Non-Existence of Subordinate Solutions for Jacobi Operators in Some Critical Cases

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Abstract. We show a method of repairing some gaps in proofs of the absolute continuity of the spectrum of Jacobi operators. Such gaps have been found in several recent papers, dealing mainly with the so-called *critical case* (i.e., *Jordan box case*). We solve the problem by proving that the subordinate solution does not exist for many cases with two linearly independent generalised eigenvectors possessing “similar” asymptotic behaviour.

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1. Introduction

Subordination theory, formulated first by Gilbert and Pearson in [7] for one dimensional Shrödinger operators, where transferred to the discrete case by [22, 34], became a very popular method used for spectral analysis of Jacobi operators in $\ell^2(\mathbb{N})$ (see, e.g., [3–5, 8–21, 23–27, 29, 30, 32, 33]). One of the typical ways of employing the subordination theory for spectral studies is related to asymptotic studies, and it contains the following three basic steps:

1. Finding asymptotics of some solutions to the generalised eigenequation (generalised eigenvectors) for the Jacobi operator J . This is often achieved with the use of some discrete Levinson type theorems—see [3, 5, 9, 11–13, 16, 18, 19, 21, 24, 25, 27, 30, 32];¹
2. Reading some information on the existence of subordinate solution (see Definition (2.1)) from the asymptotic information on solutions;
3. Obtaining some spectral information from the subordinate solution information via subordination theory ([22, Theorem 3]).

Such schema was particularly fruitful for studies of the absolute continuous spectrum, where the absence of the subordinate solution plays a crucial role (e.g., [3–5, 8, 9, 11–15, 17–19, 24–27, 33]).

In this paper we concentrate on the second step. It was usually treated as the simplest one, especially when the generalisation of the Behncke–Stolz lemma was formulated by Janas and Naboko. This generalisation (for short, GBS, see e.g., [10, Lemma 1.5]) states that for a fixed spectral parameter no subordinate solution exists (*non-subordinacy*), if all the generalised eigenvectors satisfy a certain estimate expressed in terms of the weighs of the Jacobi operator. GBS proved to be applicable to many classes of Jacobi operators. Thus, to make the use of it more “automatic”, several conditions simpler to check, yielding the estimate from GBS, have been found. One of them, formulated in terms of transfer matrices, was the so-called H -class condition (see [23]). Another one, being exactly in spirit of the above step 2., relates non-subordinacy to some asymptotic properties of the $\mathbb{C}^2$ -vector-generalised eigenvectors, i.e., of the $\mathbb{C}^2$ -vector sequences $x = \{x(n)\}_{n \geq 1}$ where

$$x(n) = \begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix}, \quad n \in \mathbb{N}, \quad (1.1)$$

and u is a (scalar) generalised eigenvector (i.e., a sequence satisfying (2.3)). The appropriate result can be formulated as follows² (compare with [24, Lemma 5.9]).

Proposition 1.1. *Suppose that Jacobi operator J satisfies (2.2), and for a spectral parameter $\lambda \in \mathbb{C}$ it possesses two $\mathbb{C}^2$ -vector-generalised eigenvectors x_+ , x_- , such that*

$$x_{\pm}(n) = \psi_{\pm}(n)y_{\pm}(n), \quad n \in \mathbb{N},$$

¹ Note that in [16, 18] some extra assumptions should be added to the formulation of the discrete Levinson type theorem (Prop. 7.1 and Prop. 4.1, respectively).

² For a stronger version see Proposition 4.1.

where $y_{\pm}$ are sequences of $\mathbb{C}^2$ vectors such that

$$y_{\pm}(n) \longrightarrow y_{\pm}^{\infty}, \quad (1.2)$$

$y_{-}^{\infty}, y_{+}^{\infty}$ are linearly independent $\mathbb{C}^2$ vectors, and $\psi_{\pm}$ are sequences of nonzero numbers satisfying

$$\psi_{+}(n) = \overline{\psi_{-}(n)}, \quad \text{for } n \text{ sufficiently large.} \quad (1.3)$$

Then no subordinate solution for J and λ exists.

Unfortunately, this fact, however well-known, it has never been explicitly formulated in the above simple form—it has rather been used as one of the “folklore type” arguments. This was probably the first reason of several further misunderstandings and gaps. Some authors forgot that the above result **does not** concern the generalised eigenvectors, but $\mathbb{C}^2$ -vector-generalised eigenvectors. In particular, the linear independence of y_{-}^{∞} and y_{+}^{∞} is quite a strong condition, which is typically satisfied for so-called non-critical case (see, e.g., [8]), and it can not be replaced by the assumption, that x_{+} and x_{-} are linearly independent sequences of $\mathbb{C}^2$ vectors. And thus it is not so strange, that the following conjecture on generalised eigenvectors is also **not true**.

Conjecture 1.2. Suppose that Jacobi operator J satisfies (2.2), and for a spectral parameter $\lambda \in \mathbb{C}$ it possesses two linearly independent generalised eigenvectors u_{+} , u_{-} , such that

$$u_{\pm}(n) = \psi_{\pm}(n)s_{\pm}(n), \quad n \in \mathbb{N}, \quad (1.4)$$

where $s_{\pm}$ are scalar sequences with

$$s_{\pm}(n) \longrightarrow 1,$$

and $\psi_{\pm}$ are sequences of nonzero numbers satisfying

$$\psi_{+}(n) = \overline{\psi_{-}(n)}, \quad \text{for } n \text{ sufficiently large.} \quad (1.5)$$

Then no subordinate solution for J and λ exists.

To check that this is **false**, one can consider a self-adjoint Jacobi matrix J with real coefficients satisfying (2.2), such that for some $\lambda \in \mathbb{R}$ there exists a subordinate solution for J and λ (surely, this is a frequent situation, e.g., J can be an arbitrary self-adjoint Jacobi operator with non-empty point spectrum, and λ —its arbitrary eigenvalue). Let u be now the generalized eigenvector for these J and λ determined by the initial conditions $u(1) = 1, u(2) = i$. Define $u_{-} := u$, and $u_{+} = \bar{u}$. Both u_{+} , u_{-} are generalised eigenvectors for J and λ , since λ and the coefficients of J are real. Moreover u_{+} , u_{-} are linearly independent, since $u_{+}(1) = 1, u_{+}(2) = -i$. So taking $\psi_{\pm} := u_{\pm}$ and $s_{\pm} \equiv 1$ we see that the assumptions of the conjecture hold, but the assertion does not.

Unluckily, this conjecture, though false, was also regarded as a “folklore type” result. And this could be the direct reason of gaps in proofs of non-subordinacy in many recent papers on spectral theory of Jacobi matrices: [4, 8, 17, 19, 26, 27, 33]. These papers concentrate mainly on the so-called Jordan box (i.e., critical) case—see [8].

The aim of the present article is to fill those gaps. We try to reach this goal, roughly speaking, by finding some extra conditions which added to the assumptions of Conjecture 1.2 change it into a theorem. And the reason why we attempt to repair just Conjecture 1.2 is that it concerns the typical form of asymptotic information obtained by the use of some difference equation theory tools (see e.g. [6]). This information, in the scalar representation, can be expressed as follows:

Certain generalised eigenvectors $u_{\pm}$ possess the form (1.4), where “the main scalar parts” $\psi_{\pm}$ of asymptotics are explicitly “computable”, and we have only some general regularity information on the implicit part $s_{\pm}$.

The main extra condition we add to the assumptions of Conjecture 1.2 is formulated here in terms of the so-called absolute γ , 2-Cesaro convergence of some scalar sequences (for short, γ , 2-convergence), which is defined in Sect. 2. The *convergence weight* γ is determined here by the absolute value of the main scalar parts, more precisely

$$\gamma := |\psi_-|^2 = |\psi_+|^2, \quad (1.6)$$

and the sequence z to be γ , 2-convergent is given by the complex signum function (with $\operatorname{sgn}(a) := \frac{a}{|a|}$ for $a \neq 0$) of this main scalar parts:

$$z := \frac{\psi_+}{\psi_-} = \operatorname{sgn}(\psi_+)^2 = \operatorname{sgn}(\psi_-)^{-2}. \quad (1.7)$$

The result is the following theorem.

Theorem 1.3. *Suppose that Jacobi operator J is self-adjoint and all the assumptions of Conjecture 1.2 hold. Then there exists a subordinate solution for J and λ iff z is γ , 2-convergent, with z, γ given by (1.6) and (1.7). Moreover, if $z \xrightarrow{\gamma, 2} g$, then $u_+ - gu_-$ is a subordinate solution for J and λ .*

Note that this result gives even sufficient **and** necessary conditions for subordinacy, under the assumptions of Conjecture 1.2 and with the self-adjointness of J . Its proof is immediate, by a slightly more general result—Theorem 4.2, where we do not assume that the main scalar parts $\psi_{\pm}$ should be mutually conjugated, and some weaker assumptions on $s_{\pm}$ are made. We need this stronger version for some cases. But its proof is also elementary. In some sense, this is rather a “simple observation” than a “theorem”. However, it is a convenient observation, since it transfers the essence of the problem from subordinacy to some more classical notion of convergence.

Studies on the absolute γ , p -convergence are the subject of Sect. 3. The main result of this section is Criterion 3.11, which gives some sufficient conditions for divergence in this sense. This will allow to prove non-subordinacy in many important cases, which are not admitted by Proposition 1.1.

Section 4, the most technical one, is devoted to the proof of Theorem 4.2, being the previously mentioned generalisation of Theorem 1.3. We prove also Theorem 4.3, which gives some necessary conditions for subordinacy in terms of γ , 2-convergence. We shall need this second result later—in some proofs of non-subordinacy which cannot be obtained by the use of Theorem 4.2.

In Sect. 5 we show how to use the tools from previous sections to prove that for certain Jacobi operators J and spectral parameters λ , no subordinate solution exists. We prove several results covering various forms of asymptotics for two linearly independent generalised eigenvectors, which have been described in the recent literature. Finally, in Remarks 5.7, we show how to fill the gaps in most of the proofs of non-subordinacy mentioned before.

Some basic notation and definitions are collected in Sect. 2. The remaining notation can be found in some particular sections.

We close the paper by Sect. 6 with some open problems.

2. Preliminaries

Let us consider the right-side infinite Jacobi matrix

$$\begin{pmatrix} q_1 & w_1 & & & \\ w_1 & q_2 & w_2 & & \\ & w_2 & q_3 & w_3 & \\ & & w_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

determined by some given real sequences $\{w_n\}_{n \geq 1}$ and $\{q_n\}_{n \geq 1}$. We study the Jacobi operator J , being the maximal operator defined by the above matrix in the Hilbert space $\ell^2(\mathbb{N})$ of square-summable complex sequences on $\mathbb{N}$. So, J is the restriction of the formal Jacobi operator $\mathcal{J}$ to the domain

$$D(J) := \{u \in \ell^2(\mathbb{N}) : \mathcal{J}u \in \ell^2(\mathbb{N})\},$$

where $\mathcal{J}$ acts in the vector space $\ell(\mathbb{N})$ of all complex sequences on $\mathbb{N}$, and it is given by

$$(\mathcal{J}u)(n) := w_{n-1}u(n-1) + q_nu(n) + w_nu(n+1), \quad n \in \mathbb{N}, \quad (2.1)$$

for any $u = \{u(n)\}_{n \geq 1} \in \ell(\mathbb{N})$, with the convention that $w_0 := 0 =: u(0)$ (note that we use both ways of sequence term notation in this paper: standard—like w_n , and functional—like $u(n)$). In this paper we assume that

$$\forall_{n \in \mathbb{N}} \quad w_n \neq 0. \quad (2.2)$$

For a fixed $\lambda \in \mathbb{C}$ we consider generalised eigenvectors of J for λ , i.e., such $u = \{u(n)\}_{n \geq 1} \in \ell(\mathbb{N})$ that

$$((\mathcal{J} - \lambda)u)(n) = 0, \quad n \geq 2, \quad (2.3)$$

and $\mathbb{C}^2$ -vector-generalised eigenvectors, related to generalised eigenvectors by (1.1). By $\text{Sol}(\lambda)$ we denote the linear space of all $u \in \ell(\mathbb{N})$ satisfying (2.3), and $\text{Sol}_*(\lambda) = \text{Sol}(\lambda) \setminus \{0\}$. Note that $\dim \text{Sol}(\lambda) = 2$ by (2.2).

Let us recall the definition of subordinate solution.

Definition 2.1. $u \in \text{Sol}_*(\lambda)$ is subordinate solution for J and λ iff there exists $v \in \text{Sol}_*(\lambda)$ such that

$$\lim_{n \rightarrow +\infty} \frac{\|u\|_n}{\|v\|_n} = 0, \quad (2.4)$$

where for $x \in \ell(\mathbb{N})$, $n \in \mathbb{N}$

$$\|x\|_n := \left(\sum_{k=1}^n |x(k)|^2 \right)^{\frac{1}{2}}.$$

The above form of the definition seems convenient for our further considerations, though note that usually it is formulated in another way.

Denote by $\text{Sub}(\lambda)$ the set consisting of the 0 sequence and of all the subordinate solutions for J and λ . Let us also recall some fundamental properties of subordinate solutions (see [22], pp. 514–515 for the proofs).

Proposition 2.2. *Suppose that $\lambda \in \mathbb{C}$.*

- (i) *$u \in \text{Sol}_*(\lambda)$ is subordinate solution for J and λ iff (2.4) holds for any $v \in \text{Sol}_*(\lambda)$ linearly independent of u . Moreover, for any $v \in \text{Sol}_*(\lambda)$ linearly independent of $u \in \text{Sol}_*(\lambda)$ we have: u is subordinate solution for J and λ iff (2.4) holds.*
- (ii) *$\text{Sub}(\lambda)$ is a zero or one dimensional linear subspace of $\text{Sol}(\lambda)$.*
- (iii) *If J is self-adjoint, then any non-zero solution of (2.3) from $\ell^2(\mathbb{N})$ is a subordinate solution for J and λ (i. e., $\text{Sol}(\lambda) \cap \ell^2(\mathbb{N}) \subset \text{Sub}(\lambda)$).*

Sometimes we shall use also a more precise notation:

$$\text{Sol}(J, \lambda), \quad \text{Sol}_*(J, \lambda), \quad \text{Sub}(J, \lambda),$$

to stress the J -dependence on the sets introduced above.

The abbreviation: ‘for k s.l. . . .’ (and ‘. . . for k s.l.’) means: ‘for k sufficiently large . . .’, i.e., there exists $N \in \mathbb{Z}$ such that for $k \geq N$ Analogically, ‘for k s.s.’ (‘for k sufficiently small’) means ‘there exists $N \in \mathbb{Z}$ such that for $k \leq N$ ’.

If x is a sequence (in this paper ‘sequence’ means always: right-side infinite sequence), then its starting index will be often denoted by $\text{st}(x)$, i.e., $x = \{x_k\}_{k \geq \text{st}(x)}$. If $K \subset \{k \in \mathbb{Z} : k \geq \text{st}(x)\}$ then by x_K we denote the “image of the set K by the sequence x ”, i.e. $x_K := \{x_k : k \in K\}$.

The set of all scalar (complex) sequences (with $\text{st}(x)$ being undetermined) we denote here by ℓ , and as usual, ℓ^p, ℓ^∞ denote its subsets consisting of power p -summable ($p > 0$), and respectively, bounded sequences. The set of scalar sequences x which are bounded and satisfy $\liminf_{n \rightarrow +\infty} |x_n| > 0$ is denoted by

$$\ell_*^\infty.$$

To shorten the notation, we shall often omit writing $k \in \mathbb{Z}, n \in \mathbb{Z}$, etc., when the integer character of the variable is clear from the context, and also, for two integers n_1, n_2 we shall write $(n_1; n_2)$, $[n_1; n_2)$, etc., to denote the intersection of $\mathbb{Z}$ with the appropriate real intervals (denoted in the same way...).

The symbol ‘ $\longrightarrow$ ’ (as well as ‘ $\lim$ ’)—without any superscripts—is reserved here **only** for the limit in the usual sense (also for the convergence in $\mathbb{C}^2$).

We shall consider several special kinds of convergence of complex sequences. They are determined by a weight sequence $\gamma = \{\gamma_k\}_{k \geq k_0}$ satisfying

$$\gamma \geq 0; \quad \gamma \notin \ell^1. \quad (2.5)$$

To distinguish such a weight sequence from the weight sequence $\{w_n\}_{n \geq 1}$ for the Jacobi operator J , we call it “convergence weight”.

The *generalized Cesaro convergence* with the convergence weight γ is denoted here by $\xrightarrow{\gamma}$, and in the basic case $\text{st}(\gamma) = \text{st}(x) = k_0 = 1$, which is sufficient for spectral goals of this paper, it is defined as follows: for $x = \{x_n\}_{n \geq 1}$ —a complex sequence, and $g \in \mathbb{C}$

$$x_n \xrightarrow{\gamma} g \quad \text{iff} \quad \lim_{n \rightarrow +\infty} \frac{\sum_{k=1}^n \gamma_k x_k}{\sum_{k=1}^n \gamma_k} = g$$

(note, that some terms of the sequence on the RHS can be not properly defined, but they are properly defined for n s.l., since by (2.5) the denominator is non-zero). For general cases of $\text{st}(\gamma)$ and $\text{st}(x)$ the definition is similar: consider $x = \{x_n\}_{n \geq n_0}$ (note that $k_0 := \text{st}(\gamma)$ and $n_0 := \text{st}(x)$ here). Denote

$$k_+ := \min\{k \geq k_0 : \gamma_k > 0\}, \quad \tilde{k} := \max\{k_0, n_0\} \quad (2.6)$$

and

$$C_\gamma(N, N', x)_n := \frac{\sum_{k=N}^n \gamma_k x_k}{\sum_{k=N'}^n \gamma_k}$$

for $N \geq \tilde{k}$, $N' \geq k_0$ and $n \geq \min\{k \geq N' : \gamma_k > 0\}$ (by (2.5), this “min”, as well as the one defining k_+ , exists). In particular, we can consider $N' = k_0$ and $N = \tilde{k}$, and we denote $C_\gamma(x)_n := C_\gamma(\tilde{k}, k_0, x)_n$ for $n \geq k_+$.

Now, the generalized Cesaro convergence with the convergence weight γ is given by

$$x_n \xrightarrow{\gamma} g \quad \Longleftrightarrow \quad \lim_{n \rightarrow +\infty} C_\gamma(x)_n = g. \quad (2.7)$$

It is easily seen by (2.5) that

$$x_n \xrightarrow{\gamma} g \quad \Longleftrightarrow \quad \lim_{n \rightarrow +\infty} C_\gamma(N, N', x)_n = g, \quad (2.8)$$

independently of the choice of $N \geq \tilde{k}$, $N' \geq k_0$.

Note that for a constant convergence weight such a convergence is the standard Cesaro convergence.

However, the key kind of convergence in this paper is *the absolute γ , p -Cesaro convergence* with $p > 0$, denoted by $\xrightarrow{\gamma, p}$, and defined as follows:

$$x_n \xrightarrow{\gamma, p} g \quad \Longleftrightarrow \quad |x_n - g|^p \xrightarrow{\gamma} 0. \quad (2.9)$$

In both names of convergence we often omit “Cesaro” and/or “absolute” for short. For our spectral applications we shall use $p = 2$. Sometimes, instead of $\longrightarrow$, $\xrightarrow{\gamma}$ and $\xrightarrow{\gamma, p}$ we shall write also $\xrightarrow[n]{\gamma}$, $\xrightarrow[n]{\gamma}$ and $\xrightarrow[n]{\gamma, p}$ to indicate the symbol used for the index of the sequence under the limit. We shall use the notions: *γ -convergence/divergence*, *γ, p -convergence/divergence* in the usual

way, i.e., convergence always means the existence of a **finite** limit (in $\mathbb{C}$), and divergence = non-convergence in the respectively considered sense.

3. The Absolute γ , p -Cesaro Convergence

In this section we study the notions of γ -convergence and of the absolute γ , p -convergence. In particular a practical criterion (i.e., a sufficient condition) for γ , p -divergence is found here.

Let us start from some additional notation. The length of interval I is denoted by $|I|$, the number of elements ($\in \mathbb{N} \cup \{0, +\infty\}$) of a set A is denoted by $\sharp A$.

The set of the finite complex limit points of a scalar sequence x will be denoted by $\text{LIM}_{\mathbb{C}}(x)$, i.e., $\text{LIM}_{\mathbb{C}}(x)$ is the set of all $g \in \mathbb{C}$ for which there exists a sequence $\{k_n\}_{n \geq 1}$ of integers such that $k_n \rightarrow +\infty$ and $x_{k_n} \rightarrow g$.

The symbol $\overline{x \wr M}$ denotes the arithmetic average of x over a finite set $M \subset \mathbb{Z}$, more precisely:

$$\overline{x \wr M} := \begin{cases} \frac{1}{\sharp M} \sum_{k \in M, k \geq \text{st}(x)} x_k & \text{for } M \neq \emptyset, \\ 0 & \text{for } M = \emptyset. \end{cases}$$

We call two convergence weight sequences γ, γ' *equivalent*, which will be denoted by $\gamma \equiv \gamma'$, iff there exist such positive constants c, C , that $c\gamma_k \leq \gamma'_k \leq C\gamma_k$ for k s.l..

A convergence weight γ is called here *shiftable* iff there exists $\epsilon > 0$, such that $\gamma_{k+1} \geq \epsilon\gamma_k$ for k s.l..

For a convergence weight $\gamma = \{\gamma_k\}_{k \geq k_0}$ satisfying (2.5) consider the measure μ_γ given for any $M \subset \mathbb{Z}$ by

$$\mu_\gamma(M) := \sum_{k \in M, k \geq k_0} \gamma_k.$$

We shall use also the following sets of indexes of complex sequence x related to terms being “far from / near to” the complex number g :

$$F_n(x, g, \delta) := \{s \in [\text{st}(x); n] : |x_s - g| \geq \delta\};$$

$$N_n(x, g, \delta) := \{s \in [\text{st}(x); n] : |x_s - g| < \delta\}$$

for $\delta > 0$ and $n \in \mathbb{Z}$.

For the Readers convenience we collect here several basic properties of $\xrightarrow{\gamma}$ and $\xrightarrow{\gamma, p}$ (see also some general properties of various convergence notions, e.g., [2]).

Proposition 3.1. Assume that $p > 0$ and γ satisfies (2.5).

1. For both $\xrightarrow{\gamma}$ and $\xrightarrow{\gamma, p}$ the convergence generalises the usual convergence ($\longrightarrow$), the limit is unique if it exists, and it depends linearly on the sequence.
2. For both $\xrightarrow{\gamma}$ and $\xrightarrow{\gamma, p}$ any change of a finite number of terms of the sequence, as well as its extrapolation or restriction (to another domain

of the form $[m; +\infty)$) has no influence on the convergence and on the value of the limit.

3. $x_n \xrightarrow{\gamma} g \iff (\operatorname{Re} x_n \xrightarrow{\gamma} \operatorname{Re} g \ \& \ \operatorname{Im} x_n \xrightarrow{\gamma} \operatorname{Im} g).$
4. The $\xrightarrow{\gamma}$ limit depends monotonically on real sequences.
5. The “3 sequences property” holds for $\xrightarrow{\gamma}$, i.e., if $x_n \leq y_n \leq z_n$ for n s.l. and $x_n, z_n \xrightarrow{\gamma} g$, then $y_n \xrightarrow{\gamma} g$.
6. $|x_n| \xrightarrow{\gamma} 0 \implies x_n \xrightarrow{\gamma} 0$ (but note: there is no “ $\Leftarrow$ ” in general).
7. If $x_n \xrightarrow{\gamma, p} g$, then $g \in \operatorname{LIM}_{\mathbb{C}}(x)$ (note that the similar fact for $\xrightarrow{\gamma}$ can be not true); if x is a real sequence and $x_n \xrightarrow{\gamma} g$, then $\liminf_{n \rightarrow +\infty} x_n \leq g \leq \limsup_{n \rightarrow +\infty} x_n$.
8. If x is bounded, $x_n \xrightarrow{\gamma, p} g$ and $p' > p$, then $x_n \xrightarrow{\gamma, p'} g$.
9. If $x_n \xrightarrow{\gamma, 1} g$, then $x_n \xrightarrow{\gamma} g$.
10. The relation $\overset{\gamma, p}{\approx}$ given by: $x \overset{\gamma, p}{\approx} y \iff (x_n - y_n) \xrightarrow{\gamma, p} 0$ is an equivalence relation. In particular, $(x \overset{\gamma, p}{\approx} y \ \& \ x_n \xrightarrow{\gamma, p} g) \implies y_n \xrightarrow{\gamma, p} g$.
11. If γ' is a convergence weight equivalent to γ , then $x_n \xrightarrow{\gamma, p} g \iff x_n \xrightarrow{\gamma', p} g$ (note that the similar fact for $\xrightarrow{\gamma}, \xrightarrow{\gamma'}$ can be not true).
12. If γ is shiftable, then $x_n \xrightarrow{\gamma, p} g \implies x_{n+1} \xrightarrow{\gamma, p} g$ (note that the similar fact for $\xrightarrow{\gamma}$ can be not true).
13. Suppose that γ is shiftable. If x is γ, p -convergent, then $(\Delta x)_n \xrightarrow{\gamma, p} 0$ and, in particular, $0 \in \operatorname{LIM}_{\mathbb{C}}(\Delta x)$.
14. If $x_n \xrightarrow{\gamma, p} g$ then for any $\delta > 0$

$$\frac{\mu_{\gamma}(F_n(x, g, \delta))}{\mu_{\gamma}([k_0; n])} \xrightarrow[n]{} 0.$$

Proof. The following parts of the above proposition can be immediately checked with no difficulty, using (2.8): the uniqueness and the linearity for $\xrightarrow{\gamma}$ in part 1.; part 2.; part 3.; part 4.; part 5.; part 11.; part 12.

Observe, that for the case of γ with all $\gamma_k > 0$, the fact that $\xrightarrow{\gamma}$ generalises $\longrightarrow$ follows from the classical Stolz theorem. But the general “non-negative” case can be easily reduced to this special case, because we have

$$\gamma_n = 0 \implies C_{\gamma}(N, N', x)_{n-1} = C_{\gamma}(N, N', x)_n.$$

By the Minkowski inequality we have

$$(C_{\gamma}(|x + y|^p)_n)^{1/p} \leq (C_{\gamma}(|x|^p)_n)^{1/p} + (C_{\gamma}(|y|^p)_n)^{1/p} \quad \text{for } p \geq 1, \quad (3.1)$$

and using $|a + b|^p \leq |a|^p + |b|^p$ when $0 < p < 1$ we get

$$C_{\gamma}(|x + y|^p)_n \leq C_{\gamma}(|x|^p)_n + C_{\gamma}(|y|^p)_n \quad \text{for } 0 < p < 1. \quad (3.2)$$

Now, applying these inequalities to $x - y$ and $y - z$ instead of x and y , we obtain part 10. Using symmetry and transitivity of $\overset{\gamma, p}{\approx}$, assuming $x_n \xrightarrow{\gamma, p} g$

and $x_n \xrightarrow{\gamma, p} g'$ we obtain $g \overset{\gamma, p}{\approx} g'$ (treating complex numbers as the appropriate constant sequences), i.e., $|g - g'|^p \xrightarrow{\gamma} 0$. Thus $g = g'$, and the uniqueness for $\xrightarrow{\gamma, p}$ is proved.

The linearity for $\xrightarrow{\gamma, p}$ also follows from (3.1) or from (3.2). To obtain that $\xrightarrow{\gamma, p}$ generalises $\longrightarrow$ it suffices to see that $x_n \longrightarrow g \implies |x_n - g|^p \longrightarrow 0$ and then, to use the proved fact that $\xrightarrow{\gamma}$ generalises $\longrightarrow$. Hence we completed the proof of part 1.

The part 6. follows from part 5.

If $x_n \xrightarrow{\gamma, p} g$ and $g \notin \text{LIM}_{\mathbb{C}}(x)$, then $|x_n - g|^p \geq \delta^p \geq 0$ for n s.l., with some $\delta > 0$, which contradicts part 5. Similarly, if $x_n \xrightarrow{\gamma} g$ and $g < \liminf_{n \rightarrow +\infty} x_n$, then $x_n - g \geq \delta \geq 0$ for n s.l., with some $\delta > 0$, which contradicts part 5. again, since $x_n - g \xrightarrow{\gamma} 0$ by the part 1. Analogically for “ $\limsup_{n \rightarrow +\infty}$ ”. Thus we get part 7.

To prove part 8. take x bounded, $x_n \xrightarrow{\gamma, p} g$, $p' > p$, and choose C such that $|x_n - g| \leq C$ for any $n \geq \text{st}(x)$. We can also choose D such that $|t|^{p'-p} \leq D$ for $|t| \leq C$, hence for any $n \geq \text{st}(x)$ we have $0 \leq |x_n - g|^{p'} \leq D|x_n - g|^p$, which gives part 8. by part 5.

The part 9. follows from parts 6. and 1.

The part 13. follows from parts 12., 1. and 7.

The part 14. follows from the estimate: for n s.l.

$$\begin{aligned} C_{\gamma}(|x - g|^p)_n &= \frac{\sum_{k \in N_n(x, g, \delta), k \geq k_0} \gamma_k |x_k - g|^p + \sum_{k \in F_n(x, g, \delta), k \geq k_0} \gamma_k |x_k - g|^p}{\mu_{\gamma}([k_0; n])} \\ &\geq \delta^p \frac{\mu_{\gamma}(F_n(x, g, \delta))}{\mu_{\gamma}([k_0; n])}. \end{aligned}$$

□

Parts 13. and 14. of Proposition 3.1 can be employed to find some tools for proving γ, p -divergence. Asymptotic information on generalised eigenvectors, that we typically get using some asymptotic methods, leads us (via Theorems 1.3, 4.2, 4.3) to studies of γ, p -divergence of some sequences possessing exponential form

$$z_n = \exp(ia_n) \quad \text{for } n \text{ s.l.}, \quad (3.3)$$

where $a = \{a_n\}$ is a real sequence given usually by a rather explicit formula. Let us show a simple example showing the possibility of the direct use of Proposition 3.1 part 13 as a divergence tool for such z -s.

Example 3.2. Assume (2.5) and (3.3) with $a_n = c \cdot n + \theta$, $c, \theta \in \mathbb{R}$. If $\exp(ic) = 1$, then z is constant and $z_n \xrightarrow{\gamma, p} \exp(i\theta)$ by Proposition 3.1. 1. But if $\exp(ic) \neq 1$ and γ is shiftable, then for n s.l. $|(\Delta z)_n| = |\exp(ic) - 1|$, and thus $0 \notin \text{LIM}_{\mathbb{C}}(\Delta z)$. In this case, by Proposition 3.1. 13., z is γ, p -divergent.

Our aim is to find other tools, working for some sequences a satisfying $(\Delta a)_n \longrightarrow 0$.

Let $A \subset \mathbb{R}$, $T > 0$ and let $a = \{a_n\}_{n \geq n_0}$ be a real sequence.

Recall that k_+ is given by (2.6) with $k_0 = \text{st}(\gamma)$.

Definition 3.3. A is γ -essential mod T for a iff $\limsup_{k \rightarrow +\infty} h_k(\gamma, A) > 0$, where

$$h_k(\gamma, A) := \frac{\mu_\gamma(\{s \in [n_0; k] : a_s \in \bigcup_{l \in \mathbb{Z}} (lT + A)\})}{\mu_\gamma([k_0; k])}$$

for $k \geq k_+$.

Observation 3.4. Any interval of the length greater than T is obviously γ essential mod T for any sequence a (under the assumption (2.5)). The same is true for any non-open interval of the length T . If $A_2 \supset A_1$ and A_1 is γ -essential mod T for a , then A_2 is also γ -essential mod T for a .

The γ -essentiality mod T for a can be used to obtain the γ, p -divergence of sequences given by (3.3) or by a more general formulae.

Proposition 3.5. Assume (2.5). Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be a T -periodic function ($T > 0$) and

$$x_n = f(a_n) \quad \text{for } n \text{ s.l.} \quad (3.4)$$

If $A \subset \mathbb{R}$, $g \in \mathbb{C}$ and (i) A is γ -essential mod T for a , (ii) $\text{dist}(g, f(A)) > 0$,

then $x_n \not\xrightarrow{\gamma p} g$. If f and a are such that for any $g \in \overline{f(\mathbb{R})}$ there exists $A \subset \mathbb{R}$ such that (i) and (ii) above hold, then x is γ, p -divergent.

Proof. We can assume that $x_n = f(a_n)$ for any $n \geq n_0$. If $s \geq n_0$ and $a_s \in \bigcup_{l \in \mathbb{Z}} (lT + A)$, then $x_s = f(a_s) \in f(A)$. Hence, assuming (ii), we see that for some $\delta > 0$

$$\left\{ s \in [n_0; n] : a_s \in \bigcup_{l \in \mathbb{Z}} (lT + A) \right\} \subset F_n(x, g, \delta)$$

and thus

$$\frac{\mu_\gamma(F_n(x, g, \delta))}{\mu_\gamma([k_0; n])} \geq \frac{\mu_\gamma(\{s \in [n_0; n] : a_s \in \bigcup_{l \in \mathbb{Z}} (lT + A)\})}{\mu_\gamma([k_0; n])}.$$

Therefore, if (i) holds, then $x_n \not\xrightarrow{\gamma p} g$ by Proposition 3.1. 14. Now, to get the second assertion, it suffices to recall that if $x \xrightarrow{\gamma p} g$, then by Proposition 3.1. 7. we should have $g \in \overline{f(\mathbb{R})}$. $\square$

Let φ, y be real sequences. We define two kinds of “growth control of y by φ ”.

Definition 3.6.

- $y \prec^I \varphi$ iff for any $C > 0$ there exist $D > 0$ and $N \geq \text{st}(\varphi), \text{st}(y)$, such that for any $s, s' \geq N$

$$|\varphi_s - \varphi_{s'}| \leq C \implies |y_s| \leq D|y_{s'}|.$$

- $y \prec^{II} \varphi$ iff for any $C \geq 1$ there exist $D > 0$ and $N \geq \text{st}(\varphi), \text{st}(y)$, such that for any $s, s' \geq N$

$$(|\varphi_s| \leq C|\varphi_{s'}| \ \& \ |\varphi_{s'}| \leq C|\varphi_s|) \implies |y_s| \leq D|y_{s'}|.$$

We shall be mainly interested in the situation with

$$|\varphi_n| \longrightarrow +\infty. \quad (3.5)$$

In such a case the condition $\prec^{II} \varphi$ is a quite strong requirement on y . In particular the following can be easily checked:

Remark 3.7. If (3.5) holds and $y \prec^{II} \varphi$, then $y \prec^I \varphi$.

Here are some elementary growth control examples.

Examples 3.8.

- (i) If $\varphi_n = C_1 n^\beta$ and $y_n = C_2 n^b$ for n s.l., with $b, \beta, C_1, C_2 \in \mathbb{R}, \beta > 0, C_1 \neq 0$, then $y \prec^{II} \varphi$ and $y \prec^I \varphi$.
- (ii) If $\varphi_n = C_1 \ln n$ and $y_n = C_2 n^b$ for n s.l., with $b, C_1, C_2 \in \mathbb{R}, C_1 \neq 0$, then $y \prec^I \varphi$, but $y \not\prec^{II} \varphi$, with $C_2, b \neq 0$.

Soon we shall need the following generalisations of these examples:

Proposition 3.9. Suppose that $\rho, h, r^{(1)}, r^{(2)}$ are real sequences, $r^{(1)}, r^{(2)} \in \ell^\infty$, $\rho, h \in \ell_\infty^*$, $b, \beta \in \mathbb{R}, \beta > 0$, and that $\eta : [1; +\infty) \longrightarrow \mathbb{R}$ is a differentiable function with

$$\liminf_{t \rightarrow +\infty} |\eta'(t)| > 0$$

- (i) If $\varphi_n = \rho_n n^\beta$ and $y_n = h_n n^b$ for n s.l., then $y \prec^{II} \varphi$ and $y \prec^I \varphi$.
- (ii) If $\varphi_n = \eta(\ln n + r_n^{(1)}) + r_n^{(2)}$ and $y_n = h_n n^b$ for n s.l., then $y \prec^I \varphi$.

Proof. (i) follows directly from Example 3.8 (i), and to prove (ii) it suffices to use Lagrange mean value Theorem: assuming that $C > 0$ we can choose $C', C'', N, \delta > 0$, such that for $s, s' \geq N$ and $|\varphi_s - \varphi_{s'}| \leq C$ we get

$$C' \geq |\eta(\ln s + r_s^{(1)}) - \eta(\ln s' + r_{s'}^{(1)})| \geq \delta |\ln s + r_s^{(1)} - \ln s' - r_{s'}^{(1)}|,$$

and next $C'' \geq |\ln s' - \ln s| = |\ln(s'/s)|$. □

One can easily check the following consequences of “the growth control” for the estimates of averages of sequences.

Lemma 3.10.

- Suppose that $y \prec^I \varphi$. Then for any $C > 0$ there exist $D > 0$ and $N \geq \text{st}(\varphi), \text{st}(y)$, such that for any non-empty finite $M, M' \subset \mathbb{Z} \cap [N; +\infty)$ satisfying

$$\forall_{s \in M, s' \in M'} |\varphi_s - \varphi_{s'}| \leq C$$

we have $\underline{|y| \wr M} \leq D \cdot \underline{|y| \wr M'}$.

- Suppose that $y \prec^{II} \varphi$. Then for any $C > 0$ there exist $D > 0$ and $N \geq \text{st}(\varphi), \text{st}(y)$, such that for any non-empty finite $M, M' \subset \mathbb{Z} \cap [N; +\infty)$ satisfying

$$\forall_{s \in M, s' \in M'} (|\varphi_s| \leq C|\varphi_{s'}| \wedge |\varphi_{s'}| \leq C|\varphi_s|)$$

we have $|y| \wr M \leq D \cdot |y| \wr M'$.

We are ready now to formulate the main result of the present section.

Criterion 3.11. Assume (2.5), $T > 0$, and suppose that $a = \{a_k\}_{k \geq n_0}$ is a real sequence and there exist real sequences $\varphi = \{\varphi_k\}_{k \geq n_0}$ and $\sigma = \{\sigma_k\}_{k \geq n_0}$ satisfying

- (i). $\varphi_k \leq a_k$ for k s.l.;
- (ii). $|(\Delta a)_k| \leq \sigma_k$ for k s.l.;
- (iii). $\sigma_k \rightarrow 0$;
- (iv). φ is not bounded from above;
- (v). there exists $C > 0$ such that $\sigma_k \leq C(\Delta \varphi)_k$ for k s.l.

If one of the following cases holds:

case I : (I.a) γ and $\sigma \prec^I \varphi$ and (I.b) $(a - \varphi) \in \ell^\infty$;

case II : γ and $\sigma \prec^{II} \varphi$;

then each interval with positive length is γ -essential mod T for a .

If moreover $f : \mathbb{R} \rightarrow \mathbb{C}$ is a continuous, non-constant, T -periodic function and $x_n = f(a_n)$ for n s.l., then x is γ, p -divergent with any $p > 0$.

Proof. Let $A \subset \mathbb{R}$ and for $n \in \mathbb{Z}$ denote

- $A_n := \{s \in [n_0; n] : a_s \in \bigcup_{l \in \mathbb{Z}} (lT + A)\}$;
- $B_n := \{s \geq n_0 : a_s \in \bigcup_{l \leq n} (lT + A)\}$;
- $C_n := \{s \geq n_0 : a_s \in (nT + A)\}$.

Suppose first that A is such, that

- (A1.) C_n is finite for any $n \in \mathbb{Z}$, $C_n = \emptyset$ for n s.s., and $C_n \neq \emptyset$ for n s.l.;
- (A2.) $(nT + A) \cap (n'T + A) = \emptyset$ for $n \neq n'$, $n, n' \in \mathbb{Z}$.

Let us assume also, that for A we can choose such sequence m of integers, that

- (A3.) $\max C_n \leq m_n \leq m_{n+1}$ for n s.l.,
- (A4.) there exists $M > 0$ such that $\mu_\gamma((m_n; m_{n+1}]) \leq M\mu_\gamma(C_n)$ for n s.l.

With all these assumptions, we can choose $M > 0$ and $N_0, N_1 \in \mathbb{Z}$, $N_0 \leq N_1$ such that:

$$B_n \subset A_{\max B_n}, \quad n \geq N_1 \tag{3.6}$$

(note that $\max B_n$ exists for n s.l. by (A1.) and by the equality below),

$$B_n = \bigcup_{l=N_0}^n C_l, \quad n \geq N_0 \quad \text{and } C_l\text{-s are mutually disjoint for } l \geq N_0, \tag{3.7}$$

$$\{\max C_n\}_{n \geq N_1} \text{ is unbounded from above, and } m_n \rightarrow +\infty \quad (3.8)$$

(this follows from (A1.), (3.7), and (A3.)),

$$\max B_n \leq m_n, \quad k_0 \leq m_n \leq m_{n+1}, \quad \text{for } n \geq N_1, \quad (3.9)$$

$$\mu_\gamma((m_n; m_{n+1}]) \leq M \mu_\gamma(C_n) \quad \text{for } n \geq N_1 \quad (3.10)$$

(this follows from (3.7), (A1.), (A3.), (A4.) and (3.8)).

Thus, by (3.6) (3.7), (3.9), (3.10) for any $N \geq N_1$ and $n \geq N$ we get:

$$\begin{aligned} h_{\max B_n}(\gamma, A) &= \frac{\mu_\gamma(A_{\max B_n})}{\mu_\gamma([k_0; \max B_n])} \geq \frac{\mu_\gamma(B_n)}{\mu_\gamma([k_0; \max B_n])} = \frac{\sum_{l=N_0}^n \mu_\gamma(C_l)}{\mu_\gamma([k_0; \max B_n])} \\ &\geq \frac{\sum_{l=N}^n \mu_\gamma(C_l)}{\mu_\gamma([k_0; m_n])} \geq M^{-1} \frac{\sum_{l=N}^n \mu_\gamma((m_l; m_{l+1}])}{\mu_\gamma([k_0; m_n])} \\ &\geq M^{-1} \frac{\mu_\gamma((m_N; m_n])}{\mu_\gamma([k_0; m_n])}. \end{aligned} \quad (3.11)$$

But by $m_n \rightarrow +\infty$ and $\gamma \notin \ell^1$ we have $\frac{\mu_\gamma((m_N; m_n])}{\mu_\gamma([k_0; m_n])} \rightarrow 1$, moreover by (3.8) and (3.7) $\{\max B_n\}_{n \geq N_1}$ is unbounded from above. Therefore (3.11) shows that A is γ -essential mod T for a .

Now, for the first part of the assertion, it remains to prove that for A being any interval with positive length the conditions (A1.)–(A4.) hold with some sequence m appropriately chosen for this interval. By Observation 3.4 it suffices to consider each $A := (b; b+3\delta)$ with $b \in \mathbb{R}, 0 < \delta \leq T/3$ (in fact, it would also suffice to check this only for shorter intervals). For (A1.)–(A3.) we shall not use the assumptions of either case I or case II. Let us start from the following lemma—simple, but convenient (we omit here the obvious proof).

Lemma 3.12. *Let $I \subset \mathbb{R}$ be an interval with $\inf I =: p \in \mathbb{R}, 0 < r \leq |I|$. If a real sequence x satisfies*

- $x_{\text{st}(x)} \leq p$, but p is not the upper bound for x ,
- $(\Delta x)_k < r$ for $k \geq \text{st}(x)$,

then x possesses a term in I .

(i) Observe that by (v) and (ii) we have $\varphi_{k+1} \geq \varphi_k$ for k s.l., and by (iv),

$$\varphi_k \longrightarrow +\infty, \quad a_k \longrightarrow +\infty. \quad (3.12)$$

In particular $C_n = \emptyset$ for n s.s. and C_n is finite for any $n \in \mathbb{Z}$. Let $\tilde{A} := (b + \delta; b + 2\delta)$. Using (i), (ii), (iii) and (v) we choose $K_1 \in \mathbb{Z}, K_1 \geq n_0$ and $C > 0$, such that

$$\sigma_k < \delta/2, \quad \varphi_k \leq a_k, \quad |(\Delta a)_k| \leq \sigma_k \leq C(\Delta \varphi)_k, \quad \varphi_k \leq \varphi_{k+1} \quad \text{for } k \geq K_1. \quad (3.13)$$

Now, by (3.12) we can apply Lemma 3.12 for the sequence $\{a_k\}_{k \geq K_1}$, and choosing appropriately large n'_0 we see that for $n \geq n'_0$ there exists a term of this sequence in $\tilde{A} + nT$. Thus let

$$k_n \geq K_1, \quad a_{k_n} \in \tilde{A} + nT \quad \text{for } n \geq n'_0. \quad (3.14)$$

So, we also have

$$k_n \in C_n, \quad n \geq n'_0,$$

which gives (A1.). Obviously (A2.) holds because $3\delta < T$.

For $n \geq n'_0$ define

$$l_n := \max \left\{ l \geq k_n : \sum_{s=k_n}^{l-1} \sigma_s < \delta \right\}. \quad (3.15)$$

Note that l_n is properly defined, because $\sum_{s=k_n}^{k_n-1} \sigma_s = 0$, and $\sum_{s=k_n}^{+\infty} \sigma_s = +\infty$ (if the sum is finite, then by (ii) a would be bounded - a contradiction with (3.12)). We have

$$a_l = a_{k_n} + \sum_{s=k_n}^{l-1} (\Delta a)_s, \quad l \geq k_n,$$

thus by (3.13), (3.14) we obtain

$$[k_n; l_n] \subset C_n, \quad n \geq n'_0, \quad (3.16)$$

and defining $j_n := l_n - k_n + 1$ for $n \geq n'_0$ we get

$$1 \leq j_n \leq \#C_n, \quad n \geq n'_0. \quad (3.17)$$

Let us define the sequence m as follows:

$$m_n := \max\{k \geq K_1 : \varphi_k \leq b + 3\delta + Tn\}, \quad n \geq n'_0. \quad (3.18)$$

Note that m_n is properly defined, because the set under 'max' is finite (by (3.12)) and nonempty (it contains k_n by (3.14)). By the definition $m_{n+1} \geq m_n$. Let $n_1 \geq n'_0$ be such, that $b + n_1 T > \max\{a_{n_0}, \dots, a_{K_1}\}$. Suppose that $n \geq n_1$ and $k \in C_n$. We have

$$k \geq n_0, \quad a_k \geq b + nT > \max\{a_{n_0}, \dots, a_{K_1}\},$$

which means that $k > K_1$. Hence, by (3.13) we also have $\varphi_k \leq a_k \leq b + 3\delta + Tn$, and this is why $k \leq m_n$. So we get

$$\max C_n \leq m_n \quad \text{for } n \geq n_1,$$

thus (A3.) holds. We have also obtained

$$k \geq K_1, \quad \varphi_k \leq b + 3\delta + Tn \quad \text{for } k \in C_n, \quad n \geq n_1. \quad (3.19)$$

From (3.18) and (3.15) we see that

$$\varphi_{m_{n+1}} > b + 3\delta + Tn, \quad \varphi_{m_{n+1}} \leq b + 3\delta + Tn + T \quad \text{for } n \geq n_1, \quad (3.20)$$

$$j_n \cdot \sigma \restriction [k_n; l_n] = \sum_{s=k_n}^{l_n} \sigma_s \geq \delta \quad \text{for } n \geq n_1. \quad (3.21)$$

By (3.13), (3.20)

$$b + 3\delta + Tn \leq \varphi_k \leq b + 3\delta + T + Tn \quad \text{for } k \in (m_n; m_{n+1}], \quad n \geq n_1. \quad (3.22)$$

The definition of C_n gives

$$b + Tn \leq a_k \leq b + 3\delta + Tn \quad \text{for } k \in C_n, \quad n \geq n_1, \quad (3.23)$$

hence, using (3.13) and (3.19), for $k \in C_n$, $n \geq n_1$

$$\begin{aligned}
 \varphi_k &= \varphi_{K_1} + \sum_{s=K_1}^{k-1} (\Delta\varphi)_s \geq \varphi_{K_1} + \frac{1}{C} \sum_{s=K_1}^{k-1} \sigma_s \\
 &= \left(\varphi_{K_1} - \frac{1}{C} a_{K_1} \right) + \frac{1}{C} \left(a_{K_1} + \sum_{s=K_1}^{k-1} \sigma_s \right) \\
 &\geq \left(\varphi_{K_1} - \frac{1}{C} a_{K_1} \right) + \frac{1}{C} \left(a_{K_1} + \sum_{s=K_1}^{k-1} (\Delta a)_s \right) = \left(\varphi_{K_1} - \frac{1}{C} a_{K_1} \right) + \frac{1}{C} a_k \\
 &\geq \left(\varphi_{K_1} + \frac{b - a_{K_1}}{C} \right) + \frac{T}{C} n.
 \end{aligned} \tag{3.24}$$

Note that **under the extra assumption (I.b)** the estimate (3.23) gives a better lower bound for φ on C_n , namely there exists a real constant c_1 such that

$$c_1 + Tn \leq \varphi_k \quad \text{for } k \in C_n, \quad n \geq n_1 \tag{3.25}$$

(note that $C \geq 1$ by (3.13), so the above is really better than (3.24) for “large” n). Using Lemma 3.10 and the estimates (3.22), (3.19), and (3.25)/(3.24)—for the case I/II, respectively, we choose c_2 such that

$$\underline{\gamma \wr (m_n; m_{n+1})} \leq c_2 \cdot \underline{\gamma \wr C_n} \quad \text{for } n \text{ s.l.}, \tag{3.26}$$

$$(m_{n+1} - m_n \geq 2) \implies \underline{\sigma \wr [k_n; l_n]} \leq c_2 \cdot \underline{\sigma \wr (m_n; m_{n+1})} \quad \text{for } n \text{ s.l.} \tag{3.27}$$

Thus, by (3.21), (3.27), (3.13) and (3.20), for n s.l., if $m_{n+1} - m_n \geq 2$, then

$$\begin{aligned}
 \frac{(\Delta m)_n}{j_n} &= j_n^{-1} + \frac{(m_{n+1} - m_n - 1)}{j_n} \\
 &\leq 1 + \delta^{-1} (m_{n+1} - m_n - 1) \cdot \underline{\sigma \wr [k_n; l_n]} \\
 &\leq 1 + \delta^{-1} c_2 (m_{n+1} - m_n - 1) \cdot \underline{\sigma \wr (m_n; m_{n+1})} \\
 &\leq 1 + \delta^{-1} c_2 C \sum_{s=m_n+1}^{m_{n+1}-1} (\Delta\varphi)_s \\
 &= 1 + \delta^{-1} c_2 C (\varphi_{m_{n+1}} - \varphi_{m_n+1}) < 1 + \delta^{-1} c_2 CT,
 \end{aligned}$$

and if $m_{n+1} - m_n < 2$, then $\frac{(\Delta m)_n}{j_n} \leq 1$. Hence we have

$$\frac{(\Delta m)_n}{j_n} \leq 1 + \delta^{-1} c_2 CT, \quad \text{for } n \text{ s.l.} \tag{3.28}$$

And finally, by (3.28), (3.26) and (3.17) for n s.l. we get

$$\begin{aligned}
 \mu_\gamma((m_n; m_{n+1}]) &= (\Delta m)_n \cdot \underline{\gamma \wr (m_n; m_{n+1})} \leq c_2 (1 + \delta^{-1} c_2 CT) j_n \cdot \underline{\gamma \wr C_n} \\
 &\leq c_2 (1 + \delta^{-1} c_2 CT) \#C_n \cdot \underline{\gamma \wr C_n} = c_2 (1 + \delta^{-1} c_2 CT) \mu_\gamma(C_n),
 \end{aligned}$$

which proves (A4.).

Now, to prove the γ, p -divergence of x it suffices to use Proposition 3.5. $\square$

To understand the formulation of Criterion 3.11 better let us explain in a few words the role of the sequences a, φ, σ . The sequence a is the main sequence defining the sequence x to be γ, p -divergent, and φ, σ are auxiliary sequences, which we should choose properly, to satisfy conditions (i)–(v) and (I or II) of Criterion. The typical idea is to choose them in such a way, that “for large entries”:

- (1°) φ is a “regularly behaving” and possibly large lower bound sequence for a ;
- (2°) σ is a “regularly behaving” and possibly small upper bound sequence for $|\Delta a|$.

We present examples of two concrete classes of x -s and γ -s, being illustrations of the use of the above criterion and of the above idea of choosing the auxiliary sequences. These examples will be important in Sect. 5.

Example 3.13. Consider $\gamma \geq 0$ and x given by

$$x_n = f(c_n n^\beta), \quad \gamma(n) = n^{-d} \quad \text{for } n \text{ s.l.,}$$

where $f: \mathbb{R} \rightarrow \mathbb{C}$ is a continuous, non-constant, periodic function,

$$d \leq 1, \quad 0 < \beta < 1, \quad (3.29)$$

and $c = \{c_n\}_{n \geq 1}$ is a real sequence satisfying

$$c \in \ell^\infty, \quad n(\Delta c)_n \rightarrow 0 \quad \text{and} \quad \left(\liminf_{n \rightarrow +\infty} c_n > 0 \quad \text{or} \quad \limsup_{n \rightarrow +\infty} c_n < 0 \right). \quad (3.30)$$

Then x is γ, p -divergent with any $p > 0$.

Proof. We can assume that $\liminf_{n \rightarrow +\infty} c_n > 0$, due to possible substitution “ c_n by $-c_n$ and $f(t)$ by $f(-t)$ ”. We use Criterion 3.11 taking: T —a positive period of f ,

$$a_n := c_n n^\beta, \quad \varphi_n := \delta n^\beta, \quad \sigma_n := S((n+1)^\beta - n^\beta), \quad n \in \mathbb{N},$$

with $\delta := \frac{1}{2} \liminf_{n \rightarrow +\infty} c_n, S := 2 \limsup_{n \rightarrow +\infty} c_n$. It is immediately clear that the assumptions (i), (iii)–(v) of the criterion hold. By Proposition 3.9 (i) we are in the case II of this criterion. We should only check the assumption (ii). But for n s.l. we have

$$\begin{aligned} |(\Delta a)_n| &= |(\Delta c)_n n^\beta + c_{n+1}((n+1)^\beta - n^\beta)| \\ &= |n(\Delta c)_n n^{\beta-1} + c_{n+1}((n+1)^\beta - n^\beta)| \\ &= |o(1) + c_{n+1}|((n+1)^\beta - n^\beta) \leq S((n+1)^\beta - n^\beta) = \sigma_n. \end{aligned}$$

$\square$

For the second example we introduce the following notation. Let $\eta: [1; +\infty) \rightarrow \mathbb{R}$. We write

$$\eta \in \Upsilon \quad (3.31)$$

iff η satisfies:

- a). for some $a > 1$ η restricted to $(a; +\infty)$ is differentiable, increasing (in the non-strict sense) and convex;
- b). $\lim_{t \rightarrow +\infty} \eta'(t) \neq 0$
- c). for any $R > 0$ there exists $D > 0$ and $M \geq 1$ such that for any $t \geq M$ $\eta'(t+R) \leq D\eta'(t)$;
- d). $\lim_{t \rightarrow +\infty} \eta'(t)e^{-t} = 0$.

Note that the limit from b) exists (finite or $+\infty$) by convexity. For instance, $\eta \in \Upsilon$ if it is given for $t \geq 1$ by the formula $\eta(t) = t$, or more generally, by $\eta(t) = t^\alpha$, with $\alpha \geq 1$.

Example 3.14. Consider $\gamma \geq 0$ and x given by

$$x_n = f\left(\eta(\ln n + r_n^{(1)}) + r_n^{(2)}\right), \quad \gamma(n) = n^{-d} \quad \text{for } n \text{ s.l.,}$$

where $d \leq 1$, $f: \mathbb{R} \rightarrow \mathbb{C}$ is a continuous, non-constant, periodic function, $\eta \in \Upsilon$, and $r^{(1)}, r^{(2)}$ are real bounded sequences satisfying:

$$\exists_{\rho < 1, C > 0} \quad -\frac{\rho}{n} \leq (\Delta r^{(1)})_n \leq \frac{C}{n} \quad \text{for } n \text{ s.l.,} \quad (3.32)$$

$$\exists_{C' > 0} \quad |(\Delta r^{(2)})_n| \leq \frac{C'}{n} \eta'(\ln n) \quad \text{for } n \text{ s.l..} \quad (3.33)$$

Then x is γ, p -divergent with any $p > 0$.

Proof. We use now the case I of Criterion 3.11 with a positive period of f as T ,

$$\begin{aligned} a_n &:= \eta(\ln n + r_n^{(1)}) + r_n^{(2)}, & \varphi_n &:= \eta(\ln n + r_n^{(1)}) + A, \\ \sigma_n &:= \frac{B}{n} \eta'(\ln n), & n &\in \mathbb{N}, \end{aligned}$$

where $A := \inf_{n \geq n_0} r_n^{(2)}$ for a fixed $n_0 \geq \text{st}(r^{(2)})$ and B will be chosen soon. Using the Lagrange mean value theorem and $\eta \in \Upsilon$, (3.32), (3.33) we can easily see that assumptions (i)–(iii) of the criterion hold with B chosen to be sufficiently large. We also obtain (iv) by the convexity of η and by b) from the definition of the class Υ . Now, using (3.32) with $\eta \in \Upsilon$ and with the boundedness of $r^{(1)}$, we obtain the assumption (v). By Proposition 3.9 (ii) we get $\gamma \prec^I \varphi$, we also have $(a - \varphi) \in \ell^\infty$, by the choice of a and φ . Thus, it remains only to prove that $\sigma \prec^I \varphi$. But again by the Lagrange mean value theorem, taking some $C > 0$ we can choose $\tilde{C}, N, \delta > 0$, such that for any $s, s' \geq N$ which satisfy $|\varphi_s - \varphi_{s'}| \leq C$ we get

$$C \geq |\eta(\ln s + r_s^{(1)}) - \eta(\ln s' + r_{s'}^{(1)})| \geq \delta |\ln s + r_s^{(1)} - \ln s' - r_{s'}^{(1)}|,$$

and consequently $\tilde{C} \geq |\ln s' - \ln s| = |\ln(s'/s)|$. Hence, using $\eta \in \Upsilon$, we can choose $\tilde{N} \geq N$ and $\tilde{D} > 0$ such that for any $s, s' \geq \tilde{N}$ which satisfy $|\varphi_s - \varphi_{s'}| \leq C$ we have:

$$\sigma_s = \frac{B}{s} \eta'(\ln s) \leq \frac{B}{s} \eta'(\ln s' + \tilde{C}) \leq \frac{\tilde{D}B}{s'} \eta'(\ln s') \frac{s'}{s} \leq \tilde{D}e^{\tilde{C}} \sigma_{s'}.$$

□

4. Asymptotics of Generalised Eigenvectors and Subordinacy

Let us recall first a simple result (following directly from [24, Lem. 5.9 and Prop. 5.5]) related to not scalar, but $\mathbb{C}^2$ -vector-generalised eigenvectors asymptotics. It is a stronger version of Proposition 1.1.

Proposition 4.1. *Suppose that Jacobi operator J satisfies (2.2), and for a spectral parameter³ $\lambda \in \mathbb{C}$ it possesses two $\mathbb{C}^2$ -vector-generalised eigenvectors x_+ , x_- , such that*

$$x_{\pm}(n) = \psi_{\pm}(n)y_{\pm}(n), \quad n \in \mathbb{N},$$

where $y_{\pm}$ are **bounded** sequences of $\mathbb{C}^2$ vectors,

$$\inf_{n \in \mathbb{N}} |\det(y_-(n), y_+(n))| > 0, \quad (4.1)$$

and $\psi_{\pm}$ are sequences of nonzero numbers satisfying

$$\inf_{n \in \mathbb{N}} \left| \frac{\psi_+(n)}{\psi_-(n)} \right| > 0, \quad \sup_{n \in \mathbb{N}} \left| \frac{\psi_+(n)}{\psi_-(n)} \right| < +\infty. \quad (4.2)$$

Then the transfer matrix sequence $\{B_n(\lambda)\}_{n \geq 2}$, with

$$B_n(\lambda) = \begin{pmatrix} 0 & 1 \\ -\frac{w_{n-1}}{w_n} & \frac{\lambda - q_n}{w_n} \end{pmatrix}, \quad (4.3)$$

is in H -class and in particular no subordinate solution for J and λ exists.

Now we formulate some results for generalised eigenvectors, i.e., for scalar solutions. These results can be treated as “repairs” of Conjecture 1.2, promised in Introduction. The first is a generalisation of Theorem 1.3.

Theorem 4.2. *Assume that J is self-adjoint, (2.2) holds and that for some $\lambda \in \mathbb{C}$ u_+ , u_- are two linearly independent vectors from $\text{Sol}(\lambda)$ such that*

$$u_{\pm}(n) = \psi_{\pm}(n) \cdot s_{\pm}(n), \quad \text{with} \quad \psi_+(n) = z(n) \cdot \psi_-(n), \quad \text{for } n \text{ s.l.,}$$

where $\psi_{\pm}$, $s_{\pm}$, z are scalar sequences, and

- (i). $z \in \ell^{\infty}$,
- (ii). $s_- \in \ell^{\infty}_*$,
- (iii). $\frac{s_+(n)}{s_-(n)} \longrightarrow \kappa \in \mathbb{C} \setminus \{0\}$.

Then there exists a subordinate solution for J and λ iff z is $\gamma, 2$ -convergent, where $\gamma := |\psi_-|^2$. Moreover, if $z \xrightarrow{\gamma, 2} g$, then $u_+ - \kappa g u_-$ is a subordinate solution for J and λ .

The second result gives only necessary conditions for subordinacy, but it can be used sometimes for some weaker assumptions on asymptotics.

If x is a sequence and $\xi = \{\xi_j\}$ is a strictly increasing sequence of integers, then $x \circ \xi$ denotes the composition of those sequences, i.e., $\text{st}(x \circ \xi) := \min\{j \geq \text{st}(\xi) : \xi_j \geq \text{st}(x)\}$, and $(x \circ \xi)(j) = x(\xi_j)$ for $j \geq \text{st}(x \circ \xi)$. For

³In [24, Prop. 5.5] $\lambda \in \mathbb{R}$ is assumed, but the argumentation works for $\lambda \in \mathbb{C}$ with no changes.

$k \in \mathbb{Z}$ we write $k \triangleright \xi$, to denote that k is a term of ξ (i.e., $k = \xi_j$ for some $j \geq \text{st}(\xi)$).

Theorem 4.3. *Assume that (2.2) holds, ξ is a strictly increasing sequence of natural numbers, $\lambda \in \mathbb{C}$, and u_+, u_- are two linearly independent vectors from $\text{Sol}(\lambda)$ such that*

$$u_{\pm}(n) = \psi_{\pm}(n) \cdot s_{\pm}(n), \quad \text{with} \quad \psi_+(n) = z(n) \cdot \psi_-(n), \quad \text{for } n \text{ s.l.,}$$

where $\psi_{\pm}, s_{\pm}, z$ are scalar sequences, and

0. $\psi_- \notin \ell^2$,
1. $s_-, s_+, z \in \ell^\infty$,
2. there is $K_0 \in \mathbb{N}$ and a positive constant C , such that for $n \geq K_0$

$$\sum_{k=K_0}^n |\psi_-(k)|^2 \leq C \sum_{k \triangleright \xi, K_0 \leq k \leq n} |\psi_-(k)|^2, \quad (4.4)$$
3. $\liminf_{j \rightarrow +\infty} |(s_- \circ \xi)(j)| > 0$,
4. $\frac{(s_+ \circ \xi)(j)}{(s_- \circ \xi)(j)} \xrightarrow{j} \kappa \in \mathbb{C} \setminus \{0\}$.

If there exists a subordinate solution for J and λ , then $z \circ \xi$ is $\tilde{\gamma}$, 2-convergent, where $\tilde{\gamma} := |\psi_- \circ \xi|^2$.

In the proofs of these two results we shall use the following technical lemma. For $u, v \in \ell$ and for $n \geq N \geq \text{st}(u), \text{st}(v)$ denote:

$$\|u\|_{N:n} := \left(\sum_{k=N}^n |u(k)|^2 \right)^{\frac{1}{2}}, \quad \text{and} \quad \|u|v\|_{N:n} := \frac{\|u\|_{N:n}}{\|v\|_{N:n}} \quad \text{when} \quad \|v\|_{N:n} \neq 0.$$

Lemma 4.4. *Assume that $u, v, u', v' \in \ell$, $N \geq \text{st}(u), \text{st}(u'), \text{st}(v), \text{st}(v')$ and $b \in [0; +\infty]$. If $v \notin \ell^2$, then*

- a). for any $N' \geq N$ $\|u|v\|_{N':n} \xrightarrow{n} b \iff \|u|v\|_{N:n} \xrightarrow{n} b$;
- a'). if u possesses infinitely many non-zero terms, then for any $N' \geq N$ $\|v|u\|_{N':n} \xrightarrow{n} b \iff \|v|u\|_{N:n} \xrightarrow{n} b$;
- b). if $b = 0$ or $+\infty$ and $u(k) = u'(k)j_u(k)$, $v(k) = v'(k)j_v(k)$ for $k \geq N$, where $j_u, j_v \in \ell$, j_u, j_v are bounded and $\inf_{k \geq N} |j_u(k)|, \inf_{k \geq N} |j_v(k)| > 0$, then $v' \notin \ell^2$ and $\|u|v\|_{N:n} \xrightarrow{n} b \iff \|u'|v'\|_{N:n} \xrightarrow{n} b$;
- c). if $\|u'|v\|_{N:n} \xrightarrow{n} 0$, then $\|(u+u')|v\|_{N:n} \xrightarrow{n} 0 \iff \|u|v\|_{N:n} \xrightarrow{n} 0$;
- d). if $|u_n| = h_n|v_n|$ for n s.l., where $h_n \rightarrow b$, then $\|u|v\|_{N:n} \xrightarrow{n} b$.

Proof. By $v \notin \ell^2$, we have $\|u|v\|_{N:n}^2 = \frac{\|u|v\|_{N':n}^2 + o(1)}{1+o(1)}$ for n s.l., so we get

a). To get a') we use a) with $\|u|v\|_{M:n} = \|v|u\|_{M:n}^{-1}$ for any $M \geq N$ and for n s.l.. The point b) is obvious. The point c) follows from the triangle inequality for $\|\cdot\|_{N:n}$. The proof of d) can be made by the use of the classical Stolz theorem, analogically as in the proof of Proposition 3.1. 1. $\square$

Proof of Theorem 4.2. Observe first that $u_- \notin \ell^2(\mathbb{N})$, because by (i)–(iii) we can choose $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that

$$|u_+(n)| \leq C|u_-(n)| \quad \text{for } n \geq N, \quad (4.5)$$

and hence the condition $u_- \in \ell^2(\mathbb{N})$ with our linear independence assumption would give $\text{Sol}(\lambda) \subset \ell^2(\mathbb{N})$, which is impossible by Proposition 2.2 (ii) and (iii). Now we prove that u_- is not subordinate (for J, λ —we omit writing it later on). On the contrary, suppose that $u_- \in \text{Sub}(\lambda)$. Then, by Proposition 2.2 (i) we have $\|u_-|u_+\|_{1:n} \rightarrow 0$, and thus $\|u_+|u_-\|_{1:n} \rightarrow +\infty$. So, using $u_- \notin \ell^2(\mathbb{N})$ and Lemma 4.4 a), we get $\|u_+|u_-\|_{N:n} \rightarrow +\infty$, which contradicts (4.5).

Therefore, by Proposition 2.2 (ii), there exists a subordinate solution iff for some $g' \in \mathbb{C}$ the vector $u_{g'} := u_+ - g'u_-$ is subordinate, that is, by Proposition 2.2 (i) and Lemma 4.4 a), iff for some $g' \in \mathbb{C}$

$$\|u_{g'}|u_-\|_{N:n} \xrightarrow[n]{} 0. \quad (4.6)$$

But for some $N' \geq N$ we have $\inf_{k \geq N'} |s_{\pm}(k)| > 0$ and for any $n \geq N'$

$$u_{g'}(n) = \psi_-(n)s_+(n) [z(n) - \kappa^{-1}g'] + \omega(n), \quad (4.7)$$

with $\omega(n) = \psi_-(n)\epsilon(n)$ and $\epsilon(n) = s_+(n)g' \left[\kappa^{-1} - \frac{s_-(n)}{s_+(n)} \right]$. By (ii), (iii) we also have $\psi_- \notin \ell^2$, and $\epsilon(n) \xrightarrow[n]{} 0$. Hence, by Lemma 4.4 a) and d), $\|\omega|\psi_-\|_{N':n} \xrightarrow[n]{} 0$. Thus, by Lemma 4.4 a), c), b) and by (4.7) we have

$$\begin{aligned} \|u_{g'}|u_-\|_{N:n} \xrightarrow[n]{} 0 &\iff \|u_{g'}|u_-\|_{N':n} \xrightarrow[n]{} 0 \iff \|u_{g'}|\psi_-\|_{N':n} \xrightarrow[n]{} 0 \\ &\iff \|\psi_-s_+(z - \kappa^{-1}g')|\psi_-\|_{N':n} \xrightarrow[n]{} 0 \\ &\iff \|\psi_-(z - \kappa^{-1}g')|\psi_-\|_{N':n} \xrightarrow[n]{} 0. \end{aligned}$$

But by (2.8) the last convergence means exactly that $z \xrightarrow{\gamma,2} \kappa^{-1}g'$. $\square$

Proof of Theorem 4.3. Using assumptions 1. and 3. we choose positive constants C_1, C_2 and an integer $K_1 \geq K_0$ such that for any $k \geq K_1$

$$|u_+(k)| = |\psi_-(k)z(k)s_+(k)| \leq C_1|\psi_-(k)|,$$

and

$$|u_-(k)| = |\psi_-(k)s_-(k)| \geq C_2|\psi_-(k)| \quad \text{for any } k \triangleright \xi, k \geq K_1. \quad (4.8)$$

Thus, $u_- \notin \ell^2(\mathbb{N})$, by assumption 2. and 0., and moreover, we can choose positive constants C_3, C_4, C_5 such that for n s.l.

$$\begin{aligned} \|u_-|u_+\|_{K_1:n}^2 &\geq \frac{\sum_{k \triangleright \xi, K_1 \leq k \leq n} |u_-(k)|^2}{C_1^2 \sum_{K_1 \leq k \leq n} |\psi_-(k)|^2} \geq \frac{C_2^2}{C_1^2} \frac{\sum_{k \triangleright \xi, K_1 \leq k \leq n} |\psi_-(k)|^2}{\sum_{K_0 \leq k \leq n} |\psi_-(k)|^2} \\ &= \frac{C_2^2}{C_1^2} \frac{\left(\sum_{k \triangleright \xi, K_0 \leq k \leq n} - \sum_{k \triangleright \xi, K_0 \leq k < K_1} \right) |\psi_-(k)|^2}{\sum_{K_0 \leq k \leq n} |\psi_-(k)|^2} \\ &\geq C_3 - \frac{C_4}{\sum_{K_0 \leq k \leq n} |\psi_-(k)|^2} \geq C_5. \end{aligned} \quad (4.9)$$

Using again assumption 0., 1., 2. and (4.8), we can find positive constant C_6 and $K_2 \geq K_1$ such that for n s.l.

$$\begin{aligned}
 \sum_{K_2 \leq k \leq n} |u_-(k)|^2 &\leq C_6 \sum_{K_0 \leq k \leq n} |\psi_-(k)|^2 \leq CC_6 \sum_{k \triangleright \xi, K_0 \leq k \leq n} |\psi_-(k)|^2 \\
 &= CC_6 \left(\sum_{k \triangleright \xi, K_2 \leq k \leq n} + \sum_{k \triangleright \xi, K_0 \leq k < K_2} \right) |\psi_-(k)|^2 \\
 &\leq 2CC_6 \sum_{k \triangleright \xi, K_2 \leq k \leq n} |\psi_-(k)|^2 \leq 2CC_2^{-2}C_6 \sum_{k \triangleright \xi, K_2 \leq k \leq n} |u_-(k)|^2. \quad (4.10)
 \end{aligned}$$

Suppose that $u_- \in \text{Sub}(\lambda)$. Then, by Proposition 2.2 (i) we have $\|u_-|u_+\|_{1:n} \rightarrow 0$ and thus, using Lemma 4.4 a') (note that u_+ is a non-zero vector from $\text{Sol}(\lambda)$ and thus by (2.2) it possesses infinitely many non-zero terms), we get $\|u_-|u_+\|_{K_1:n} \rightarrow 0$, a contradiction with (4.9). Hence $u_- \notin \text{Sub}(\lambda)$.

Therefore, by Proposition 2.2 (ii), if there exists a subordinate solution, then for some $g' \in \mathbb{C}$ the solution $u_{g'} := u_+ - g'u_-$ is subordinate, that is, by Proposition 2.2 (i) and Lemma 4.4 a)

$$\|u_{g'}|u_-\|_{K_2:n} \xrightarrow[n]{} 0. \quad (4.11)$$

Applying (4.10), for some positive constant C_7 and for n s.l. we have

$$\|u_{g'}|u_-\|_{K_2:n}^2 \geq C_7 \frac{\sum_{k \triangleright \xi, K_2 \leq k \leq n} |u_{g'}(k)|^2}{\sum_{k \triangleright \xi, K_2 \leq k \leq n} |u_-(k)|^2} = C_7 \|u_{g'} \circ \xi | u_- \circ \xi\|_{j_0:t_n}^2,$$

where $j_0 := \min\{j \geq \text{st}(\xi) : K_2 \leq \xi_j\}$ and $t_n := \max\{j \geq \text{st}(\xi) : \xi_j \leq n\}$ for $n \geq \xi_{\text{st}(\xi)}$. Thus from (4.11) $\|u_{g'} \circ \xi | u_- \circ \xi\|_{j_0:t_n} \xrightarrow[n]{} 0$. But ξ is a strictly increasing sequence of natural numbers, so we see that $t_{\xi_n} = n$ for $n \geq \text{st}(\xi)$ and $\xi_n \rightarrow +\infty$, which gives

$$\|u_{g'} \circ \xi | u_- \circ \xi\|_{j_0:n} \xrightarrow[n]{} 0. \quad (4.12)$$

By assumptions 3. and 4., for some $j_1 \geq j_0$ we have $\inf_{j \geq j_1} |(s_{\pm} \circ \xi)(j)| > 0$ and for any $j \geq j_1$

$$(u_{g'} \circ \xi)(j) = (\psi_- \circ \xi)(j) \cdot (s_+ \circ \xi)(j) [(z \circ \xi)(j) - \kappa^{-1}g'] + (\psi_- \circ \xi)(j) \cdot \epsilon(j), \quad (4.13)$$

with $\epsilon(j) = g' \cdot (s_+ \circ \xi)(j) \left[\kappa^{-1} - \frac{(s_- \circ \xi)(j)}{(s_+ \circ \xi)(j)} \right]$.

By assumptions 0.—3. we have also $u_- \circ \xi, \psi_- \circ \xi \notin \ell^2$, and $\epsilon(j) \xrightarrow[j]{} 0$. Hence, by Lemma 4.4 d), $\|(\psi_- \circ \xi) \cdot \epsilon | \psi_- \circ \xi\|_{j_1:n} \xrightarrow[n]{} 0$. Thus, by Lemma 4.4 a), c), b) and by (4.12), (4.13) we consecutively obtain: $\|u_{g'} \circ \xi | u_- \circ \xi\|_{j_1:n} \xrightarrow[n]{} 0$, then $\|u_{g'} \circ \xi | \psi_- \circ \xi\|_{j_1:n} \xrightarrow[n]{} 0$, next $\|(\psi_- s_+ (z - \kappa^{-1}g')) \circ \xi | \psi_- \circ \xi\|_{j_1:n} \xrightarrow[n]{} 0$, and finally $\|(\psi_- (z - \kappa^{-1}g')) \circ \xi | \psi_- \circ \xi\|_{j_1:n} \xrightarrow[n]{} 0$. But, by (2.8), the last convergence means exactly that $z \circ \xi \xrightarrow[\tilde{\gamma}^2]{} \kappa^{-1}g'$. $\square$

5. Proving Non-Subordinacy

We show here how to use the technical tools introduced in previous sections to prove, that for certain Jacobi operators J and spectral parameters λ , no subordinate solution exists, i.e., $\text{Sub}(\lambda) = \{0\}$. We study various forms of asymptotics for two linearly independent generalised eigenvectors, found by the authors of [4, 8, 17, 19, 26, 27, 33] through some asymptotic tricks for difference equations. The results described below are not mutually independent—we start from some simpler cases to show the restrictions in use of some tools formulated in previous sections, and the reasons for formulating some others. For all the cases we shall assume the existence of two linearly independent vectors u_+, u_- from $\text{Sol}(\lambda)$, possessing the general form

$$u_{\pm}(n) = r(n) \exp(\pm ia_n) \cdot s_{\pm}(n), \quad n \in \mathbb{N}, \quad (5.1)$$

with various conditions on sequences r —the positive *explicit modulus*, a —the real *explicit phase*, $s_{\pm}$ —the complex *implicit terms* (note that even under quite strong restrictions on these sequences, they are not uniquely determined).

We concentrate here mainly on the so-called *Jordan box* (or *critical*) case—see e.g. [8], because this is the main case when GBS method ([10, Lemma 1.5]) does usually not work. In particular, we try to repair the gaps in the proofs of non-subordinacy from [4, 8, 17, 19, 26, 27, 33].

In the following cases we shall always assume that $r(n) := n^{-b}$ with $b \leq \frac{1}{2}$, but the form of a_n and $s_{\pm}(n)$ will become gradually more and more complicated.

Proposition 5.1. *Assume that (2.2) holds for $J, \lambda \in \mathbb{C}$ and that there exist two linearly independent vectors u_+, u_- from $\text{Sol}(J, \lambda)$, which satisfy (5.1) with*

$$r(n) := n^{-b}, \quad a_n := cn^{\beta}, \quad s_{\pm}(n) := 1 + \epsilon_{\pm}(n) \quad \text{for } n \text{ s.l.} \quad (5.2)$$

where $\epsilon_{\pm}(n) \rightarrow 0$ and

$$b \leq \frac{1}{2}, \quad 0 < \beta < 1, \quad c \in \mathbb{R} \setminus \{0\}. \quad (5.3)$$

Then J is self-adjoint and $\text{Sub}(J, \lambda) = \{0\}$.

Proof. Observe that J is self-adjoint by [1, Sect. VII, Lem. 1.5], since u_- (nor u_+) is not in $\ell^2(\mathbb{N})$ by $b \leq \frac{1}{2}$. So, defining

$$\psi_{\pm}(n) := r(n) \exp(\pm ia_n), \quad n \in \mathbb{N}, \quad (5.4)$$

we apply Theorem 1.3. It remains only to check that z is $\gamma, 2$ -divergent, with

$$z(n) = \exp(2ia_n), \quad \gamma(n) := r(n)^2, \quad n \in \mathbb{N}.$$

The above is a special case of Example 3.13 with $f(t) := \exp(2it), t \in \mathbb{R}$. $\square$

We can also consider a slightly more general case of the terms $s_{\pm}$.

Proposition 5.2. Assume that (2.2) holds for $J, \lambda \in \mathbb{C}$ and that there exist two linearly independent vectors u_+, u_- from $\text{Sol}(J, \lambda)$, which satisfy (5.1) with

$$r(n) := n^{-b}, \quad a_n := cn^\beta, \quad s_\pm(n) := p_n + \epsilon_\pm(n) \quad \text{for } n \text{ s.l.} \quad (5.5)$$

where $\{p_n\}_{n \geq 1} \in \ell_*^\infty$, $\epsilon_\pm(n) \rightarrow 0$ and (5.3) holds. Then J is self-adjoint and $\text{Sub}(J, \lambda) = \{0\}$.

Proof. The proof is almost the same as for Proposition 5.1; the only difference is the use of Theorem 4.2 (with $\kappa = 1$ in (iii)) instead of Theorem 1.3. $\square$

Let us consider now a case with the sequence a being more complicated. Here c from the formula for a_n is no longer constant.

Proposition 5.3. Assume that (2.2) holds for $J, \lambda \in \mathbb{C}$ and that there exist two linearly independent vectors u_+, u_- from $\text{Sol}(J, \lambda)$, which satisfy (5.1) with

$$r(n) := n^{-b}, \quad a_n := c_n n^\beta, \quad s_\pm(n) := p_n + \epsilon_\pm(n) \quad \text{for } n \text{ s.l.} \quad (5.6)$$

where $\{p_n\}_{n \geq 1} \in \ell_*^\infty$, $\epsilon_\pm(n) \rightarrow 0$, $c = \{c_n\}_{n \geq 1}$ is a real sequence satisfying

$$c \in \ell^\infty, \quad n(\Delta c)_n \rightarrow 0, \quad (\liminf_{n \rightarrow +\infty} c_n > 0 \quad \text{or} \quad \limsup c_n < 0), \quad (5.7)$$

and

$$b \leq \frac{1}{2}, \quad 0 < \beta < 1. \quad (5.8)$$

Then J is self-adjoint and $\text{Sub}(J, \lambda) = \{0\}$.

Proof. The argumentation for the self-adjointness is the same as for Proposition 5.1 and we also apply Theorem 4.2 with (5.4).

It remains only to check that z is $\gamma, 2$ -divergent, with

$$z(n) = \exp(2ia_n), \quad \gamma(n) := r(n)^2, \quad n \in \mathbb{N}.$$

But (5.7)=(3.30), hence this is also the case of Example 3.13 with $f(t) := \exp(2it)$, $t \in \mathbb{R}$. $\square$

Below we describe a frequently encountered form of the sequence a , which can be represented by some c and β as in Proposition 5.3.

Example 5.4. Fix

$$0 < \beta_1 < \dots < \beta_M < 1, \quad d_j \in \mathbb{R} \quad \text{for } j = 1, \dots, M, \quad d_M \neq 0,$$

with some $M \in \mathbb{N}$ and consider the sequence a given by

$$a_n := \sum_{j=1}^M d_j n^{\beta_j}, \quad n \in \mathbb{N}.$$

Taking $\beta := \beta_M$ and c given by $c_n := d_M + \sum_{j=1}^{M-1} \frac{d_j}{n^{(\beta - \beta_j)}}$ we see that $a_n := c_n n^\beta$ and (5.7) holds. A slightly more general form of a can be found in Remarks 5.7. 6—see (5.11).

Let us consider now a situation similar to the previous case, but with essentially different assumptions on the terms $s_{\pm}$. Namely, in Proposition 5.3 it was important that $\{p_n\}_{n \geq 1} \in \ell_*^\infty$, and here $\{p_n\}_{n \geq 1}$ is periodic, but some zero terms are not forbidden!

Proposition 5.5. *Assume that (2.2) holds for $J, \lambda \in \mathbb{C}$ and that there exist two linearly independent vectors u_+, u_- from $\text{Sol}(J, \lambda)$, which satisfy (5.1) and (5.6) with (5.8), $\epsilon_{\pm}(n) \rightarrow 0$, $c = \{c_n\}_{n \geq 1}$ being a real sequence satisfying (5.7) and with $\{p_n\}_{n \geq 1}$ being periodic, but not constantly zero sequence. Then J is self-adjoint and $\text{Sub}(J, \lambda) = \{0\}$.*

Proof. The argumentation is similar to that in Proposition 5.3 with the following important change. Instead of Theorem 4.2 we apply Theorem 4.3 with (5.4) and with ξ given by

$$\xi_n := \tau n + \nu,$$

where $\tau > 0$ is a period of the sequence p and $\nu \in \{0, \dots, \tau - 1\}$ is such, that $p_\nu \neq 0$. The use of Theorem 4.3 easily reduces the problem to the $\tilde{\gamma}$, 2-divergence of x , where

$$\tilde{\gamma}(n) := |\psi_-(\xi_n)|^2 = (\tau n + \nu)^{-2b}, \quad x(n) := \exp(2ic_{(\tau n + \nu)}(\tau n + \nu)^\beta), \quad n \in \mathbb{N}.$$

By Proposition 3.1 11. we can consider the convergence weight γ given by $\gamma(n) := n^{-2b}$, instead of $\tilde{\gamma}$. We can also express x in the form $x(n) = \exp(2ic'_n n^\beta)$, with

$$c'_n := c_{(\tau n + \nu)} \left(\frac{\tau n + \nu}{n} \right)^\beta = \tau^\beta c_{(\tau n + \nu)} \left(1 + \frac{\tau^{-1}\nu}{n} \right)^\beta.$$

Now we use again Example 3.13 with f as before, and it suffices to check that $n(\Delta c')_n \rightarrow 0$. We have

$$\begin{aligned} n(\Delta c')_n &= \tau^\beta \left(\sum_{k=0}^{\tau-1} \frac{n}{\tau n + \nu + k} (\tau n + \nu + k)(\Delta c)_{(\tau n + \nu + k)} \right) \left(1 + \frac{\tau^{-1}\nu}{n} \right)^\beta \\ &\quad + \tau^\beta c_{(\tau n + \tau + \nu)} \left[\left(1 + \frac{\tau^{-1}\nu}{n+1} \right)^\beta - \left(1 + \frac{\tau^{-1}\nu}{n} \right)^\beta \right] n, \end{aligned}$$

hence by (5.7) we obtain $n(\Delta c')_n \rightarrow 0$. □

The last case deals with a different—“logarithmic” type of the phase a . The concrete form of it is given by a function η from the class Υ (see (3.31)).

Proposition 5.6. *Assume that (2.2) holds for $J, \lambda \in \mathbb{C}$ and that there exist two linearly independent vectors u_+, u_- from $\text{Sol}(J, \lambda)$, which satisfy (5.1) with*

$$r(n) := n^{-b}, \quad a_n := \eta(\ln n + r_n^{(1)}) + r_n^{(2)}, \quad s_{\pm}(n) := p_n + \epsilon_{\pm}(n) \quad \text{for } n \text{ s.l.}, \quad (5.9)$$

where $b \leq \frac{1}{2}$, $\eta \in \Upsilon$, $r^{(1)}, r^{(2)}$ are real bounded sequences satisfying (3.32), (3.33) and $\{p_n\}_{n \geq 1} \in \ell_*^\infty$, $\epsilon_\pm(n) \rightarrow 0$. Then J is self-adjoint and $\text{Sub}(J, \lambda) = \{0\}$.

Proof. The self-adjointness follows as in Proposition 5.3. We also apply Theorem 4.2 with (5.4). It remains only to check that z is $\gamma, 2$ -divergent, with

$$z(n) = \exp(2ia_n), \quad \gamma(n) := n^{-2b}, \quad n \in \mathbb{N}.$$

But this is the case of Example 3.14 with $f(t) := \exp(2it)$, $t \in \mathbb{R}$. $\square$

We are ready now to show the ways of repairing the gaps in proofs mentioned in Introduction.

Remarks 5.7. We list here proofs to repair and we show how the gaps can be repaired via the propositions of this section. For the last case 7. we give only some suppositions or suggestions how the problem could be solved.

1. In the proof of [17, Corollary 3.2., p. 393] there is no proper explanation of non-existence of subordinate solutions. But [17, Theorem 3.1.] gives the solutions $u_\pm$ as in our Proposition 5.1 with $b = \frac{1}{4}$, $\beta = \frac{1}{2}$, $c = 2\sqrt{\lambda + 1}$ ($\lambda > -1$). Hence, this proposition solves the problem.
2. In [33, Theorem 2.3. (a), (b), (c), p. 194] the author argues for non-existence of subordinate solutions using “ $|u_+(n)| \sim |u_-(n)|$ ”, being an improper way in general (see Conjecture 1.2). But for the cases (b) and (c) of the theorem (for $\lambda < \frac{1}{2}$, $\lambda > \frac{1}{2}$, respectively) [33, Lemma 2.2.] gives (after unifying the formulae for even and odd entries) the solutions $u_\pm$ as in our Proposition 5.2 with $b = \frac{1}{4}$, $\beta = \frac{1}{2}$, and with p being a 4-periodic sequence with all the terms being non-zero. Thus the gap is filled. The case (a) of the theorem is a “non-critical” case, and the simplest way of solving the non-subordinacy problem is here to use GBS (see Introduction)—e.g. in the form of our Proposition 4.1. However, proceeding similarly as above in the cases (b), (c), it is easy to see that the (a)-gap can also be filled by our Proposition 5.3.
3. In the proof of [8, Theorem 5.3., p. 617] the author refers to [11] for non-existence of subordinate solutions. Unfortunately, this case cannot be covered by [11] (dealing only with “non-Jordan” cases). But [8, Theorem 4.1.] gives the solutions $u_\pm$ as in our Proposition 5.3 with the phase a having the form from our Example 5.4, which solves the problem.
4. The proof of [19, Theorem 5.1., p. 427] contains a gap—the second inequality in the estimate for $\|\vec{u}(n)\|$ (in the second line of the proof) is false in the considered case, and thus GBS is not applicable. But by [19, Theorem 4.1.], this gap can be easily repaired exactly as in remark 3 above.
5. In the proof of [4, Theorem 3., p. 228] the argumentation, leading to the lower estimates for $\sum_{n=1}^N |u_n|^2$ (in the penultimate line of the proof) does not explain the result, moreover the estimate seems to be not true (probably, the authors forgot about the case of u being a linear combination with both coefficients possessing the same absolute value). Using

the asymptotics from [4, Theorem 2. and 3.], and unifying the formulae for even and odd entries of both scalar solutions, one can easily see that they have the form as in Proposition 5.5 with sequence c being constant, and with p being a 4-periodic sequence possessing some non-zero terms. This fills the gap.

6. In [26, Cases 2 and 3 of Section 5, p. 124, 125] also Conjecture 1.2 is used to argue for non-existence of subordinate solutions. But the paper is mainly devoted to computations of asymptotic formulae of eigenvectors, and the details of asymptotic formula for those cases are contained in [26, Theorem 4]. After some calculations and estimates based on the fact that for any $\alpha > 0$

$$\sum_{k=1}^{n-1} \frac{1}{k^\alpha} = \int_1^n \frac{1}{t^\alpha} dt + \vartheta_\alpha(n), \quad n \in \mathbb{N}, \quad (5.10)$$

with some ϑ_α being a convergent sequence, one can check that Proposition 5.3 works also here. However, the form of the phase a from our Example 5.4 can be not general enough, because of the possible presence of the LHS term of (5.10) with $\alpha = 1$. Thus, “moving the term $\exp(\pm i\Theta(n))$ ” of the solution $u_\pm$, with Θ being the appropriate convergent sequence, to the term $s_\pm$ (see (5.1)) we can get the form:

$$a_n := d_0 \ln(n) + \sum_{j=1}^M d_j n^{\beta_j}, \quad n \in \mathbb{N}, \quad (5.11)$$

with some $M \in \mathbb{N}$ and

$$0 < \beta_1 < \dots < \beta_M < 1, \quad d_j \in \mathbb{R} \text{ for } j = 0, \dots, M, \quad d_M \neq 0.$$

Obviously, taking $\beta := \beta_M$ and c given by $c_n := d_M + \frac{d_0 \ln(n)}{n^\beta} + \sum_{j=1}^{M-1} \frac{d_j}{n^{(\beta-\beta_j)}}$ we see that $a_n := c_n n^\beta$ and (5.7) holds. Hence Proposition 5.3 can be applied.

7. The spectral results [27, Wniosek 4.8, p. 70 and Wniosek 4.12, p. 79] were also formulated assuming that Conjecture 1.2 is true. The first one is just a reformulation of the results from [26] mentioned above in remark 6, hence our methods can repair the gap. But [27, Wniosek 4.12] is more problematic. The asymptotic background for this spectral result is formulated in [27, Twierdzenie 4.11]. However, the formula [27, (4.3.44) with (4.3.45)] is rather not sufficiently explicit and detailed to allow the use of our methods. The main problem is the presence of some implicit terms “ $\mathcal{O}(\dots)$ ” in [27, (4.3.45)], which can be not l^1 terms. Such information, with only $\mathcal{O}(\dots)$ form, seems to be rather too weak to answer the question of existence of subordinate solutions on the base of propositions formulated above. We can expect that the proof of [27, Wniosek 4.12] will be possible via our methods (e.g., by our Proposition 5.6), if some more information on terms “ $\mathcal{O}(\dots)$ ” is found. Using the results of this section, sufficient information can be obtained, if some conditions stronger than [27, (4.3.15)–(4.3.19)] are assumed. An alternative solution of the problem will be probably presented soon in [28].

6. Open Problems

The following open problems related to this article seem to be interesting.

1. One can see, that all the results of Sect. 5 work only for explicit phase sequences a being “weakly increasing”, i.e., “not faster than $\mathcal{O}(n^\alpha)$ ” with $\alpha < 1$. An “intermediate case”, with $\alpha = 1$, appears e.g. in Example 3.2. But there is no result here for “strongly increasing” phase sequences a , e.g., for $a(n) = \text{const} \cdot n^\alpha$ with $\alpha > 1$. The reason is hidden in Criterion 3.11, more precisely, in its assumption (iii), which forces the condition

$$(\Delta a)(n) \longrightarrow 0.$$

It is natural to look for a similar kind of criterion working for some “strongly increasing” explicit phase sequences.

2. According to my knowledge, in the existing literature dealing with asymptotics of base vectors u_+, u_- of $\text{Sol}(\lambda)$ with the general form 5.1 (in the self-adjoint case), we can find only the “weakly increasing” phase sequences a mentioned above, if we limit ourselves to all λ -s from some “large” (e.g., open non-empty) subsets of $\mathbb{R}$. Hence the problem is: construct (by explicit formulae on weights and diagonals) a self-adjoint Jacobi operator for which similar asymptotics of base vectors u_+, u_- exist, but with the “strongly increasing” phase sequences a for all λ -s from some “large” subset of $\mathbb{R}$.

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References

- [1] Berezanskii, J.M.: Expansions in Eigenfunctions of Self-adjoint Operators. Translations of Mathematical Monographs, vol. 17. American Mathematical Society, Providence (1968)
- [2] Boos, J.: Classical and Modern Methods in Summability. Oxford Mathematical Monographs. Oxford University Press, New York (2000)
- [3] Boutet de Monvel, A., Janas, J., Naboko, S.: Unbounded Jacobi matrices with a few gaps in the essential spectrum: constructive examples. Integr. Equ. Oper. Theory **69**(2), 151–170 (2010)
- [4] Damanik, D., Naboko, S.: Unbounded Jacobi matrices at critical coupling. J. Approx. Theory **145**(2), 221–236 (2007)
- [5] Dombrowski, J., Janas, J., Moszyński, M., Pedersen, S.: Spectral gaps resulting from periodic perturbations of a class of Jacobi operators. Constr. Approx. **20**, 585–601 (2004)
- [6] Elaydi, S.N.: An Introduction to Difference Equations. Springer, New York (1999)

- [7] Gilbert, D.J., Pearson, D.B.: On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators. *J. Math. Anal. Appl.* **128**(1), 30–56 (1987)
- [8] Janas, J.: The asymptotic analysis of generalized eigenvectors of some Jacobi operators. Jordan box case. *J. Differ. Equ. Appl.* **12**(6), 597–618 (2006)
- [9] Janas, J.: Asymptotic of solutions of some linear difference equations and applications to unbounded Jacobi matrices in Operator theory live. *Theta Ser. Adv. Math.* **12**, 81–88 (2010)
- [10] Janas, J., Moszyński, M.: Alternative approaches to the absolute continuity of Jacobi matrices with monotonic weights. *Integr. Equ. Oper. Theory* **43**, 397–416 (2002)
- [11] Janas, J., Moszyński, M.: Spectral properties of Jacobi matrices by asymptotic analysis. *J. Approx. Theory* **120**, 309–336 (2003)
- [12] Janas, J., Moszyński, M.: New discrete Levinson type asymptotics of solutions of linear systems. *J. Differ. Equ. Appl.* **12**(2), 133–163 (2006)
- [13] Janas, J., Moszyński, M.: Spectral analysis of unbounded Jacobi operators with oscillating entries. *Stud. Math.* **209**(2), 107–133 (2012)
- [14] Janas, J., Naboko, S.: Jacobi matrices with power-like weights—grouping in blocks approach. *J. Funct. Anal.* **166**(2), 218–243 (1999)
- [15] Janas, J., Naboko, S.: Asymptotics of generalized eigenvectors for unbounded Jacobi matrices with power-like weights, Pauli matrices, commutation relations and Cesaro averaging. *Oper. Theory: Adv. Appl.* **117**, 165–186 (2000)
- [16] Janas, J., Naboko, S.: Multithreshold spectral phase transitions for a class of Jacobi matrices. *Oper. Theory: Adv. Appl.* **124**, 267–285 (2001)
- [17] Janas, J., Naboko, S.: Spectral properties of selfadjoint Jacobi matrices coming from birth and death processes. In: *Recent Advances in Operator Theory and Related Topics* (Szeged, 1999). *Oper. Theory Adv. Appl.*, vol. 127, 387–397. Birkhäuser, Basel (2001)
- [18] Janas, J., Naboko, S.: Spectral analysis of selfadjoint Jacobi matrices with periodically modulated entries. *J. Funct. Anal.* **191**(2), 318–342 (2002)
- [19] Janas, J., Naboko, S., Sheronova, E.: Asymptotic behaviour of generalized eigenvectors of Jacobi matrices in the critical (“double root”) case. *Z. Anal. Anwend.* **28**(4), 411–430 (2009)
- [20] Janas, J., Naboko, S., Stolz, G.: Spectral theory for a class of periodically perturbed unbounded Jacobi matrices: elementary methods. *J. Comput. Appl. Math.* **171**(1-2), 265–276 (2004)
- [21] Janas, J., Naboko, S., Stolz, G.: Decay bounds on eigenfunctions and the singular spectrum of unbounded Jacobi matrices. *Int. Math. Res. Not.* **29**(4), 736–764 (2009)
- [22] Khan, S., Pearson, D.B.: Subordinacy and spectral theory for infinite matrices. *Helv. Phys. Acta* **65**(4), 505–527 (1992)
- [23] Moszyński, M.: Spectral properties of some Jacobi matrices with double weights. *J. Math. Anal. Appl.* **280**, 400–412 (2003)
- [24] Moszyński, M.: Slowly oscillating perturbations of periodic Jacobi operators in $\ell^2(\mathbb{N})$. *Stud. Math.* **192**(3), 259–279 (2009)
- [25] Motyka, W.: Absolute continuity for Jacobi matrices with power-like weights. *Colloq. Math.* **107**(2), 179–190 (2007)

- [26] Motyka, W.: The asymptotic analysis of a class of self-adjoint second-order difference equations: Jordan box case. *Glasgow Math. J.* **51**, 109–125 (2009)
- [27] Motyka, W.: Analiza spektralna wybranych klas nieograniczonych macierzy Jacobiego. unpublished work (2010, in Polish)
- [28] Motyka, W.: Selfadjoint Jacobi matrices in the double root case. Preprint (in preparation)
- [29] Naboko, S., Pchelintseva, I., Silva, L.O.: Discrete spectrum in a critical coupling case of Jacobi matrices with spectral phase transitions by uniform asymptotic analysis. *J. Approx. Theory* **161**(1), 314–336 (2009)
- [30] Naboko, S., Simonov, S.: Spectral analysis of a class of Hermitian Jacobi matrices in a critical (double root) hyperbolic case. *Proc. Edinb. Math. Soc.* (2) **53**(1), 239–254 (2010)
- [31] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics, vol. I.* Academic Press, New York-London (1972)
- [32] Silva, L.O.: Uniform and smooth Benzaid–Lutz type theorems and applications to Jacobi matrices. In: *Operator Theory, Analysis and Mathematical Physics. Operator Theory: Advances and Applications*, vol. 174, pp. 173–186. Birkhäuser, Boston (2007)
- [33] Simonov, S.: An example of spectral phase transition phenomenon in a class of Jacobi matrices with periodically modulated weights. In: *Operator Theory, Analysis and Mathematical Physics. Operator Theory: Advances and Applications*, vol. 174, pp. 187–203. Birkhäuser, Boston (2007)
- [34] Stolz, G.: Spectral theory for slowly oscillating potentials. I. Jacobi matrices. *Manuscr. Math.* **84**(3–4), 245–260 (1994)

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