Linear extensions of some Baire-one functions

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Abstract. Let X be a Hausdorff topological space, and let $\mathcal{B}_1(X)$ denote the space of all real Baire-one functions defined on X . Let A be a nonempty subset of X endowed with the topology induced from X , and let $\mathcal{F}(A)$ be the set of functions $A \rightarrow \mathbb{R}$ with a property $\mathcal{F}$ making $\mathcal{F}(A)$ a linear subspace of $\mathcal{B}_1(A)$. We give a sufficient condition for the existence of a linear extension operator $T_A: \mathcal{F}(A) \rightarrow \mathcal{F}(X)$, where $\mathcal{F}$ means *to be piecewise continuous on a sequence of closed and G_δ subsets of X* and is denoted by $\mathcal{P}_0$. We show that T_A restricted to bounded elements of $\mathcal{F}(A)$ endowed with the supremum norm is an isometry. As a consequence of our main theorem, we formulate the conclusion about existence of a linear extension operator for the classes of Baire-one-star and piecewise continuous functions.

Mathematics Subject Classification. 26A15, 54C08, 54C20, 54C30.

Keywords. Baire one function, Baire one star function, Piecewise continuous function, Extension, Zero set, Normal space, Perfectly normal space, Tietze extension theorem, Kuratowski extension theorem, Borsuk-Dugundji extension.

1. Introduction. We consider real functions. Let X , A , $\mathcal{B}_1(X)$, $\mathcal{F}(X)$, and $\mathcal{P}_0(X)$ have the same meaning as in the abstract. By $\mathcal{F}^b(A)$ we denote the subspace of $\mathcal{F}(A)$ consisting of all bounded functions. We shall consider $\mathcal{F}^b(A)$ with the norm

$$\|f\|_A = \sup_{x \in A} |f(x)|, \quad f \in \mathcal{F}^b(A).$$

This paper deals with the following extension problem which is inspired by the classical Tietze extension theorem.

Let A be a fixed (e.g., closed, G_δ , etc.) nonempty subset of a topological space X , and let $f_0 \in \mathbb{R}^A$ be a function with a certain property $\mathcal{F}$. Can f_0 be extended to a function $f \in \mathbb{R}^X$ with the same property $\mathcal{F}$?

The Tietze extension theorem is of course right for a class of continuous functions. It is well known that if X is a metric space and A is a closed subset of X , the Tietze theorem can be significantly strengthened. In 1933 Borsuk [2] proved that there is a positive linear extension operator from $\mathcal{C}(A)$ into $\mathcal{C}(X)$. In 1951 Dugundji [6] generalized Borsuk's theorem for continuous mappings into locally convex linear spaces. In 1933 Kuratowski [13] obtained a result for functions of the first class defined on G_δ -subsets of a metric space. Furthermore, in 2005 Kalenda-Spurný [11] and Shatery-Zafarani [19] extended Kuratowski's theorem to completely regular and perfectly normal spaces, respectively.

The natural problem to consider is whether an analogy to Kuratowski's theorem can hold for a given subspace of $\mathcal{B}_1(X)$. In 1990, Császár [3, Theorem 3.1] obtained a positive result in this direction: applying the method of the proof of the metric-case of the Tietze theorem, he obtained a Kuratowski-type result for the property $\mathcal{F} := C^d =$ *to be a discrete limit of a sequence of continuous functions*.

In this paper we consider a larger property than the $\mathcal{C}^d$ - the property $\mathcal{B}_1^*$ (defined in 1976 for $X = \mathbb{R}$ by O'Malley [17, p. 187] and extended in 1985 to Hausdorff spaces by Peek [18, p. 577]): for X a Hausdorff space we say that a real function f on X has property $\mathcal{B}_1^*$ if for every nonempty closed subset $H \subset X$, there is an open $U \subset X$ such that $U \cap H \neq \emptyset$ and $f|_{U \cap H}$ is continuous. Peek noticed that $\mathcal{B}_1^*$ is larger than $\mathcal{C}^d$; indeed: with the notation of Section 1,

$$\mathcal{B}_1^*(X) \supset \mathcal{C}^d(X) \quad \text{for every Hausdorff space } X. \quad (1)$$

Our main result, included in Theorem 1 in Section 4 is a general Dugundji-type theorem for property $\mathcal{P}_0$ in the class of normal spaces; it reduces to property $\mathcal{B}_1^*$ for the class of complete metric spaces (Corollary 1): we show that if A is a zero-subset of X a normal space, then there is a linear extension operator $\mathcal{P}_0(A) \rightarrow \mathcal{P}_0(X)$.

2. Preliminaries. Let A be a nonempty subset of X . The set of all *real functions with a closed graph on X* is denoted by $\mathcal{U}(X)$. It is known (cf. [8, Th. 3.6] and [20, p. 196]) that

Fact 1. For every $f \in \mathcal{U}(X)$ and every compact subset $F \subset \mathbb{R}$, the set $f^{-1}(F)$ is closed.

The symbol $\mathcal{U}^+(X)$ stands for the set of all non-negative elements of $\mathcal{U}(X)$. In 1985 Doboš [5] proved that the sum of two non-negative functions with a closed graph is a function with a closed graph. Since $0 \in \mathcal{U}^+(X)$, we have

$$\mathcal{U}^+(X) + \mathcal{U}^+(X) = \mathcal{U}^+(X), \quad (2)$$

thus $\mathcal{U}^+(X)$ is a cone in $\mathbb{R}^X$. In this paper, we use the following characterization of closedness of the graph (see [1]).

Lemma 1. *Let X be a Hausdorff topological space, and let $f: X \rightarrow \mathbb{R}$. The graph of f is closed if and only if for every $x \in X$ and every $m \in \mathbb{N}$ there is an open neighborhood V of x such that*

$$f(V) \subset (-\infty, -m) \cup \left(f(x) - \frac{1}{m}, f(x) + \frac{1}{m}\right) \cup (m, \infty). \quad (3)$$

A function $f: X \rightarrow \mathbb{R}$ is *piecewise continuous* on X if there is a sequence $(X_n(f))_n$ in X , depending on f , of nonempty closed sets such that $X = \bigcup_{n=1}^{\infty} X_n(f)$ and every restriction $f|_{X_n(f)}$ is continuous. The set of all real piecewise continuous functions on X is denoted by $\mathcal{P}(X)$.¹

According to the notation of Section 1, the set of all real piecewise continuous functions on X for which every $X_n(f)$ is additionally G_δ in X , is denoted by $\mathcal{P}_0(X)$. Obviously, $\mathcal{P}_0(X) \subset \mathcal{P}(X)$ for every X (Hausdorff) with the identity $\mathcal{P}_0(X) = \mathcal{P}(X)$ for X perfectly normal. Moreover, by the Tietze theorem,

$$\mathcal{P}(X) \subset \mathcal{B}_1(X) \quad \text{for } X \text{ normal,}^2 \quad (4)$$

and $\mathcal{B}_1^*(X) \subset \mathcal{P}_0(X)$ for X metrizable [12, Theorem 2.3]. Hence

Fact 2. If X is a complete metric space, then $\mathcal{B}_1^*(X) = \mathcal{P}(X) = \mathcal{P}_0(X)$.

3. The family $\mathcal{P}_0(X)$. In this section, we give some significant properties of the family $\mathcal{P}_0(X)$. The lemma below defines a relationship between that class and the class of functions with a closed graph.

Lemma 2. If X is a Hausdorff topological space, then $\mathcal{U}(X) \subset \mathcal{P}_0(X)$.

Proof. Let $f \in \mathcal{U}(X)$ and let $(W_n)_{n=1}^{\infty}$ be an increasing sequence of subsets of X of the form $W_n = f^{-1}([-n, n])$. Obviously $\bigcup_{n=1}^{\infty} W_n = X$ and (by Fact 1) every set W_n is closed. Thus, every restriction $f|_{W_n}$ is continuous [10, Theorem 3, p. 202]. Furthermore, every W_n is a G_δ subset of X because

$$X \setminus W_n = f^{-1}\left((-\infty, -n)\right) \cup f^{-1}\left((n, \infty)\right), \quad (5)$$

and from the identities

$$(-\infty, -n) = \bigcup_{k=1}^{\infty} \left[-n-k, -n-\frac{1}{k}\right] \text{ and } (n, \infty) = \bigcup_{k=1}^{\infty} \left[n+\frac{1}{k}, n+k\right],$$

we obtain

$$X \setminus W_n = \bigcup_{k=1}^{\infty} f^{-1}\left(\left[-n-k, -n-\frac{1}{k}\right]\right) \cup \bigcup_{k=1}^{\infty} f^{-1}\left(\left[n+\frac{1}{k}, n+k\right]\right); \quad (6)$$

now, by (5), (6), and Fact 1, $X \setminus W_n$ is F_σ in X . □

Lemma 3. Let X be a normal topological space. Then $\mathcal{P}_0(X) - \mathcal{P}_0(X) = \mathcal{P}_0(X)$. In particular, $\mathcal{P}_0(X)$ is a linear subspace of $\mathbb{R}^X$.

Proof. Let $f, g: X \rightarrow \mathbb{R}$ be two elements of $\mathcal{P}_0(X)$. Then there are sequences $(W_k)_{k=1}^{\infty}$ and $(H_j)_{j=1}^{\infty}$ of closed and G_δ sets in X such that

- $\bigcup_k W_k = X = \bigcup_j H_j$,
- the restrictions $f|_{W_k}$ and $g|_{H_j}$ are continuous.

¹ Thus, the symbol $\mathcal{P}$ can mean the property *to be continuous on a sequence of closed sets*.

² The sequence of Tietze continuous extensions $\tilde{f}|_{X_n(f)}$ is pointwise convergent to $f \in \mathcal{P}(X)$.

Thus, the intersections $W_k \cap H_j$ are closed and G_δ in X . Moreover

$$\bigcup_{j,k=1}^{\infty} (W_k \cap H_j) = \bigcup_{k=1}^{\infty} W_k \cap \bigcup_{j=1}^{\infty} H_j = X \cap X = X$$

and every restriction $(f - g)|_{(W_k \cap H_j)}$ is continuous. Thus $f - g \in \mathcal{P}_0(X)$, whence $\mathcal{P}_0(X) - \mathcal{P}_0(X) \subset \mathcal{P}_0(X)$.

On the other hand, let $f \in \mathcal{P}_0(X)$. Since $0 \in \mathcal{P}_0(X)$, $f = f - 0 \in \mathcal{P}_0(X) - \mathcal{P}_0(X)$. Therefore $\mathcal{P}_0(X) \subset \mathcal{P}_0(X) - \mathcal{P}_0(X)$. $\square$

In 2002, Borsik [1, Theorem 2] proved that if X is perfectly normal, then $\mathcal{P}(X)(= \mathcal{P}_0(X)) = \mathcal{U}^+(X) - \mathcal{U}^+(X)$. The (key) lemma below is an extended version of this result. We use it in the proof of our Theorem 1. Since the justification for this lemma is time consuming, it is located at the end of the paper. To show equality (7) below, we use some ideas from Borsik's proof. Some gaps in Borsik's justification have been completed.

Lemma 4. *Let X be a normal topological space. Then $\mathcal{U}^+(X)$ generates $\mathcal{P}_0(X)$, i.e.*

$$\mathcal{P}_0(X) = \mathcal{U}^+(X) - \mathcal{U}^+(X). \quad (7)$$

4. The main result. Our main result presented below is a solution to the extension problem for mappings from $\mathcal{P}_0(X)$.

Theorem 1. *Let X be a normal topological space, and let A be a closed and G_δ subset of X . Let also $f \in \mathcal{P}_0(A)$. Then the mapping $T_A: \mathcal{P}_0(A) \rightarrow \mathcal{P}_0(X)$ given by the formula $T_A(f) = \bar{f}$, where*

$$\bar{f}(x) = \begin{cases} f(x) & : x \in A, \\ 0 & : x \in X \setminus A \end{cases} \quad (8)$$

is a linear extension operator such that its restriction to $\mathcal{P}_0^b(A)$ is an isometry into $\mathcal{P}_0^b(X)$.

Proof. Since A is a closed and G_δ -subset of X , $A = [g = 0]$, where g is a nonnegative continuous function on X (see [7, p. 62]). Let us fix $f \in \mathcal{P}_0(A)$. By Lemma 4, there are mappings $p, q \in \mathcal{U}^+(A)$ such that $f(x) = p(x) - q(x)$ for every $x \in A$. Furthermore (see [21, Theorem 1]), p and q have closed graph extensions $\bar{p}_{(A,g)}, \bar{q}_{(A,g)}: X \rightarrow \mathbb{R}^+$ given by the formulas

$$\bar{p}_{(A,g)}(x) = \begin{cases} p(x) & : x \in A, \\ \frac{1}{g(x)} & : x \notin A \end{cases}$$

and

$$\bar{q}_{(A,g)}(x) = \begin{cases} q(x) & : x \in A, \\ \frac{1}{g(x)} & : x \notin A, \end{cases}$$

respectively. By Lemma 4, $\bar{p}_{(A,g)} - \bar{q}_{(A,g)} \in \mathcal{P}_0(X)$, and thus the formula

$$T_A(f) = \begin{cases} f(x) & : x \in A, \\ 0 & : x \notin A \end{cases} = \bar{p}_{(A,g)}(x) - \bar{q}_{(A,g)}(x)$$

defines a linear extension mapping T_A from $\mathcal{P}_0(A)$ into $\mathcal{P}_0(X)$. It is now obvious that for $f \in \mathcal{P}_0^b(A)$, $T_A(f) \in \mathcal{P}_0^b(X)$ with

$$\|T_A(f)\|_X = \|f\|_A;$$

thus $T_A|_{\mathcal{P}_0^b(A)}$ is an isometry. $\square$

From the above theorem and Fact 2, we immediately obtain the following

Corollary 1. *Let X be a complete metric space [resp. perfectly normal topological space], and let A be its nonempty closed subset. Then the mapping T_A given by (8) is a linear extension operator $\mathcal{B}_1^*(A) \rightarrow \mathcal{B}_1^*(X)$ [resp. $\mathcal{P}(A) \rightarrow \mathcal{P}(X)$].*

5. The proof of Lemma 4.

Proof. By equality (2), $\mathcal{U}^+(X)$ is a cone in $\mathbb{R}^X$. By Lemma 2, $\mathcal{U}^+(X) \subset \mathcal{P}_0(X)$. Thus, by Lemma 3, we have

$$\mathcal{U}^+(X) - \mathcal{U}^+(X) \subset \mathcal{P}_0(X) - \mathcal{P}_0(X) = \mathcal{P}_0(X). \quad (9)$$

We shall show that the inverse inclusion to (9) is true. Our argumentation is a refinement the proof of Borsuk's result [1, Theorem 1]. Gaps that appeared in it we have completed in italics.

Let $f \in \mathcal{P}_0(X)$ and let $(W_k)_{k=1}^\infty$ be an increasing sequence of closed and G_δ subsets of X such that $\bigcup_{k=1}^\infty W_k = X$ and the restriction $f|_{W_k}$ is continuous for every $k \in \mathbb{N}$.

It is obvious that we only have to consider the case $W_k \subsetneq W_{k+1}$ for all k 's (otherwise f would be continuous on X).

Set $W_0 = \emptyset$ and $E_k = W_k \setminus W_{k-1}$, $k=1,2,\dots$

Thus $E_k \neq \emptyset$ for all k 's. Notice that

$$\bigcup_{k=1}^\infty E_k = X. \quad (10)$$

Let us fix $k \in \mathbb{N}$.

Since X is normal and every W_k is G_δ and closed in X , W_k is a zero-set.

Put $g_1 \equiv 1$. For $k = 2, 3, \dots$, there are continuous functions $g_k : X \rightarrow [0, 1]$ such that

$$g_k^{-1}(0) = W_{k-1}. \quad (11)$$

Moreover, since $W_k \subsetneq W_{k+1}$,

$$g_k(x) > 0 \text{ for } x \in W_k \setminus W_{k-1} = E_k. \quad (12)$$

Let $h_k = \min\{g_1, g_2, \dots, g_k\}$. Every h_k is a non-negative continuous function such that $h_{k+1} \leq h_k$ for every $k \in \mathbb{N}$.

Furthermore, from the definition of h_k , by (11) and by the fact that the sequence $(W_k)_{k=1}^\infty$ is strictly increasing, it follows that

$$h_k^{-1}(0) = \bigcup_{i=1}^k g_i^{-1}(0) = \bigcup_{i=1}^k W_{i-1} = W_{k-1} = g_k^{-1}(0). \quad (13)$$

Combining (12) and (13) we obtain a sharp inequality for h_k :

$$h_k(x) > 0 \text{ for } x \in E_k; \quad k = 1, 2, \dots \quad (14)$$

Since all the sets E_k are pairwise disjoint, the function $t: X \rightarrow \mathbb{R}$ of the form

$$t(x) = \frac{1}{h_k(x)} \quad \text{for } x \in E_k, \quad (15)$$

is [by (14)] well-defined and strictly positive.

Set $f_1 = f^+ + t$ and $f_2 = f^- + t$, where $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. The functions f_1 and f_2 are obviously nonnegative. Furthermore $f_1 - f_2 = f$. The proof of our lemma will be completed once we show the function f_1 has a closed graph. For this purpose we shall apply Lemma 1. Let us fix $x \in X$ and $m \in \mathbb{N}$.

From (10) it follows that there is a number $k_0 \in \mathbb{N}$ such that

$$x \in E_{k_0} = W_{k_0} \setminus W_{k_0-1}. \text{ Since [see (13)] } h_{k_0+1} \text{ vanishes on } W_{k_0}, h_{k_0+1}(x) = 0.$$

Hence (and because h_{k_0+1} is continuous on X), for every $m \in \mathbb{N}$ there is a neighborhood $V_x^{(m)}$ of x such that

$$0 \leq h_{k_0+1}(y) < \frac{1}{m}, \quad (16)$$

for every $y \in V_x^{(m)}$.

Note that for x an isolated point, we may set $V_x^{(m)} = \{x\}$.

Thus let x be non-isolated. Without loss of generality we can assume that

$$V_x^{(m)} \cap W_{k_0-1} = \emptyset. \quad (17)$$

From (13) and (17) it follows that $h_{k_0}(x) > 0$.

By the continuity of h_{k_0} , there is a neighborhood $S_x^{(m)}$ of x such that

$$\left| \frac{1}{h_{k_0}(y)} - \frac{1}{h_{k_0}(x)} \right| < \frac{1}{2m}, \quad \text{for every } y \in S_x^{(m)}. \quad (18)$$

Furthermore, since f^+ is continuous on W_{k_0} , there is a neighborhood $U_x^{(m)}$ of x such that

$$|f^+(x) - f^+(y)| < \frac{1}{2m}, \quad \text{for every } y \in U_x^{(m)} \cap W_{k_0}. \quad (19)$$

Set $V(x; m) := V_x^{(m)} \cap S_x^{(m)} \cap U_x^{(m)}$. It is obviously an open neighborhood of x . We shall show that inclusion (3) holds true with $V = V(x; m)$. Let us consider two cases:

(a) $y \in E_{k_0} \cap V(x; m)$,

(b) $y \in V(x; m) \setminus E_{k_0}$.

Case (a). Let us fix $y \in E_{k_0} \cap V(x; m)$. Thus $y \in W_{k_0} \cap V(x; m)$. Furthermore, from (18) and (19) we obtain

$$\begin{aligned} |f_1(x) - f_1(y)| &= |f^+(x) + t(x) - f^+(y) - t(y)| = \left| [f^+(x) - f^+(y)] \right. \\ &\quad \left. + \left[\frac{1}{h_{k_0}(x)} - \frac{1}{h_{k_0}(y)} \right] \right| \leq |f^+(x) - f^+(y)| + \left| \frac{1}{h_{k_0}(x)} - \frac{1}{h_{k_0}(y)} \right| \\ &< \frac{1}{2m} + \frac{1}{2m} = \frac{1}{m}. \end{aligned}$$

Hence

$$f_1(x) - \frac{1}{m} < f_1(y) < f_1(x) + \frac{1}{m}, \quad \text{for every } y \in E_{k_0} \cap V(x; m). \quad (20)$$

Case (b). Note that in this case

$$y \in V(x; m) \setminus W_{k_0}. \quad (21)$$

Indeed, since $E_{k_0} = W_{k_0} \setminus W_{k_0-1}$, we have

$$y \in V(x; m) \setminus (W_{k_0} \setminus W_{k_0-1})$$

and thus

$$y \in V(x; m) \text{ and } y \notin W_{k_0} \setminus W_{k_0-1}.$$

Therefore

$$y \in V(x; m) \text{ and } (y \notin W_{k_0} \text{ or } y \in W_{k_0-1}) \quad (22)$$

From (17) and (22) we get (21).

Moreover (since $W_j \subset W_{j+1}$ for every $j \in \mathbb{N}$), we also have $y \notin W_j$ for $j \leq k_0$. Hence $y \notin E_j = W_j \setminus W_{j-1}$ for $j \leq k_0$. On the other hand [see (10)], there is a number $p > k_0$ such that $y \in E_p$. Since the sequence (h_j) is non-increasing,

$$h_p(y) \leq h_{k_0+1}(y). \quad (23)$$

Now, from (15), (23), and inequality (16), we obtain

$$f_1(y) = f^+(y) + t(y) = f^+(y) + \frac{1}{h_p(y)} \geq \frac{1}{h_{k_0+1}(y)} > m, \quad (24)$$

for every $y \in V(x, m) \setminus E_{k_0}$.

Thus we have shown [see (20) and (24)] that, for every $x \in X$ and $m \in \mathbb{N}$, there is a neighborhood $V = V(x; m)$ of x such that

$$f_1(y) \in \left(f_1(x) - \frac{1}{m}, f_1(x) + \frac{1}{m} \right) \cup (m, \infty),$$

for every $y \in V$. By Lemma 1, f_1 has a closed graph. $\square$

6. Open problems. Since we do not know if $\mathcal{P}_0(X) = \mathcal{P}(X)$ for X an arbitrary normal space, the following two problems seem to be natural.

Problem 1. Characterize the family $\mathcal{P}_{00}(X) \stackrel{\text{def}}{=} \mathcal{U}^+(X) - \mathcal{U}^+(X)$ for X a completely regular space, at least for X a normal space.

Problem 2. Solve the Extension Problem $\mathcal{P}_{00}(A) \rightarrow \mathcal{P}_{00}(X)$, for X a normal (or completely regular) space and A a closed or G_δ subset of X .

Acknowledgements. The author wishes to express his gratitude to Professor Marek Wójtcowicz for his helpful attention in preparation of this manuscript.

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Received: 30 June 2017