Hadamard Multipliers on Spaces of Holomorphic Functions

Maria Trybuła

Dedicated to the memory of Professor Paweł Domański.

Abstract. Several representation theorems of multipliers are derived: in terms of analytic functionals, and germs of holomorphic functions. The co-induced topology via the representation theorems is discussed. As an application of the representation theorems the density of non-invertible multipliers is proved. Moreover, Euler differential operators are distinguished among all multipliers.

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1. Notations and Main Statements

Let $\Omega \subset \mathbb{C}$ be a Runge open set. A linear continuous map $M : H(\Omega) \rightarrow H(\Omega)$ is called a *Hadamard*, or *coefficient type multiplier* if every monomial is an eigenvector of M . The corresponding sequence of eigenvalues $(m_n)_{n \in \mathbb{N}}$ will be called the *multiplier sequence*.

The set of all multipliers on $H(\Omega)$, denoted by $M(\Omega)$, is a linear subspace of the space of all continuous operators $L_b(H(\Omega))$ on the Fréchet space $H(\Omega)$. Recall that the space $L_b(H(\Omega))$ is endowed with the strong topology τ_b , i.e., the topology of uniform convergence on bounded subsets of $H(\Omega)$. We equip $M(\Omega)$ with the induced topology from $L_b(H(\Omega))$. Observe that $M(\Omega)$ is a closed subspace of $L_b(H(\Omega))$, thus complete.

Abel's Theorem shows that monomials form a Schauder basis of $H(\Omega)$ if and only if $\Omega = r\mathbb{D}$ for some $r > 0$, or $\Omega = \mathbb{C}$. Thus, multipliers are not just diagonal operators. On the other hand, since Ω is a Runge open set, every multiplier acting on $H(\Omega)$ is unambiguously determined by its behaviour on monomials.

Example. Hadamard multiplication operators. (cf. [6, 11, 16–18], as well as the survey article [20]) For any function f holomorphic around 0 let G_f stand for the maximal starlike domain to which f is analytically continuable. Following Hadamard define the *Hadamard multiplication* on the space of germs of holomorphic functions around the origin:

$$*: H(\{0\}) \times H(\{0\}) \rightarrow H(\{0\}), \quad \sum_n a_n z^n * \sum_n b_n z^n := \sum_n a_n b_n z^n.$$

Assume that $U, \tilde{U}$ are arbitrary domains, $0 \in U \cap \tilde{U}$. An operator $L_g : H(U) \rightarrow H(\tilde{U})$ given by the formula $L_g(f) = f * g$, for some $g \in H(\{0\})$ is called the *Hadamard multiplication operator*. If $U \subset G_{f*g}$ for every $f \in H(U)$, and U is Runge open, then $L_g \in \mathbf{M}(U)$. We shall prove later that every multiplier is in fact a Hadamard multiplication operator. However, this requires to extend the Hadamard product to a larger class of sets than those which contain the origin.

Example. Euler differential operators, i.e., operators of the form

$$\sum_{n=0}^{\infty} a_n \theta^n, \quad \text{where } \theta f(z) := z f'(z)$$

$a_n \in \mathbb{C}$. The interested reader is invited to get acquainted with the references given in [9].

In the last section we shall provide a characterization of Euler type differential among $\mathbf{M}(\Omega)$.

Example. Hardy averaging operator $\mathcal{H} : H(r\mathbb{D}) \rightarrow H(r\mathbb{D})$, $r \in (0, \infty]$, is the operator that corresponds to the multiplier sequence $\{\frac{1}{n+1}\}_{n=0}^{\infty}$. It is not hard to check that

$$H(f)(z) = \frac{1}{z} \int_{[0,z]} f(\zeta) d\zeta.$$

The main aim of the paper is to find a representation of all multipliers on the space $H(\Omega)$. Our work is motivated by the recent paper [9] where an analogous problem in the real analytic case has been considered. Similarly as Domański and Langenbruch we give the description of all Hadamard multipliers on the space $H(\Omega)$ via analytic functionals $H(V(\Omega))'$ on the space of holomorphic germs on the dilation set of Ω , i.e.,

$$V(\Omega) := \{z \in \mathbb{C} : z\Omega \subset \Omega\}.$$

In the one real variable case the core of the problem was hidden in two results: the so-called Köthe–Grothendieck Duality which represents an analytic functional as a holomorphic function on the complement of its support; and the fact which says that every analytic functional $T \in H(S)'$, $S \subset \mathbb{R}$ open, has the minimal real carrier contained in S . In general it does not make sense to talk about the minimal carrier, due to the simple reason: intersection of two carriers might be empty. Nevertheless, on the assumption that we made about Ω , there is a simple way to omit this obstacle. Namely, intersection of two polynomially convex carriers is a carrier. Consequently, there is the

smallest polynomially convex carrier for every $T \in H(\mathbb{C})'$ (see Sect. 3). Because of this and some other properties of dilation sets (all are collected in Sect. 2), we are able to conclude:

Theorem 1.1. (Representation theorem) *Let $\Omega \subset \mathbb{C}$ be a Runge open set. Then the map*

$$\Phi : H(V)' \rightarrow M(\Omega),$$

$$\Phi(T)(g)(z) := T(g_z), \quad g \in H(\Omega), \quad z \in \Omega, \quad (1.1)$$

is an isomorphism, where $V = V(\Omega)$, $g_s(w) := g(sw)$. The inverse is defined as follows: for a given $M \in M(\Omega)$ and $z^{-1} \in \Omega_*$, the analytic functional

$$T_z := \delta_1 \circ M_{z^{-1}} \circ M \circ M_z : H(z\Omega) \rightarrow \mathbb{C},$$

(where $M_s(f)(w) := f(sw)$, $\delta_1(f) := f(1)$), does not depend on z and has a polynomially convex carrier contained in $V(\Omega)$; furthermore, T_z has a unique extension T acting on $H(V)$ that solves the equation $\Phi(T) = M$.

The celebrated Hadamard's Theorem provides the answer to the question concerning the domain of existence of the Hadamard product of Taylor series around the origin (for the classical formulation cf. [4, Chapter 1.4], or [19, Chapter 6.3]; for the extended one see [18]). However, it says nothing about the possible holomorphic extension of $f * g$ beyond its star product. On the other hand, we feel tempted to describe the set

$$\left\{ \varphi : \varphi \text{ is holomorphic around } 0, \quad M_\varphi \in M(\Omega) \right\}, \quad (1.2)$$

where $M_\varphi h := h * \varphi$, $\Omega \subset \mathbb{C}$ is a Runge open set containing the origin, and $*$ denotes the Hadamard product. Surprisingly, (1.2) is well defined for every Runge open set if “ $*$ ” is understood properly.

Let $f \in H(\Omega_f)$ and $g \in H(\Omega_g)$, where $\Omega_f, \Omega_g \subset \hat{\mathbb{C}}$ are open (!) sets such that $\Omega_f * \Omega_g \neq \emptyset$, where

$$\Omega_f * \Omega_g := \hat{\mathbb{C}} \setminus \left((\hat{\mathbb{C}} \setminus \Omega_f) \cdot (\hat{\mathbb{C}} \setminus \Omega_g) \right).$$

For every $z \in \Omega_f * \Omega_g$ there exists $\gamma \subset \Omega_f \setminus [z(\hat{\mathbb{C}} \setminus \frac{1}{\Omega_g}) \cup \{0, \infty\}]$, a finite union of closed curves such that γ separates $\hat{\mathbb{C}} \setminus \Omega_f$ and $z(\hat{\mathbb{C}} \setminus \frac{1}{\Omega_g})$, and γ satisfies certain conditions (see [18, Theorem 2.4]). Define

$$(f * g)(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta} g\left(\frac{z}{\zeta}\right) d\zeta. \quad (1.3)$$

Then $f * g \in H(\Omega_f * \Omega_g)$ is called the (extended) Hadamard product of f and g . In case $0 \in \Omega_f \cap \Omega_g$, (1.3) coincides with the classical definition of the Hadamard product. Thus, with no confusion we might simply write the Hadamard product. In particular, the Hadamard product is well defined for every pair (Ω_1, Ω_2) , $\Omega_1 \subset \hat{\mathbb{C}}_*$ an open set and $\Omega_2 \subset \mathbb{C}$ a Runge open set. Then γ surrounds $z(\mathbb{C} \setminus \frac{1}{\Omega_2})$ (compact and connected) in Ω_1 .

Theorem 1.2. ([18, Theorem 2.9(4)]) *The map*

$$*: H(\Omega_1) \times H(\Omega_2) \longrightarrow H(\Omega_1 * \Omega_2)$$

is bilinear continuous provided Ω_1, Ω_2 are open sets satisfying $\Omega_1 * \Omega_2 \neq \emptyset$.

The preliminaries together with the Köthe–Grothendieck Duality lead to the following conclusion

Theorem 1.3. (Representation theorem') *Let Ω be a Runge open set, $V = V(\Omega)$. Then $H(\hat{\mathbb{C}} \setminus \frac{1}{V})$ with the Hadamard multiplication is isomorphic (as algebras) with $\mathbf{M}(\Omega)$. The isomorphism is given as follows:*

$$\varphi: H(\hat{\mathbb{C}} \setminus \frac{1}{V}) \ni f \longmapsto \left(H(\Omega) \ni g \rightarrow f * g \in H(\Omega) \right) \in \mathbf{M}(\Omega).$$

Let $S \subset \hat{\mathbb{C}}$, $\infty \in S$. Define

$$H_0(S) := \{h \in H(S) : \lim_{z \rightarrow \infty} h(z) = 0\}.$$

Theorem 1.3 can be formulated equivalently as follows:

Theorem 1.4. (Representation theorem'') *Let $\Omega \subset \mathbb{C}$ be a Runge open set and $V = V(\Omega)$. Then the map*

$$\tilde{\varphi}: \mathbf{M}(\Omega) \ni M \longmapsto f_M \in H_0(\hat{\mathbb{C}} \setminus V),$$

where f_M is defined by the equation:

$$f_M(z) = \sum_{n \in \mathbb{N}} \frac{m_n}{z^{n+1}} \quad \text{if } Mz^n = m_n z^n, \quad n \in \mathbb{N},$$

is an algebra isomorphism between the multipliers on $H(\Omega)$ with composition of operators, and the space of holomorphic germs on $\hat{\mathbb{C}} \setminus V$ vanishing at ∞ with multiplication of Laurent series, i.e.,:

$$\sum_{n \in \mathbb{N}} \frac{a_n}{z^{n+1}} \hat{*} \sum_{n \in \mathbb{N}} \frac{b_n}{z^{n+1}} := \sum_{n \in \mathbb{N}} \frac{a_n b_n}{z^{n+1}}.$$

The second problem that we look at is related to the co-induced topology on $H(V)'$ by Φ from $\mathbf{M}(\Omega)$, i.e., the topology on $H(V)'$ that makes Φ defined in Theorem 1.1 a topological isomorphism. Before we give the precise statement define

$$\tau_k := \text{proj}_{K \subset \Omega_* \text{ compact}} H(V_K(\Omega))',$$

where $V_K(\Omega) := \{\zeta \in \mathbb{C} : \zeta K \subset \Omega\}$. Section 4 is entirely devoted to the proof of the fact that

$$\Phi: (H(V)', \tau_k) \rightarrow (\mathbf{M}(\Omega), \tau_b) \tag{1.4}$$

defined by (1.1) is a topological isomorphism. The proof of (1.4) relies on methods explored in [10, 21].

The paper ends with two applications of the results from Sects. 3 and 4: a density property of noninvertible multipliers and a characterization of Euler differential operators.

Before we proceed to the main part of the paper we explain some notation which we use throughout the paper.

$\hat{\mathbb{C}}$ denotes the Riemann sphere.

For $\Omega \subset \hat{\mathbb{C}}$ let $H(\Omega)$ denote the space of holomorphic functions on Ω . The space $H(\Omega)$ is endowed with the topology of uniform convergence on compact sets. For an arbitrary non-empty set $S \subset \mathbb{C}$ put $H(S)$ for the space of germs of holomorphic functions on S , i.e.,

$$H(S) := \bigcup_{S \subset \Omega \text{ open}} H(\Omega).$$

The natural topology considered on $H(S)$ is the finest locally convex topology for which all the restrictions $H(\Omega) \ni f \mapsto f|_S \in H(S)$ are continuous. This topology is called *inductive*.

Unless otherwise stated, the dual space to $H(\Omega)$ is equipped with the strong topology $H(\Omega)'_b$, i.e., the topology of uniform convergence on bounded sets in $H(\Omega)$.

Recall that $\Omega \subset \mathbb{C}$ is called a Runge open set if $\hat{\mathbb{C}} \setminus \Omega$ is connected. Monomials are linearly dense in $H(\Omega)$ if and only if Ω is Runge.

For $S \subset \mathbb{C}$ put $S_* := S \setminus \{0\}$.

$\mathbb{D}(z, r) := \{w : |w - z| < r\}$, $\mathbb{D}(r) := \mathbb{D}(0, r)$, $\mathbb{D} := \mathbb{D}(1)$.

We denote by M_z , $z \in \mathbb{C}$ the dilation operator

$$M_z(f) := f_z,$$

where $f_z(w) := f(zw)$, $w \in \mathbb{C}$.

We assume that $0 \in \mathbb{N}$.

For non-explained notions from Functional Analysis we refer to the book [15].

2. Dilation Set

For an open set $\Omega \subset \mathbb{C}$ define

$$V(\Omega) := \{z \in \mathbb{C} : z\Omega \subset \Omega\}.$$

$V(\Omega)$ will be called the *dilation set* of Ω . Clearly, it is always non-empty since $1 \in V(\Omega)$.

The next proposition collects some basic facts about $V(\Omega)$. In contrast to the real case we add one condition which seems to be the most important.

Proposition 2.1. *For an open set $\Omega \subset \mathbb{C}$ the dilation set $V = V(\Omega)$ satisfies the following conditions:*

- d-1) $(V_*, \cdot)$ is a multiplicative semigroup with unity;
- d-2) $\overline{V_*}^{\mathbb{C}} = V_*$;
- d-3) if V is unbounded and contains zero, then $V = \Omega = \mathbb{C}$.
- d-4) if Ω is a Runge open set, then $\hat{\mathbb{C}} \setminus V$ is connected.

Before we indicate the proposition we make two remarks:

- $0 \in V$ if and only if $0 \in \Omega$; Consequently,

$$V = \{z \in \mathbb{C} : z\Omega_* \subset \Omega\} = \bigcap_{z \in \Omega_*} z^{-1}\Omega. \quad (2.1)$$

- being a Runge set is invariant under dilation, i.e., $\hat{\mathbb{C}} \setminus \Omega$ has only one connected component if and only if $\hat{\mathbb{C}} \setminus w\Omega$ has only one connected component for every $w \in \mathbb{C}_*$.

Proof. *d-1)* is clear. Proof of *d-2)*: let $\{z_n\} \subset V_*$, $z_n \rightarrow z \in \mathbb{C}_*$. Fix $w \in \Omega$. Because $wz = (w \frac{z}{z_n})z_n$, and the factor $w \frac{z}{z_n}$ is in Ω for $n \gg 1$, we conclude that $wz \in \Omega$.

d-3): Take $r > 0$ so that $\mathbb{D}(r) \subset \Omega$, and $\{\alpha_n : n \in \mathbb{N}\} \subset V$, where $|\alpha_n| \rightarrow \infty$. The definition of the dilation set implies

$$\Omega \supset \bigcup_n \alpha_n \mathbb{D}(r) = \mathbb{C}.$$

d-4): If Ω is a Runge open set, then by (2.1) set $\hat{\mathbb{C}} \setminus V$ is a union of connected sets with non-empty intersection. Hence, $\hat{\mathbb{C}} \setminus V$ is connected. $\square$

Remark 1. (1) If $V = V(\Omega)$, where $\Omega \subset \mathbb{C}$ is open, and $0 \in V$, then $\bar{V} = V$. (2) In the real case properties d-1) – d-3) give the precise characterization of dilation sets (see [9, Proposition 2.1]). However, in the complex case, it is no longer true as the example below indicates.

(3) If V is connected, $0 \in V$, and satisfies all four conditions from Proposition 2.1, then there exists a Runge open set Ω such that $V(\Omega) = V$. Indeed, define

$$\Omega := \begin{cases} \mathbb{C} & \text{if } V = \mathbb{C}, \\ \mathbb{C} \setminus \frac{1}{V_*} & \text{otherwise.} \end{cases}$$

Then Ω has the required properties. Indeed, by *d-2)* Ω is open. So, it suffices to show that $V(\Omega) = V$. Clearly, we may assume that $\Omega \neq \mathbb{C}$. First, let us observe that

$$a \notin V \text{ if and only if } \frac{1}{a} \in \Omega.$$

To get $V \subset V(\Omega)$, assume the contrary. Take $v \in V, y \in \Omega$ so that $vy \notin \Omega$. Hence, $vy = \frac{1}{z}$ for some $z \in V_*$. By *d-1)* $zv \in V$. From the observation we conclude that $y \notin \Omega$; a contradiction. Using once more the observation, the condition *d-1)*, and $a \cdot \frac{1}{a} = 1 \in V$, we obtain: $a \notin V \Rightarrow a \notin V(\Omega)$, i.e., the opposite inclusion.

Example. (1) Let $V_1 := \{\frac{1}{2^n} : n \in \mathbb{N}\}$. An easy observation shows that V_1 is a dilation set of a Runge open set Ω , where

$$\Omega := \left\{ z : 0 < \text{Arg } z < \frac{\pi}{2}, |z| < 2 \right\} \setminus \left\{ z : |z| = 2^{-n} \text{ for some } n \in \mathbb{N}, \frac{\pi}{4} \leq \text{Arg } z < \frac{\pi}{2} \right\}.$$

(2) Fix $\theta \in [0, 2\pi)$ such that $\frac{\theta}{\pi} \notin \mathbb{Q}$. As a candidate for a dilation set consider

$$V_2 := \bigcup_{n \in \mathbb{N}} \left(0, \left(\frac{e^{i\theta}}{2} \right)^n \right].$$

There is no Runge open set Ω with V_2 as its dilation set despite V_2 satisfies all the conclusions of Proposition 2.1. Indeed, assume that such Ω exists.

Because $(0, 1] \subset V_2$ and Ω is Runge, some halfline l containing the origin is disjoint from Ω (otherwise for every $\psi \in [0, 2\pi)$ there is a non-empty segment $(0, r_\psi e^{i\psi}]$ contained in Ω , $r_\psi > 0$; hence, $\hat{\mathbb{C}} \setminus \Omega$ is a union of two disjoint closed non-empty sets: one that contains the origin, and the other containing infinity). But this contradicts the fact that $\{e^{in\theta} : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$. Indeed, because Ω is a non-empty open set, we may find a non-empty arc $\{re^{i\xi} : \xi \in (\alpha, \beta)\} =: A$ contained in Ω ($r > 0$). Finally, notice that there is $n_0 \in \mathbb{N}$ such that $\emptyset \neq l \cap \left(\frac{e^{i\theta}}{2}\right)^{n_0} A \subset l \cap \Omega = \emptyset$. A contradiction.

(3) Let $\mathbb{H}$ denote a halfplane. It is easy to check the following:

$$V(\mathbb{H}) := \begin{cases} (0, \infty) & \text{if } 0 \in \partial\mathbb{H}, \\ [0, 1] & \text{if } 0 \in \mathbb{H}, \\ [1, \infty) & \text{if } 0 \notin \overline{\mathbb{H}}. \end{cases}$$

(4) $V(r\mathbb{D}) = \overline{\mathbb{D}}$ for any $r > 0$.

Proposition 2.2. *Let V be a planar set. Then V has a neighborhood basis of Runge open sets if V_* is non-empty, closed in $\mathbb{C}_*$, and its complement in $\hat{\mathbb{C}}$ is connected (e.g., $V = V(\Omega)$ for some Runge open set Ω).*

Proof. We must show that for every choice of a neighborhood U of V we may find a Runge neighborhood W of V that is contained in U .

First, let us assume that $\overline{V} = V$. Our aim is to construct an ascending sequence of Runge open sets $(W_n)_{n \in \mathbb{N}}$ contained in U whose union covers V . If so, by Runge's Theorem, for W we may take the union (cf. [1, Theorem 3.1.1]).

We proceed to the construction. There exists an open Runge set W_1 such that

$$V \cap \overline{\mathbb{D}} \subset W_1 \Subset U \cap \mathbb{D}(2)$$

(cf. [2, Lemma 1.3.6]). In the n th step the compactness of $V \cap \overline{\mathbb{D}}(n)$ guarantees the existence of an open Runge set W_n such that

$$(V \cap \overline{\mathbb{D}}(n)) \cup \overline{W}_{n-1} \subset W_n \Subset U \cap \mathbb{D}(n+1)$$

(we applied [2, Lemma 1.3.6] to the pair $((V \cap \overline{\mathbb{D}}(n)) \cup \overline{W}_{n-1}, U \cap \mathbb{D}(n+1))$). The construction might stop after a finite number of steps. If it does not, we carry it on inductively.

Now, assume that $V \neq \overline{V}$ and V is bounded. Replacing V by its image under the map $z \mapsto \frac{1}{z}$, we reduce the situation to the previous one.

Finally, only the case when V is neither bounded nor closed is left. We shall apply the former cases. Fix $0 < \epsilon \ll \frac{1}{2}$. Observe that $V \cap (\mathbb{C} \setminus \mathbb{D}(1 - \epsilon))$ is closed, and $V \cap \overline{\mathbb{D}}(1 + \epsilon)$ is bounded but closed. Thus, there exist Runge open sets $U_\pm$ such that

$$V \cap \overline{\mathbb{D}}(1 + \epsilon) \subset U_+ \subset U \cap \mathbb{D}(1 + 2\epsilon),$$

and

$$V \cap (\mathbb{C} \setminus \mathbb{D}(1 - \epsilon)) \subset U_- \subset U \setminus \overline{\mathbb{D}}(1 - 2\epsilon).$$

Directly from the construction we get that $\infty \in (\hat{\mathbb{C}} \setminus U_+) \cap (\hat{\mathbb{C}} \setminus U_-)$. So, $W := U_+ \cup U_-$ is a Runge open neighborhood of V contained in U . $\square$

3. Topology on $H(V)$ and its Dual

By an *analytic functional* we mean any element T in the dual space $H(\mathbb{C})'$. T is said to be carried by a compact K if for every neighborhood Ω of K there is a constant C_Ω such that

$$|Th| \leq C_\Omega \|h\|_\Omega, \quad h \in H(\mathbb{C}). \quad (3.1)$$

From the definition of the topology in $H(\mathbb{C})$ it follows that every analytic functional has a carrier.

Observe that if K carries T and S is any polynomially convex set containing K , then T might actually be considered as an element of $H(S)'$. Indeed, it is enough to take the extension of T to $H(S)$. By the convexity the extension is unique.

For $T \in H(\mathbb{C})'$ let $Tz^n := t_n$, $n \in \mathbb{N}$. We call $(t_n)_{n \in \mathbb{N}}$ the *moment sequence* of T .

Every $T \in H(K)'$, where $K \subset \mathbb{C}$ compact, corresponds to a holomorphic function $f_T \in H_0(\hat{\mathbb{C}} \setminus K)$ defined by the equation

$$f_T(z) = T\left(\frac{1}{z - \cdot}\right), \quad z \notin K. \quad (3.2)$$

The correspondence between f_T and T is given by the so-called *Köthe–Grothendieck Duality*

$$T(h) = \frac{1}{2\pi i} \int_\gamma h(\zeta) f_T(\zeta) d\zeta, \quad h \in H(K), \quad (3.3)$$

where γ is a finite union of closed curves contained in $\Omega \setminus K$ if $h \in H(\Omega)$, where $\Omega \supset K$ is open and such that the index $n(\gamma, z) = 1$ for any $z \in K$ (cf. [2, Theorem 1.3.5]). In fact, the map

$$H(K)'_b \ni T \mapsto f_T \in H_0(\hat{\mathbb{C}} \setminus K)$$

is a topological isomorphism. Of course, since both $H(K)$ and $H(\Omega)$ are reflexive this gives also an analogous representation of the dual of $H(\Omega)$. However, for our purposes we wish to allow the sets in the representations theorem to be not necessarily open nor compact.

Let $\emptyset \neq S \subset \mathbb{C}$. Let us recall the definition of the *inductive topology*:

$$H_i(S) := \text{ind}_{U \supset S \text{ open}} H(U).$$

Proposition 3.1. *Assume V is a dilation set of some Runge open set. Then*

$$H_i(V)' = H_0(\hat{\mathbb{C}} \setminus V)$$

algebraically.

Proof. First observe that by Proposition 2.2

$$\text{ind}_{U \supset V \text{ open}} H(U) = \text{ind}_{U \supset V \text{ Runge open}} H(U).$$

Fix $T \in H_i(V)'$. Let U be a Runge neighborhood of V . Since the composition

$$H(U) \xrightarrow{i_{U,V}} H(V) \xrightarrow{T} \mathbb{C}$$

is continuous we may apply the Köthe–Grothendieck Duality described above. We get a compact set $K_{U,V}$ (= the intersection of all polynomially convex carriers of T) and $f_{U,V} \in H_0(\hat{\mathbb{C}} \setminus K_{U,V})$ such that

$$T \circ i_{U,V}(h) = \frac{1}{2\pi i} \int_{\gamma} h(\zeta) f_{U,V}(\zeta) d\zeta, \quad h \in H(U), \quad (3.4)$$

γ surrounds $K_{U,V}$ in U . Take another Runge neighborhood $\tilde{U}$ of V and put it, as U , in the Köthe–Grothendieck Duality. Therefore, for $z \in \hat{\mathbb{C}} \setminus (K_{U,V} \cup K_{\tilde{U},V})$:

$$f_{U,V}(z) = T_{\zeta} \left(\frac{1}{z - \zeta} \right) = f_{\tilde{U},V}(z).$$

Then, (3.4) holds with $f := f_{U,V} \cup f_{\tilde{U},V}$ ($\in H_0(\hat{\mathbb{C}} \setminus (K_{U,V} \cap K_{\tilde{U},V}))$) in place of $f_{U,V}$. Thus, by the minimality $K_{U,V} \subset K_{U,V} \cap K_{\tilde{U},V}$. Changing the roles of $U, \tilde{U}$, we obtain that $K_{U,V} = K_{\tilde{U},V}$. Consequently, T has a minimal polynomially convex carrier K_T contained in V , and there is $f_T \in H_0(\hat{\mathbb{C}} \setminus K_T)$ such that (3.3) holds for $h \in H(V)$.

For the opposite inclusion, observe that (3.3) defines an analytic functional with carrier in V for every $f_T \in H_0(\hat{\mathbb{C}} \setminus V)$.

The linearity of the correspondence follows from the uniqueness. $\square$

While the topology on the dilation set V is clear, it is just the induced topology from $\mathbb{C}$, we have two candidates for a topology on $H(V)$:

- inductive $\text{ind}_{U \supset V \text{ open Runge}} H(U) =: H_i(V)$ (see Proposition 2.2),
- projective $\text{proj}_{K \subset V \text{ compact}} H(K) =: H_p(V)$.

Define:

$$K_n := \begin{cases} V & \text{if } V \text{ compact,} \\ V \cap \overline{\mathbb{D}}(n) & \text{if } V \text{ is unbounded, } 0 \in V, \quad n \in \mathbb{N}_*. \\ V \cap \overline{\mathbb{D}}(n) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}(\frac{1}{n})) & \text{otherwise.} \end{cases} \quad (3.5)$$

Then the sequence $\{K_n\}_{n \geq 1}$ gives the exhaustion of V by polynomially convex compact sets, i.e.,

$$K_n \Subset \text{int}_V K_{n+1}, \quad \bigcup_{n \in \mathbb{N}_*} K_n = V, \quad \hat{\mathbb{C}} \setminus K_n \text{ connected.}$$

Hence, $H_p(V) = \text{proj}_n H(K_n)$. Moreover

Proposition 3.2.

$$H_p(V)' = \text{ind}_n H(K_n)' \quad \text{algebraically.}$$

Proof. Since polynomials are dense in every $H(K_n)$, it suffices to use the result on a reduced projective spectrum (cf. [6, p. 57]). $\square$

In general the inductive topology is finer than the productive one. However, we will show that they coincide if V is a dilation set. For that purpose recall:

Theorem 3.3. ([14, Theorem 1.1]) *Let $S \subset \mathbb{C}^n$ be a locally closed set (i.e., every point in S has a neighborhood $U \subset \mathbb{C}^n$ such that $U \cap S$ is closed in U). The following conditions are equivalent:*

1. $H_p(S) = H_i(S)$;
2. $H_p(S)$ is (ultra-)bornological;
3. $H_p(S)' = H_i(S)'$ algebraically.

We will show that the third condition in Theorem 3.3 holds for V .

Theorem 3.4. *Let $V = V(\Omega)$ where Ω is a Runge open set. Then*

$$H_i(V) = H_p(V). \quad (3.6)$$

Proof. To complete the demonstration let us note that:

$$H_p(V)' \stackrel{\text{Prop. 3.2}}{=} H_0(\hat{\mathbb{C}} \setminus V) \stackrel{\text{Prop. 3.1}}{=} H_i(V)'$$

in algebraic sense.

For a different proof see [14, Proposition 1.9]. □

Corollary 3.5. *Let V be as in Theorem 3.4. Then:*

1. *The inductive (projective) topology on $H(V)$ is regular.*
2. $\text{proj}_{U \supset V \text{ Runge open}} H(U)'_b = H(V)'_b$ *algebraically and topologically.*

Proof. Ad. 1: Fix a bounded set U in $H_p(V)$. By the definition of projective topology U is contained in a bounded set B of the form

$$B = \left\{ f : \sup_{U_n} |f| < C_n, n \in \mathbb{N}_* \right\},$$

where $U_n \subset \mathbb{C}$ is a Runge neighborhood of K_n and $C_n > 0, n \in \mathbb{N}_*$ (see (3.5)). Observe that $\bigcup_n U_n$ is a Runge open set (since $\infty \notin U_n$ for every n , we get $\hat{\mathbb{C}} \setminus \bigcup_n U_n$ is connected), and obviously it contains V . Thus, B is a locally bounded family of holomorphic functions on $\bigcup_n U_n$. Ad. 2: It follows from (1) and [3, Proposition 2.1]. □

4. Proofs of the Representation Theorems

Proof of Representation Theorem. Let $T \in H(V)'$. Take an arbitrary $g \in H(\Omega)$. By the openness of Ω the function g_w is holomorphic near V provided $w \in \Omega$. Therefore, $\Phi(T)(g)(w)$ is well defined.

Fix $w_0 \in \Omega$. By the proof of Proposition 3.1 we can find a polynomially convex compact set $K_T \subset V$ and a holomorphic function $f_T \in H_0(\hat{\mathbb{C}} \setminus K_T)$ such that (3.3) holds. In particular,

$$T(g_{w_0}) = \frac{1}{2\pi i} \int_{\gamma} g_{w_0}(\zeta) f_T(\zeta) d\zeta \quad \text{if } h_{w_0} \in H(\tilde{\Omega}),$$

where $\gamma \subset \tilde{\Omega} \setminus K_T$ is a suitably chosen union of curves and $\tilde{\Omega} \supset V$ is a Runge open set. Plainly, we might find open $U \ni w_0$ in a such way that h_w is defined on γ for $w \in U$. So,

$$T(g_w) = \frac{1}{2\pi i} \int_{\gamma} g_w(\zeta) f_T(\zeta) d\zeta, \quad w \in U,$$

and $\Phi(T)(g) \in H(\Omega)$.

Φ is well-defined: We shall show that $\Phi(T)$ is sequentially continuous. Because $H(\Omega)$ is a Frechet space it will imply the continuity of $\Phi(T)$. Let $g_n \rightarrow g$ in $H(\Omega)$ and $w_0 \in \Omega$. Choose $\gamma \subset \{\zeta : w_0\zeta \in \Omega\}$ as in the Köthe–Grothendieck Duality. There is $r > 0$ such that $\mathbb{D}(w_0, 2r) \cdot \gamma \subset \Omega$. Hence:

$$\Phi(T)(g_n)(w) = \frac{1}{2\pi i} \int_{\gamma} g_n(w\zeta) f_T(\zeta) d\zeta \quad \text{for } w \in \mathbb{D}(w_0, r).$$

The conclusion follows from the compactness of $\mathbb{D}(w_0, r) \cdot \gamma \subset \Omega$. Furthermore,

$$\begin{aligned} \Phi(T)(z^n)(w) &= \frac{1}{2\pi i} \int_{\gamma} (w\zeta)^n f_T(\zeta) d\zeta \\ &= w^n \frac{1}{2\pi i} \int_{\gamma} \zeta^n f_T(\zeta) d\zeta = w^n T(z^n). \end{aligned}$$

So, z^n is an eigenvector corresponding to the eigenvalue $T(z^n)$.

Surjectivity of Φ : Fix $M \in \mathbf{M}(\Omega)$. For $w \in \frac{1}{\Omega_*}$ define

$$T_w := \delta_1 \circ (M_w)^{-1} \circ M \circ M_w : H(w\Omega) \rightarrow \mathbb{C},$$

where

$$M_w(f) : H(w\Omega) \rightarrow H(\Omega), \quad M_w(f) := M(f_w), \quad f \in H(w\Omega).$$

Observe that T_w is an analytic functional whose moment sequence is the same as the multiplier sequence of M , say $(m_n)_{n \in \mathbb{N}}$. Because of the proof of Proposition 3.1, T_w has the minimal polynomially convex carrier $K_{T_w} \subset w\Omega$ and there is a holomorphic function $f_{T_w} \in H_0(\hat{\mathbb{C}} \setminus K_{T_w})$ such that

$$T_w(h) = \frac{-1}{2\pi i} \int_{-\gamma} h(\zeta) f_{T_w}(\zeta) d\zeta, \quad h \in H(w\Omega) \quad (4.1)$$

for a suitably chosen γ ($w\Omega$ is Runge and $-\gamma$ means the negatively oriented curve). Taking in (4.1) $h = z^n$ for $n \in \mathbb{N}$ we see that

$$f_{T_w}(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \quad \text{near } \infty.$$

Taking into account (3.2) and (3.3) we conclude that

$$f_T := \bigcup_{w^{-1} \in \Omega_*} f_{T_w}$$

is a well defined holomorphic function on $\hat{\mathbb{C}} \setminus K_T$, where

$$K_T := \bigcap_{w^{-1} \in \Omega_*} K_{T_w} \subset \bigcap_{w^{-1} \in \Omega_*} w^{-1}\Omega = V.$$

Hence, every T_w extends to an analytic functional $T \in H(V)'$ given by the formula

$$T(g) = \frac{1}{2\pi i} \int_{\gamma} g(\zeta) f_T(\zeta) d\zeta.$$

Further, Cauchy's Theorem gives $\Phi(T) = M$. □

Proof of Representation Theorem''. Proposition 3.1 with the proof of Theorem 1.1 give that ϕ is a bijection. It remains to show that ϕ is an algebra homomorphism. Fix $S, M \in \mathbf{M}(\Omega)$. Clearly, $S \circ M \in \mathbf{M}(\Omega)$ with $(s_n m_n)_{n \in \mathbb{N}}$ as its multiplier sequence. Because $f_S * f_M \in H_0(\{\infty\})$, and $f_S * f_M = f_{S \circ M}$ near ∞ , we conclude that $f_{S \circ M}$ is a holomorphic extension of $f_S * f_M$ to $\hat{\mathbb{C}} \setminus K_{S \circ M}$. □

Proof of Representation Theorem'. The map defined by

$$\Gamma(f)(z) := \frac{1}{z} f\left(\frac{1}{z}\right)$$

determines an algebra isomorphism between $H_0(\hat{\mathbb{C}} \setminus V)$ and $H(\hat{\mathbb{C}} \setminus \frac{1}{V})$. Since $\infty \notin V$, it remains to apply Theorem 1.4. □

5. Continuity of Φ and Φ^{-1}

A natural question is to ask what is the coinduced topology on $H(V)'$ by Φ , i.e., topology that makes Φ an isomorphism while on $\mathbf{M}(\Omega)$ we have the strong topology t_b . There is one natural candidate: τ_k ,

$$\tau_k := \text{proj}_{K \subset \Omega \text{ compact}} H(V_K^0(\Omega))'_b,$$

where $V_K^0(\Omega)$ is the union of all connected components of $V_K(\Omega)$ which have a nonempty intersection with $V(\Omega)$,

$$V_K(\Omega) := \{\zeta \in \mathbb{C} : \zeta K \subset \Omega\} = \bigcap_{z \in K_*} z^{-1}\Omega, \quad K \text{ compact.} \quad (5.1)$$

For the origin of the definition of τ_k we refer to [21].

The main goal of this section is to demonstrate the following:

Theorem 5.1. *Let $\Omega \subset \mathbb{C}$ be a Runge open set. Then*

$$\Phi : (H(V)', \tau_k) \rightarrow (\mathbf{M}(\Omega), t_b)$$

defined in Theorem 1.1 is a topological isomorphism.

Since Ω is a Runge open set, we have

$$V_K(\Omega) = V_K \cup \bigcup \{\text{bounded components of } \mathbb{C} \setminus K\}(\Omega)$$

for every compact $K \subset \Omega$ (if K' is a bounded component of $\mathbb{C} \setminus K$ and $zK' \subset \Omega$ for some $z \neq 0$, then $zK' \subset \Omega$ as well—here we have used the fact that $\hat{\mathbb{C}} \setminus \Omega$ is connected). And, finally by the same cause

$$\tau_k = \text{proj}_{K \subset \Omega \cap \mathbb{C}_* \text{ compact}} H(V_K^0(\Omega))'_b. \quad (5.2)$$

Without loss of generality we might assume that $K \subset \mathbb{C}_*$ is compact. Moreover, since dilations do not change $V(\Omega)$ suppose $1 \in \Omega$ and $1 \in K$. Hence, $V_K(\Omega) \subset \Omega$ [by (5.1)].

Define

$$\begin{aligned} \mathbf{M}(\Omega, K) &:= \{M \in \mathbf{L}_b(H(\Omega), H(K)) : M \text{ admits} \\ &\quad \text{all monomials as 'eigenvectors'}\}, \\ \mathbf{MC}(\Omega, K) &:= \{M \in \mathbf{L}_b(H(\Omega), \mathcal{C}(K)) : M \text{ admits} \\ &\quad \text{all monomials as 'eigenvectors'}\}. \end{aligned}$$

Proposition 5.2. $\mathbf{M}(\Omega, K) = \mathbf{MC}(\Omega, K)$ as sets and their equicontinuous sets coincide. The map $\Phi : T \mapsto M_T$ (as in Theorem 1.1) is a topological isomorphism from $H(V_K(\Omega))'_b$ onto $\mathbf{M}(\Omega, K)$. Its inverse is given by $M \mapsto T_M$, $T_M f = Mf(1)$. Moreover, both maps send equicontinuous sets to equicontinuous sets.

Proof. Φ is into: First of all, we point out that Φ is well defined, i.e., $\Phi(T)(f)$ is holomorphic near K for every $f \in H(\Omega)$. Indeed, since K is compact and all dilations are continuous, the map

$$K \ni z \mapsto Tf_z \in \mathbb{C}$$

is well defined on some neighbourhood U of K such that $UV_K(\Omega) \subset \Omega$. Consequently, by the Köthe–Grothendieck Duality $\Phi(T)(f)$ can be locally represented on K by

$$\frac{1}{2\pi i} \int_{\gamma} f(z\zeta) f_T(\zeta) d\zeta,$$

and we might differentiate under the integral sign. On the other hand, $M_T z^n = (Tw^n)z^n$. Only the continuity of $\Phi(T)$ is left. Once more we shall apply the Closed Graph Theorem, but this time in a more general context. $H(K)$ is a countable inductive limit of Fréchet spaces, so has a web, and de Wilde's Closed Graph Theorem holds for $(H(\Omega), H(K))$ (cf. [15, Theorem 24.31]). Assume $(f_\alpha, M_T f_\alpha) \rightarrow (f, g)$. Then

$$g(z) = \lim_{\alpha} M_T(f_\alpha)(z) = \lim_{\alpha} \frac{1}{2\pi i} \int_{\gamma} f_\alpha(z\zeta) f_T(\zeta) d\zeta = Tf_z = M_T f(z).$$

Thus, $\Phi(T)(f) = g$.

Φ is onto $\mathbf{MC}(\Omega, K)$: Let T_w be as in the proof of Theorem 1.1. Observe, that $T_w = T_{w'}$ on the set of polynomials for every $w, w' \in K^{-1}$, i.e., they represent the same analytic functional T , and the minimal polynomially convex carrier of T is contained in every $\frac{1}{z}\Omega$, $z \in K$, so as well in $V_K(\Omega)$.

Φ, Φ^{-1} sends equicontinuous sets into equicontinuous sets: The first part is quite obvious because if the minimal polynomially convex carrier of T is contained in $U \Subset V_K(\Omega)$ satisfying $|Tf| \leq C\|f\|_U$ for some $C > 0$, then

$$\|M_T f\|_K = \sup_{z \in K} |Tf_z| \leq C \sup_{z \in K} \|f_z\|_U = C\|f\|_{KU}.$$

Assume now that $\emptyset \neq \mathcal{M} \subset \text{MC}(\Omega, K)$, and there are a positive constant C and an open set $U \Subset \Omega$, $1 \in U$ such that

$$\|Mf\|_K \leq C\|f\|_U.$$

Fix $f \in H(\mathbb{C})$ and $\epsilon^{-1} \in K$. We have

$$|T_M f| = |Mf(1)| = |Mf_\epsilon(\epsilon)| \leq C\|f_\epsilon\|_U = C\|f\|_{\epsilon U}.$$

Hence, the minimal polynomial convex carrier for T_M is a compact set contained in $V_K(U) \subset \bigcap_{\epsilon \in K} \epsilon^{-1}\overline{U} \Subset V_K(\Omega)$. Since every $\epsilon^{-1}\overline{U}$ is compact, the minimal polynomially convex carrier of T_M is contained in $\bigcap_{j=1}^n \epsilon_j^{-1}U$, and the latter set does not depend on the choice of $M \in \mathcal{M}$. Now, the equicontinuity of $\{T_M : M \in \mathcal{M}\}$ easily follows.

Continuity of Φ : Recall that $H(V_K(\Omega))'_b$ is a reflexive Fréchet space so ultrabonological (cf. [15, Chapter 23, 24]). Thus, every bounded set is equicontinuous. Consequently, Φ is locally bounded, hence continuous ([15, Proposition 23.27, 24.10]).

$\text{MC}(\Omega, K) = \text{M}(\Omega, K)$: We have proved that $i \circ \Phi$ is surjective and continuous, here $i : \text{M}(\Omega, K) \hookrightarrow \text{MC}(\Omega, K)$ is an embedding. Hence, $\text{MC}(\Omega, K) = \text{M}(\Omega, K)$.

Continuity of Φ^{-1} : We shall recall the main result from [21].

Let X be a Banach space and $d \in \mathbb{N}_*$. We put

$$F_k := \{(x_\alpha)_{\alpha \in \mathbb{N}^d} \in X^{\mathbb{N}^d} : \|x\|_k := \sup_{\alpha \in \mathbb{N}^d} \|x_\alpha\| k^{-|\alpha|} < \infty\}$$

and

$$F := \text{ind}_k F_k,$$

where $|\alpha| := \alpha_1 + \dots + \alpha_d$. On F we consider for any positive null-sequence $\delta = (\delta_n)_{n \in \mathbb{N}}$ the continuous norm

$$|x|_\delta := \sup_\alpha \|x_\alpha\| \delta_{(|\alpha|)},$$

where $\delta_{(|\alpha|)} := \delta_0 \delta_1 \dots \delta_{|\alpha|}$.

Lemma 5.3. ([21]) *The norms δ_n are a fundamental system of seminorms on F .*

Lemma 5.3 was the key observation that led Vogt in [21] to discover a canonical system of seminorms on $H(K)$, where $K \subset \mathbb{R}$ compact. It turns out that after a small modification this proof works in our setting.

Since we are interested in the case $d = 1$, for simplicity we put $\delta_{(n)} := \delta_{(n)}^{(n)}$, $n \in \mathbb{N}$.

Proposition 5.4. *Let $V \subset \mathbb{C}$ be a non-empty polynomially convex open set. Then a fundamental system of seminorms on $H_0(\hat{\mathbb{C}} \setminus V)$ is given by*

$$|f|_\delta := \max \left\{ \sup_{z \in \partial V, n \in \mathbb{N}} \frac{|f^{(n)}(z)|}{n!} \delta_{(n)}, \sup_{n \in \mathbb{N}} \frac{|(f \circ g)^{(n)}(0)|}{n!} \delta_{(n)} \right\},$$

where $g(w) := \frac{1}{w}$, and δ is any strictly positive null sequence.

Proof. The case $V = \mathbb{C}$, is covered by Lemma 5.3. Let V be a proper subset of $\mathbb{C}$.

Define F as in Lemma 5.3 with

$$X := \mathcal{C}_0(\partial V) \times \mathbb{C},$$

and the norm $|(f, x)| := \max \{ \sup_{\partial V} |f|, |x| \}$, where $\mathcal{C}_0(\partial V) := \{f \in \mathcal{C}(\partial V) : \lim_{z \rightarrow \infty} f(z) = 0\}$. Let

$$L : H_0(\hat{\mathbb{C}} \setminus V) \longrightarrow F,$$

$$L(f) := \left(\frac{1}{n!} f^{(n)}|_{\partial V}, \frac{(f \circ g)^{(n)}(0)}{n!} \right)_{n \in \mathbb{N}}.$$

First of all, let us observe that F is an imbedding spectrum of Banach spaces. Hence, in particular F is a webbed (DF)-space, and the family

$$\{\mathbb{B}_n | n \in \mathbb{N}\}, \mathbb{B}_n := \{f \in F_n : |f|_n \leq 1\}$$

is a fundamental system of bounded sets in F (note that $\mathbb{B}_n$ is closed in F and apply [15, Lemma 25.16]). Next, because $\hat{\mathbb{C}} \setminus V$ is a compact set, the space $H_0(\hat{\mathbb{C}} \setminus V)$ is ultrabornological and Montel (because it is topologically isomorphic with $H(L)$ for some compact $L \subset \mathbb{C}$). Moreover, since $H_0(\hat{\mathbb{C}} \setminus V)$ is an inductive limit (even regular), continuity can be proved on the step spaces. Hence, let us fix an open set $U \supset \hat{\mathbb{C}} \setminus V$. It is enough to show that the graph of $L|_{H(U)}$ is sequentially closed, according to [13, Theorem 13.3.4]. But, this easily follows from the Weierstrass's Theorem. Finally, we claim that L is actually an injective topological embedding. By Baernstein's Lemma this is so if L^{-1} is locally bounded. But, the preimage of $\mathbb{B}_n$ by Weierstrass's Theorem is closed in the locally uniform topology since it consists all those $f \in H_0(\hat{\mathbb{C}} \setminus V)$ that are holomorphic near the set $g(\mathbb{D}(\frac{1}{n})) \cup \bigcup_{z \in \partial V} \mathbb{D}(z, \frac{1}{n})$. So, the functions in $L^{-1}(\mathbb{B}_n)$ are uniformly bounded on $(\partial V + \mathbb{D}(\epsilon)) \cup (\hat{\mathbb{C}} \setminus \mathbb{D}(\frac{1}{\epsilon}))$ for some $\epsilon > 0$. $\square$

Fix an arbitrary strictly positive null-sequence δ . Let $f_T \in H_0(\hat{\mathbb{C}} \setminus V_K(\Omega))$ be the representation of $T \in H(V_K(\Omega))'_b$ given by the Köthe–Grothendieck Duality.

We check easily that

$$\begin{aligned} \sup_{n \in \mathbb{N}} \frac{|(f_T \circ g)^{(n)}(0)|}{n!} \delta_{(n)} &= \sup_{n \in \mathbb{N}_*} |T(z^{n-1})| \delta_{(n)} \\ &\leq \sup_{n \in \mathbb{N}_*, y \in K} |M_T(\delta_{(n)} z^{n-1})(y)|. \end{aligned}$$

Since δ is a null sequence, for every compact S there is a positive number C such that $\delta_{(n)} \leq C \|z\|_S^n$ for all $n \in \mathbb{N}$. Thus, the family $\{z^n \delta_{(n+1)} : n \in \mathbb{N}\}$ is bounded in $H(\mathbb{C})$.

On the other hand, since for every $z \in \partial V_K(\Omega)$ there is $w_z \in K$ such that $zw_z \notin \Omega$. The remaining part of $|\cdot|_\delta$ we might estimate as follows:

$$\begin{aligned} \sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \frac{|f_T^{(k)}(z)|}{k!} \delta_{(k)} &= \sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \left| \zeta T\left(\frac{1}{z - \zeta}\right)^{k+1} \right| \delta_{(k)} \\ &= \sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \left| M_{\zeta T}\left(\frac{1}{z - \frac{\zeta}{w_z}}\right)^{k+1}(w_z) \right| \delta_{(k)} \leq \sup_{h \in \mathcal{B}, y \in K} |M_T h(y)|, \end{aligned}$$

where

$$\mathcal{B} := \left\{ h = h_{z,k} : \zeta \mapsto \frac{\delta_{(k)}}{(z - \frac{\zeta}{w_z})^{k+1}} \mid z \in \partial V, k \in \mathbb{N} \right\}. \quad (5.3)$$

It remains to show that $\mathcal{B}$ is bounded in $H(\Omega)$. For this fix a compact $J \subset \Omega$. The conclusion will follow if we show that

$$C_L := \inf_{z \in \partial V, \zeta \in J} \left| z - \frac{\zeta}{w_z} \right| > 0. \quad (5.4)$$

Assume this is not the case, i.e., there are $z_n \in \partial V$, $\zeta_n \in J$ such that $|z_n - \frac{\zeta_n}{w_{z_n}}| \rightarrow 0$, and $w_{z_n} \rightarrow w \in K \subset \mathbb{C}_*$, $\zeta_n \rightarrow \zeta \in J$. Clearly, we have the following

$$z_n w_{z_n} - \zeta_n = w_{z_n} \left(z_n - \frac{\zeta_n}{w_{z_n}} \right) \rightarrow 0. \quad (5.5)$$

The sequence z_n must be bounded. Say $z_n \rightarrow z \in \partial V$. Hence,

$$L \ni \zeta = \lim_n \zeta_n = \lim_n z_n w_{z_n} \notin \Omega,$$

a contradiction, and the proof is completed. $\square$

Proof of Theorem 5.1. We have the following

$$\mathbf{M}(\Omega) = \text{proj}_{K \in \Omega} \mathbf{M}(\Omega, K) \stackrel{\Omega \text{ is Runge open}}{=} \text{proj}_{K \in \Omega_*} \mathbf{M}(\Omega, K)$$

Hence, by Proposition 5.2 τ_k is the co-induced topology. $\square$

6. Further Remarks and Comments

We open the section by presenting one application of Theorems 1.4 and 5.1. In the proof we shall apply some known facts concerning algebras on the spaces of holomorphic functions on admissible domains as well, i.e., these which are closed under Hadamard multiplication—cf. the survey article [20].

Proposition 6.1. *Let Ω be a Runge open set. Then the set of non-invertible multipliers is dense in $\mathbf{M}(\Omega)$.*

Proof. Fix $f \in H_0(\hat{\mathbb{C}} \setminus V(\Omega))$. Without loss of generality we may assume that f is invertible, equivalently, the equation $f * g = \frac{1}{1-z}$ has a solution in $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$. Put $\gamma_n(z) := \frac{1}{1-z} - \frac{1}{n} \frac{1}{(1-z)^2}$, $n \in \mathbb{N}_*$. [18, Theorem 2.7] guarantees that the Hadamard multiplication on $H_0(\hat{\mathbb{C}} \setminus V(\Omega))^2$ coincides with the Hadamard multiplication on $H_0(\hat{\mathbb{C}} \setminus \{1\})$ and $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$. Moreover, for large R the integral $\int_{-\partial \mathbb{D}(R)} \gamma_n(\zeta) \zeta^n d\zeta$ is zero for all $n \in \mathbb{N}$. Hence, γ_n is not invertible in $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$. On the other hand, $\gamma_n \rightarrow \frac{1}{1-z}$ in $H_0(\hat{\mathbb{C}} \setminus \{1\})$. Thus,

by Theorem 1.2 we get: $\gamma_n * f \rightarrow f$ in $H_0((\hat{\mathbb{C}} \setminus \{1\}) * (\hat{\mathbb{C}} \setminus V(\Omega))) = H_0(\hat{\mathbb{C}} \setminus V(\Omega))$. Now, because $\hat{*} = -*$ on $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$, it remains to see that $\gamma_n * f$ does not have an inverse because γ_n has no inverse.

So far we have shown that the set of non-invertible elements in $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$ endowed with the inductive topology is dense. By Theorem 5.1 the co-induced topology by φ on $H_0(\hat{\mathbb{C}} \setminus V(\Omega))$ is weaker than the inductive one (see Theorem 1.4), and that completes the proof. $\square$

Secondly, we would like to single out Euler differential operators among multipliers. Recall that an operator of the form

$$\sum_{n=0}^{\infty} a_n \theta^n$$

is called an *Euler differential operator*, where $a_n \in \mathbb{C}$, and the differential operator θ is defined recursively as follows:

$$\begin{cases} \theta^0(f) := f, \\ \theta(f) := z f', \\ \theta^{n+1} := \theta \circ \theta^n, \quad n \in \mathbb{N}. \end{cases}$$

It acts on a class of functions whenever the series $\sum_{n=0}^{\infty} a_n \theta^n(f)(z)$ converges pointwise for functions f from the considered class.

Domański and Langenbruch have shown that every Euler operator acting on $H(I)$, where $I \subset \mathbb{R}$ is open, is associated with a unique entire function of exponential type zero (see [9, Theorem 4.1]). Consequently, an Euler type operator acts on $H(I)$ if and only it acts on $H(\mathbb{R})$. As the example below indicates, the situation in $\mathbb{C}$ is slightly different.

Example. Let $M := \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n$. Clearly, if f is entire, then

$$\theta^n f(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} k^n z^k.$$

Hence,

$$\sum_{n=0}^m \frac{1}{n!} \theta^n f(z) = \sum_{k=0}^{\infty} \left(\sum_{n \leq m} \frac{k^n}{n!} \right) \frac{f^{(k)}(0)}{k!} z^k \xrightarrow{\mathbb{C}} \sum_{k=0}^{\infty} e^k \frac{f^{(k)}(0)}{k!} z^k = f(ze).$$

This shows that though M is associated with a function which is not of exponential type zero, M is a multiplier on $H(\mathbb{C})$.

In fact, the above example indicates something more:

Proposition 6.2. *Let a series $\sum_{n \in \mathbb{N}} a_n z^n$ defines an entire function of exponential type 1, i.e., there exist $M, \tau > 0$ so that $|f(z)| \leq M \exp(\tau|z|)$ for all $z \in \mathbb{C}$. Then $M = \sum_{n=0}^{\infty} a_n \theta^n$ is an Euler differential operator. In particular, M is a Hadamard multiplier on $H(\mathbb{C})$.*

We presume that the converse to Proposition 6.2 is true. However, we are able to show only a partial result.

Remark 2. Assume $M = \sum_n^\infty a_n \theta^n$ is an Euler operator acting on $H(\mathbb{C})$ with values in $H(\mathbb{C})$. Then the function $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function of order at most 1, i.e., $|f(z)| \leq \exp(|z|^{1+\epsilon})$ as $|z| \gg 1$ for every $\epsilon > 0$. Indeed, let P_n be defined by the equation

$$\theta^n \exp(z) := P_n(z) \exp(z).$$

(P_n is known as the n th Touchard polynomial.) Consider the numbers $P_n(1) =: B_n$, $n \in \mathbb{N}$. The asymptotic behaviour of B_n was established by de Bruijn ([8, Section 6.2]). Namely,

$$\begin{aligned} \frac{\ln B_n}{n} &= \ln n - \ln \ln n - 1 + \frac{\ln \ln n}{\ln n} + \frac{1}{\ln n} \\ &\quad + \frac{1}{2} \left(\frac{\ln \ln n}{\ln n} \right)^2 + O\left(\frac{\ln \ln n}{(\ln n)^2} \right) \quad (n \rightarrow \infty). \end{aligned} \quad (6.1)$$

Hence,

$$\limsup_{n \rightarrow \infty} \frac{n \ln n}{-\ln |a_n|} \leq \limsup_{n \rightarrow \infty} \frac{n \ln n}{\ln B_n} = 1,$$

or equivalently, f is of order at most 1 (cf. [5, Chapter 2]).

On the other hand, considering the function $\sum_{n \geq 3} \left(\frac{\ln n}{n} \right)^n z^n$ and using (6.1) one might show that not every entire function of order 1 induces an Euler operator on $H(\mathbb{C})$.

Theorem 6.3. *Let $M = \sum_{n=0}^\infty a_n \theta^n$ be an Euler differential operator. The following assertions are equivalent:*

1. M acts on $H(\Omega)$ for every Runge open set $\Omega \subsetneq \mathbb{C}$.
2. M acts on $H(I)$ for every open set $I \subset \mathbb{R}$.
3. M acts on $H(\Omega)$ for some Runge open set $\Omega \subsetneq \mathbb{C}$.
4. M acts on $H(I)$ for some open set $I \subset \mathbb{R}$.
5. The function $f(z) = \sum_{n=0}^\infty a_n z^n$ is an entire function of exponential type zero.
6. $M : H(I) \rightarrow H(I)$ is a continuous multiplier for every open set $I \subset \mathbb{R}$.
7. $M : H(\Omega) \rightarrow H(\Omega)$ is a continuous multiplier for every Runge open set $\Omega \subsetneq \mathbb{C}$.

Proof. The equivalence of (2), (4), (5), and (6) was proven in [9, Theorem 4.1]. Thus, it remains to show that (3) $\Rightarrow$ (5) $\Rightarrow$ (7). First, since $\theta^n f = \sum_{1 \leq k \leq n} c_{n,k} z^k f^{(k)}$ for some $c_{n,k} \in \mathbb{N}$, notice that without loss of generality we might assume that $1 \in \partial\Omega$. So, for any $0 < \epsilon \ll \frac{1}{2}$ there is $\alpha_\epsilon \in \Omega \cap \mathbb{D}(1, \frac{\epsilon}{2})$. Furthermore, $\frac{1}{1-z} \in H(\Omega)$, and around the origin

$$\theta^n \left(\frac{1}{1-z} \right) = \sum_{k=0}^\infty k^n z^k = \frac{A_n(z)}{(1-z)^{n+1}},$$

where A_n are the so-called Euler polynomials defined recursively by:

$$A_0(z) = 1, \quad A_{n+1}(z) = A'_n(z)(z - z^2) + (n+1)A_n(z)z. \quad (6.2)$$

By (6.2) we easily deduce that $\deg A_n = n$ and $A_n(1) = n!$. Furthermore, A_n has only real non-positive zeros (cf. [7, p. 292]). Thus,

$$\left| \theta^n \left(\frac{1}{1-z} \right) (\alpha_\epsilon) \right| = \left| \frac{A_n(\alpha_\epsilon)}{(1-\alpha_\epsilon)^{n+1}} \right| \geq \frac{2n!}{\epsilon^{n+1}},$$

where the last inequality follows from the fact that $|\alpha_\epsilon - z_j| \geq \frac{1}{2}|1 - z_j|$ for any zero z_j of A_n , what completes the proof of (3) $\Rightarrow$ (5).

If (5) holds, then $M : H(\Omega) \rightarrow H(\Omega)$, $M = \sum_n a_n \theta^n$ is well defined and has a closed graph for every domain $\Omega \subset \mathbb{C}$ (cf. [12, Theorem 11.2.3]). So, (7) holds (cf. [13, Theorem 13.3.4]). $\square$

Remark 3. (1) Theorem 1.3 rediscovers a well know fact:

$$\mathbf{M}(\mathbb{D}) \cong H(\mathbb{D}) \quad (6.3)$$

via the map

$$f \mapsto M_f. \quad (6.4)$$

It is not hard to see that the unit disc is the only open set for which (6.3) with (6.4) holds.

(2) $M(\mathbb{P}) \cong H(\mathbb{C} \setminus \{1\})$, $\mathbb{P} := \{z : |\Re z| < m\}$, $m > 0$, and the isomorphism is given by (6.4).

(3) Putting together Theorem 1.1 and [9, Theorem 2.4] we conclude that the algebra $\mathbf{M}(\mathbb{C})$ ($\mathbf{M}(\mathbb{R})$) is not isomorphic with $\mathbf{M}(U)$, for any $U \subset \mathbb{R}$ open or $U \subsetneq \mathbb{C}$ Runge open set ($U \subset \mathbb{C}$ Runge open set).

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References

- [1] Berenstein, C., Gay, R.: Complex Variables: An Introduction. Springer, Berlin (1991)
- [2] Berenstein, C., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, Berlin (1995)
- [3] Bierstedt, K.D.: An introduction to locally convex inductive limits. In: Hogbenlend, H. (ed.) Functional Analysis and Its Applications. World Science, Singapore, pp. 35–133 (1988)
- [4] Bieberbach, L.: Analytische Fortsetzung. Springer, Berlin (1955)

- [5] Boas, R.P.: Entire Functions. Academic Press Inc., Cambridge (1954)
- [6] Brück, R., Render, H.: Invertibility of holomorphic functions with respect to the Hadamard product. *Complex Var.* **42**, 207–223 (2000)
- [7] Comptet, L.: Advanced Combinatorics. Reidel, Dordrecht (1974)
- [8] De Bruijn, N.G.: Asymptotic Methods in Analysis. North Holland Publishing Co., Amsterdam (1958)
- [9] Domański, P., Langenbruch, M.: Representation of multipliers on spaces of real analytic functions. *Analysis* **32**, 137–162 (2012)
- [10] Domański, P., Langenbruch, M., Vogt, D.: Hadamard type operators on spaces of real analytic functions in several variables. *J. Funct. Anal.* **269**, 3868–3913 (2015)
- [11] Frerick, L.: Coefficient multipliers with closed range. *Note Mat. (Lecce)* **17**, 61–70 (1997)
- [12] Hille, E.: Analytic Function Theory, vol. II. Chelsea, New York (1987)
- [13] Jarchow, H.: Locally Convex Spaces. B.G. Teubner, Leipzig (1981)
- [14] Martineau, A.: Sur la topologie des espaces de fonctions holomorphes. *Math. Ann.* **163**, 62–88 (1966)
- [15] Meise, R., Vogt, D.: Introduction to Functional Analysis. Clarendon Press, Oxford (2004)
- [16] Müller, J.: The Hadamard multiplication theorem and applications in summability theory. *Complex Var.* **18**, 155–166 (1992)
- [17] Müller, J.: Coefficient multipliers from $H(G_1)$ into $H(G_2)$. *Arch. Math.* **61**, 75–81 (1993)
- [18] Müller, J., Pohlen, T.: The Hadamard product on open sets in the extended plane. *Complex Anal. Oper. Theory* **6**(1), 257–274 (2012)
- [19] Segal, S.: Nine Introductions in Complex Analysis, Revised edn. Elsevier, Amsterdam (2008)
- [20] Render, H.: Hadamard’s multiplication theorem-recent developments. *Colloq. Math.* **74**, 79–92 (1997)
- [21] Vogt, D.: A Fundamental System of Seminorms for $A(K)$, preprint 2013. [arXiv:1309.6292](https://arxiv.org/abs/1309.6292)

Maria Trybula (✉)
Faculty of Mathematics and Informatics
Adam Mickiewicz University
Umultowska 87
61-614 Poznan
Poland
e-mail: maria.h.trybula@gmail.com

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