On g -functions for countable state subshifts

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Abstract. This note revisits the problem of finding necessary and sufficient conditions for a subshift to admit a continuous g -function. Results obtained by Krieger (in: IMS lecture notes monograph series, vol 48, pp 306–316, 2006) on finite alphabet subshifts are generalized to countable state subshifts.

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1. Introduction. Let $\mathcal{A}$ be a finite or countably infinite alphabet provided with the discrete topology, let $\mathbb{Z}_- = \{0, -1, -2, \dots\}$, and let Θ be the shift on $X = \mathcal{A}^{\mathbb{Z}_-}$. Let $\mathcal{G}$ denote the set of continuous g -functions on X . That is, $\mathcal{G}$ is the set of all continuous $g: X \rightarrow [0, 1]$ with the property that

$$\sum_{x' \in \Theta^{-1}(x)} g(x') = 1 \text{ for every } x = (\dots, x_{-1}, x_0) \in X. \quad (1.1)$$

For a subset $K \subset X$ and $g \in \mathcal{G}$, we say that K is g -invariant or invariant for g if

$$g(x) = 0 \text{ for all } x \in K^c \text{ such that } \Theta(x) \in K. \quad (1.2)$$

We consider the following problem: given a subset $K \subset X$, are there $g \in \mathcal{G}$ for which K is invariant? Gundy [6, p. 79] raises the special case of this question when $\mathcal{A} = \{0, 1\}$ and K is a subshift, that is, a closed Θ -invariant subset. Krieger [12] considers the case when $\mathcal{A}$ is finite and gives a necessary and sufficient condition for subshifts of the two-sided space $\mathcal{A}^{\mathbb{Z}}$. As in [12], we refer to this condition as *property G*. For a subshift of the one-sided space, property G can be stated as the requirement that there is no $x \in K$ such that $(x, a) \in \overline{\{x' \in K^c: \Theta(x') \in K\}}$ for every $a \in \mathcal{A}$.¹ For a nonempty subset

¹Here and throughout, K^c and $\overline{K}$ denote complement and closure, respectively.

$K \subset X$, we let K_e denote the set $\{x \in K^c : \Theta(x) \in K\}$ and we refer to points of K_e as *points of exit* from K .

The notions of g -function and g -measure have their origin in the study of a class of random processes known as *chains of infinite order* or *chains with complete connections* (see [4, 8, 15]). These processes later appeared as part of the formalism introduced by Keane [11], who used the term g -function and g -measure for transition probabilities and stationary distributions, respectively. Subsequently, a large literature on g -functions emerged (see, e.g., [1, 9, 10, 17, 18]) and the formalism was used in a variety of contexts, including wavelet analysis [2, 3, 5, 7] and statistical mechanics [13, 16, 19]. The basic questions concerning g -functions can typically be stated on sequence spaces with finite or countably infinite alphabets (cf. [9]), but it is in general a non-trivial problem to determine whether results for finite alphabets carry over to the infinite alphabet case, where compactness arguments are not available. We consider the countable alphabet version of the problem studied in [12] to be a problem worth pursuing.

The results presented in this note are organized as follows: In Section 3 we give some examples of subshifts with property G and show that if $\mathcal{A}$ is finite and $K \subset X$ is a subshift, then $\overline{K_e} \cap K = \emptyset$ if and only if K is a subshift of finite type (see Section 2 for the definition). From this result it follows that subshifts that admit strictly positive g -functions ((1.2) holds and $g(x) > 0$ for all $x \in K$) must be of finite type if $\mathcal{A}$ is finite (cf. [12]). In Section 4 we consider the case when $\mathcal{A}$ is countably infinite. We show that property G remains to be necessary and sufficient for a subshift to be invariant for some $g \in \mathcal{G}$, but that subshifts with strictly positive g -functions need not be of finite type. With $\mathcal{A}$ finite or countably infinite, a subshift $K \subset X$ admits a strictly positive g -function if and only if $\overline{K_e}$ and K do not intersect.

2. Notation. The notation in this paper follows [6]. The alphabet $\mathcal{A}$ is assumed finite or countably infinite, and X denotes the set $\mathcal{A}^{\mathbb{Z}_-}$ of all sequences $x = (\dots, x_{-1}, x_0)$ of symbols from $\mathcal{A}$. Let $\mathcal{W}$ denote the collection of all finite words over $\mathcal{A}$, and let $\mathcal{W}_n \subset \mathcal{W}$ denote the words of length n , $n \geq 1$. The word w is said to appear in $x \in X$ if $x_{[i,j]} := (x_i, \dots, x_j) = w$ for some $i \leq j \leq 0$.

The cylinders

$$C(w) := \{x \in X : x = (x', w) \text{ for some } x' \in X\}, \quad w \in \mathcal{W},$$

give a basis for the product topology (each copy of $\mathcal{A}$ is given the discrete topology) on X . We will use the fact that cylinders are clopen (that is, closed and open), that X is metrizable with

$$\rho(x, x') = \begin{cases} 0 & \text{if } x = x', \\ 2^{-l(x, x')} & \text{if } x \neq x', \quad l(x, x') = \min\{|j| : x_j \neq x'_j\}, \end{cases} \quad (2.1)$$

and that X is compact if $\mathcal{A}$ is finite. By a subshift of X , we mean a closed nonempty subset $K \subset X$ with $K = \Theta(K)$, where $\Theta(x)_j := x_{j-1}$, $j \in \mathbb{Z}_-$. Subshifts are defined by forbidden words (see [14, Ch. 1.2]). For $\mathcal{F} \subset \mathcal{W}$, we let $X_{\mathcal{F}}$ be the set of all $x \in X$ in which no word of $\mathcal{F}$ appears. We say that

$w \in \mathscr{W}$ is admissible in K if w appears in a point of K . A subshift is said to be of finite type if it can be written $X_{\mathcal{F}}$ for some finite $\mathcal{F} \subset \mathscr{W}$.

3. Exit points and property G . As in [12], we say that a subshift $K \subset X$ has a g -function if K is invariant for some $g \in \mathcal{G}$, that is, if there exists a $g \in \mathcal{G}$ with the property that $g(x) = 0$ for all $x \in K_e := \{x \in K^c : \Theta(x) \in K\}$. That K has a *strict* g -function means that K is invariant for some $g \in \mathcal{G}$ with $\{x \in X : g(x) = 0\} = \overline{K_e}$. Finally, that K has a *strictly positive* g -function means that K is invariant for some $g \in \mathcal{G}$ such that $g(x) > 0$ for all $x \in K$. Invariance of arbitrary subsets of X with respect to g -functions is defined in the same way.

Krieger [12, pp. 306–307] uses the following notation (translated from the two-sided setting) to formulate his necessary and sufficient condition for a finite alphabet subshift $K \subset X$ to have a g -function: for $x \in X, w \in \mathscr{W}$, let

$$\Gamma_{\infty}^{-}(w) = \{x' \in K : (x', w) \in K\}, \quad (3.1)$$

$$\omega_1^{+}(w) = \bigcap_{x' \in \Gamma_{\infty}^{-}(w)} \{a \in \mathcal{A} : (x', w, a) \in K\}, \quad (3.2)$$

$$\Delta_1^{+}(x) = \bigcup_{n \in \mathbb{N}} \omega_1^{+}(x_{[-n, 0]}). \quad (3.3)$$

Property G is the condition that

$$\Delta_1^{+}(x) \neq \emptyset \text{ for every } x \in X. \quad (3.4)$$

In the present notation, (3.4) says that

$$\text{there is no } x \in X \text{ with } (x, a) \in \overline{K_e} \text{ for every } a \in \mathcal{A}. \quad (3.5)$$

Remark 3.1. Assuming that $K \subset X$ is a subshift, (3.5) is equivalent to that there are no $x \in K$ with $(x, a) \in \overline{K_e}$ for every $a \in \mathcal{A}$.

Conditions (3.4) and (3.5) have formulations for general subsets of X with $\mathcal{A}$ finite or countably infinite. The proof of the following result shows that they are easily translated into each other.

Proposition 3.2. *For $K \subset X$, (3.4) is satisfied if and only if (3.5) is satisfied.*

Proof. For $x \in X, a \in \mathcal{A}$, we have $(x, a) \in \overline{K_e}$ if and only if $a \notin \Delta_1^{+}(x)$. (If $(x, a) \in \overline{K_e}$, then there is for every $n \in \mathbb{N}$ some $x' \in K$ with $x'_{[-n, 0]} = x_{[-n, 0]}$ and $(x', a) \in K^c$. Then $a \notin \omega_1^{+}(x_{[-n, 0]})$ for all $n \in \mathbb{N}$, so that $a \notin \Delta_1^{+}(x)$. Reversing the argument, we see that $a \notin \Delta_1^{+}(x)$ implies $(x, a) \in \overline{K_e}$.) So $(x, a) \in \overline{K_e}$ for all $a \in \mathcal{A}$ if and only if $\Delta_1^{+}(x) = \emptyset$. Thus (3.4) is equivalent to that there is no $x \in X$ with $(x, a) \in \overline{K_e}$ for every $a \in \mathcal{A}$. $\square$

We will see that subshifts of finite type have property G regardless of whether $\mathcal{A}$ is finite or countably infinite (see Remark 3.5; Theorem 4.2). For a subshift of $\{0, 1\}^{\mathbb{Z}}$ with property G that is not of finite type, let K be the even shift from [14, p. 6], where $x \in K$ if and only if between any two 1s there is an even number of 0s. If $(x, a) \in K_e$, then $a = 1$. So no $x \in \{0, 1\}^{\mathbb{Z}}$ can have the property that $(x, 0)$ and $(x, 1)$ are both in $\overline{K_e}$.

The even shift has the property that as w ranges over the admissible words of K , the number of predecessor sets $P(w) := \{x \in K : (x, w) \in K\}$ is finite. This property is characteristic of sofic subshifts (cf. [14, Def. 3.1.3]). We show below that if $\mathcal{A}$ is finite, then every sofic subshift has property G . An example of a subshift with property G that is not sofic is given by the context free shift from [14, p. 7]: take $\mathcal{A} = \{0, 1, 2\}$ and let $\mathcal{F} = \{w = (0, 1^k, 2^m, 0) : m \neq k\}$. That $K = X_{\mathcal{F}}$ has property G follows from that $a = 0$ if $(x, a) \in K_e$.

Proposition 3.3. *For $\mathcal{A}$ finite, every sofic subshift $K \subset X$ has property G .*

Proof. Assume that $\mathcal{A}$ is finite and let $K = X_{\mathcal{F}}$ for some $\mathcal{F} \subset \mathcal{W}$. Without loss of generality, we may assume that if $w = (w_{-n}, \dots, w_0) \in \mathcal{F}$ and $n \geq 1$, then $(w_{-n+1}, \dots, w_0)$ and $(w_{-n}, \dots, w_{-1})$ are both admissible.

Suppose that K is sofic. We show that there is a finite $\mathcal{F}' \subset \mathcal{F}$ with

$$\overline{K_e} \subset \bigcup_{w \in \mathcal{F}'} C(w). \quad (3.6)$$

For $w = (w_{-n+1}, \dots, w_0) \in \mathcal{W}$, let

$$\tilde{w} := (w_{-n+1}, \dots, w_{-1}),$$

where $\tilde{w}$ is understood to be the empty word if $n = 1$. Every point of $\Theta(K_e)$ can be written $(x, \tilde{w})$ for some $x \in K$ and $w \in \mathcal{F}$. So $\Theta(K_e) \subset \bigcup_{w \in \mathcal{F}} P(\tilde{w})$, where $P(\tilde{w}) = \{x \in K : (x, \tilde{w}) \in K\}$ is the predecessor set of the (admissible) word $\tilde{w}$. Since K is sofic, $\bigcup_{w \in \mathcal{F}''} P(\tilde{w}) = \bigcup_{w \in \mathcal{F}} P(\tilde{w})$ for some finite $\mathcal{F}'' \subset \mathcal{F}$. If we now let $\mathcal{F}' := \{(\tilde{w}, a) : w \in \mathcal{F}'', a \in \mathcal{A}\} \cap \mathcal{F}$, then $K_e \subset \bigcup_{w \in \mathcal{F}'} C(w)$. Since $\bigcup_{w \in \mathcal{F}'} C(w)$ is closed, (3.6) holds.

That (3.6) holds implies that K has property G . For suppose that there existed some $x \in K$ such that $(x, a) \in \overline{K_e}$ for every $a \in \mathcal{A}$, by (3.6), we would then have $(x, a) \in \bigcup_{w \in \mathcal{F}'} C(w)$ for every $a \in \mathcal{A}$. Take $n \geq 2$ such that the length of every word in $\mathcal{F}'$ is at most n , and let $w := x_{[-n+1, 0]}$. Since $(x, a) \in \bigcup_{w \in \mathcal{F}'} C(w)$ for every $a \in \mathcal{A}$, (w, a) contains a word of $\mathcal{F}'$ for every a . In particular, (w, a) is inadmissible for every $a \in \mathcal{A}$. But then w is inadmissible, contradicting that $x \in K$. $\square$

To a given $g \in \mathcal{G}$ we can associate two random processes, $\mathbf{x}_t, t \in \mathbb{Z}$, and $\mathbf{z}_t, t \in \mathbb{Z}$. The first process is Markov and has state space X . Conditioned on $\mathbf{x}_t$, it moves from $\mathbf{x}_t$ to $(\mathbf{x}_t, a)$ with probability $g((\mathbf{x}_t, a)), a \in \mathcal{A}$ (cf. [7, 10]). The first coordinate process $\mathbf{z}_t = (\mathbf{x}_t)_0, t \in \mathbb{Z}$, of $\mathbf{x}_t$ will typically (depending on g) be of infinite order. That is, the transition probabilities for this process depend on its entire past history. For a subshift $K \subset X$, points of K_e might be called points of exit from K : if $\mathbf{x}_t, t \in \mathbb{Z}$, leaves K at time $t \geq 1$, then we have $\mathbf{x}_t \in K^c$ and $\Theta(\mathbf{x}_t) \in K$. Paths that start in K remain in K with probability one if and only if $g(x) = 0$ for all $x \in K_e$. When $\mathcal{A} = \{0, 1\}$, the last condition is equivalent to that $g(x) = 1$ for all $x \in K$ with $(\dots, x_{-2}, x_{-1}, 1 - x_0) \in K^c$. (In [6], such points are called *barrier points* of K .) If $\mathcal{A}$ has more than two symbols, then we may have $g(x) = 0$ for all $x \in K_e$ and $g(x) < 1$ for every $x \in X$.

If $K \subset X$ is g -invariant, then $g(x) = 0$ for all $x \in \overline{K_e}$. So for K to have a strictly positive g -function, we must have $K \cap \overline{K_e} = \emptyset$. We have the following for finite alphabet subshifts (cf. [12, Corollary 2.2]).

Proposition 3.4. *For $\mathcal{A}$ finite, the following are equivalent for a subshift $K \subset X$: (i) K is of finite type. (ii) K_e is closed. (iii) $K \cap \overline{K_e} = \emptyset$.*

Remark 3.5. As the proof shows, (i) implies (ii) and (iii) also in the case when $\mathcal{A}$ is countably infinite.

Proof. Write $K = X_{\mathcal{F}}$ for some $\mathcal{F} \subset \mathcal{W}$. As in the proof of Proposition 3.3, may assume that if $w = (w_{-n+1}, \dots, w_0) \in \mathcal{F}$ is a word of length $n \geq 2$, then $(w_{-n+2}, \dots, w_0)$ and $(w_{-n+1}, \dots, w_{-1})$ are both admissible.

(i) implies (ii): We can write

$$K_e = \Theta^{-1}(K) \bigcap \left(\bigcup_{w \in \mathcal{F}} C(w) \right). \quad (3.7)$$

Here $\Theta^{-1}(K)$ is closed since K is closed and Θ is continuous. Since $C(w)$ is closed for every $w \in \mathcal{F}$, $\bigcup_{w \in \mathcal{F}} C(w)$ is closed if $\mathcal{F}$ is finite. From this and (3.7) it follows that K_e is closed if $\mathcal{F}$ is finite.

(ii) implies (iii) since $K_e \subset K^c$ by definition.

(iii) implies (i): Assume that $K \cap \overline{K_e} = \emptyset$ and suppose for contradiction that $\mathcal{F}$ is infinite. For $i > 1$, choose a word $w = (w_1, \dots, w_{n_i}) \in \mathcal{F}$ of length $n_i \geq i$. By our assumption on $\mathcal{F}$, $(w_2, \dots, w_{n_i})$ and $(w_1, \dots, w_{n_i-1})$ are both admissible. Take $x(i) \in K, i > 1$, such that $(w_1, \dots, w_{n_i-1})$ is the initial word in $x(i)$ of length $n_i - 1$. Then the initial word in $x'(i) := (x(i), w_{n_i})$ of length n_i is admissible, but the initial word of length n_i is forbidden. That is, $x'(i) \in K_e$ and $\rho(x'(i), K) \leq 2^{-n_i+1}$. Since X is compact, $x'(i)$ has a convergent subsequence $x'(i_k), k \geq 1$, and since $\rho(x'(i_k), K) \rightarrow 0$, we have $\lim_{k \rightarrow \infty} x'(i_k) \in K \cap \overline{K_e}$. $\square$

4. Strict g -functions and strictly positive g -functions. This section provides necessary and sufficient conditions for subshifts to admit strict g -functions and strictly positive g -functions, respectively. We begin by showing that property G remains to be necessary and sufficient for a subset $K \subset X$ to have a g -function if $\mathcal{A}$ is allowed to be countably infinite.

Theorem 4.1. *For $\mathcal{A}$ finite or countably infinite, the following are equivalent for a subset $K \subset X$: (i) K has property G . (ii) K has a g -function. (iii) K has a strict g -function.*

The finite alphabet part of this result was proved in [12]. Here is a slight modification of the construction in [12, p. 310]: for $x \in X$ and $a \in \mathcal{A}$, denote

$$x^{*,a} = (\dots, x_{-2}, x_{-1}, a), \quad (4.1)$$

$$n(x) = \begin{cases} \min\{n \in \mathbb{N} : C(x_{[-n,0]}) \subset (\overline{K_e})^c\} & \text{if } x \in (\overline{K_e})^c, \\ \infty & \text{if } x \in \overline{K_e}, \end{cases} \quad (4.2)$$

and let (here $1/\infty := 0$)

$$g(x) = \frac{1/n(x)}{\sum_{a \in \mathcal{A}} 1/n(x^{*,a})}. \quad (4.3)$$

Property G assures that for every $x \in X$, $n(x^{*,a})$ is finite for at least one $a \in \mathcal{A}$. Thus g is well defined on all of X . That $\mathcal{A}$ is finite makes $\sum_{a \in \mathcal{A}} 1/n(x^{*,a})$ finite for every $x \in X$, which means that $\sum_{a \in \mathcal{A}} g(x^{*,a}) = 1$ for all $x \in X$. Since $g(x) = 0$ if $x \in K_e$, K is g -invariant.

The function (4.3) is still well defined if $\mathcal{A}$ is infinite. But it will not generally be the case that (1.1) holds and there is also the issue of continuity. The construction below defines $g(x)$ in terms of $\rho(x, K_e) := \inf_{x' \in K_e} \rho(x, x')$.² The g -function is essentially a weighted sum of translates of $\rho(\cdot, K_e)$, the partial sums of which converge uniformly.

Proof of Theorem 4.1. We only show that (i) implies (iii), the other implications being obvious.

Fix a sequence $\lambda_a, a \in \mathcal{A}$, of weights with $\lambda_a > 0$ and $\sum_{j \in \mathcal{A}} \lambda_a = 1$. For $m > 1$, let $\mathcal{E}_m$ be the collection of $w \in \mathcal{W}_m$ for which $C(w)$ contains a point of $\overline{K_e}$. Given $w = (w_{-m+1}, \dots, w_{-1}, w_0) \in \mathcal{W}_m$ and $a \in \mathcal{A}$, denote

$$w^{*,a} = (w_{-m+1}, \dots, w_{-1}, a). \quad (4.4)$$

That K has property G means that we can fix m such that if $w \in \mathcal{E}_m$, then $w^{*,b} \notin \mathcal{E}_m$ for some $b = b(w) \in \mathcal{A}$. Given $w \in \mathcal{E}_m$ and $b = b(w) \in \mathcal{A}$, define g on

$$U(w) := \bigcup_{a \in \mathcal{A}} C(w^{*,a})$$

by setting

$$g(x) = \begin{cases} \lambda_{x_0} \rho(x, K_e) & \text{if } x_0 \neq b, \text{ i.e., if } x \in \bigcup_{a \in \mathcal{A} \setminus \{b\}} C(w^{*,a}), \\ 1 - \sum_{a \in \mathcal{A} \setminus \{b\}} \lambda_a \rho(x^{*,a}, K_e) & \text{if } x_0 = b, \text{ i.e., if } x \in C(w^{*,b}). \end{cases}$$

If $w, w' \in \mathcal{E}_m$, then we either have $U(w) = U(w')$ or $U(w) \cap U(w') = \emptyset$. Thus

$$U_{\mathcal{E}_m} := \bigcup_{w \in \mathcal{E}_m} U(w) \quad (4.5)$$

is a disjoint union of sets $U(w), w \in \mathcal{E}_m$, which means that g is well defined on $U_{\mathcal{E}_m}$. If x is in $(U_{\mathcal{E}_m})^c = \bigcup_{w \in \mathcal{W}_m \setminus \mathcal{E}_m} U(w)$, then so is $x^{*,a}$ for every $a \in \mathcal{A}$. So if we define $g(x) = \lambda_{x_0}$ for $x \in (U_{\mathcal{E}_m})^c$, then we have $\sum_{a \in \mathcal{A}} g(x^{*,a}) = 1$ for all $x \in X$.

It remains to verify that: (I) g is continuous, (II) $g \geq 0$, and (III) $\{x \in X : g(x) = 0\} = \overline{K_e}$.

I: That g is continuous on $C(w)$ for each w in (4.5) follows from that $\rho(x, K_e)$ is continuous and that the partial sums of $\sum_{a \in \mathcal{A} \setminus \{b\}} \lambda_a \rho(x^{*,a}, K_e)$ converge

²This technique is used in [6] to construct g -functions for a class of shifts of finite type.

uniformly. Since $C(w)$ is open for each $w \in \mathcal{E}_m$, g is continuous on $U(w)$. Since each $U(w)$ is open, g is continuous on $U_{\mathcal{E}_m}$. Since $(U_{\mathcal{E}_m})^c$ is open, g is continuous on X .

II: That $g \geq 0$ follows from that $\rho \geq 0$, $\lambda_a \geq 0$, and $\sum_{a \in \mathcal{A}} \lambda_a = 1$.

III: If $x \in \overline{K_e}$, then $x \in C(w)$ for some $w \in \mathcal{E}_m$. Since $C(w^{*,b}) \cap \overline{K_e} = \emptyset$, we must have $x \in \cup_{a \in \mathcal{A} \setminus \{b\}} C(w^{*,a})$, so that $g(x) = \lambda_{x_0} \rho(x, K_e) = 0$. Conversely, if $g(x) = 0$, then it cannot be that $g(x) = 1 - \sum_{a \in \mathcal{A} \setminus \{b\}} \lambda_a \rho(x^{*,a}, K_e)$, because $\lambda_b > 0$ and $\rho \leq 1$. Thus $g(x) = \lambda_{x_0} \rho(x, K_e)$. Since $\lambda_{x_0} > 0$, we must have $\rho(x, K_e) = 0$ and therefore $x \in \overline{K_e}$.

From Theorem 4.1 we see that a subshift $K \subset X$ with property G has a strictly positive g -function if $K \cap \overline{K_e} = \emptyset$. The latter condition implies property G and is therefore both necessary and sufficient:

Theorem 4.2. *For $\mathcal{A}$ finite or countably infinite, the following are equivalent for a subshift $K \subset X$: (i) K has a strictly positive g -function. (ii) $K \cap \overline{K_e} = \emptyset$. (iii) K_e is closed.*

Proof. (i) implies (ii) since $g(x) = 0$ for all $x \in \overline{K_e}$ if K is g -invariant.

(ii) implies (iii): Suppose that K_e is not closed. We show that the frontier of K_e lies in K , that is, $\overline{K_e} \setminus K_e \subset K$. Take $x \in \overline{K_e} \setminus K_e$ and a sequence x^n in K_e with $x^n \rightarrow x$. That x^n is in K_e means that $\Theta(x^n) \in K$. Since Θ is continuous and K is closed, $\Theta(x^n) \rightarrow \Theta(x) \in K$. Because x was selected from the complement of K_e , we must have $x \in K$.

(iii) implies (i): Assume that K_e is closed. Property G then says that there are no $x \in X$ with $(x, a) \in K_e$ for every $a \in \mathcal{A}$. Since $(x, a) \notin K_e$ if $x \in K^c$, this amounts to the condition that there are no $x \in K$ with $(x, a) \in K_e$ for every $a \in \mathcal{A}$, which is another way of saying that $K \subset \Theta(K)$. Since $K = \Theta(K)$, we can conclude that K has property G . By Theorem 4.1 there exists a $g \in \mathcal{G}$ with $\{x \in X : g(x) = 0\} = \overline{K_e}$. Because K_e is closed, $\{x \in X : g(x) = 0\} = K_e \subset K^c$. $\square$

If $\mathcal{A}$ is countably infinite, then it is still true that $K \cap \overline{K_e} = \emptyset$ if K is a subshift of finite type (see Remark 3.5). To see that the reverse implication does not hold, let K be defined by proscribing all but one of the symbols of $\mathcal{A}$. Then K is not of finite type, yet $K \cap \overline{K_e} = \emptyset$. More generally, if $K = X_{\mathcal{F}}$ where the length of every $w \in \mathcal{F}$ is less than M for some $M < \infty$ (that is, if K is a M -step shift), then the distance between K and K_e is at least 2^{-M} . If no such M exists there is a sequence x^n in K_e with $\rho(x^n, K) \rightarrow 0$, but since X is not compact, the sequence may not have a limit point. For example, if no two words of $\mathcal{F}$ have a symbol in common and $K = X_{\mathcal{F}}$, then $K \cap \overline{K_e} = \emptyset$ holds even if $\mathcal{F}$ contains arbitrarily long words.

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