the analogous result in a bounded domain. We shall follow closely the proof in [3] for the one dimensional case, referring to that paper for some technical Lemmata. The term $\lambda \tilde{G}$, coming from λh in (1.1), is the source of the difficulties. Since we cannot estimate the L^p norm of h in terms of the L^p norm m , the necessary *a priori* bounds are not a straightforward extension of that in [3]. To overcome this problem we shall estimate an appropriate negative Sobolev norm of h in terms of the L^p norm of m , see Lemmata 2.4 and 2.5 below.

We finally remark that the quartic double well potential, $V(m) = m^4/4 - m^2/2$ has been chosen only for notation simplicity; the proof works for any polynomial of even degree with positive leading coefficient.

Since the parameter λ will be kept fixed throughout all the paper we omit to indicate the dependence on it. We shall denote by C a generic positive constant whose numerical value may change from line to line.

2 Finite volume approximations

Let $C_K^\infty(\mathbb{R})$ be the space of infinitely differentiable functions on $\mathbb{R}$ with compact support and introduce $\mathcal{L} = \{\Lambda \in C_K^\infty(\mathbb{R}) : 0 \leq \Lambda \leq 1\}$. For $\Lambda \in \mathcal{L}$, we introduce the following finite volume approximations of the problem (1.5)

$$m_\Lambda = \Lambda R - G[\Lambda U'(m_\Lambda)] - \lambda \tilde{G}[\Lambda m_\Lambda] . \quad (2.1)$$

In this section we establish a global existence result for (2.1) together with some bounds uniform for $\Lambda \in \mathcal{L}$. To this end we need to introduce some more notation.

For $\alpha > 0$, let $\varrho(x) := e^{-\alpha|x|}$. We shall denote by ϱ also the measure $\varrho(x)dx$ on $\mathbb{R}$. We introduce the following finite measures on $\mathbb{R}_+ \times \mathbb{R}$, omitting the dependence on $\alpha > 0$ from the notation

$$\begin{aligned} \mu(dt, dx) &:= e^{-\alpha^2 t/2} \varrho(x) dx dt \\ \mu_T(dt, dx) &:= \chi_{[0, T]}(t) \mu(dt, dx) \\ \mu_\Lambda(dt, dx) &:= \Lambda(x) \mu(dt, dx) \\ \mu_{T, \Lambda}(dt, dx) &:= \chi_{[0, T]}(t) \Lambda(x) \mu(dt, dx) \end{aligned}$$

where $\chi_{[0, T]}$ denotes the characteristic function of $[0, T]$. For ν a measure and f a function use the notation $\nu(f) = \int d\nu f$.

For $p \in (1, \infty)$, we introduce the Sobolev space $H_1^p(\varrho)$ obtained by completing $C_K^\infty(\mathbb{R})$ with respect to the norm

$$\|\varphi\|_{H_1^p(\varrho)}^p := \|\nabla \varphi\|_{L^p(\varrho)}^p + \|\varphi\|_{L^p(\varrho)}^p$$

where ∇ denotes the derivative with respect to x . Since $H_1^p(\varrho) \subset L^p(\varrho)$, for $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$ we introduce the dual space $H_{-1}^q(\varrho)$ by completing

$L^q(\varrho)$ with respect to the norm

$$\|\ell\|_{H_{-1}^q(\varrho)} := \sup_{\varphi: \|\varphi\|_{H_1^p(\varrho)} \leq 1} \varrho(\ell \varphi) .$$

For $f = f(t, x)$ we shall use the notation

$$\|f\|_{H_{-1}^p(\mu_T)}^p := \int_0^T dt e^{-\alpha^2 t/2} \|f(t)\|_{H_{-1}^p(\varrho)}^p$$

omitting to write T on the l.h.s. if $T = \infty$.

From Lemma 1.1 it follows that for each $\alpha > 0$ and $p \in [1, \infty)$ we have $Z \in L^p(\mu)$. Furthermore for each $T > 0$ and $\Lambda \in \mathcal{L}$ we have

$$\sup_{t \in [0, T]} \sup_{x \in \mathbb{R}} |\Lambda(x) Z(t, x)| < \infty .$$

Moreover, by the assumptions on the initial data, the same holds for

$$R(t) = p_t m(0) + \lambda t p_t q(0) + Z(t) .$$

We next state a local existence result for the problem (2.1). Since $\tilde{G}$ has a kernel which can be estimated as the one of G , the proof of the next result, which is based on Picard iterations, is the same as [3, Prop. 1] and we omit it.

Lemma 2.1 *For each $\Lambda \in \mathcal{L}$ and each realization of Z in $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ there is a time $T_0 > 0$ such that there exists a unique $\mathcal{F}_t$ -adapted continuous and bounded solution of (2.1) on $[0, T_0] \times \mathbb{R}$.*

To show the existence of a global solution, it is enough to prove that if m_Λ is a continuous solution of (2.1) on $[0, T^*) \times \mathbb{R}$ then

$$\sup_{t \in [0, T^*)} \sup_{x \in \mathbb{R}} |m_\Lambda(t, x)| < \infty . \quad (2.2)$$

The key ingredient for proving (2.2) is the following *a priori* bound on the $L^p(\mu_{T^*, \Lambda})$ norm of solutions.

Proposition 2.2 *Let m_Λ be a continuous solution of (2.1) on $[0, T^*) \times \mathbb{R}$. For each $\alpha > 0$ and $p \in [1, \infty)$ there exists a constant $C = C(\alpha, p, \|R\|_{L^p(\mu)}) < \infty$, independent of $T^* > 0$ and of $\Lambda \in \mathcal{L}$, such that*

$$\|m_\Lambda\|_{L^p(\mu_{T^*, \Lambda})} \leq C . \quad (2.3)$$

The proof of the proposition is split in several Lemmata, the first one, which is proven integrating by parts, is [3, Lemma 7].

Lemma 2.3 *If both $f(t, x)$ and $(\partial_t - \frac{1}{2}\Delta)f$ are continuous on $(0, T) \times \mathbb{R}$, $f(0) = 0$ and $|f|^{2n+2}$ and $|\nabla f|^{2n+2}$ are in $L^1(\mu_T)$, then, for any $n = 0, 1, \dots$,*

$$\mu_T \left(f^{2n+1} \left(\partial_t - \frac{1}{2}\Delta \right) f \right) \geq (2n+1) \mu_T (f^{2n} |\nabla f|^2) . \quad (2.4)$$

Let m_Λ be a continuous solution of (2.1). We define

$$\begin{aligned} u_\Lambda &:= m_\Lambda - \Lambda R, \\ q_\Lambda &:= \left(\partial_t - \frac{1}{2}\Delta \right) \tilde{G}[\Lambda m_\Lambda] . \end{aligned} \quad (2.5)$$

From (2.1), if m_Λ is continuous and bounded on $[0, T] \times \mathbb{R}$, then $u_\Lambda \in C^\infty((0, T] \times \mathbb{R})$ by the regularizing properties of G and $\tilde{G}$. Moreover, for $t \in [0, T]$, $u_\Lambda(t, x)$ (together with its derivative) is exponentially decaying as $x \rightarrow \infty$.

Lemma 2.4 *Let m_Λ be a continuous solution of (2.1) on $[0, T^*) \times \mathbb{R}$; then for each $n = 0, 1, \dots$ and $\beta > 0$ there exists a constant $C = C(n, \beta)$, independent of $T^* > 0$ and $\Lambda \in \mathcal{L}$, such that*

$$\mu_{T^*, \Lambda} \left(e^{-(2n+2)\beta t} u_\Lambda^{2n+1} U'(m_\Lambda) \right) \leq C \|e^{-\beta t} q_\Lambda\|_{H_{-1}^{2n+2}(\mu_{T^*})}^{2n+2} . \quad (2.6)$$

Proof. We apply Lemma 2.3 with $f = e^{-\beta t} u_\Lambda$ and $T < T^*$. From (2.1) we get

$$\left(\partial_t - \frac{1}{2}\Delta \right) e^{-\beta t} u_\Lambda = -\beta e^{-\beta t} u_\Lambda - e^{-\beta t} [\Lambda U'(m_\Lambda) + \lambda q_\Lambda]$$

hence, by (2.4)

$$\begin{aligned} \mu_T \left(e^{-(2n+2)\beta t} \left[(2n+1) u_\Lambda^{2n} (\nabla u_\Lambda)^2 + \beta u_\Lambda^{2n+2} + u_\Lambda^{2n+1} \Lambda U'(m_\Lambda) \right] \right) \\ \leq -\lambda \mu_T \left(e^{-(2n+2)\beta t} u_\Lambda^{2n+1} q_\Lambda \right) . \end{aligned} \quad (2.7)$$

Let $p, q \in (1, \infty)$ such that $p^{-1} + q^{-1} = 1$; then for each $\gamma > 0$ there exists a constant $C = C(\gamma, p)$ such that for any $a, b \in \mathbb{R}$

$$|a b| \leq \gamma |a|^p + C |b|^q \quad (2.8)$$

therefore, by the duality between $H_1^p(\varrho)$ and $H_{-1}^q(\varrho)$, we have

$$|\varrho(fg)| \leq \gamma \|f\|_{H_1^p(\varrho)}^p + C \|g\|_{H_{-1}^q(\varrho)}^q . \quad (2.9)$$

By applying (2.9) we get

$$\begin{aligned}
\left| \mu_T \left(e^{-(2n+2)\beta t} u_\Lambda^{2n+1} q_\Lambda \right) \right| &\leq \int_0^T dt e^{-\alpha^2 t/2 - (2n+2)\beta t} \left| \varrho \left(u_\Lambda^{2n+1}(t) q_\Lambda(t) \right) \right| \\
&\leq \int_0^T dt e^{-\alpha^2 t/2 - (2n+2)\beta t} \left[\gamma(2n+1)^{\frac{2n+2}{2n+1}} \varrho \left(u_\Lambda(t)^{2n \frac{2n+2}{2n+1}} (\nabla u_\Lambda(t))^{\frac{2n+2}{2n+1}} \right) \right. \\
&\quad \left. + \gamma \varrho \left(u_\Lambda^{2n+2}(t) \right) + C \|q_\Lambda(t)\|_{H_{-1}^{2n+2}(\varrho)}^{2n+2} \right] \\
&\leq \int_0^T dt e^{-\alpha^2 t/2 - (2n+2)\beta t} \left[\gamma(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) \varrho \left(u_\Lambda(t)^{2n} (\nabla u_\Lambda(t))^2 + u_\Lambda(t)^{2n+2} \right) \right. \\
&\quad \left. + \gamma \varrho \left(u_\Lambda^{2n+2}(t) \right) + C \|q_\Lambda(t)\|_{H_{-1}^{2n+2}(\varrho)}^{2n+2} \right] \\
&= \gamma(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) \mu_T \left(e^{-(2n+2)\beta t} u_\Lambda^{2n} (\nabla u_\Lambda)^2 \right) \\
&+ \gamma \left[(2n+1)^{\frac{2n+2}{2n+1}} c_1(n) + 1 \right] \mu_T \left(e^{-(2n+2)\beta t} u_\Lambda^{2n+2} \right) + C \|e^{-(2n+2)\beta t} q_\Lambda\|_{H_{-1}^{2n+2}(\mu_T)}^{2n+2}
\end{aligned} \tag{2.10}$$

where we used, in the third step, Hölder inequality in the form (as follows from (2.8))

$$\varrho(fg) \leq c_1(n) \left[\varrho \left(|f|^{\frac{2n+1}{n+1}} \right) + \varrho \left(|g|^{\frac{2n+1}{n}} \right) \right]$$

with $f = [u_\Lambda(t)^n \nabla u_\Lambda(t)]^{\frac{2n+2}{2n+1}}$ and $g = u_\Lambda(t)^n u_\Lambda^{\frac{2n+2}{2n+1}}$.

Choosing $\gamma = \gamma(n, \lambda, \beta)$ small enough and taking $T \uparrow T^*$ the lemma now follows from (2.7) and (2.10). $\square$

Lemma 2.5 For each $p \in (1, \infty)$ there exists a constant $C = C(p)$ such that, for any $\alpha, \beta, T > 0$ and $F \in L^p(\mu_T)$,

$$\left\| e^{-\beta t} \left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F] \right\|_{H_{-1}^p(\mu_T)} \leq C \left(1 + \frac{1}{\alpha} \right) \|e^{-\beta t} F\|_{L^p(\mu_T)} . \tag{2.11}$$

Proof. We can write

$$\tilde{G}[F](t) = \int_0^t ds p_{t-s} \int_0^s ds' \frac{1}{2} \Delta p_{s-s'} F(s') \tag{2.12}$$

so that (in distribution sense)

$$\left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F](t) = \int_0^t ds \frac{1}{2} \Delta p_{t-s} F(s) . \tag{2.13}$$

We thus get

$$\left\| \left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F](t) \right\|_{H_{-1}^p(\varrho)} \leq \frac{1}{2} \int_0^t ds \|\Delta p_{t-s} F(s)\|_{H_{-1}^p(\varrho)} . \tag{2.14}$$

For $q^{-1} + p^{-1} = 1$ and $\varphi \in C_K^\infty(\mathbb{R})$ we now have

$$\begin{aligned} (\varphi, \Delta p_{t-s} F(s))_{\varrho} &= (\varrho \varphi, \Delta p_{t-s} F(s))_{L^2(dx)} \\ &= - \left(\varrho \left[\nabla \varphi + \frac{\varphi \nabla \varrho}{\varrho} \right], \nabla p_{t-s} F(s) \right)_{L^2(dx)} \\ &\leq \left[\|\nabla \varphi\|_{L^q(\varrho)} + \|\varphi\|_{L^q(\varrho)} \sup_{x \in \mathbb{R}} |\nabla \log \varrho(x)| \right] \|\nabla p_{t-s} F(s)\|_{L^p(\varrho)} \end{aligned} \quad (2.15)$$

since $\sup_{x \in \mathbb{R}} |\nabla \log \varrho(x)| = \alpha$ we conclude

$$\|\Delta p_{t-s} F(s)\|_{H_{-1}^p(\varrho)} \leq (1 + \alpha) \|\nabla p_{t-s} F(s)\|_{L^p(\varrho)} . \quad (2.16)$$

As $\exp\{-\alpha|x|\} \leq \exp\{-\alpha|y|\} \exp\{\alpha|x-y|\}$,

$$|\nabla p_{t-s} F(s)(x)| e^{-\alpha|x|/p} \leq \int dy e^{-\alpha|y|/p} |F(s, y)| |\nabla p_{t-s}(x-y)| e^{\alpha|x-y|/p}, \quad (2.17)$$

by applying Young's inequality for convolutions we then obtain

$$\|\nabla p_{t-s} F(s)\|_{L^p(\varrho)} \leq \|F(s)\|_{L^p(\varrho)} \int dx |\nabla p_{t-s}(x)| e^{\alpha|x|/p}. \quad (2.18)$$

We define

$$\psi_{\alpha,p}(t) := \int dx |\nabla p_t(x)| e^{\alpha|x|/p} \leq \frac{2}{\sqrt{2\pi t}} + \frac{2\alpha}{p} \exp\left\{\frac{\alpha^2 t}{2p^2}\right\}. \quad (2.19)$$

Then, from (2.14), (2.16) and (2.18),

$$\begin{aligned} \chi_{[0,T]}(t) e^{-\frac{\alpha^2 t}{2p} - \beta t} &\left\| \left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F](t) \right\|_{H_{-1}^p(\varrho)} \\ &\leq \frac{1+\alpha}{2} \int ds \chi_{[0,T]}(s) e^{-\frac{\alpha^2 s}{2p} - \beta s} \|F(s)\|_{L^p(\varrho)} \chi_{[0,T]}(t-s) \\ &\quad \times e^{-\frac{\alpha^2(t-s)}{2p} - \beta(t-s)} \psi_{\alpha,p}(t-s). \end{aligned}$$

Again by Young's inequality for convolutions we get

$$\begin{aligned} &\left\| e^{-\beta t} \left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F] \right\|_{H_{-1}^p(\mu_T)} = \\ &\quad \left\{ \int_0^T dt e^{-\frac{\alpha^2 t}{2} - \beta p t} \left\| \left(\partial_t - \frac{1}{2} \Delta \right) \tilde{G}[F](t) \right\|_{H_{-1}^p(\varrho)}^p \right\}^{1/p} \\ &\leq \frac{1+\alpha}{2} \left\{ \int_0^T dt e^{-\frac{\alpha^2 t}{2p}} \psi_{\alpha,p}(t) \right\} \left\{ \int_0^T dt e^{-\frac{\alpha^2 t}{2} - \beta p t} \|F(t)\|_{L^p(\varrho)}^p \right\}^{1/p}. \end{aligned}$$

By using the estimate in (2.19) and recalling $p > 1$, it is easy to show there is a constant $C = C(p) > 0$ so that

$$\frac{1+\alpha}{2} \int_0^T dt e^{-\frac{\alpha^2 t}{2p}} \psi_{\alpha,p}(t) \leq C \left(1 + \frac{1}{\alpha}\right).$$

The lemma is proved. $\square$

Proof of Proposition 2.2. Recalling (2.5) and that $U'(m) = m^3 - (1 - \lambda)m$, by expanding the l.h.s. of (2.6), using Lemma 2.5, and $\Lambda^2 \leq \Lambda$, we get there exists a constant $C = C(\alpha, n, \beta)$ such that

$$\begin{aligned} \mu_{T^*, \Lambda} \left(e^{-(2n+2)\beta t} u_{\Lambda}^{2n+4} \right) \\ \leq C \mu_{T^*, \Lambda} \left(e^{-(2n+2)\beta t} \left\{ |u_{\Lambda}|^{2n+1} [u_{\Lambda}^2 |\Lambda R| + |u_{\Lambda}| (\Lambda R)^2 + |\Lambda R|^3] \right. \right. \\ \left. \left. + u_{\Lambda}^{2n+2} + |u_{\Lambda}|^{2n+1} |\Lambda R| + |\Lambda R|^{2n+2} \right\} \right). \end{aligned} \quad (2.20)$$

Let

$$M := \left(\mu_{T^*, \Lambda} \left(e^{-(2n+2)\beta t} u_{\Lambda}^{2n+4} \right) \right)^{1/(2n+4)}$$

by using repeatedly Hölder inequality in (2.20) we get

$$\begin{aligned} M^{2n+4} \leq C \left\{ M^{2n+3} \|\Lambda R\|_{L^{2n+4}(\mu)} + M^{2n+2} [1 + \|\Lambda R\|_{L^{2n+4}(\mu)}^2] \right. \\ \left. + M^{2n+1} \|\Lambda R\|_{L^{2n+4}(\mu)} [1 + \|\Lambda R\|_{L^{2n+4}(\mu)}^2] + \|\Lambda R\|_{L^{2n+4}(\mu)}^{2n+2} \right\} \end{aligned}$$

we then conclude that M is bounded by some constant $C = C(\alpha, n, \beta, \|\Lambda R\|_{L^{2n+4}(\mu)})$.

Recalling $m_{\Lambda} = u_{\Lambda} + \Lambda R$, by using triangular and Cauchy–Schwartz inequalities, we have

$$\|m_{\Lambda}\|_{L^p(\mu_{\Lambda, T^*})} \leq \|\Lambda R\|_{L^p(\mu_{\Lambda, T^*})} + \|e^{\beta t}\|_{L^{2p}(\mu_{\Lambda, T^*})} \|e^{-\beta t} u_{\Lambda}\|_{L^{2p}(\mu_{\Lambda, T^*})}.$$

Since, for each $p \in [1, \infty)$, $\|\Lambda R\|_{L^p(\mu_{\Lambda, T^*})} \leq \|R\|_{L^p(\mu)} < \infty$, by choosing $\beta = \beta(\alpha, p)$ small enough and $n = n(p)$ large enough, the proposition follows. $\square$

Proof of (2.2). Let $Y^* := \sup\{|x| : \Lambda(x) > 0\}$; writing explicitly the kernels in the integral equation (2.1), and using Young's inequality for convolutions, it follows

$$\begin{aligned} \sup_{t \in [0, T^*]} \sup_{x \in \mathbb{R}} |m_{\Lambda}(t, x)| &\leq \sup_{t \in [0, T^*]} \sup_{x \in \mathbb{R}} |\Lambda(x) R(t, x)| \\ &+ e^{\frac{\alpha^2}{4} T^* + \frac{\alpha}{2} Y^*} \times \left\{ \left[\int_0^{T^*} dt \int dz p_t(z)^2 \right]^{\frac{1}{2}} \left[\int \mu_{T^*, \Lambda}(dt, dx) U'(m_{\Lambda}(t, y))^2 \right]^{\frac{1}{2}} \right. \\ &\left. + \lambda \left[\int_0^{T^*} dt \int dz p_t(z)^2 \left[\frac{z^2}{2t} + \frac{1}{2} \right]^2 \right]^{\frac{1}{2}} \left[\int \mu_{T^*, \Lambda}(dt, dx) m_{\Lambda}(t, y)^2 \right]^{\frac{1}{2}} \right\} \end{aligned} \quad (2.21)$$

where we used also that $\Lambda^2 \leq \Lambda$. By using that, as follows from Proposition 2.2, $U'(m_\Lambda)$ is in $L^2(\mu_{T^*,\Lambda})$, and the properties of the heat kernel, it is easy to see that the r.h.s. of (2.21) is bounded. $\square$

The following lemma states an *a priori* uniform bound for the solutions of (2.1), which by (2.2) are defined for any $t \geq 0$.

Lemma 2.6 *Let m_Λ be a continuous solution of (2.1) on $\mathbb{R}_+ \times \mathbb{R}$. Then for each $\alpha > 0$, $p \in [1, \infty)$ there exists a constant $C = C(\alpha, p, \|R\|_{L^p(\mu)}) < \infty$ independent of $\Lambda \in \mathcal{L}$ such that $\|m_\Lambda\|_{L^p(\mu)} \leq C$.*

Proof. Since the continuous solution of equation (2.1) exist globally in time, the inequality (2.3) may be extended for $T^* \uparrow \infty$; we get

$$\|\Lambda m_\Lambda\|_{L^p(\mu)} \leq \|m_\Lambda\|_{L^p(\mu_\Lambda)} \leq C < \infty. \quad (2.22)$$

Note that for $p > 1$ both G and $\tilde{G}$ are bounded operators in $L^p(\mu)$ (see [3, Lemma 9] for G and the same proof also works for $\tilde{G}$). From the integral equation (2.1), the bound (2.22), and $R \in L^p(\mu)$ the lemma follows. $\square$

3 Infinite volume equation

In this section we conclude the proof of Theorem 1.3 by removing the truncation Λ in (2.1); following [3], we prove first that, provided β is chosen large enough, $\exp(-\beta t) \Lambda m_\Lambda$ converges in $L^p(\mu)$.

Lemma 3.1 *Let m_Λ be a solution of (2.1). For each $p \in [1, \infty)$ there exists a positive constant $k_p < \infty$ such that, for any $\alpha > 0$ and $\beta \geq k_p(1 + \alpha^{-p})$, $\{\exp(-\beta t) \Lambda m_\Lambda, \Lambda \in \mathcal{L}\}$ is Cauchy in $L^p(\mu)$ as $\Lambda \uparrow 1$.*

Proof. Recalling (2.5), if we consider $f = e^{-\beta t}[u_\Lambda - u_{\bar{\Lambda}}]$, from equation (2.1), we have

$$\left(\partial_t - \frac{1}{2}\Delta\right)f = -\beta f - e^{-\beta t} [\Lambda U'(m_\Lambda) - \bar{\Lambda} U'(m_{\bar{\Lambda}})] - \lambda e^{-\beta t} [q_\Lambda - q_{\bar{\Lambda}}].$$

By the same computations as in Lemma 2.4 with f as above, (see equations (2.7) and (2.10)), but choosing $\gamma = \gamma(n, \lambda)$ independent of β and taking $T \uparrow \infty$, there exists a constant $c_1 = c_1(n)$ independent of α, β such that

$$\begin{aligned} & \mu \left(e^{-(2n+2)\beta t} [u_\Lambda - u_{\bar{\Lambda}}]^{2n+1} [\Lambda U'(m_\Lambda) - \bar{\Lambda} U'(m_{\bar{\Lambda}})] \right) \\ & + (\beta - 1) \mu \left(e^{-(2n+2)\beta t} [u_\Lambda - u_{\bar{\Lambda}}]^{2n+2} \right) \leq c_1 \|e^{-\beta t} (q_\Lambda - q_{\bar{\Lambda}})\|_{H_{-1}^{2n+2}(\mu)}^{2n+2}. \end{aligned} \quad (3.1)$$

By using Lemma 2.5 there is a constant $c_2 = c_2(n)$ such that

$$\begin{aligned} & \|e^{-\beta t} (q_\Lambda - q_{\bar{\Lambda}})\|_{H_{-1}^{2n+2}(\mu)}^{2n+2} \\ & \leq c_2 \left(1 + \alpha^{-(2n+2)}\right) \mu \left(e^{-(2n+2)\beta t} [\Lambda m_\Lambda - \bar{\Lambda} m_{\bar{\Lambda}}]^{2n+2}\right). \end{aligned} \quad (3.2)$$

On the other hand, since for any $a, b \in \mathbb{R}$ we have $(a^3 - b^3)(a - b) \geq (a - b)^4/4$, there exists a constant $c_3 = c_3(\lambda)$ such that

$$\begin{aligned} & (u_\Lambda - u_{\bar{\Lambda}}) [\Lambda U'(m_\Lambda) - \bar{\Lambda} U'(m_{\bar{\Lambda}})] \\ & = (m_\Lambda - m_{\bar{\Lambda}} - (\Lambda - \bar{\Lambda})R) (\Lambda m_\Lambda^3 - \bar{\Lambda} m_{\bar{\Lambda}}^3) - (1 - \lambda) (u_\Lambda - u_{\bar{\Lambda}}) (\Lambda m_\Lambda - \bar{\Lambda} m_{\bar{\Lambda}}) \\ & \geq \frac{1}{4} (\Lambda m_\Lambda - \bar{\Lambda} m_{\bar{\Lambda}})^4 - c_3 (u_\Lambda - u_{\bar{\Lambda}})^2 \\ & - c_3 (|1 - \Lambda| + |1 - \bar{\Lambda}|) [m_\Lambda^2(1 + m_\Lambda^2) + m_{\bar{\Lambda}}^2(1 + m_{\bar{\Lambda}}^2) + R^2(1 + R^2)] \end{aligned} \quad (3.3)$$

By plugging the bounds (3.2) and (3.3) into (3.1) and using Hölder inequality, we find there exists a $c_4 = c_4(n, \alpha, \lambda)$ such that

$$\begin{aligned} & \frac{1}{4} \mu \left(e^{-(2n+2)\beta t} (\Lambda m_\Lambda - \bar{\Lambda} m_{\bar{\Lambda}})^{2n+4}\right) \\ & + \left(\beta - 1 - c_3 - c_1 c_2 \left(1 + \alpha^{-(2n+2)}\right)\right) \mu \left(e^{-(2n+2)\beta t} (u_\Lambda - u_{\bar{\Lambda}})^{2n+2}\right) \\ & \leq c_4 \left(\|1 - \Lambda\|_{L^2(\mu)} + \|1 - \bar{\Lambda}\|_{L^2(\mu)}\right) \\ & \quad \times \left\{1 + \|m_\Lambda\|_{L^{4n+8}(\mu)}^{2n+4} + \|m_{\bar{\Lambda}}\|_{L^{4n+8}(\mu)}^{2n+4} + \|R\|_{L^{4n+8}(\mu)}^{2n+4}\right\}. \end{aligned}$$

Given $p \geq 2$, let $n = n(p) = [p/2] - 1$ and k_p such that $1 + c_3 + c_1 c_2 (1 + \alpha^{-(2n+2)}) \leq k_p (1 + \alpha^{-p})$ for any $\alpha > 0$. The lemma now follows from Lemma 2.6 and $R \in L^p(\mu)$ for any $p \in [1, \infty)$. $\square$

The proof of Theorem 1.3 can now be completed as in [3], we shall just sketch the argument.

Proof of Theorem 1.3. To prove existence of a continuous solution of (1.5), we first note that, for each $p \in [1, \infty)$ and α so large that $\alpha^2 > k_p (1 + \alpha^{-p})$, m_Λ is Cauchy in $L^p(\mu)$. This follows from Lemmata 2.6, 3.1, and Hölder inequality; moreover the limit m satisfies equation (1.5). Since G and $\tilde{G}$ map $L^2(\mu)$ into $C_\alpha(\mathbb{R}_+ \times \mathbb{R})$, see [3, Lemma 12], we also have $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$.

In order to show $m \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ for any $\alpha > 0$, we note that, by Lemma 2.6, m_Λ is uniformly bounded in $L^p(\mu)$; we can thus find a weakly convergent subsequence $m_{\Lambda_k} \rightarrow m'$. On the other hand, by Lemma 3.1, $e^{-\beta t} \Lambda_k m_{\Lambda_k}$ converges strongly in $L^p(\mu)$ for $\beta \geq k_p (1 + \alpha^{-p})$, hence $m = m'$ μ -a.s. Since $m' \in L^p(\mu)$ for any $\alpha > 0$ and $p \in [1, \infty)$, by the same argument as above, we get $m' \in C_\alpha(\mathbb{R}_+ \times \mathbb{R})$ and $m = m'$.

To prove uniqueness, let m_1 and m_2 be two continuous solutions of (1.5). By applying Lemma 2.3 to the function $f = e^{-\beta t} [m_1 - m_2]$ and repeating the same computations as in Lemma 3.1 it is easy to show $m_1 = m_2$. $\square$

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References

- [1] L. Bertini, S. Brassesco, P. Buttà and E. Presutti, Front fluctuations in one dimensional stochastic phase field equations, *Ann. Henri Poincaré* **3**, 29–86 (2002).
- [2] G. Caginalp and X. Chen, Convergence of the phase field model to its sharp interface limits, *European J. Appl. Math.* **9**, 417–445 (1998).
- [3] C. R. Doering, Nonlinear parabolic stochastic differential equations with additive colored noise on $\mathbb{R}^d \times \mathbb{R}_+$: a regulated stochastic quantization, *Comm. Math. Phys.* **109**, 537–561 (1987).
- [4] W. G. Faris and G. Jona-Lasinio, Large fluctuations for a nonlinear heat equation with noise, *J. Phys. A* **15**, 3025–3055 (1982).
- [5] P. C. Hohenberg and B. I. Halperin, Theory of dynamic critical phenomena, *Rev. Mod. Phys.* **49**, 435–479 (1977).
- [6] H. M. Soner, Convergence of the phase-field equations to the Mullins–Sekerka problem with kinetic undercooling, *Arch. Rational Mech. Anal.* **131**, 139–197 (1995).

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