



Regularizing quadrangles in the Möbius plane

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Abstract. For any given cross ratio $\delta \in \mathbb{C}$ we define a non-linear Möbius-invariant procedure creating vertices of a new quadrangle from a given quadrangle. Reiterating the process we discover a remarkable regularizing property. The values $\delta \in \mathbb{C}$ for which this phenomenon of regularization develops are examined. The regularizing effect also depends upon the shape of the starting quadrangle. The investigation of such shapes leads into the field of complex dynamics in one variable. Methods of discrete dynamics are employed to explore the regularizing behaviour of the procedure.

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1. Introduction

Euclidean and affine iterative procedures creating sequences of polygons have been studied by several authors (e.g. [5, 8, 11, 12] and [6]). At times, these procedures show some regularizing effect. This paper is devoted to the study of a procedure based on Möbius geometry (see [3]). This seems to be the first attempt to prove such a regularizing effect on quadrangles created by a non-linear procedure.

Let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$; the transformations $M : \mathbb{C}^* \rightarrow \mathbb{C}^*$ given by $M(\mathbf{z}) := \frac{\alpha_0 \mathbf{z} + \alpha_1}{\alpha_2 \mathbf{z} + \alpha_3}$ with complex numbers $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ and $\alpha_0 \alpha_3 \neq \alpha_1 \alpha_2$ are called *Möbius transformations*. Along with this group $PGL(\mathbb{C})$ of Möbius transformations \mathbb{C}^* is called the *Möbius plane*.

A quadrangle is an ordered set $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ of four points such that its *cross ratio*¹

$$\text{cr}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) := \frac{(\mathbf{p}_0 - \mathbf{p}_3)(\mathbf{p}_1 - \mathbf{p}_2)}{(\mathbf{p}_0 - \mathbf{p}_2)(\mathbf{p}_1 - \mathbf{p}_3)} \in \mathbb{C}^* \tag{1.1}$$

is defined. For the cross ratio $z^* := \text{cr}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ we have the following identities:

$$\begin{aligned} \text{cr}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_0) &= 1 - z^* & \text{cr}(\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_0, \mathbf{p}_1) &= z^* \\ \text{cr}(\mathbf{p}_3, \mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2) &= 1 - z^* & \text{cr}(\mathbf{p}_3, \mathbf{p}_2, \mathbf{p}_1, \mathbf{p}_0) &= z^* \end{aligned} \tag{1.2}$$

Let $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ be a quadrangle and M be any Möbius transformation. Then the quadrangle of the image points $M(\mathbf{p}_j)$ has the same cross ratio $\text{cr}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \text{cr}(M(\mathbf{p}_0), M(\mathbf{p}_1), M(\mathbf{p}_2), M(\mathbf{p}_3))$. On the other hand, if $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ are quadrangles with the same cross ratio there exists a unique Möbius transformation M such that $M(\mathbf{p}_j) = \mathbf{q}_j$ for $j = 0, 1, 2, 3$. The complex number $z^* = \text{cr}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) \in \mathbb{C}^*$ characterizes a quadrangle $Q = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ w.r.t. Möbius geometry. The cross ratio z^* is called the *characteristic cross ratio* of Q or the *shape*² of Q . According to (1.2) we have

Lemma 1.1. *For two quadrangles $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and $(\mathbf{q}_0, \mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ whose shapes are z^* and $1 - z^*$, respectively, there exists a Möbius transformation M such that $M(\mathbf{p}_0) = \mathbf{q}_1, M(\mathbf{p}_1) = \mathbf{q}_2, M(\mathbf{p}_2) = \mathbf{q}_3, M(\mathbf{p}_3) = \mathbf{q}_0$.*

The quadrangle Q is called *cyclic* if and only if $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ lie on a Möbius circle. A cyclic quadrangle is characterized by a real cross ratio z^* .

2. New generations of quadrangles

In this section we define a procedure generating a new quadrangle from a given quadrangle.

Definition 2.1. We select a complex number $\delta \in \mathbb{C} \setminus \{0, 1\}$. For a given quadrangle $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ the *next generation quadrangle* $(\mathbf{p}_0^*, \mathbf{p}_1^*, \mathbf{p}_2^*, \mathbf{p}_3^*)$ is determined by

$$\delta = \text{cr}(\mathbf{p}_j, \mathbf{p}_{j+1}, \mathbf{p}_{j+2}, \mathbf{p}_j^*) \text{ for } j = 0, 1, 2, 3. \tag{2.1}$$

Here and in the following paragraphs the index j is to be interpreted mod 4. The complex number δ is referred to as the *construction cross ratio*.

We can also offer an explicit formula for the vertices of the next generation quadrangle:

$$\mathbf{p}_j^* = \frac{\mathbf{p}_j(\mathbf{p}_{j+2} - \mathbf{p}_{j+1}) - \delta \mathbf{p}_{j+1}(\mathbf{p}_{j+2} - \mathbf{p}_j)}{(\mathbf{p}_{j+2} - \mathbf{p}_{j+1}) - \delta (\mathbf{p}_{j+2} - \mathbf{p}_j)} \text{ for } j = 0, 1, 2, 3 \tag{2.2}$$

¹In fact, there are six ways to define a cross ratio of four points (see [4, p. 62]). For $\sigma = \text{cr}(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ the other ways yield one of the following values $\frac{1}{\sigma}, 1 - \sigma, \frac{1}{1 - \sigma}, \frac{\sigma}{\sigma - 1}$ and $\frac{\sigma - 1}{\sigma}$.

²This definition can be viewed as a Möbius-geometric extension of the *shape* defined in [1].

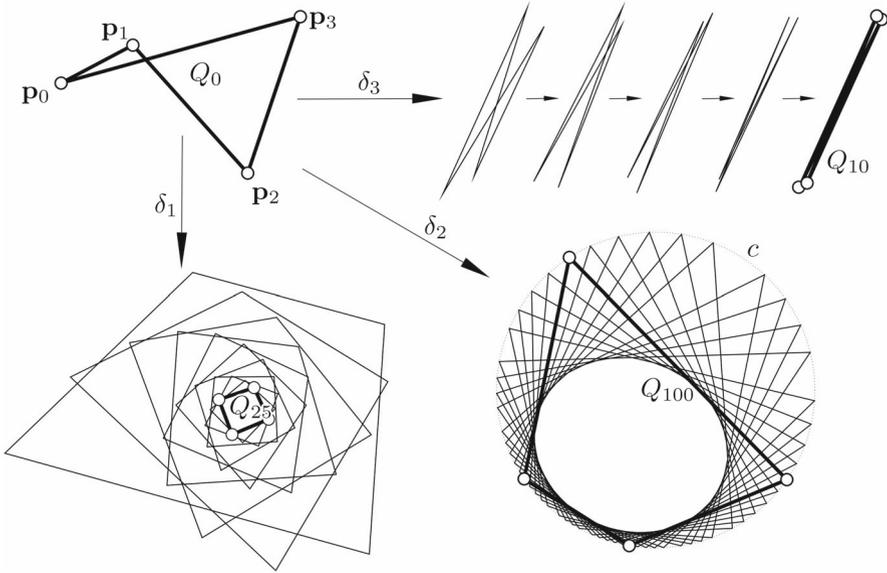


FIGURE 1 The initial quadrangle $Q_0 = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and some of the following generations (scaled and shifted) for three different procedures defined by the construction cross ratios $\delta_1 = 2.5 + i$, $\delta_2 = 2.5$ and $\delta_3 = 0.75 + 0.25 i$

Owing to the invariance of cross ratios with respect to Möbius transformations, Definition 2.1 yields:

Lemma 2.2. *Let $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ be a quadrangle and M be a Möbius transformation. Then the next generation quadrangle to $(M(\mathbf{p}_0), M(\mathbf{p}_1), M(\mathbf{p}_2), M(\mathbf{p}_3))$ yields the same characteristic cross ratio z^* as the next generation quadrangle to $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$: The procedure (2.2) is Möbius invariant.*

This procedure can be applied iteratively to a starting quadrangle (see Fig. 1). The shape z_n of the next generation quadrangle $Q_n = (\mathbf{p}_0^*, \mathbf{p}_1^*, \mathbf{p}_2^*, \mathbf{p}_3^*)$ is fully determined by the chosen construction cross ratio δ and the shape z_{n-1} of the quadrangle $Q_{n-1} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$. So we can compute the characteristic cross ratio z_n of Q_n straight from the characteristic cross ratio z_{n-1} of Q_{n-1} : For pairwise distinct points $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ Lemma 2.2 tells us that we are entitled to replace the quadrangle $Q_{n-1} = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ by some prototype $(0, \infty, 1, z_{n-1})$ with the same characteristic cross ratio z_{n-1} . For this prototype the vertices of the next generation quadrangle Q_n are:

$$\mathbf{p}_0^* = \delta, \quad \mathbf{p}_1^* = \frac{\delta + z_{n-1} - 1}{\delta}, \quad \mathbf{p}_2^* = \frac{z_{n-1}(1 - \delta)}{z_{n-1} - \delta}, \quad \mathbf{p}_3^* = \frac{z_{n-1}}{1 - \delta} \tag{2.3}$$

Its corresponding characteristic cross ratio $f_\delta(z_{n-1}) := z_n$ is

$$f_\delta(z_{n-1}) = \frac{(1 - z_{n-1})(\delta^2 - \delta + z_{n-1})^2}{(\delta^2 - 2\delta z_{n-1} + z_{n-1})(\delta^2 - 2\delta(1 - z_{n-1}) + 1 - z_{n-1})}. \tag{2.4}$$

The same result can be derived if $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2$ are not pairwise distinct. This formula can also be viewed as an iteration rule enabling us to compute $z_n = f_\delta^n(z_0)$ from the shape z_0 of a starting quadrangle Q_0 . According to our construction it is clear that the following functional equations apply:

$$f_\delta(1 - z) = 1 - f_\delta(z) \quad \text{and} \quad f_{1-\delta}(z) = f_\delta(z) \quad \forall (\delta, z) \in \mathbb{C} \times \mathbb{C}^* \tag{2.5}$$

3. Regular quadrangles

A quadrangle $(\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ is called *regular* (in the sense of Möbius Geometry) if the values (1.2) are equal: $z^* = 1 - z^*$. Obviously, this yields two kinds of regular quadrangles: $z^* = \frac{1}{2}$ and $z^* = \infty$. For the cross ratio $z^* = \frac{1}{2}$ we can think of the example $(1, i, -1, -i)$ which reminds us of a square. This first kind of regular quadrangles is sometimes referred to as *harmonic quadrangle*³ (see [4, p. 63]). As for $z^* = \infty$ (second kind), we can think of the example $(1, -1, 1, -1)$. The next generation quadrangle Q_n of a regular quadrangle Q_{n-1} is again regular of the same kind.

Definition 3.1. An infinite sequence of quadrangles Q_0, Q_1, \dots created by (2.2) is said to be *regularizing* if the sequence (2.4) of the corresponding shapes z_0, z_1, \dots converges to $\frac{1}{2}$ or ∞ : $\lim_{n \rightarrow \infty} z_n \in \{\frac{1}{2}, \infty\}$.

The explicit formula (2.2) enables us to implement the procedure and to apply it to particular examples of quadrangles. Mind that the image of a regular quadrangle will not necessarily appear *regular* in terms of Euclidean geometry in \mathbb{R}^2 ; after all, the setting is Möbius Geometry. Apart from this, *Regularization* (see Definition 3.1) does not mean that the quadrangle Q_n tends towards some regular limit quadrangle. It may well be that the sequence Q_0, Q_1, \dots of quadrangles shrinks towards a degenerate quadrangle $(\mathbf{p}, \mathbf{p}, \mathbf{p}, \mathbf{p})$ with $\mathbf{p} \in \mathbb{C}^*$. Even in such a case the regularizing property in terms of Definition 3.1 might still apply⁴; see also Fig. 1.

In Fig. 1 the initial quadrangle $Q_0 = (\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ has the shape $z_0 \approx 1.168 - 0.345i$. The three illustrated procedures are defined by the construction cross ratios $\delta_1 = 2.5 + i$, $\delta_2 = 2.5$ and $\delta_3 = 0.75 + 0.25i$. They behave differently:

- For δ_1 we get the shape $z_{25} \approx 0.526 - 0.007i$ for the highlighted quadrangle Q_{25} ; the procedure regularizes towards the shape $z^* = \frac{1}{2}$. The quadrangles, while getting more and more regular of first kind, keep shrinking from generation to generation towards a point.
- For δ_2 the quadrangle Q_{100} is highlighted; its shape is $z_{100} \approx 0.5 + 1.87 \times 10^{-11}i$. The procedure again regularizes towards the shape $z^* = \frac{1}{2}$. The circumcircles of any three distinct points of the quadrangles tend towards

³If we went for another definition of the cross ratio where z^* is replaced by $\frac{z^*-1}{z^*}$, the value turns into -1 , which evokes the term *harmonic*.

⁴The point \mathbf{p} can even be the element $\infty \in \mathbb{C}^*$; this is when the quadrangle apparently ‘explodes’ from the Euclidean point of view.

one limit circle c . The points of the quadrangles asymptotically approach that Möbius circle c , but still there does not exist any limit quadrangle.

- For δ_3 the procedure regularizes towards $z^* = \infty$. The quadrangle Q_{10} is highlighted; its shape is $z_{10} \approx -715.48 + 637.32 i$.

Whether or not the process (2.2) regularizes may depend on the construction cross ratio δ as well as on the characteristic cross ratio (shape) z_0 of the starting quadrangle. In order to examine the behaviour we need not compute the points of the sequence of quadrangles via (2.2) explicitly. According to Definition 3.1 we instead study the function (2.4); this way we zero in on the behaviour of the shape.

4. The rational iteration and its fixed points

Along with the emergence of $f_\delta(z)$ (see Equation (2.4)) we have arrived in the realm of complex-valued rational iteration (see also [2] and [7]). The functional equations (2.5) inspire the substitutions

$$\zeta := 2z - 1, \quad \gamma := (2\delta - 1)^2 \quad \text{and} \quad R_\gamma(\zeta) := (f_\delta(z) + 1)/2. \tag{4.1}$$

In terms of ζ and γ the iteration (2.4) now reads:

$$R_\gamma(\zeta) = \frac{\zeta [(3 - \gamma)(1 + \gamma) - 4\zeta^2]}{(1 + \gamma)^2 - 4\gamma\zeta^2} \tag{4.2}$$

Properties of the discrete dynamical system defined by R_γ have a direct impact on the geometry of our iterative procedure defined in (2.1) and (2.2). This is why we focus on the behaviour of this system in the next few sections.

The function R_γ is rational of degree $d = 3$ with $R_\gamma(-\zeta) = -R_\gamma(\zeta)$ for all $\gamma \in \mathbb{C} \setminus \{\pm 1, 0\}$ and $\zeta \in \mathbb{C}^*$. For $\gamma = \pm 1$ we get $R_{\pm 1}(\zeta) = \pm\zeta \quad \forall \zeta \in \mathbb{C}^*$; we exclude the cases $\gamma = \pm 1$ from now on. For $\gamma = 0$ the function is even a cubic polynomial $R_0(\zeta) = \zeta(3 - 4\zeta^2)$.

Every limit value ζ of a rational iteration (4.2) has to be a fixed point. So we first go for the set of fixed points of R_γ :

$$t_0 = \infty, \quad t_1 = 0, \quad t_{2,3} = \pm\sqrt{(1 + \gamma)/2} \tag{4.3}$$

t_0 and t_1 are the values relating to the shape-values ∞ and $1/2$ of regular quadrangles. The two fixed points $t_{2,3}$, however, are different from t_0, t_1 due to $\gamma \neq -1$; they also characterize quadrangles Q_0 that keep their shape during all subsequent steps of iteration.

According to [2, p. 99] and [7, p. 45] we check whether one of the fixed points can be attracting. A fixed point $t \in \mathbb{C}$ of R_γ is called *attracting* if and only if $|R'_\gamma(t)| < 1$ (*repelling* if and only if $|R'_\gamma(t)| > 1$). The infinite fixed point $t = \infty$ of R_γ is called *attracting (repelling)* if and only if $G_\gamma(\zeta) := 1/R_\gamma(1/\zeta)$ is attracting (repelling) at $\zeta = 0$. The *basin of attraction* \mathcal{A}_t of an attracting fixed point $t \in \mathbb{C}^*$ is the set of points $\zeta \in \mathbb{C}^*$ with $\lim_{n \rightarrow \infty} R_\gamma^n(\zeta) = t$.

For any attracting fixed point t there exists a neighbourhood $U \subset \mathbb{C}^*$ of t such that for all $\zeta \in U$ we have $\lim_{n \rightarrow \infty} R_\gamma^n(\zeta) = t$. In order to determine the local behaviour of $R_\gamma(\zeta)$ at the fixed points we consider the derivatives

$$R'_\gamma(t_0) = G'_\gamma(0) = \gamma, \quad R'_\gamma(t_1) = \frac{3 - \gamma}{1 + \gamma} \quad \text{and} \quad R'_\gamma(t_{2,3}) = -3. \tag{4.4}$$

From (4.4) we infer that for all $\gamma \in \mathbb{C} \setminus \{\pm 1\}$ the two fixed points $t_{2,3}$ are repelling. t_0 is attracting if and only if $\gamma \in C_0$, t_1 is attracting if and only if $\gamma \in C_1$ with:

$$\begin{aligned} C_0 &:= \{\gamma \in \mathbb{C} \mid |\gamma| < 1\} \\ C_1 &:= \{\gamma \in \mathbb{C} \mid \left| \frac{3 - \gamma}{1 + \gamma} \right| < 1\} = \{\gamma \in \mathbb{C} \mid \operatorname{Re}(\gamma) > 1\} \end{aligned} \tag{4.5}$$

The two domains C_0 and C_1 are disjoint.

It can also happen that an iteratively generated sequence of points is heading towards a periodic orbit; these are *cycles of length $n \geq 2$* relating to fixed points of $R_\gamma^n(z)$ that are not fixed points of $R_\gamma^m(z)$ for any $m \in \{1, \dots, n - 1\}$. Such a cycle is called *attracting (repelling)* if and only if it contains an attracting (a repelling) fixed point of R_γ^n .

Due to $R_\gamma(\pm 1) = \mp 1$ the pair $(-1, 1) \in \mathbb{C}^{*2}$ is one example of a cycle of length 2. For this cycle we have

$$(R_\gamma^2)'(-1) = (R_\gamma^2)'(1) = ((3 + \gamma)/(1 - \gamma))^4. \tag{4.6}$$

Hence, this cycle is attracting if and only if $\gamma \in \tilde{C} := \{\gamma \in \mathbb{C} \mid \operatorname{Re}(\gamma) < -1\}$; this domain \tilde{C} is disjoint from both, C_0 and C_1 . Additionally, we have $(R_\gamma^n)'(\pm 1) = (R'_\gamma(\pm 1))^n$ which will be of importance in Sect. 6.

In the following sections we lock a value γ in one of the domains C_0 or C_1 . Then, either $t_0 = \infty$ or $t_1 = 0$, is an attracting fixed point of the rational function R_γ given by (4.2). The other fixed points are repelling, though there might well exist further attracting cycles of R_γ .

5. Critical points and basins of attraction

According to [2, p. 43] a point ζ is called a *critical point* of $R_\gamma(\zeta)$ if and only if R_γ fails to be injective in any neighbourhood of ζ and R_γ is not constant. As R_γ is rational of degree $d = 3$ there are at most $2d - 2 = 4$ critical points in \mathbb{C}^* . Candidates for critical points are the zeroes of R'_γ . The poles of R_γ and the point ∞ deserve particular care. For the polynomial case $\gamma = 0$ the function $R_0(\zeta)$ has only the critical points $c_{0,1} = \pm \frac{1}{2}$ and $c_{2,3} = \infty$. In all other cases ($\gamma \notin \{0, \pm 1\}$) we get the following four critical points:

$$c_{0,1} = \pm(1 + \gamma)/2, \quad c_{2,3} = \pm\sqrt{(3 + 2\gamma - \gamma^2)/(4\gamma)} \tag{5.1}$$

As we excluded $\gamma = -1$ we have $c_0 \neq c_1$ and $R_\gamma(c_{0,1}) = \pm 1$. Apart from the polynomial case $\gamma = 0$ we have one further exceptional case $\gamma = 3$ where we

get $c_2 = c_3 = 0 = t_1$; the fixed point t_1 happens to be a critical point. For the polynomial case $\gamma = 0$ the same thing occurs to $t_0 = c_2 = c_3 = \infty$.

For all $\gamma \neq \pm 1$ the points c_0, \dots, c_3 are the four critical points for R_γ (counted with multiplicity). Any basin of attraction for attracting cycles or fixed points has to contain at least one critical point of R_γ (see [2, p. 194] or [7, p. 82]). As the critical points c_0 and c_1 are mapped into the points of the fixed cycle $(-1, 1)$ of length 2 the first two critical points cannot belong to basins of attraction for the fixed points $t_0 = \infty$ or $t_1 = 0$. The remaining two critical points $c_{2,3}$ are symmetric with respect to the point 0. From the symmetry of R_γ we can see: For $\lim_{n \rightarrow \infty} R_\gamma^n(c_2) = q$ we have $\lim_{n \rightarrow \infty} R_\gamma^n(c_3) = -q$. For values q with $q = -q$ we can conclude that c_2 and c_3 indulge in the very same behaviour: If one of them lies in the basin of attraction of t_0 , so will the other one. The same holds for t_1 . We arrive at one of the following three cases:

1. Let γ be locked in C_0 ; then t_0 is a point of attraction and c_2, c_3 are both contained in the basin of attraction \mathcal{A}_∞ of $t_0 = \infty$.
2. Let γ be locked in C_1 ; then t_1 is a point of attraction and c_2, c_3 are both contained in the basin of attraction \mathcal{A}_0 of $t_1 = 0$.
3. If γ is locked anywhere outside of $C_0 \cup C_1 \cup \{\pm 1\}$ the *same behaviour statement* from above does not apply any more: c_2 and c_3 could well lie in different basins of attraction. These basins cannot belong to fixed points as the fixed points are repelling in this case.

According to [2, p. 194] we can be sure that in the cases 1 and 2 there are no further attracting cycles of any length. Thus, we have

Theorem 5.1. *For $\gamma \in C_0 \subset \mathbb{C}$ the only point of attraction of R_γ is $t_0 = \infty$. For $\gamma \in C_1 \subset \mathbb{C}$ the only point of attraction is $t_1 = 0$. In any of these cases there exist no further attracting fixed points or cycles of any length.*

In each of the cases 1 and 2 there is only one basin of attraction of the corresponding fixed point. This does not mean, though, that any point $\zeta \in \mathbb{C}^*$ converges to t_0 or t_1 ; the exceptions will be described further on.

6. The sets of Fatou and Julia

We follow [2, p. 50] and define: The family of functions $\{R_\gamma^n \mid n \in \mathbb{N}\}$ is called *equicontinuous* at \tilde{z} if and only if

$$\forall \varepsilon > 0 \exists \alpha > 0 \text{ and } N_0 \in \mathbb{N} : \forall z \in \mathbb{C}^*, \forall n \geq N_0$$

$$\text{such that } |z - \tilde{z}| < \alpha \Rightarrow |R_\gamma^n(z) - R_\gamma^n(\tilde{z})| < \varepsilon. \tag{6.1}$$

The *Fatou set* \mathcal{F}_γ of R_γ is the maximal open subset of \mathbb{C}^* (see [2, p. 50] or [7, p. 40]) on which the sequence R_γ^n is equicontinuous. The *Julia set* \mathcal{J}_γ is the complementary set $\mathcal{J}_\gamma = \mathbb{C}^* \setminus \mathcal{F}_\gamma$.

Obviously, the sequence R_γ^n is not equicontinuous at repelling fixed points or repelling cycles: Such points are contained in the Julia set \mathcal{J}_γ .

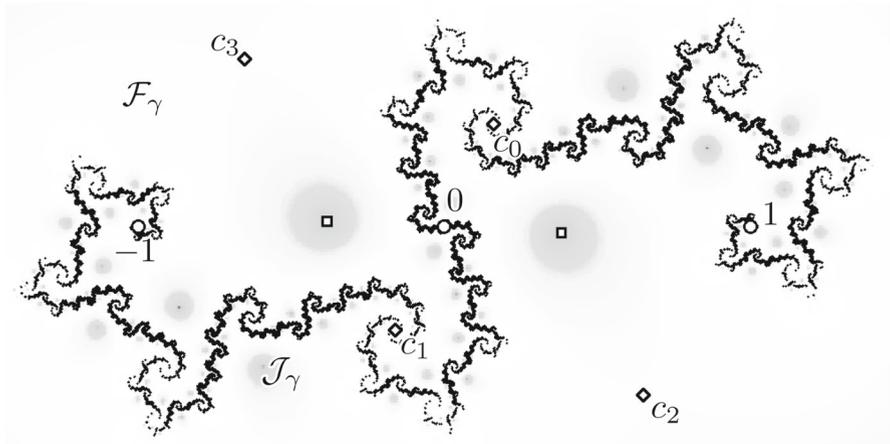


FIGURE 2 The Julia set \mathcal{J}_γ for $\gamma = 0.96(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4})$. Here $t_0 = \infty$ is the attracting fixed point, the three marked points $-1, 0$ and 1 belong to \mathcal{J}_γ . The diamond-shaped symbols indicate the critical points c_0, \dots, c_3 , the squares mark the two finite preimages of ∞ . According to Theorem 6.1 \mathcal{J}_γ has a diameter $\leq 2 \times 4.6$

If γ lies in C_0 the set $\{-1, 0, 1\}$ is contained in the Julia set \mathcal{J}_γ and $\mathcal{F}_\gamma = \mathcal{A}_\infty$. If γ lies in C_1 the set $\{-1, 1, \infty\}$ is contained in the Julia set \mathcal{J}_γ and $\mathcal{F}_\gamma = \mathcal{A}_0$. The following result sheds some light on the Fatou set \mathcal{F}_γ and the Julia set \mathcal{J}_γ for $\gamma \in C_0$; the proof is straightforward:

Theorem 6.1. For $\gamma \in C_0$ and $|z| > M_\gamma := \max\{\sqrt{\frac{3}{2}}, \sqrt{\frac{|1+\gamma|(|1+\gamma|+|3-\gamma|)}{4(1-|\gamma|)}}\}$ we have $|R_\gamma(z)| \geq q|z|$ with some real $q > 1$.

One property of the Fatou set \mathcal{F}_γ is that, for $\gamma \in C_0$, it contains the subset $\{z \in \mathbb{C}^* \mid |z| > M_\gamma\}$ - the respective Julia set \mathcal{J}_γ is bounded. The Julia set \mathcal{J}_γ (see [2, p. 71]) is the closure of the backward orbit $O^-(z) := \{w \in \mathbb{C}^* \mid \exists n > 0 \text{ with } R_\gamma^n(w) = z\}$ for any $z \in \mathcal{J}_\gamma$. This property can be used to visualize the Julia set \mathcal{J}_γ in any of the two aforementioned cases $\gamma \in C_0$ and $\gamma \in C_1$. Figures 2 and 3 refer to $\gamma \in C_0$ and $\gamma \in C_1$, respectively; the Julia set \mathcal{J}_γ is displayed in black.⁵

A brief look at the Figs. 2 and 3 implores the question: Are the Julia sets \mathcal{J}_γ connected, disconnected or do we encounter the phenomenon of a Cantor Set (compare [2, p. 227])? Due to numerical observations for particular values of γ we can imply that the preconditions for [9, Theorem 2.2] are fulfilled: For the considered values of γ the Julia set \mathcal{J}_γ is connected. Without being able

⁵The shading in the Fatou set \mathcal{F}_γ in both Figs. 2 and 3 was generated by means of Wolfram Mathematica 11.0. The centers of the spots in \mathcal{F}_γ in Fig. 2 are the reverse iterates of the point ∞ . The grey spots in Fig. 3 relate to the reverse iterates of 0.

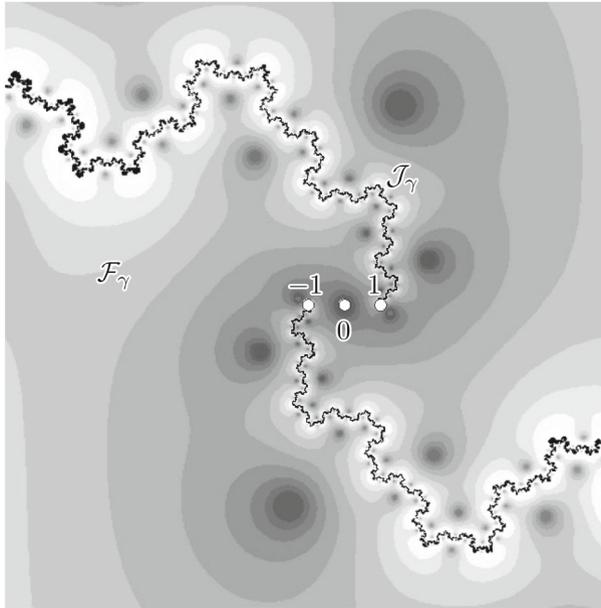


FIGURE 3 The Julia set \mathcal{J}_γ for $\gamma = 2.25(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$. Here $t_1 = 0$ is the attracting point, the points $-1, 1$ and ∞ belong to \mathcal{J}_γ

to deliver a general proof, we conjecture that the Julia set \mathcal{J}_γ is connected for all $\gamma \in C_0 \cup C_1$.

The critical values of R_γ are the images $v_j := R(c_j)$ ($j = 0, \dots, 3$) of critical points c_j (see (5.1)). According to [10, Theorem B] a sufficient condition for the Julia set \mathcal{J}_γ to be of Lebesgue measure 0 is

$$\sum_{n \geq 0} |(R_\gamma^n)'(v)|^{-1} < \infty \tag{6.2}$$

for those critical values $v \in \mathcal{J}_\gamma$ that are not in the backward orbit of a critical point. In our case the critical values to be considered are $v_{0,1} = \pm 1$. With (4.6) the expression (6.2) is a geometric series. In fact, it is convergent exactly for all $\gamma \in \mathbb{C}$ that are not contained in the closure of \tilde{C} . As this holds for all $\gamma \in C_0 \cup C_1$ we can be sure that the Lebesgue measure of the respective Julia set \mathcal{J}_γ is zero.

Now we remember our primary issue of regularization. We transcribe the results on the discrete dynamical system R_γ and interpret them accordingly:

Theorem 6.2. *The procedure established in Definition 2.1, employed iteratively, has the following regularization properties:*

- *If the construction cross ratio δ delivers $\gamma = (2\delta - 1)^2$ in $C_0 \subset \mathbb{C}$ and the shape z_0 of the initial quadrangle provides a value $\zeta = 2z_0 - 1$ lying*

in the Fatou set \mathcal{F}_γ , the procedure regularizes towards the shape $z^* = \infty$. This is the shape of regular quadrangles of second kind with $\mathbf{p}_0 = \mathbf{p}_2$ and $\mathbf{p}_1 = \mathbf{p}_3$.

- If the construction cross ratio δ delivers $\gamma = (2\delta - 1)^2$ in $C_1 \subset \mathbb{C}$ and the shape z_0 of the initial quadrangle provides a value $\zeta = 2z_0 - 1$ lying in the Fatou set \mathcal{F}_γ , the procedure regularizes towards $\zeta = 0$, i.e. $z^* = \frac{1}{2}$. This is the shape of regular quadrangles of first kind.

In both cases the exceptional shapes z_0 (where no regularization happens) are exactly those with $\zeta = 2z_0 - 1$ lying in the corresponding Julia set \mathcal{J}_γ . The sets of exceptional values ζ have Lebesgue measure zero for any $\gamma \in C_0 \cup C_1$ (see also Figs. 2, 3). For the respective values δ any quadrangle with generic shape z_0 is regularized by our procedure.

7. Conclusions

We have created a procedure (Definition 2.1) based on a parameter $\delta \in \mathbb{C}$ which – if applied iteratively – is capable of regularizing a quadrangle. We proved a regularization theorem: If the parameter δ delivers a value $\gamma = (2\delta - 1)^2$ lying in $C_0 \subset \mathbb{C}$ or $C_1 \subset \mathbb{C}$ (see (4.5)) then the procedure regularizes any generic quadrangle.

The results on our procedure regularizing quadrangles obviously begs the question whether adequate considerations can be applied to n -gons for $n \geq 5$; this question leads to complex discrete dynamics in dimension $n - 3$.

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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