ERRATUM

Erratum to: The Space $B_{\infty,\infty}^{-1}$, Volumetric Sparseness, and 3D NSE

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The proof of Lemma 3.3 presented in the paper contains an error. Here, we prove a slightly different form of the lemma that suffices for our purposes. In fact, this is exactly what is needed to prove the main result, Theorem 1.1.

1. The Correct Form of Lemma 3.3

The following lemma is a vector-valued, Besov space version of a scalar-valued, Sobolev space lemma in [2]. All the norms to appear in the statement of the lemma are ∞ -type norms.

Lemma 3.3. *Let $\epsilon \in (0, 1]$, $r \in (0, 1]$ and u a vector-valued function in L^∞ . Then, for any pair λ, δ , $\lambda \in (0, 1)$ and $\delta \in (\frac{1}{1+\lambda}, 1)$, there exists an explicit constant $c = c(\lambda, \delta)$ such that if*

$$\|u\|_{B_{\infty,\infty}^{-\epsilon}} \leq c(\lambda, \delta) r^\epsilon \|u\|_\infty,$$

then each of the six super-level sets $A_\lambda^{i,\pm} := \{x \in \mathbb{R}^3 : u_i^\pm > \lambda \|u\|_\infty\}$ is r -semi-mixed with the ratio δ . (Here, as customary, for a real-valued function g , $g^+(x) = \max(g(x), 0)$ and $g^-(x) = -\min(g(x), 0)$.)

Proof. Arguing by contradiction, assume that there exists $i \in \{1, 2, 3\}$ such that either $A_\lambda^{i,+}$ or $A_\lambda^{i,-}$ is not r -semi-mixed with the ratio δ . Without loss of generality, assume that it is $A_\lambda^{i,+}$ (if it were $A_\lambda^{i,-}$, the only modification to the proof would be replacing the function f below with $-f$).

Then, there exists $x_0 \in \mathbb{R}^3$ such that

$$\frac{m(A_\lambda^{i,+} \cap B(x_0, r))}{m(B(x_0, r))} > \delta;$$

equivalently,

$$m(A_\lambda^{i,+} \cap B(x_0, r)) > \delta \Pi(3) r^3,$$

where $\Pi(3)$ denotes the volume of the unit ball in $\mathbb{R}^3$.

Let $f \in B_{1,1}^\varepsilon$ be a smooth radial cut-off equal to 1 in $B(x_0, r)$, and vanishing outside $B(x_0, (1 + \eta)r)$ for some $\eta > 0$ (the value to be determined at the end of the proof). Then,

$$\|u\|_{B_{\infty,\infty}^{-\varepsilon}} \geq \frac{c}{\|f\|_{B_{1,1}^\varepsilon}} \left| \int_{\mathbb{R}^3} u_i(x) f(x) dx \right| \quad (3.1)$$

for a positive constant c .

An explicit calculation of the $B_{1,1}^\varepsilon$ -norm of f via the finite differences (using the finite differences of order two for the endpoint case $\varepsilon = 1$; c.f. chapter II in [1]) yields

$$\|f\|_{B_{1,1}^\varepsilon} \leq c(\eta) r^{3-\varepsilon} \quad (3.2)$$

for some $c(\eta) > 0$.

Next, write

$$\left| \int_{\mathbb{R}^3} u_i(x) f(x) dx \right| \geq \int_{\mathbb{R}^3} u_i(x) f(x) dx \geq I - |II| - |III|,$$

where

$$I = \int_{A_\lambda^{i,+} \cap B(x_0, r)} u_i(x) f(x) dx,$$

$$II = \int_{B(x_0, r) \setminus A_\lambda^{i,+}} u_i(x) f(x) dx,$$

and

$$III = \int_{(B(x_0, (1+\eta)r) \setminus B(x_0, r))} u_i(x) f(x) dx.$$

It is plain that

$$\begin{aligned} I &= \int_{A_\lambda^{i,+} \cap B(x_0, r)} u_i(x) dx = \int_{A_\lambda^{i,+} \cap B(x_0, r)} u_i^+(x) dx \\ &> \lambda \|u\|_\infty m(A_\lambda^{i,+} \cap B(x_0, r)) \geq \lambda \delta \Pi(3) r^3 \|u\|_\infty, \end{aligned} \quad (3.3)$$

$$\begin{aligned} |II| &= \left| \int_{B(x_0, r) \setminus A_\lambda^{i,+}} u_i(x) dx \right| \leq \|u\|_\infty (m(B(x_0, r)) - m(A_\lambda^{i,+} \cap B(x_0, r))) \\ &\leq \|u\|_\infty (\Pi(3) r^3 - \delta \Pi(3) r^3) \\ &= (1 - \delta) \Pi(3) r^3 \|u\|_\infty, \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} |III| &\leq \left| \int_{(B(x_0, (1+\eta)r) \setminus B(x_0, r))} u_i(x) dx \right| \\ &\leq \|u\|_\infty (m(B(x_0, (1+\eta)r)) - m(B(x_0, r))) \\ &\leq ((1+\eta)^3 - 1) \Pi(3) r^3 \|u\|_\infty. \end{aligned} \quad (3.5)$$

It follows from (3.1), (3.2) and (3.3)–(3.5) that

$$\|u\|_{B_{\infty,\infty}^{-\varepsilon}} > c^*(\eta) \Pi(3) r^\varepsilon \|u\|_\infty (\lambda \delta + \delta - (1 + \eta)^3).$$

Since $\delta > \frac{1}{1+\lambda}$, we can define $\eta = \eta(\lambda, \delta)$ to be the solution to the equation $(1 + \eta)^3 = \frac{\delta(1+\lambda)+1}{2}$; this in turn yields

$$\|u\|_{B_{\infty,\infty}^{-\varepsilon}} > c(\lambda, \delta) \Pi(3) r^\varepsilon \|u\|_\infty \quad (3.6)$$

with $c(\lambda, \delta) = c^*(\eta) \frac{\delta(1+\lambda)-1}{2}$, which is positive since $\delta > \frac{1}{1+\lambda}$. This contradicts the statement in the lemma. $\square$

Remark 3.4. The counterexample to the converse of the lemma constructed in the paper is also a counterexample to the converse of the form of the lemma as exposed here.

Remark 3.5. As far as utilizing the current form of the lemma to prove the main result, Theorem 1.1., the reason that we do not need the full vectorial super-level set $A_\lambda := \{x \in \mathbb{R}^3 : |u(x)| > \lambda \|u\|_\infty\}$ to be r -semi-mixed is that the argument utilizing the harmonic measure is *completely local* in the space, i.e., we are considering a bound on $|u(x_0, s_0)|$ at a *particular point* x_0 , which is simply one of the six component values $u_i^\pm(x_0, s_0)$ (considering the max-norm of a vector in $\mathbb{R}^3$); hence, the r -semi-mixedness of the component super-level sets provided by the new form of the lemma suffices. Shortly, this does not alter the proof of the main result at all.

The only slight modification needed is caused by the constraint $\delta \in (\frac{1}{1+\lambda}, 1)$; however, this only requires re-sizing a number of absolute constant with the end result that the small absolute constant in Theorem 1.1 becomes a little smaller.

References

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