Erratum

Erratum to: On the Dunford Property (C) for Bounded Linear Operators RS and SR

Pietro Aiena and Manuel González

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In [1, Theorem 3.3] it has been proved the following theorem:

Theorem 0.1. *Let $S, R \in L(X)$ be such that $RSR = R^2$ and $SRS = S^2$. Then any one of R, S, RS , and SR has property (C) (respectively, property (Q)) implies they all have property (C) (respectively, property (Q)).*

The proof of this theorem relies on [1, Lemma 3.1], where for $x \in X$ the following inclusions of local spectra $\sigma_{SR}(SRx)$ and $\sigma_R(x)$ have been proved.

- (i) $\sigma_{SR}(SRx) \subseteq \sigma_R(x) \subseteq \sigma_{SR}(SRx) \cup \{0\}$.
- (ii) $\sigma_{RS}(RSx) \subseteq \sigma_S(x) \subseteq \sigma_{RS}(RSx) \cup \{0\}$.

Unfortunately, the proof of the second inclusions in (i) and (ii) is incorrect. However, the result of Theorem 0.1 is correct and a proof of it may be obtained by replacing Lemma 3.1 of [1] by the following lemma.

Lemma 0.2. *Let $S, R \in L(X)$ be such that $RSR = R^2$ and $SRS = S^2$. For each $x \in X$ we have*

- (i) $\sigma_R(R^2x) \subseteq \sigma_{SR}(SRx) \subseteq \sigma_R(x)$.
- (ii) $\sigma_S(S^2x) \subseteq \sigma_{RS}(RSx) \subseteq \sigma_S(x)$.

Proof. (i) We have only to show $\sigma_R(R^2x) \subseteq \sigma_{SR}(SRx)$. Suppose that $\lambda_0 \notin \sigma_{SR}(SRx)$. Then there is a neighborhood U_0 of λ_0 and an analytic function $f : U_0 \rightarrow X$ such that $(\lambda I - SR)f(\lambda) = SRx$ for all $\lambda \in U_0$. We have

$$\begin{aligned} R(\lambda I - SR)f(\lambda) &= (\lambda R - RSR)(f(\lambda)) = (\lambda R - R^2)f(\lambda) \\ &= (\lambda I - R)Rf(\lambda) = RSRx = R^2x, \end{aligned}$$

for all $\lambda \in U_0$. Thus $\lambda_0 \in \rho_R(R^2x)$ and hence $\lambda_0 \notin \sigma_R(R^2x)$.

The proof of the first inclusion in (ii) is similar. □

We correct now the proof Theorem 0.1.

Proof of Theorem 0.1. The equivalence of property (C) for RS and SR has been proved in corollary [1, Corollary 2.6]. By theorem [1, Theorem 3.2] property (C) for R implies property (C) for SR . We show the converse.

Suppose that SR has property (C). To show that R has property (C), let (x_n) be a sequence of $X_R(\mathcal{F}) = \mathcal{X}_R(\mathcal{F})$ which converges to x and, for every $n \in \mathbb{N}$, let $f_n : \mathbb{C} \setminus \mathcal{F} \rightarrow X$ be an analytic function such that $(\lambda I_X - R)f_n(\lambda) = x_n$ for all $\lambda \in \mathbb{C} \setminus \mathcal{F}$. Then

$$\begin{aligned} SR(\lambda I_X - R)f_n(\lambda) &= (\lambda SR - SR^2)f_n(\lambda) = (\lambda SR - (SR)^2)f_n(\lambda) \\ &= (\lambda I_X - SR)SRf_n(\lambda) = SRx_n, \end{aligned}$$

for all $\lambda \in \mathbb{C} \setminus \mathcal{F}$. This implies that $SRx_n \in X_{SR}(\mathcal{F})$, and since $X_{SR}(\mathcal{F})$ is closed it then follows that $SRx \in X_{SR}(\mathcal{F})$, or equivalently $\sigma_{SR}(SRx) \subseteq \mathcal{F}$. Consider the two different cases $0 \in \mathcal{F}$ and $0 \notin \mathcal{F}$. Suppose $0 \in \mathcal{F}$. By Lemma 0.2 we have $\sigma_R(R^2x) \subseteq \sigma_{SR}(SRx) \subseteq \mathcal{F}$, thus $R^2x = R(Rx) \in X_R(\mathcal{F})$ and hence, by Aiena and González [1, Lemma 2.3], we have $Rx \in X_R(\mathcal{F})$, and the same reasoning entails that $x \in X_R(\mathcal{F})$, so $X_R(\mathcal{F})$ is closed.

Finally, consider the case $0 \notin \mathcal{F}$ and again proceed as in the proof of theorem [1, Theorem 2.5], if $Z := X_R(\mathcal{F} \cup \{0\})$ then Z is closed by the first part of the proof. If $A := R|_Z$ then $X_R(\mathcal{F}) = Z_A(\mathcal{F})$. Again, if $0 \notin \sigma(A)$ then $X_R(\mathcal{F}) = Z_A(\mathcal{F}) = Z$ is closed. In the other case, $0 \in \sigma(A)$ if $\mathcal{F}_0 = \mathcal{F} \cap \sigma(A)$ then $X_R(\mathcal{F}) = Z_A(\mathcal{F}) = Z_A(\mathcal{F}_0)$ and the last subspace is closed. The equivalence S has property (C) if and only if RS has property (C) may be proved in a similar way. The proof relative to property (Q) is analogous to that of property (C). $\square$

Reference

- [1] Aiena, P., González, M.: On the Dunford property (C) for bounded linear operators RS and SR . Integr. Equ. Oper. Theory. doi:[10.1007/s00020-011-1875-2](https://doi.org/10.1007/s00020-011-1875-2)

Pietro Aiena (✉)

Dipartimento di Metodi e Modelli Matematici

Facoltà di Ingegneria, Università di Palermo

Palermo, Italy

e-mail: paiena@unipa.it

Manuel González

Departamento de Matemáticas

Facultad de Ciencias, Universidad de Cantabria

39071 Santander, Spain

e-mail: gonzalem@unican.es