Generalized Absolute Values and Polar Decompositions of a Bounded Operator

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Abstract. Generalized absolute values as well as corresponding to them generalized polar decompositions of a bounded linear operator T of a Hilbert space $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ are defined, motivated by the inequality $|\langle Tx, y \rangle_{\mathcal{K}}|^2 \leq \langle |T|x, x \rangle_{\mathcal{H}} \langle |T^*|y, y \rangle_{\mathcal{K}}$. It is shown that there is a natural bijection between generalized absolute values of T and of T^* which sends $|T|$ to $|T^*|$. For a bounded nonnegative operator A on $\mathcal{H}$ and a bounded Borel function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, equivalent conditions for A and $f(|T|)$ to be generalized absolute values of T are established and corresponding to them generalized absolute values of T^* are determined.

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1. Introduction

The polar decomposition

$$T = Q|T| \quad (1)$$

(where $T = \sqrt{T^*T}$ and Q is a partial isometry with $\mathcal{N}(Q) = \mathcal{N}(T)$) of a bounded linear operator $T: \mathcal{H} \rightarrow \mathcal{K}$ acting between Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ is a very useful tool in theory of Hilbert space operators. There are also other classical representations of T , closely related to the latter, namely

$$T = |T^*|Q \quad (2)$$

and

$$T = \sqrt{|T^*|}Q\sqrt{|T|} \quad (3)$$

(or, more generally, $T = |T^*|^p Q |T|^{1-p}$ for every $p \in (0, 1)$; see e.g. [7, Theorem 2.7]). However, (3) is a consequence of (1) and (2), and the intertwining rule for the squares of nonnegative operators (that is, $Q|T| = |T^*|Q$ and

therefore $Q\sqrt{|T|} = \sqrt{|T^*|}Q$; cf. [7, Remark 2.8]). One may ask whether it is just a coincidence that these three connections hold true. Or, is there any aspect, other than mutual ‘duality’ of their formulas, which joins $|T|$ and $|T^*|$ in such a way they form a specific pair associated with T ? Whatever it was, every such an aspect would enable to look for similar pairs of operators and to collect properties common for all such pairs, which could explain some ‘unexpected’ or ‘inexplicable’ features known for the pair $(|T|, |T^*|)$.

In the present paper we concentrate on the following inequality

$$|\langle Tx, y \rangle_{\mathcal{K}}|^2 \leq \langle |T|x, x \rangle_{\mathcal{H}} \langle |T^*|y, y \rangle_{\mathcal{K}} \quad (4)$$

(satisfied for every $x \in \mathcal{H}$ and $y \in \mathcal{K}$). It turns out that if (4) is fulfilled when $|T|$ is replaced by a nonnegative bounded operator A on $\mathcal{H}$, then $A \geq |T|$ (this will be shown in the sequel) and, similarly, if (4) is fulfilled when $|T^*|$ is replaced by a nonnegative bounded operator B on $\mathcal{K}$, then $B \geq |T^*|$. This means that the pair $(|T|, |T^*|)$ is minimal in the set $\mathcal{S}(T)$ in the sense of the following

Definition 1.1. Let $\mathcal{S}(T)$ consist of all pairs (A, B) such that $A: \mathcal{H} \rightarrow \mathcal{H}$ and $B: \mathcal{K} \rightarrow \mathcal{K}$ are bounded and nonnegative operators and satisfy the inequality

$$|\langle Tx, y \rangle_{\mathcal{K}}|^2 \leq \langle Ax, x \rangle_{\mathcal{H}} \langle By, y \rangle_{\mathcal{K}} \quad (5)$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. We equip $\mathcal{S}(T)$ with the coordinatewise (partial) order:

$$(A, B) \leq (C, D) \iff A \leq C \text{ and } B \leq D.$$

A pair $(A, B) \in \mathcal{S}(T)$ is said to be *minimal* provided there is no pair $(A', B') \in \mathcal{S}(T) \setminus \{(A, B)\}$ such that $(A', B') \leq (A, B)$.

The observation preceding the above definition leads to a generalization of the notion of the absolute value of T . Namely,

Definition 1.2. Whenever $(C, D) \in \mathcal{S}(T)$, C and D are said to be a *T -bound* and a *T^* -bound*, respectively. An operator A is said to be a *generalized absolute value of T* if there is a minimal pair in $\mathcal{S}(T)$ whose first entry is A . The set of all generalized absolute values of T is denoted by $\text{GAV}(T)$.

The reader should notice that $\mathcal{S}(T^*) = \{(B, A): (A, B) \in \mathcal{S}(T)\}$ and that if (A, B) is minimal in $\mathcal{S}(T)$, then $A \in \text{GAV}(T)$ as well as $B \in \text{GAV}(T^*)$ (because (B, A) is a minimal pair in $\mathcal{S}(T^*)$).

The following result is rather surprising

Theorem 1.3. *For every T -bound A there is a (necessarily unique) operator $A^\# = A_T^\#$ such that $(A, A_T^\#) \in \mathcal{S}(T)$ and $A_T^\# \leq B$ for each operator B such that $(A, B) \in \mathcal{S}(T)$. What is more, $A_T^\# = S^*S$ where*

$$S = (\sqrt{A} \Big|_{\overline{\mathcal{R}(A)}})^{-1} \circ T^* \in \mathcal{B}(\mathcal{K}, \mathcal{H}). \quad (6)$$

It follows from the above theorem that for every T^* -bound B there is a least operator $^\#B (= B_{T^*}^\#)$ among all C with $(C, B) \in \mathcal{S}(T)$.

Theorem 1.3 is useful in producing generalized absolute values, as it is shown by

- Proposition 1.4.** (A) Let A be a T -bound and $C = \#(A^\#)$. Then C is a generalized absolute value of T and $C^\# = A^\#$. In particular, $A^\# \in \text{GAV}(T^*)$, $(C, C^\#)$ is a minimal member of $\mathcal{S}(T)$ and $(C, C^\#) \leq (A, B)$ whenever $(A, B) \in \mathcal{S}(T)$.
- (B) The functions $\text{GAV}(T) \ni X \mapsto X^\# \in \text{GAV}(T^*)$ and $\text{GAV}(T^*) \ni Y \mapsto \#Y \in \text{GAV}(T)$ are mutually inverse bijections and the set of all minimal pairs in $\mathcal{S}(T)$ coincides with $\{(A, A^\#): A \in \text{GAV}(T)\}$.

The (standard) absolute value of an operator plays a fundamental role in the polar decomposition. It turns out that the latter representation may be generalized to the context of arbitrary generalized absolute values:

Theorem 1.5. For every generalized absolute value A of T there is a unique partial isometry Q_A of $\mathcal{H}$ into $\mathcal{K}$ such that

$$\mathcal{N}(Q_A) = \mathcal{N}(A), \quad \mathcal{N}(Q_A^*) = \mathcal{N}(A^\#) \quad (7)$$

and

$$T = \sqrt{A^\#} Q_A \sqrt{A}. \quad (8)$$

Definition 1.6. The representation (8) (with partial isometry Q_A satisfying (7)) is called the *generalized polar decomposition of T corresponding to A* .

With this approach the representation (3) seems to be the main one as corresponding to $|T|$ (while (1) and (2) correspond to rather odd generalized absolute values of T , namely T^*T and the orthogonal projection onto the closure of the range of T , respectively). It also turns out that the relation (8) with Q_A being a partial isometry satisfying (7) characterizes generalized absolute values:

- Proposition 1.7.** (A) An operator $A \in \mathcal{B}_+(\mathcal{H})$ is a generalized absolute value of T if and only if $\mathcal{R}(T^*) \subset \mathcal{R}(\sqrt{A})$ and the inverse image of $\mathcal{R}(T^*)$ under $\sqrt{A}$ is dense in $\mathcal{H}$.
- (B) For any $A \in \mathcal{B}_+(\mathcal{H})$ and $B \in \mathcal{B}_+(\mathcal{K})$ the following conditions are equivalent:
- $A \in \text{GAV}(T)$ and $B = A^\#$,
 - there is a partial isometry $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\mathcal{N}(V) = \mathcal{N}(A)$, $\mathcal{N}(V^*) = \mathcal{N}(B)$ and $T = \sqrt{B}V\sqrt{A}$.

It seems to be interesting to know when $Q_A = Q_{|T|}$, i.e. when the partial isometry (Q_A) appearing in the generalized polar decomposition of T corresponding to $A \in \text{GAV}(T)$ coincides with the partial isometry appearing in the standard polar decomposition (1). The full answer to this problem is contained in

Theorem 1.8. For $A \in \text{GAV}(T)$ the following conditions are equivalent:

- $Q_A = Q$,
- $A|T| = |T|A$,
- $A^\#|T^*| = |T^*|A^\#$.

With use of the generalized absolute values one may generalize other notions, e.g. of normal operators: a bounded operator $N: \mathcal{H} \rightarrow \mathcal{H}$ is a *generalized normal operator* if and only if there is a generalized absolute value A of N such that $A^\# = A$ (because N is normal if and only if $|N| = |N^*|$ and always $|N|^\# = |N^*|$). However, it is not of our interest.

The paper is organized as follows. Section 2 deals with T -bounds and generalized absolute values. In that part we prove Theorem 1.3, Proposition 1.4 and point (A) of Proposition 1.7. In Section 3 generalized polar decompositions are discussed and Theorem 1.5 and point (B) of Proposition 1.7 are proved. The last part discusses generalized absolute values A for which $Q_A = Q_{|T|}$. It contains the proof of Theorem 1.8. As a special case of these investigations we obtain the formula for $[f(T^*T)]^\#$ for certain bounded Borel functions $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$, which is closely related to [7, Theorem 2.7].

Notation. In this paper $\mathcal{H}$ and $\mathcal{K}$ are complex Hilbert spaces and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ denote their inner products. By an *operator* we mean a linear function between Hilbert spaces. All considered operators are bounded. $\mathcal{B}(\mathcal{H}, \mathcal{K})$ is the Banach space of all bounded operators of $\mathcal{H}$ into $\mathcal{K}$, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{B}_+(\mathcal{H})$ is the set of all nonnegative members of $\mathcal{B}(\mathcal{H})$. Whenever T is an operator of $\mathcal{H}$ into $\mathcal{K}$, $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\overline{\mathcal{R}}(T)$ stand for, respectively, the kernel, the image and the closure of the image of T . The *polar decomposition* of T has the form $T = Q|T|$ where $|T| = \sqrt{T^*T}$ and Q is a unique partial isometry of $\mathcal{H}$ into $\mathcal{K}$ such that $\mathcal{N}(Q) = \mathcal{N}(T)$. Every operator of norm no greater than 1 is called a *contraction*. For two selfadjoint operators A and B on the Hilbert space $\mathcal{H}$, we write $A \leq B$ if and only if $\langle Ax, x \rangle_{\mathcal{H}} \leq \langle Bx, x \rangle_{\mathcal{H}}$ for every $x \in \mathcal{H}$. The set of all nonnegative real numbers is denoted by $\mathbb{R}_+$.

For basic facts on Hilbert spaces and Hilbert space operators the reader is referred to any textbook on these subjects, e.g. [1–3, 5, 8–10] or [11].

2. Generalized Absolute Values

From now on, T is a fixed bounded operator of $\mathcal{H}$ into $\mathcal{K}$.

One may easily show, using Zorn's lemma, that for every pair (A, B) of $\mathcal{S}(T)$ there is at least one minimal pair (C, D) in $\mathcal{S}(T)$ such that $(C, D) \leq (A, B)$. In the sequel we shall prove this with no use of axiom of choice and we shall give an explicit formula for a minimal element less than a fixed member of $\mathcal{S}(T)$.

We begin with a characterization of T -bounds. The next result may be seen as a variation of the well known result of Douglas [4] (see also [6, Theorem 2.1]) and thus the proof is omitted.

Proposition 2.1. *For an operator $A \in \mathcal{B}_+(\mathcal{H})$ the following conditions are equivalent:*

- (a) A is a T -bound,
- (b) there is a positive real constant M such that $\|Tx\|_{\mathcal{K}}^2 \leq M \langle Ax, x \rangle_{\mathcal{H}}$ for every $x \in \mathcal{H}$,
- (c) there is a positive real constant c such that $T^*T \leq cA$,
- (d) $T = C\sqrt{A}$ for some $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$,

(e) $\mathcal{R}(T^*) \subset \mathcal{R}(\sqrt{A})$.

We are mainly interested in generalized absolute values of T . The main tool for investigating them is Theorem 1.3 which we now prove.

Proof of Theorem 1.3. Suppose $B \in \mathcal{B}_+(\mathcal{K})$. We want to know when $(A, B) \in \mathcal{S}(T)$. For this, fix $y \in \mathcal{K}$. By Proposition 2.1(e), $\mathcal{N}(A) = \mathcal{N}(\sqrt{A}) \subset \mathcal{N}(T)$ and hence if $x_0 \in \mathcal{N}(A)$ and $x_1 \in \mathcal{N}(A)^\perp$, then $T(x_0 + x_1) = T(x_1)$ and $\langle A(x_0 + x_1), x_0 + x_1 \rangle_{\mathcal{H}} = \langle Ax_1, x_1 \rangle_{\mathcal{H}}$. This yields that (5) will be satisfied for every $x \in \mathcal{H}$ if and only if it will be such for each $x \in E := \mathcal{N}(A)^\perp = \overline{\mathcal{R}(A)}$. So, we must have

$$\langle By, y \rangle_{\mathcal{K}} \geq \sup \left\{ \frac{|\langle Tx, y \rangle_{\mathcal{K}}|^2}{\|\sqrt{A}x\|_{\mathcal{H}}^2} : x \in E_* \right\}$$

where $E_* = E \setminus \{0\}$. Observe that $\sqrt{A}|_E$ is one-to-one and $\mathcal{R}(T^*) \subset \sqrt{A}(E)$ (by Proposition 2.1(e)) and thus the operator S given by (6) is well defined. Its boundedness follows from the Closed Graph Theorem. Notice that $\sqrt{A}S = T^*$. We infer from this that $\langle Tx, y \rangle_{\mathcal{K}} = \langle x, \sqrt{A}Sy \rangle_{\mathcal{H}} = \langle \sqrt{A}x, Sy \rangle_{\mathcal{H}}$. This, combined with the facts that S takes values in E and $\sqrt{A}(E)$ is dense in E , yields

$$\begin{aligned} \sup \left\{ \frac{|\langle Tx, y \rangle_{\mathcal{K}}|^2}{\|\sqrt{A}x\|_{\mathcal{H}}^2} : x \in E_* \right\} &= \sup \left\{ \left| \left\langle \frac{\sqrt{A}x}{\|\sqrt{A}x\|_{\mathcal{H}}}, Sy \right\rangle_{\mathcal{H}} \right|^2 : x \in E_* \right\} \\ &= \|Sy\|_{\mathcal{H}}^2 = \langle S^*Sy, y \rangle_{\mathcal{H}} \end{aligned}$$

which finishes the proof. $\square$

As a consequence of Theorem 1.3 we get Proposition 1.4:

Proof of Proposition 1.4. Let A and C be as in point (A) of the proposition. Since $(A, A^\#)$, $(C, A^\#)$ and $(C, C^\#)$ belong to $\mathcal{S}(T)$, we infer from the definition of C (and definitions of $X^\#$ and $^\#Y$ for T -bounds X and T^* -bounds Y) that $C \leq A$ and $C^\# \leq A^\#$. Consequently, $(A, C^\#) \in \mathcal{S}(T)$ (because $(C, C^\#) \in \mathcal{S}(T)$) and thus $A^\# \leq C^\#$ as well. This shows that $C^\# = A^\#$. Let us now show that $(C, C^\#)$ is minimal in $\mathcal{S}(T)$. Suppose $(X, Y) \in \mathcal{S}(T)$ is such that $X \leq C$ and $Y \leq C^\#$. Then $(C, Y) \in \mathcal{S}(T)$ and hence $C^\# \leq Y$. So, $Y = C^\# = A^\#$ and consequently $(X, A^\#) \in \mathcal{S}(T)$ which implies that $C \leq X$. This proves minimality of $(C, C^\#)$. Thus, $C \in \text{GAV}(T)$ and $A^\# = C^\# \in \text{GAV}(T^*)$. Finally, if $(A, B) \in \mathcal{S}(T)$, then $A^\# \leq B$ and therefore $(C, C^\#) \leq (A, B)$, which finishes the proof of (A).

We now pass to (B). The above argument shows that $^\#(A^\#) \leq A$ for every T -bound A . Now if $A \in \text{GAV}(T)$, there is B such that (A, B) is a minimal pair in $\mathcal{S}(T)$. But then $(^\#(A^\#), A^\#) \in \mathcal{S}(T)$ and $(^\#(A^\#), A^\#) \leq (A, B)$ which yields $A = ^\#(A^\#)$. Similarly, $(^\#B)^\# = B$ for every $B \in \text{GAV}(T^*)$. In particular, we conclude from (A) that $(A, A^\#)$ is minimal in $\mathcal{S}(T)$ for every $A \in \text{GAV}(A)$. Conversely, if (A, B) is a minimal pair in $\mathcal{S}(T)$, then $A \in \text{GAV}(T)$ (by definition) and $B = A^\#$ (by Theorem 1.3 and minimality of (A, B)). $\square$

Observe that the function Φ from the set of all T -bounds onto $\text{GAV}(T)$ given by $\Phi(A) = \#(A^\#)$ is a surjection such that $\Phi(A) = A$ for every $A \in \text{GAV}(T)$. It turns out that

Theorem 2.2. *For every T -bound A ,*

$$\#(A^\#) = \sqrt{A}P\sqrt{A}$$

where P is the orthogonal projection onto the closure of $\{x \in \overline{\mathcal{R}}(A) : \sqrt{A}x \in \mathcal{R}(T^*)\}$.

Proof. Let S be given by (6) and let $S = V|S|$ be the polar decomposition of S . Denote by E the range of P . Observe that

$$\overline{\mathcal{R}}(S) = \mathcal{R}(V) = E. \quad (9)$$

By Theorem 1.3, $A^\# = S^*S$ and thus

$$\sqrt{A^\#} = |S| = V^*S. \quad (10)$$

Further, we infer from (6) that $\mathcal{N}(S) = \mathcal{N}(T^*)$ and hence

$$\overline{\mathcal{R}}(S^*) = \mathcal{R}(V^*) = \overline{\mathcal{R}}(|S|) = \overline{\mathcal{R}}(A^\#) = \overline{\mathcal{R}}(T). \quad (11)$$

In order to compute $\#(A^\#)$ we have to apply Theorem 1.3 with T and A replaced by T^* and $A^\#$. So, $\#(A^\#) = L^*L$ where

$$L = (\sqrt{A^\#} \Big|_{\overline{\mathcal{R}}(A^\#)})^{-1} \circ T \in \mathcal{B}(\mathcal{H}, \mathcal{K}).$$

It follows from (10), (11) and the definition of S that

$$(\sqrt{A^\#} \Big|_{\overline{\mathcal{R}}(A^\#)})^{-1} = (T^* \Big|_{\overline{\mathcal{R}}(T)})^{-1} \circ \sqrt{A}V \Big|_{\mathcal{R}(S^*)}$$

which yields

$$L = (T^* \Big|_{\overline{\mathcal{R}}(T)})^{-1} \circ \sqrt{A}VT. \quad (12)$$

But from (6) it follows that $\sqrt{A}S = T^*$ and therefore $\sqrt{A}VT = \sqrt{A}VS^*\sqrt{A} = \sqrt{A}|S^*|\sqrt{A} = \sqrt{A}SV^*\sqrt{A} = T^*V^*\sqrt{A}$. So, if P_0 is the orthogonal projection onto $\overline{\mathcal{R}}(T)$, the latter calculation and (12) give $L = P_0V^*\sqrt{A}$ and finally

$$\#(A^\#) = \sqrt{A}VP_0V^*\sqrt{A} = \sqrt{A}VV^*\sqrt{A} = \sqrt{A}P\sqrt{A},$$

because $\mathcal{R}(V^*) = \mathcal{R}(P_0)$ (by (11)) and $VV^* = P$ by (9). $\square$

Proof of point (A) of Proposition 1.7. First note that a T -bound A is a generalized absolute value of T if and only if $\#(A^\#) = A$ and that the inverse image of $\mathcal{R}(T^*)$ under $\sqrt{A}$ is dense in $\mathcal{H}$ if and only if the closure E of $\{x \in \overline{\mathcal{R}}(A) : \sqrt{A}x \in \mathcal{R}(T^*)\}$ coincides with $\overline{\mathcal{R}}(A)$. Hence the assertion follows from Proposition 2.1(e), Theorem 2.2 and the fact that the function $X \mapsto \sqrt{A}X\sqrt{A}$ is an injection on the set of all selfadjoint operators X such that $\mathcal{N}(A) \subset \mathcal{N}(X)$. $\square$

3. Generalized Polar Decompositions

From now on to the end of the paper, Q stands for the partial isometry appearing in the polar decomposition (1) of T .

The following is quite an easy analog of Douglas result [4] and therefore the proof is left as an exercise.

Proposition 3.1. *For $A \in \mathcal{B}_+(\mathcal{H})$ and $B \in \mathcal{B}_+(\mathcal{K})$ the following conditions are equivalent:*

- (a) $(A, B) \in \mathcal{S}(T)$,
- (b) *there is a contraction $K \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that*

$$T = \sqrt{B}K\sqrt{A}. \quad (13)$$

What is more, if $(A, B) \in \mathcal{S}(T)$, there is a unique contraction K satisfying (13) and such that $\mathcal{N}(A) \subset \mathcal{N}(K)$ and $\mathcal{N}(B) \subset \mathcal{N}(K^)$.*

Now for $(A, B) \in \mathcal{S}(T)$ we denote by $K_{A,B} = K_{A,B}^T$ the unique contraction such that $\mathcal{N}(A) \subset \mathcal{N}(K_{A,B})$, $\mathcal{N}(B) \subset \mathcal{N}(K_{A,B}^*)$ and $T = \sqrt{B}K_{A,B}\sqrt{A}$.

For every T -bound A let $Q_A = K_{A,A^\#}$. We now have

Proposition 3.2. *For every T -bound A the contraction Q_A is a partial isometry with $\mathcal{N}(Q_A^*) = \mathcal{N}(A^\#) = \mathcal{N}(T^*)$. If A is a generalized absolute value of T , then $\mathcal{N}(Q_A) = \mathcal{N}(A) = \mathcal{N}(T)$.*

Proof. Let S be given by (6) and let $S = V|S|$ be the polar decomposition of S . Note that $\mathcal{N}(V) = \mathcal{N}(S) = \mathcal{N}(T^*)$ and $\mathcal{N}(V^*) \supset \mathcal{N}(A)$ (because $\mathcal{R}(V) \subset \overline{\mathcal{R}(A)}$). We claim that $Q_A = V^*$. Indeed, $\sqrt{A}S = T^*$ (thanks to (6)), $\sqrt{A^\#} = |S|$ and hence $T = \sqrt{A^\#}V^*\sqrt{A}$. The uniqueness of $K_{A,A^\#}$ implies that in fact $Q_A = V^*$. This proves the first assertion. To show the remainder, just apply Proposition 1.4 and the first claim of the proposition for the T^* -bound $A^\#$ (with T^* instead of T). $\square$

Proof of Theorem 1.5. Just apply Propositions 3.1 and 3.2. $\square$

Proof of point (B) of Proposition 1.7. We only need to show that (i) is implied by (ii). By Proposition 3.1, $(A, B) \in \mathcal{S}(T)$. We conclude from the assumptions of (ii) that $\mathcal{R}(V) = \overline{\mathcal{R}(B)}$ and $\mathcal{R}(V^*) = \overline{\mathcal{R}(A)}$ and hence $V^*V\sqrt{A} = \sqrt{A}$ and $VV^*\sqrt{B} = \sqrt{B}$. Now if S is given by (6), then $S = V^*\sqrt{B}$ and thus $A^\# = S^*S = B$. The same argument shows that $^\#B = A$ and we are done. $\square$

Now Proposition 1.7 and (3) yield

Corollary 3.3. *$|T|$ is a generalized absolute value of T , $|T|^\# = |T^*|$ and $Q_{|T|} = Q$.*

Let P and P_* denote the orthogonal projections onto $\overline{\mathcal{R}(T)}$ and $\overline{\mathcal{R}(T^*)}$, respectively. We infer from Proposition 1.7 and (1) that $T^*T \in \text{GAV}(T)$ and $(T^*T)^\# = P$. Similarly, from (2) it follows that $P_* \in \text{GAV}(T)$ and $(P_*)^\# = TT^*$. P_* is the most elementary operator which is a generalized absolute value of T . What is more, P_* has a closed range. As a consequence

of (8) we obtain a general result on generalized absolute values with closed ranges:

Proposition 3.4. *Let $A \in \text{GAV}(T)$.*

- (i) $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(\sqrt{A^\#}) = \mathcal{R}(T)$,
- (ii) $\mathcal{R}(A^\#)$ is closed if and only if $\mathcal{R}(\sqrt{A}) = \mathcal{R}(T^*)$.

Proof. By symmetry (thanks to Proposition 1.4), we only need to show (i). We deduce from (8) and (7) that $\mathcal{R}(T) = \sqrt{A^\#}(Q_A(\mathcal{R}(A)))$ and Q_A is a partial isometry with initial and final spaces $\overline{\mathcal{R}(A)}$ and $\overline{\mathcal{R}(A^\#)}$ (respectively). This gives the assertion, since for any $E \subset \overline{\mathcal{R}(A^\#)}$ one has: $\sqrt{A^\#}(E) = \mathcal{R}(\sqrt{A^\#}) \iff E = \overline{\mathcal{R}(A^\#)}$. $\square$

Remark 3.5. Generalized polar decompositions may be produced in the following way, starting from an arbitrary T -bound A . Apply Proposition 2.1(d) and take a (unique) operator $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $T = C\sqrt{A}$ and $\mathcal{N}(A) \subset \mathcal{N}(C)$. Let $C = |C^*|V$ be the polar decomposition of C . Then $C = |C^*|V$ and thus $T = |C^*|V\sqrt{A}$. Finally, put $D = V\sqrt{A}$ and consider the polar decomposition $D = W|D|$ of D . One may show that $A^\# = CC^*$, $\#(A^\#) = D^*D$ and $Q_{\#(A^\#)} = W$. Hence the representation $T = |C^*|W|D|$ is the generalized polar decomposition of T corresponding to $\#(A^\#)$.

4. The Case ‘ $Q_A = Q$ ’

In this part we investigate those generalized absolute values A of T for which $Q_A = Q$ (recall that $Q = Q_{|T|}$).

Proof of Theorem 1.8. It suffices to prove the equivalence of (a) and (b), since $T^* = \sqrt{A}Q_A^*\sqrt{A^\#}$ is the generalized polar decomposition of T^* corresponding to $A^\#$.

If $Q_A = Q$, then for every $x \in \mathcal{H}$ we get

$$\begin{aligned} \langle \sqrt{A}|T|x, x \rangle_{\mathcal{H}} &= \langle |T|x, \sqrt{A}x \rangle_{\mathcal{H}} = \langle Q|T|x, Q\sqrt{A}x \rangle_{\mathcal{K}} \\ &= \langle Tx, Q\sqrt{A}x \rangle_{\mathcal{K}} = \langle \sqrt{A^\#}(Q\sqrt{A}x), Q\sqrt{A}x \rangle_{\mathcal{K}} \geq 0 \end{aligned}$$

which shows that $\sqrt{A}|T|$ is selfadjoint and therefore $\sqrt{A}$ commutes with $|T|$. This yields (b).

Now suppose that A commutes with $|T|$. Then

$$\sqrt{A}|T| = |T|\sqrt{A} \tag{14}$$

as well. By the uniqueness of Q_A , it suffices to check that $T = \sqrt{A^\#}Q\sqrt{A}$. By Proposition 2.1(e), $\mathcal{R}(|T|) = \mathcal{R}(T^*) \subset \mathcal{R}(\sqrt{A})$ and thus the operator $C = (\sqrt{A}|_{\overline{\mathcal{R}(A)}})^{-1} \circ |T|$ is well defined and bounded. Moreover, by (14), $C \in \mathcal{B}_+(\mathcal{H})$. Now let, as usual, S be given by (6). Note that $S = CQ^*$, since $T^* = |T|Q^*$, and that $\mathcal{N}(Q) = \mathcal{N}(|T|) = \mathcal{N}(C)$. So, $S^* = QC$ is the polar decomposition of S^* . We infer from this that $S = Q^*|S|$. Finally we have $T = (\sqrt{A}S)^* = S^*\sqrt{A} = |S|Q\sqrt{A} = \sqrt{A^\#}Q\sqrt{A}$. $\square$

A very special case of the situation ‘ $Q_A = Q$ ’ appears when $A = f(|T|)$ for some bounded Borel function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$. First we give a simple characterization of generalized absolute values of T of this type.

Proposition 4.1. *Let $A \in \mathcal{B}_+(\mathcal{H})$ be such that $\mathcal{R}(\sqrt{A}) = \mathcal{R}(T^*)$ and let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bounded Borel function. The operator $f(A)$ is a generalized absolute value of T if and only if there is a bounded Borel function $\hat{f}: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\hat{f}^{-1}(\{0\}) = \{0\}$, $\sup_{t>0} \frac{t}{\hat{f}(t)} < +\infty$ and f and $\hat{f}$ are equal almost everywhere with respect to the spectral measure of A (that is, $\hat{f}(A) = f(A)$).*

Proof. First observe that if $f(A) \in \text{GAV}(T)$, then, thanks to Proposition 3.2 and Proposition 2.1(e),

$$\mathcal{N}(f(A)) = \mathcal{N}(A) \quad (15)$$

and $\mathcal{R}(\sqrt{A}) = \mathcal{R}(T^*) \subset \mathcal{R}(\sqrt{f(A)})$. The latter inclusion implies, by Douglas’ result [4] (cf. [6, Theorem 2.1]), that

$$A \leq c f(A) \quad (16)$$

for some constant $c \in \mathbb{R}_+$. It is now easily inferred from (15) and (16) that a suitable function $\hat{f}$ indeed exists.

To prove the converse implication, we may assume that $f = \hat{f}$. Let $f^\#: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by

$$f^\#(0) = 0 \quad \text{and} \quad f^\#(t) = \frac{t}{f(t)} \quad \text{for } t > 0. \quad (17)$$

Note that $f^\#$ is bounded and Borel. Further, let P be the orthogonal projection onto $\overline{\mathcal{R}(A)}$. Since $f^{-1}(\{0\}) = (f^\#)^{-1}(\{0\}) = \{0\}$, we get $\mathcal{N}(f(A)) = \mathcal{N}(f^\#(A)) = \mathcal{N}(P) = \mathcal{N}(A)$. We also have $\sqrt{A} = \sqrt{f^\#(A)}P\sqrt{f(A)}$ (because $f(t) \cdot f^\#(t) = t$ for every $t \in \mathbb{R}_+$). This yields that $f(A) \in \text{GAV}(\sqrt{A})$, thanks to Proposition 1.7. But it follows from point (A) of the latter result that $\text{GAV}(\sqrt{A}) = \text{GAV}(T)$ (because $\mathcal{R}(\sqrt{A}) = \mathcal{R}(T^*)$) and we are done. $\square$

We end the paper with the following

Theorem 4.2. *Let $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a bounded Borel function with $f^{-1}(\{0\}) = \{0\}$ and $\sup_{t>0} \frac{t}{f(t)} < +\infty$. Let $f^\#: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by (17). Then $f(T^*T) \in \text{GAV}(T)$ and $[f(T^*T)]^\# = f^\#(TT^*)$.*

Proof. We conclude from Proposition 4.1 that $f(T^*T) \in \text{GAV}(T)$. The remainder follows from the well known decomposition

$$T = \sqrt{f^\#(TT^*)}Q\sqrt{f(T^*T)} \quad (18)$$

(see e.g. [7, Theorem 2.7]; (18) may also be concluded from the relation $|T^*|Q = Q|T|$ and the well known result on intertwining between normal operators: if N and M are normal and $NX = XM$, then $g(N)X = Xg(M)$ for every Borel function $g: \mathbb{C} \rightarrow \mathbb{C}$ and Proposition 1.7. $\square$

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